Variation of the uncentered maximal characteristic Function

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Abstract

Let M be the uncentered Hardy-Littlewood maximal operator or the dyadic maximal operator and $d \geq 1$. We prove that for a set $E \subset \mathbb{R}^d$ of finite perimeter the bound $\operatorname{var} \mathrm{M} 1_E \leq C_d \operatorname{var} 1_E$ holds. We also prove this for the local maximal operator.

Introduction

The uncentered Hardy-Littlewood maximal function of a nonnegative locally integrable function f is given by

$$\mathrm{M} f(x) = \sup_{B \ni x} \frac{1}{\mathcal{L}(B)} \int_B f$$

where the supremum is taken over all open balls $B \subset \mathbb{R}^d$ that contain x. Various versions of this maximal operator have been investigated. There is the (centered) Hardy-Littlewood maximal operator, where the supremum is taken only over those balls that are centered in x, or the dyadic maximal operator which maximizes over dyadic cubes instead of balls. Those operators also have local versions, where for some open set $\Omega \subset \mathbb{R}^d$ the supremum is taken only over those balls or cubes that are contained in Ω. For example the local dyadic maximal function with respect to Ω of $f \in L^1_{\mathrm{loc}}(\Omega)$ at $x \in \Omega$ is given by

$$\mathrm{M} f(x) = \sup_{x \in Q \subset \Omega} \frac{1}{\mathcal{L}(Q)} \int_Q f$$

where the supremum is taken over all half open dyadic cubes $Q \subset \mathbb{R}^d$ with $x \in Q \subset \Omega$.

It is well known that many maximal operators are bounded on $L^p(\mathbb{R}^d)$ if and only if $p > 1$. The regularity of the maximal operator was first studied in [17], where Kinnunen proved for the Hardy-Littlewood maximal operator that for $p > 1$ and $f \in W^{1,p}(\mathbb{R}^d)$ also the bound

$$\|\nabla \mathrm{M} f\|_p \leq C_{d,p} \|\nabla f\|_p$$

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holds, from which it follows that the Hardy-Littlewood maximal operator is bounded on $W^{1,p}(\mathbb{R}^d)$. The proof combines the pointwise bound $|\nabla \mathrm{M}f| \le \mathrm{M}|\nabla f|$ with the $L^p(\mathbb{R}^d)$-bound of the maximal operator. Since the maximal operator is not bounded on $L^1(\mathbb{R}^d)$, this approach fails for $p = 1$. For $p > 1$ the gradient $L^p(\mathbb{R}^d)$-bound or some corresponding version is valid for most maximal operators. However so far no counterexample has been found for $p = 1$. So in 2004, Hajłasz and Onninen posed the following question in [15]: For the Hardy-Littlewood maximal operator M, is $f \mapsto |\nabla \mathrm{M}f|$ a bounded mapping $W^{1,1}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$? This question for various maximal operators has since become a well known problem and has been the subject of lots of research. In one dimension for $L^1(\mathbb{R})$ the gradient bound has already been proven in [26] by Tanaka for the uncentered maximal function, and later in [21] by Kurka for the centered Hardy-Littlewood maximal function. The latter proof turned out to be much more complicated. In [22] Luiro has proven the gradient bound for radial functions in $L^1(\mathbb{R}^d)$ for the uncentered maximal operator. More research on this question, and also more generally on the endpoint regularity of maximal operators can be found in [1, 2, 3, 7, 8, 9, 14, 24]. However, so far the question has been essentially unsolved in dimensions larger than one for any maximal operator.

In this paper we prove that for M being the dyadic or the uncentered Hardy Littlewood maximal operator and $E \subset \mathbb{R}^d$ being a set with finite perimeter, we have

$$\operatorname{var} \mathrm{M}1_E \le C_d \operatorname{var} 1_E.$$

This answers the question of Hajłasz and Onninen in a special case, and is the first truly higher dimensional result for $p = 1$ to the best of our knowledge. We furthermore prove a localized version, as is stated in Theorems 1.2 and 1.3. The Hardy-Littlewood uncentered maximal function and the dyadic maximal function have in common, that their levels sets $\{\mathrm{M}f > \lambda\}$ can be written as the union of all balls/dyadic cubes X with $\int_X f > \lambda \mathcal{L}(X)$. Our proof relies on this. Since this is not true for the centered Hardy-Littlewood maximal function, a different approach has to be found for that maximal operator.

Also related topics for various exponents $1 \le p \le \infty$ have been studied, such as the continuity of the maximal operator in Sobolev spaces [5] and bounds for the gradient of other maximal operators, such as fractional, convolution, discrete, local and bilinear maximal operators [6, 10, 11, 16, 19, 20, 23, 25].

