Endpoint regularity of the dyadic and the fractional maximal function

Julian Weigt

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Background

For $f : \mathbb{R}^d \to \mathbb{R}$ the uncentered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}f(x) = \sup_{B \ni x} f_B \qquad \text{with} \qquad f_B = \frac{1}{\mathcal{L}(B)} \int_B |f|.$$

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The Hardy-Littlewood maximal function theorem:

$$\|\mathrm{M}f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|f\|_{L^p(\mathbb{R}^d)} \qquad \text{if and only if } p > 1$$

Juha Kinnunen (1997):

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Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}f\|_{L^1(\mathbb{R}^d)} \leq C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}?$$

Progress on classical Hardy-Littlewood

$d = 1$	[Tanaka 2002, Aldaz+Pérez Lázaro 2007]
block decreasing f	[Aldaz+Pérez Lázaro 2009]
centered M, $d = 1$	[Kurka 2015]
radial f	[Luiro 2018]
characteristic f	[W 2020]

The dyadic and the fractional maximal function

For $\alpha \in (0, d)$ define the uncentered fractional Hardy-Littlewood maximal function by

$$\mathrm{M}_\alpha f(x) = \sup_{B \ni x} r(B)^\alpha f_B,$$

where $r(B)$ is the radius of B.

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$$\|\nabla \mathrm{M}_\alpha f\|_{L^{d/(d-\alpha)}(\mathbb{R}^d)} \leq C_{d,\alpha} \|\nabla f\|_1$$

The version with $p > 1$ follows the same way as for $\alpha = 0$ in Kinnunen (1997).

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The corresponding bound is

$$\operatorname{var} \mathrm{M}^{\mathrm{d}} f \le C_d \operatorname{var} f.$$

Progress on related operators

convolution, $d = 1$	[Carneiro+Svaiter 2013]
uncentered fractional, $\alpha > 1$	[Kinnunen+Saksman 2003, Carneiro+Madrid 2017]
fractional smooth convolution	[Beltran+Ramos+Saari 2018]
fractional lacunary	[Beltran+Ramos+Saari 2018]
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related: Continuity of $f \mapsto \nabla \mathrm{M} f$ as $W^{1,1}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ (does not follow from boundedness), boundedness on other spaces, local maximal operator.

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$$\operatorname{var} \mathrm{M}^{\mathrm{d}} f = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{\mathrm{M}^{\mathrm{d}} f > \lambda\}) \, \mathrm{d}\lambda$$

1. The Dyadic Maximal Operator

By the coarea formula and $\{f > \lambda\} \subset \{\mathrm{M}^{\mathrm{d}} f > \lambda\}$ up to measure zero

$$\begin{aligned}\operatorname{var} \mathrm{M}^{\mathrm{d}} f &= \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{\mathrm{M}^{\mathrm{d}} f > \lambda\}) \,\mathrm{d}\lambda \\ &\leq \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{\mathrm{M}^{\mathrm{d}} f > \lambda\} \setminus \overline{\{f > \lambda\}}) + \mathcal{H}^{d-1}(\partial\{f > \lambda\}) \,\mathrm{d}\lambda\end{aligned}$$

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Since

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{f > \lambda\}) \,\mathrm{d}\lambda = \operatorname{var} f$$

it suffices to bound the first summand.

Definition

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$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{\mathrm{M}^{\mathrm{d}} f > \lambda\} \setminus \overline{\{f > \lambda\}}) \,\mathrm{d}\lambda$$
$$= \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \{\text{maximal } Q : f_Q > \lambda\} \setminus \overline{\{f > \lambda\}}) \,\mathrm{d}\lambda$$

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&\le \int_{\mathbb{R}} \sum_{\text{maximal } Q : f_Q > \lambda} \mathcal{H}^{d-1}(\partial Q \setminus \overline{\{f > \lambda\}})\,\mathrm{d}\lambda
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Given Q, let λ_Q be the smallest such λ.

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&= \int_{\mathbb{R}} \sum_{Q : \lambda_Q < \lambda < f_Q} \mathcal{H}^{d-1}(\partial Q \setminus \overline{\{f > \lambda\}})\,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \Big(\sum_{Q : \lambda_Q < \lambda < \tilde{\lambda}_Q} + \sum_{Q : \tilde{\lambda}_Q < \lambda < f_Q} \Big) \mathcal{H}^{d-1}(\partial Q \setminus \overline{\{f > \lambda\}})\,\mathrm{d}\lambda
\end{aligned}$$

where "$\mathcal{L}(Q \cap \{f > \tilde{\lambda}_Q\}) = 2^{-d-2}\mathcal{L}(Q)$".

High density case

$$\int_{\mathbb{R}} \sum_{Q:\lambda_Q<\lambda<\tilde{\lambda}_Q} \mathcal{H}^{d-1}(\partial Q \setminus \overline{\{f > \lambda\}})\,\mathrm{d}\lambda$$

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&\leq \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{f>\lambda\})\,\mathrm{d}\lambda \\
&= \operatorname{var} f
\end{aligned}
$$

High density case

Proposition (High density)

For $\mathcal{L}(Q \cap E) \geq \varepsilon \mathcal{L}(Q)$ we have

$$\mathcal{H}^{d-1}(\partial Q \setminus \overline{E}) \lesssim_\varepsilon \mathcal{H}^{d-1}(Q \cap \partial E).$$

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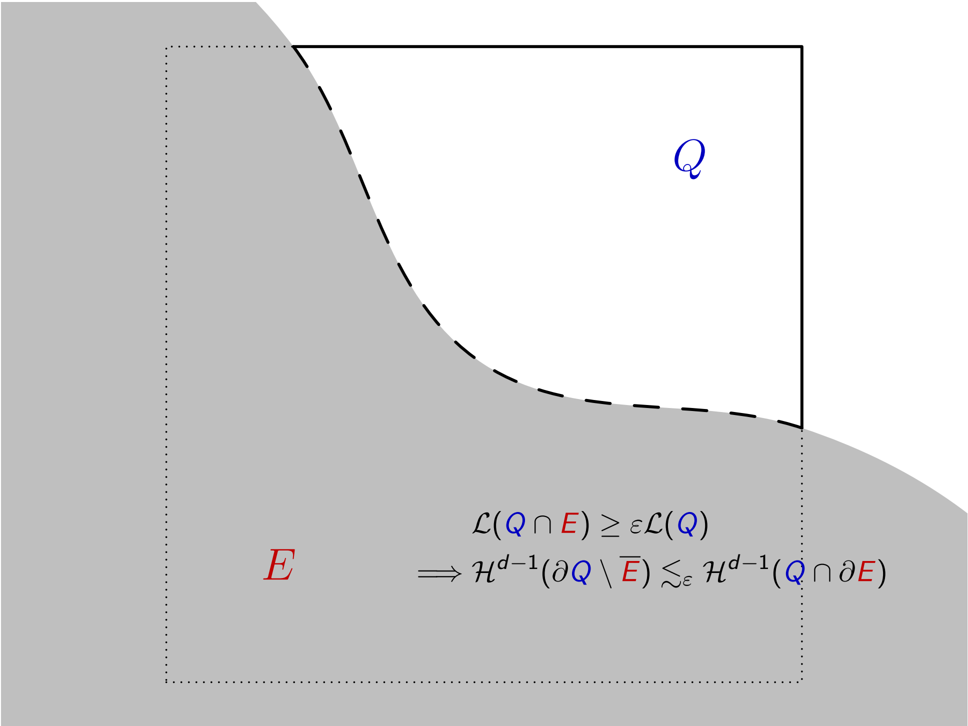

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$\mathcal{L}(Q \cap E) \geq \varepsilon \mathcal{L}(Q)$
E
$\Longrightarrow \mathcal{H}^{d-1}(\partial Q \setminus \overline{E}) \lesssim_\varepsilon \mathcal{H}^{d-1}(Q \cap \partial E)$

Low density case

Estimate

$$\int_{\mathbb{R}} \sum_{Q : \tilde{\lambda}_Q < \lambda < f_Q} \mathcal{H}^{d-1}(\partial Q \setminus \overline{\{f > \lambda\}}) \, \mathrm{d}\lambda$$

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$$= \sum_{Q}(f_Q-\tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q)$$

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$$= \sum_{Q} (f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q)$$

We have to estimate this by var f.

Relative isoperimetric inequality

For $\mathcal{L}(Q \cap E) \leq \mathcal{L}(Q)/2$ the relative isoperimetric inequality states

$$\mathcal{L}(Q \cap E)^{\frac{d-1}{d}} \lesssim \mathcal{H}^{d-1}(Q \cap \partial E)$$