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1 Preliminaries and main result

We work in the setting of sets of finite perimeter, as in Evans-Gariepy [12], Section 5. For a measurable set $E \subset \mathbb{R}^d$ we denote by $\mathcal{L}(E)$ its Lebesgue measure and by $\mathcal{H}^{d-1}(E)$ its $d-1$-dimensional Hausdorff measure. For an open set $\Omega \subset \mathbb{R}^d$, a function $f \in L^1_{\mathrm{loc}}(\Omega)$ is said to have locally bounded variation if for each open and compactly supported $U \subset \Omega$ we have

$$\sup\Big\{\int_U f \operatorname{div} \varphi : \varphi \in C^1_{\mathrm{c}}(U;\mathbb{R}^d),\ |\varphi| \le 1\Big\} < \infty.$$

Such a function comes with a measure μ and a function $\nu : \Omega \to \mathbb{R}^d$ that has $|\nu| = 1$ μ-a.e. such that for all $\varphi \in C^1_{\mathrm{c}}(\Omega; \mathbb{R}^d)$ we have

$$\int_\Omega f \operatorname{div} \varphi = \int_\Omega \varphi \nu \, \mathrm{d}\mu.$$

We define the variation of f in Ω by

$$\operatorname{var}_\Omega f = \mu(\Omega).$$

For a measurable set $E \subset \mathbb{R}^d$ we define the measure theoretic boundary by

$$\partial_* E = \Big\{ x : \limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \setminus E)}{r^d} > 0, \ \limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \cap E)}{r^d} > 0 \Big\}.$$

The following is our strategy to approach the variation of the maximal function.

Lemma 1.1 (Theorem 5.9 in [12])**.** Let $\Omega \subset \mathbb{R}^d$ be open. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then

$$\operatorname{var}_\Omega f = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial_* \{ f > \lambda \} \cap \Omega) \, \mathrm{d}\lambda.$$

We say that measurable set $E \subset \mathbb{R}^d$ has locally finite perimeter if 1_E has locally bounded variation. For $f = 1_E$ we call $\operatorname{var}_\Omega 1_E$ the perimeter of E and ν from above the outer normal of E. Lemma 1.1 implies

$$\operatorname{var}_\Omega 1_E = \mathcal{H}^{d-1}(\partial_* E \cap \Omega).$$

Recall the definition of the set of dyadic cubes

$$\bigcup_{n \in \mathbb{Z}} \{ [x_1, x_1 + 2^n) \times \ldots \times [x_d, x_d + 2^n) : i = 1, \ldots, n, \ x_i \in 2^n \mathbb{Z} \}.$$

The maximal characteristic function can be written as

$$\mathrm{M} 1_E(x) = \sup_{X \ni x} \frac{\mathcal{L}(E \cap X)}{\mathcal{L}(X)},$$

where X ranges over all balls for the uncentered maximal operator, and over all dyadic cubes in Ω for the dyadic maximal operator. Now we are ready to state the main results of this paper.

Theorem 1.2. *Let* M *be the local dyadic maximal operator with respect to an open set* $\Omega \subset \mathbb{R}^d$*. Let* $E \subset \mathbb{R}^d$ *be a set with locally finite perimeter. Then*

$$\operatorname{var}_\Omega \mathrm{M} 1_E \le C_d \mathcal{H}^{d-1}(\partial_* E \cap \Omega)$$

where C_d *depends only on the dimension* d*.*

Theorem 1.3. *Let* M *be the local uncentered maximal operator with respect to an open set* $\Omega \subset \mathbb{R}^d$*. Let* $E \subset \mathbb{R}^d$ *be a set with locally finite perimeter. Then*

$$\operatorname{var}_\Omega \mathrm{M} 1_E \le C_d \mathcal{H}^{d-1}(\partial_* E \cap \Omega)$$

where C_d *depends only on the dimension* d*.*

We can take $\Omega = \mathbb{R}^d$. We denote $\{\mathrm{M}1_E > \lambda\} = \{x \in \Omega : \mathrm{M}1_E(x) > \lambda\}$. We reduce Theorems 1.2 and 1.3 to the following results.

Proposition 1.4. Let M be the local dyadic maximal operator with respect to some open set $\Omega \subset \mathbb{R}^d$. Let $E \subset \mathbb{R}$ be a set with locally finite perimeter and $\lambda \in (0,1)$. Then

$$\mathcal{H}^{d-1}(\partial_*\{\mathrm{M}1_E > \lambda\} \cap \Omega) \leq C_d \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \Omega).$$

By Lemma 2.6 we have $\overline{E}^* \cap \Omega \subset \overline{\{\mathrm{M}1_E > \lambda\}}^*$ so that we might intersect the right-hand side with $\overline{\{\mathrm{M}1_E > \lambda\}}^*$.

Proposition 1.5. Let M be the local uncentered maximal operator. Let $E \subset \mathbb{R}^d$ be a set with locally finite perimeter and $\lambda \in (0,1)$. Then

$$\mathcal{H}^{d-1}(\partial_*\{\mathrm{M}1_E > \lambda\} \cap \Omega) \leq C_d \lambda^{-\frac{d-1}{d}} (1 - \log \lambda) \mathcal{H}^{d-1}(\partial_* E \cap \{\mathrm{M}1_E > \lambda\}).$$

The constants C_d that appear in Theorems 1.2 and 1.3 and Propositions 1.4 and 1.5 are not equal. Since the proofs of Theorems 1.2 and 1.3 are almost the same we do them simultaneously.