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Proposition

$$(f_Q - \tilde{\lambda}_Q)\mathcal{L}(Q) \lesssim \int_{\mathbb{R}} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\, \mathrm{d}\lambda$$

where P is maximal above $\bar{\lambda}_P$ and

$$"\mathcal{L}(P \cap \{f > \bar{\lambda}_P\}) = 2^{-1}\mathcal{L}(P)"$$
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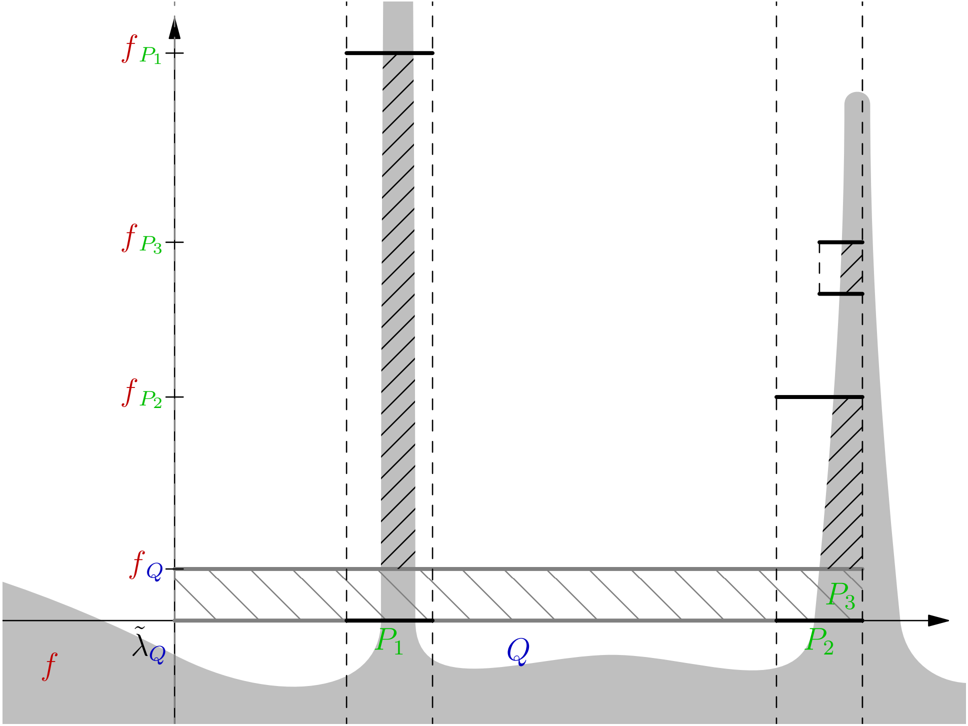

f_{P_1}
f_{P_3}
f_{P_2}
f_Q
P_3
$\tilde{\lambda}_Q$
P_1
Q
P_2
f

$$\sum_{Q}(f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q)$$

$$\lesssim \int_{\mathbb{R}} \sum_{Q} l(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathrm{d}\lambda$$

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Recall $0 < \alpha < d$ and

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Kinnunen and Saksman (2003)

For an optimal ball B for x we have

$$|\nabla \mathrm{M}_\alpha f(x)| \leq (d - \alpha) r(B)^{\alpha - 1} f_B.$$

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Conclude

$$|\nabla \mathrm{M}_\alpha f(x)| \lesssim \sup_{B \in \mathcal{B}_\alpha,\ x \in \overline{B}} r(B)^{\alpha-1} f_B =: \mathrm{M}_{\alpha,-1} f(x).$$

1. Make disjoint

$$\int (\mathrm{M}_{\alpha,-1} f)^{\frac{d}{d-\alpha}} = \int \sup_{B \in \mathcal{B}_\alpha} (r(B)^{\alpha-1} f_B)^{\frac{d}{d-\alpha}} 1_B$$

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$$\lesssim_{\alpha,c_1,c_2} \int \sum_{B \in \widetilde{\mathcal{B}}_\alpha} (r(B)^{\alpha-1} f_B)^{\frac{d}{d-\alpha}} 1_B$$

where $\widetilde{\mathcal{B}}_\alpha \subset \mathcal{B}_\alpha$ such that for two balls $B, C \in \widetilde{\mathcal{B}}$ we have $c_1 B \cap c_1 C = \emptyset$, or $r(C) < r(B)$ and $f_C > c_2 f_B$.

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$$\lesssim_\alpha \left(\sum_{Q\in\tilde{\mathcal{Q}}_\alpha} f_Q \mathcal{H}^{d-1}(\partial Q)\right)^{\frac{d}{d-\alpha}}$$

where $\mathrm{l}(Q) \sim_\alpha r(B)$ and $f_Q \sim_\alpha f_B$ so that also $c_\alpha Q \cap c_\alpha P = \emptyset$, or $\mathrm{l}(P) < \mathrm{l}(Q)$ and $f_P > 2f_Q$.

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Thank you