Proofs of Theorems 1.2 and 1.3. By X we denote a ball in Ω for the uncentered maximal operator and a cube in Ω for the local dyadic maximal operator. By Lemma 1.1 and Propositions 1.4 and 1.5 we have

$$\begin{aligned}
\mathrm{var}_\Omega \mathrm{M}1_E &= \int_0^1 \mathcal{H}^{d-1}(\partial_*\{\mathrm{M}1_E > \lambda\} \cap \Omega) \,\mathrm{d}\lambda \\
&\leq C_d \int_0^1 \lambda^{-\frac{d-1}{d}} (1 - \log \lambda) \mathcal{H}^{d-1}(\partial_* E \cap \Omega) \,\mathrm{d}\lambda \\
&= d(d+1) C_d \mathcal{H}^{d-1}(\partial_* E \cap \Omega).
\end{aligned}$$

□

In Sections 2 to 4 we prove Propositions 1.4 and 1.5. In Section 5 we prove Proposition 5.1 which is Proposition 1.5 without the factor $1 - \log \lambda$. The rate $\lambda^{-\frac{d-1}{d}}$ is optimal.

We introduce some notation we will use throughout the paper. By $a \lesssim b$ we mean that there exists a constant C_d that depends only on the dimension d such that $a \leq C_d b$. For a set $\mathcal{B}$ of subsets of $\mathbb{R}^d$ we write

$$\bigcup \mathcal{B} = \bigcup_{B \in \mathcal{B}} B$$

as is commonly used in set theory. For a ball $B = B(x,r) \subset \mathbb{R}^d$ and $c > 0$ we denote $cB = B(x, cr)$. If $\mathcal{B}$ is a set of balls we denote

$$c\mathcal{B} = \{cB : B \in \mathcal{B}\}.$$

We also need more measure theoretic quantities. We define the measure theoretic interior by

$$\mathring{E}^* = \Big\{ x : \limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \setminus E)}{r^d} = 0 \Big\}$$

and the measure theoretic exterior by

$$E^{\complement} = \Big\{x : \limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\cap E)}{r^d} = 0\Big\}.$$

Then we have for the measure theoretic boundary that $\partial_* E = \mathbb{R}^d \setminus (\mathring{E}^* \cup E^{\complement})$. We further define the measure theoretic closure by

$$\overline{E}^* = \mathring{E}^* \cup \partial_* E = \Big\{x : \limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\cap E)}{r^d} > 0\Big\}.$$

Lemma 1.6. Let $A, B \subset \mathbb{R}^d$ be measurable. Then

$$\partial_*(A\cup B) \subset (\partial_* A \setminus \overline{B}^*) \cup (\partial_* B \setminus \overline{A}^*) \cup (\partial_* A \cap \partial_* B).$$

Proof. Let $x \in \partial_*(A\cup B)$. Then

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\cap (A\cup B))}{r^d} > 0,$$
$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\setminus (A\cup B))}{r^d} > 0.$$

By symmetry it suffices to consider the case that

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\cap A)}{r^d} > 0.$$

Then

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\setminus A)}{r^d} \geq \limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\setminus (A\cup B))}{r^d} > 0$$

which means $x \in \partial_* A$. Analogously, if

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\cap B)}{r^d} > 0$$

then $x \in \partial_* B$ so we get $x \in \partial_* A \cap \partial_* B$. Otherwise

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\cap B)}{r^d} = 0$$

and we can conclude $x \in \partial_* A \setminus \overline{B}^*$. □

We define the reduced boundary as the set of all points x such that for all $r > 0$ we have $\mu(B(x,r)) > 0$,

$$\lim_{r\to 0} \fint_{B(x,r)} \nu \,\mathrm{d}\mu = \nu(x),$$

and $|\nu(x)| = 1$. This is Definition 5.4 in [12]. As in the remark after Definition 5.4 we have $\partial^* E \subset \partial_* E$. By Lemma 5.5 in [12] we have that $\mathcal{H}^{d-1}$ restricted to $\partial_* E$ is equal to $\mathcal{H}^{d-1}$ restricted to $\partial^* E$. Thus it suffices to consider only the reduced boundary when estimating the perimeter of a set. But most of the time we will formulate the results for the measure theoretic boundary. The exception is Lemma 2.6, which we could only prove for the reduced boundary because there we make use of Theorem 5.13 in [12], which states the following.

Lemma 1.7 (Theorem 5.13 in [12])**.** Let $E \subset \mathbb{R}^d$ be a measurable set. Assume $0 \in \partial^* E$ with $\nu(0) = (1, 0, \dots, 0)$. Then for $r \to 0$ we have $1_{\frac{1}{r}E} \to 1_{\{x:x_1<0\}}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

A central tool used here is the relative isoperimetric inequality, see Theorem 5.11 in [12]. It states that for a ball B and any set E we have

$$\min\{\mathcal{L}(E \cap B), \mathcal{L}(B \setminus E)\}^{d-1} \lesssim \mathcal{H}^{d-1}(\partial_* E \cap B)^d. \tag{1}$$

However we need the relative isoperimetric inequality also for other sets than balls. An open bounded set A is called a John domain if there is a constant K and point $x \in A$ from which every other point $y \in A$ can be reached via a path γ such that for all t we have

$$\mathrm{dist}(\gamma(t), \partial A) \geq K^{-1}|y - \gamma(t)|. \tag{2}$$

This is called the cone condition, see Figure 2. Theorem 107 in the lecture notes [13] by Piotr Hajłasz states the following:

Lemma 1.8. Let $A \subset \mathbb{R}^d$ be a John domain with constant K. Then A satisfies a relative isoperimetric inequality with constant only depending on K, i.e.

$$\min\{\mathcal{L}(E \cap A), \mathcal{L}(A \setminus E)\}^{d-1} \lesssim_K \mathcal{H}^{d-1}(\partial_* E \cap A)^d.$$

For example a ball and a cube are John domains.

Another basic tool is the Vitali covering lemma, see for example Theorem 1.24 in [12].

Lemma 1.9 (Vitali covering lemma)**.** Let $\mathcal{B}$ be a set of balls in $\mathbb{R}^d$ with diameter bounded by some $R \in \mathbb{R}$. Then it has a countable subset $\tilde{\mathcal{B}}$ of disjoint balls such that

$$\bigcup \mathcal{B} \subset \bigcup 5\tilde{\mathcal{B}}.$$

Instead of considering $\{\mathrm{M}1_E > \lambda\}$ we will only consider a finite union of balls/cubes. In order to pass from there to the whole $\{\mathrm{M}1_E > \lambda\}$ we will use an approximation result. We say that a sequence $(A_n)_n$ of sets in $\mathbb{R}^d$ converges to some set A in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ if $(1_{A_n})_n$ converges to 1_A in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Lemma 1.10 (Theorem 5.2 in [12] for characteristic functions)**.** Let $\Omega \subset \mathbb{R}^d$ be an open set and let $(A_n)_n$ be subsets of $\mathbb{R}^d$ of locally finite perimeter that converge to A in $L^1_{\mathrm{loc}}(\Omega)$. Then

$$\mathcal{H}^{d-1}(\partial_* A \cap \Omega) \leq \liminf_{n\to\infty} \mathcal{H}^{d-1}(\partial_* A_n \cap \Omega).$$

2 Tools for both maximal operators

We start with a couple of tools that are used for both maximal operators.

Lemma 2.1. There is a constant N depending only on the dimension such that for any open ball or cube X in $\mathbb{R}^d$ and any ball C that is centered on the boundary of X and with $\mathrm{diam}\, C \lesssim \mathrm{diam}\, X$ and $L \leq \frac{1}{4}$ and $A = X \cap C \cap \{y : \mathrm{dist}(y, X^{\complement}) > L \,\mathrm{diam}\, C\}$ we have the relative isoperimetric inequality

$$\min\{\mathcal{L}(E \cap A), \mathcal{L}(A \setminus E)\}^{d-1} \leq N\mathcal{H}^{d-1}(\partial_* E \cap A)^d.$$

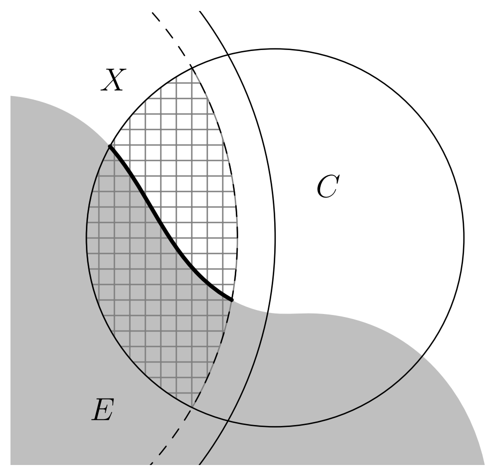

Figure 1: The regions in Lemmas 2.1 and 2.5.

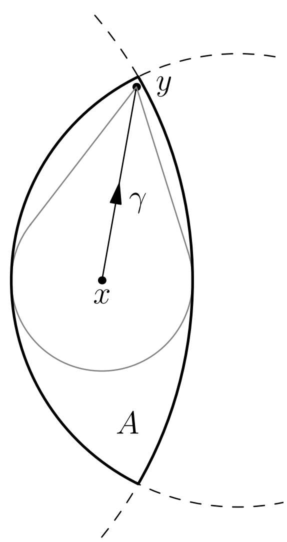

Figure 2: A in Lemma 2.1 is a John domain.

Proof. By Lemma 1.8 it suffices to show that A is a John domain. We pick as our point x the one with the largest distance to ∂A. Note that then there is a K such that

$$\frac{\sup_{y\in\partial A}\operatorname{dist}(y,x)}{\operatorname{dist}(x,\partial A)} \le K.$$

For any $y \in A$ we take our path to be the straight line between x and y. Since A is convex, for every $y \in A$ it contains the convex hull of $B(x, \operatorname{dist}(x, \partial A)) \cup \{y\}$, which implies (2), the cone condition. □

Lemma 2.2. Let $X \subset \mathbb{R}^d$ be a set with finite measure which satisfies a relative isoperimetric inequality, for example an open ball, an open cube or the set A from Lemma 2.1. Let $0 < \varepsilon < 1$. Let $E \subset \mathbb{R}^n$ be a measurable set with $\mathcal{L}(E \cap X) \le (1-\varepsilon)\mathcal{L}(X)$. Then

$$\mathcal{L}(E\cap X)^{d-1} \lesssim_\varepsilon \mathcal{H}^{d-1}(\partial_* E \cap X)^d.$$

Proof. For $\mathcal{L}(E\cap X) \le \frac{\mathcal{L}(X)}{2}$ the claim follows directly from the relative isoperimetric inequality for X. It remains to consider $\frac{\mathcal{L}(X)}{2} \le \mathcal{L}(E \cap X) \le (1-\varepsilon)\mathcal{L}(X)$. Then by the isoperimetric inequality

$$\mathcal{H}^{d-1}(\partial_* E\cap X)^d \gtrsim \mathcal{L}(X\setminus E)^{d-1} \ge \varepsilon^{d-1}\mathcal{L}(X)^{d-1} \ge \varepsilon^{d-1}\mathcal{L}(E\cap X)^{d-1}.$$

□

Eventually we only need the following consequence.

Corollary 2.3. Let $X \subset \mathbb{R}^d$ be an open ball, an open cube, or the set A from Lemma 2.1. Let $\varepsilon > 0, \lambda, E$ such that $\lambda \le \mathcal{L}(E\cap X)/\mathcal{L}(X) \le 1-\varepsilon$. Then

$$\mathcal{H}^{d-1}(\partial_* E\cap X) \gtrsim_\varepsilon \lambda^{\frac{d-1}{d}}\mathcal{H}^{d-1}(\partial X).$$

Proof. The set X satisfies the premise of Lemma 2.2 and furthermore $\mathcal{H}^{d-1}(\partial X)^d \lesssim \mathcal{L}(X)^{d-1}$. Thus

$$\mathcal{H}^{d-1}(\partial_* E\cap X) \gtrsim_\varepsilon \mathcal{L}(E\cap X)^{\frac{d-1}{d}} \ge \lambda^{\frac{d-1}{d}}\mathcal{L}(X)^{\frac{d-1}{d}} \gtrsim \lambda^{\frac{d-1}{d}}\mathcal{H}^{d-1}(\partial X).$$

□

Lemma 2.4 (Boxing inequality, c.f. Theorem 3.1 in Kinnunen, Korte, Shanmugalingam, Tuominen [18])**.** Let $E \subset \mathbb{R}^d$ be a set with finite measure that is contained in the union of a set $\mathcal{B}$ of balls B with $\mathcal{L}(E\cap B) \le \frac{\mathcal{L}(B)}{2}$. Then there is a set $\mathcal{F}$ of balls F with $\mathcal{L}(F\cap E) = \mathcal{L}(F)/2$ that covers almost all of E. Furthermore each $F \in \mathcal{F}$ is contained in a $B \in \mathcal{B}$ and each $B \in \mathcal{B}$ contains an $F \in \mathcal{F}$.

Proof. It suffices to show that for every ball $B(x_1, r_1) \in \mathcal{B}$ every Lebesgue point $x \in \mathring{E}^*$ with $x \in B(x_1, r_1)$ is contained in a ball $F \subset B(x_1, r_1)$ with $\mathcal{L}(F \cap E) = \mathcal{L}(F)/2$. By assumption

$$\mathcal{L}(E\cap B(x_1,r_1)) \le \frac{\mathcal{L}(B(x_1,r_1))}{2}$$

and since x is a Lebesgue point there is a ball $B(x_0, r_0)$ with $x \in B(x_0, r_0) \subset B(x_1, r_1)$ and

$$\mathcal{L}(E\cap B(x_0,r_0)) \ge \frac{\mathcal{L}(B(x_0,r_0))}{2}.$$

Define $x_t = (1-t)\cdot x_0 + t\cdot x_1$ and $r_t = (1-t)\cdot r_0 + t\cdot r_1$ so that $t \mapsto B(x_t, r_t)$ is a continuous transformation of balls. That means there is a t with

$$\mathcal{L}(E \cap B(x_t, r_t)) = \frac{\mathcal{L}(B(x_t, r_t))}{2}.$$

Since $x \in B(x_0, r_0) \subset B(x_t, r_t) \subset B(x_1, r_1)$ that means we have found the right ball. □

We also need a more specialized version of Lemma 2.4. It has a similar proof.

Lemma 2.5. Let X be a cube or ball in $\mathbb{R}^d$ and E a set with $\mathcal{L}(E \cap X) \geq \lambda \mathcal{L}(X)$. Then there is a cover $\mathcal{C}$ of $\partial_* X \setminus \overline{E}^*$ consisting of balls C with $\operatorname{diam} C \leq 2 \operatorname{diam} X$ and

$$\mathcal{H}^{d-1}\Big(\partial_* E \cap C \cap \Big\{y : \operatorname{dist}(y, X^\complement) > \frac{\lambda \operatorname{diam} C}{4d^{\frac{d}{2}-1}}\Big\}\Big) \gtrsim \lambda^{\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial C). \tag{3}$$

The constants in Lemma 2.5 are not optimal and one could also impose a stronger bound on the diameter of the balls $C \in \mathcal{C}$ for large λ.

Proof of Lemma 2.5. It suffices to show that for each $x \in \partial X \setminus \overline{E}^*$ there is a ball C centered in x that satisfies (3). So let $x \in \partial X \setminus \overline{E}^*$ and $0 < r \leq \operatorname{diam} X$. Then

$$\begin{aligned}
\mathcal{L}\Big(\Big\{y \in B(x,r) \cap X : \operatorname{dist}(y, X^\complement) \leq \frac{\lambda r}{2d^{\frac{d}{2}-1}}\Big\}\Big) &\leq \mathcal{L}\Big(\Big\{y \in X \cap B(x,r) : \operatorname{dist}(y, (B(x,r)\cap X)^\complement) \leq \frac{\lambda r}{2d^{\frac{d}{2}-1}}\Big\}\Big) \\
&\leq \frac{\lambda r}{2d^{\frac{d}{2}-1}} \mathcal{H}^{d-1}(\partial(B(x,r) \cap X)) \\
&\leq \frac{\lambda r}{2d^{\frac{d}{2}-1}} \mathcal{H}^{d-1}(\partial B(x,r)) \\
&= \frac{\lambda}{2d^{\frac{d}{2}}} \mathcal{L}(B(x,r)) \\
&\leq \frac{\lambda}{2} \mathcal{L}(B(x,r) \cap X).
\end{aligned} \tag{4}$$

For $r > 0$ define

$$A(r) = B(x,r) \cap \Big\{y : \operatorname{dist}(y, X^\complement) > \frac{\lambda r}{2d^{\frac{d}{2}-1}}\Big\}.$$

Then from

$$\frac{\mathcal{L}(X \cap E)}{\mathcal{L}(X)} \geq \lambda$$

and (4) with $r = \operatorname{diam} X$ we get

$$\frac{\mathcal{L}(A(\operatorname{diam} X) \cap E)}{\mathcal{L}(A(\operatorname{diam} X))} \geq \frac{\mathcal{L}(A(\operatorname{diam} X) \cap E)}{\mathcal{L}(X)} \geq \lambda - \frac{\lambda}{2} = \frac{\lambda}{2}.$$

Since $x \notin \overline{E}^*$ we have $\mathcal{L}(E \cap B(x,r))/r^d \to 0$ for $r \to 0$. That implies that in particular there is an r_0 with

$$\frac{\mathcal{L}(A(r_0) \cap E)}{\mathcal{L}(A(r_0))} \leq \frac{\lambda}{2}.$$

By continuity that means there is an $r_0 \leq r \leq \operatorname{diam} X$ such that

$$\frac{\mathcal{L}(A(r) \cap E)}{\mathcal{L}(A(r))} = \frac{\lambda}{2}.$$

Then we use Corollary 2.3 for $X = A(r)$ and $\varepsilon = \frac{1}{2}$ to get

$$\mathcal{H}^{d-1}(\partial B(x,r)) \lesssim \mathcal{H}^{d-1}(\partial A(r)) \lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap A(r)),$$

which is (3). □

Note that the following Lemma 2.6 addresses the reduced boundary $\partial^* E$ and not the measure theoretic boundary $\partial_* E$.

Lemma 2.6. Let $\Omega \subset \mathbb{R}^d$ be an open set and $E \subset \mathbb{R}^d$ be measurable. Then for both maximal operators and every $\lambda \in (0,1)$ we have $\mathring{E}^* \cap \Omega \subset \{\mathrm{M}1_E > \lambda\}$, and for the uncentered maximal operator also $\partial^* E \subset \{\mathrm{M}1_E > \lambda\}$.

Note that this is a version of $\mathrm{M}f \geq f$ almost everywhere.

Proof. Let $x \in \mathring{E}^* \cap \Omega$. Then for every $\varepsilon > 0$ there is a ball $B \subset \Omega$ with center x with $\mathcal{L}(B \setminus E) \leq \varepsilon \mathcal{L}(B)$ and a dyadic cube $x \in Q \subset B$ with $\mathcal{L}(Q) \gtrsim \mathcal{L}(B)$. That means $\mathcal{L}(Q \setminus E) \leq \varepsilon \mathcal{L}(B) \lesssim \varepsilon \mathcal{L}(Q)$.

Let $x \in \partial^* E$. It suffices to consider $x = 0$ and

$$\lim_{r \to 0} \fint_{B(0,r)} \nu_E = (1, 0, \ldots, 0).$$

Denote by B a translate of the unit ball that contains the origin and with

$$\mathcal{L}(\{y \in B : y_1 < 0\}) > \lambda \mathcal{L}(B).$$

Denote by B_r the same ball scaled by r with respect to the origin. Then by Lemma 1.7 we have

$$\lim_{r \to 0} \fint_{B_r} 1_E = \frac{\mathcal{L}(\{y \in B : y_1 < 0\})}{\mathcal{L}(B)} > \lambda \mathcal{L}(B),$$

which means $\mathrm{M}1_E(0) > \lambda$. □

3 The dyadic maximal function

In this section we discuss the argument for the dyadic maximal operator. It already showcases the main idea of the proof for the uncentered maximal operator. We have

$$\{\mathrm{M}1_E > \lambda\} = \bigcup \{\text{dyadic cube } Q : \mathcal{L}(E \cap Q) > \lambda \mathcal{L}(Q)\}.$$

The first step in the proof of Proposition 1.4 is to consider only a finite set $\mathcal{Q}$ of cubes Q with $\mathcal{L}(E \cap Q) > \lambda \mathcal{L}(Q)$ instead of the whole set, because this allows to write

$$\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{Q}) \leq \sum_{Q \in \mathcal{Q}} \mathcal{H}^{d-1}(\partial_* Q \cap \partial_* \bigcup \mathcal{Q}).$$

From there we use approximation results to extend to the union of all cubes Q with $\mathcal{L}(E \cap Q) > \lambda \mathcal{L}(Q)$. The strategy for the uncentered maximal operator is precisely the same, with cubes replaced by balls.

The main argument is Proposition 3.1, which is more or less Proposition 1.4 for the case that $\{\mathrm{M}1_E > \lambda\}$ consists of only one cube. Proposition 3.1 readily implies Proposition 1.4 because $\{\mathrm{M}1_E > \lambda\}$ is a disjoint union of such cubes. Two balls however can have nontrivial intersections, which is why the proof for the uncentered Hardy-Littlewood maximal operator is much more complicated than the proof for the dyadic maximal operator.

Proposition 3.1. Let $E \subset \mathbb{R}^d$ be measurable and Q a cube with $\mathcal{L}(E \cap Q) = \lambda \mathcal{L}(Q)$. Then

$$\mathcal{H}^{d-1}(\partial Q \setminus \overline{E}^*) \lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \mathring{Q}).$$

Proof. We apply Lemma 2.5 to $X = \mathring{Q}$ and for the resulting cover use Lemma 1.9 to extract a disjoint subcollection $\mathcal{C}$ such that $5\mathcal{C}$ still covers $\partial Q \setminus \overline{E}^*$. Then by Lemma 2.5 we have

$$\begin{aligned}\mathcal{H}^{d-1}(\partial Q \setminus \overline{E}^*) &\leq \sum_{C \in \mathcal{C}} \mathcal{H}^{d-1}(\partial 5C) \\ &\lesssim \lambda^{-\frac{d-1}{d}} \sum_{C \in \mathcal{C}} \mathcal{H}^{d-1}(\partial_* E \cap C \cap \mathring{Q}) \\ &\leq \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \mathring{Q}).\end{aligned}$$

□

Remark 3.2. For $\lambda \leq \frac{1}{2}$ Proposition 3.1 also follows directly from the relative isoperimetric inequality (1) for Q. Proposition 3.1 also holds for Q being a ball.

Proof of Proposition 1.4. For each $x \in \{\mathrm{M}1_E > \lambda\} \cap \Omega$ there is a dyadic cube $Q \subset \Omega$ with $x \in Q$ and $\mathcal{L}(E \cap Q) > \lambda \mathcal{L}(Q)$. Since there are only countably many dyadic cubes we can enumerate them $Q_1, Q_2, \ldots$. For each n let $\mathcal{Q}_n$ be the subset of maximal cubes of $Q_1, \ldots, Q_n$. We want to approximate the boundary of $\{\mathrm{M}1_E > \lambda\}$ by the boundary of $\bigcup \mathcal{Q}_n$. We have

$$\bigcup_n Q_n = \{\mathrm{M}1_E > \lambda\}$$

and by Lemma 2.6

$$\bigcup \mathcal{Q}_n \subset \bigcup \mathcal{Q}_n \cup \mathring{E}^* \subset \{\mathrm{M}1_E > \lambda\}.$$

Therefore, as E and $\mathring{E}^*$ agree up to measure zero, $\bigcup \mathcal{Q}_n \cup E$ approaches $\{\mathrm{M}1_E > \lambda\}$ in $L^1_{\mathrm{loc}}(\Omega)$. Thus by Lemma 1.10 we get

$$\mathcal{H}^{d-1}(\partial_* \{\mathrm{M}1_E > \lambda\} \cap \Omega) \leq \limsup_{n \to \infty} \mathcal{H}^{d-1}(\partial_* (\bigcup \mathcal{Q}_n \cup E) \cap \Omega).$$

Then we use that the boundary of the union of two sets is contained in the union of the boundaries of the sets, but supported away from their interiors, i.e. we apply Lemma 1.6

$$\mathcal{H}^{d-1}(\partial_* (\bigcup \mathcal{Q}_n \cup E) \cap \Omega) \leq \mathcal{H}^{d-1}((\partial_* \bigcup \mathcal{Q}_n \setminus \overline{E}^*) \cap \Omega) + \mathcal{H}^{d-1}(\partial_* E \cap \Omega). \tag{5}$$

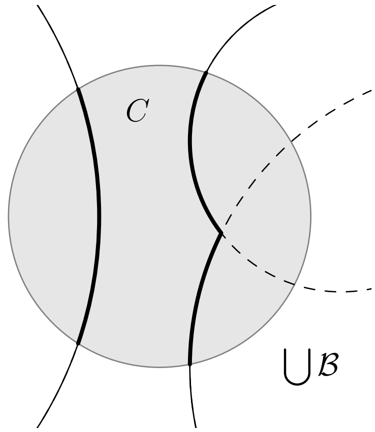

Figure 3: The objects in Lemma 4.1.

Even though this is not necessary, in the line corresponding to (5) in the proof for the uncentered Hardy-Littlewood maximal function we can actually eliminate the term $\mathcal{H}^{d-1}(\partial_* E \cap \Omega)$ thanks to Lemma 2.6; see (7) in Section 4 and the subsequent comment. Here this is not so clear because for the dyadic maximal function Lemma 2.6 is weaker. But in any case, it suffices to estimate the first term on the right hand side of (5). We invoke Proposition 3.1 and use the disjointness of the cubes in $\mathcal{Q}_n$. This implies

$$\begin{aligned}\mathcal{H}^{d-1}((\partial_* \bigcup \mathcal{Q}_n \setminus \overline{E}^*) \cap \Omega) &\leq \sum_{Q \in \mathcal{Q}_n} \mathcal{H}^{d-1}((\partial_* Q \setminus \overline{E}^*) \cap \Omega) \\ &\lesssim \sum_{Q \in \mathcal{Q}_n} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap Q) \\ &\leq \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \Omega \cap \{\mathrm{M}1_E > \lambda\}).\end{aligned}$$

□

4 The uncentered maximal function

In this section we prove Proposition 1.5. The main step is Proposition 4.4. It is Proposition 3.1 for a set $\mathcal{B}$ of finitely many balls B with $\mathcal{L}(B \cap E) > \lambda \mathcal{L}(B)$ instead of one cube.

Lemma 4.1. Let $K > 0$ and C be a ball and $\mathcal{B}$ a finite set of balls B with $\operatorname{diam}(B) \geq K \operatorname{diam}(C)$. Then

$$\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B} \cap C) \lesssim (K^{-d} + 1) \mathcal{H}^{d-1}(\partial C).$$

The rate K^{-d} does not play a role in the application. We need a short computation before we can prove Lemma 4.1.

Lemma 4.2. There is a number N large enough such that the following holds. Let $C \subset \mathbb{R}^d$ be a ball centered in the origin. Then for any two points $y_1, y_2 \in C$ and $x_1, x_2 \in \mathbb{R}^d$ with $|x_1|, |x_2| \geq$

$(N+1)\operatorname{diam}(C)/2$ and $\sphericalangle(x_1,x_2)\le \pi/4$ we have

$$\sphericalangle(y_1-x_1,y_2-x_2)\le \pi/2.$$

Proof. Since $|y_i|\le \operatorname{diam}(C)/2$ we have

$$|y_i-x_i|-\operatorname{diam}(C)/2<|x_i|<|y_i-x_i|+\operatorname{diam}(C)/2.$$

Thus for $N\to\infty$ we have

$$|y_i-x_i|/|x_i|\to 1 \tag{6}$$

uniformly. For simplicity assume $|x_1|\le|x_2|$. Then

$$\frac{\langle y_1-x_1,y_2-x_2\rangle}{|x_1||x_2|}=\frac{\langle y_1,y_2\rangle}{|x_1||x_2|}-\frac{\langle y_1,x_2/|x_2|\rangle}{|x_1|}-\frac{\langle y_2,x_1/|x_1|\rangle}{|x_2|}+\langle x_1/|x_1|,x_2/|x_2|\rangle.$$

The first three summands vanish uniformly for $N\to\infty$ and by assumption $\langle x_1/|x_1|,x_2/|x_2|\rangle\ge\cos(\pi/4)$. Thus by (6) there is an N such that

$$\frac{\langle y_1-x_1,y_2-x_2\rangle}{|y_1-x_1||y_2-x_2|}\ge\cos(\pi/2).$$

□

Proof of Lemma 4.1. It suffices to consider the case that the center of C is the origin. Take N from Lemma 4.2. First consider the case $K\ge N$. Then Lemma 4.2 says that for any two balls in $\mathcal{B}$ whose centers have an angle of at most $\pi/4$, the angles of their surface normals at any two points inside C differs by at most $\pi/2$. That means for any unit vector e we have that

$$\partial_*\bigcup\{B(x,r)\in\mathcal{B}:\sphericalangle(e,x)\le\pi/4\}\cap C$$

is a graph with Lipschitz constant 1 which thus has perimeter bounded by $\mathcal{H}^{d-1}(\partial C)$. So take a finite $\pi/4$-net of directions A. Then

$$\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B}\cap C)\le\sum_{e\in A}\mathcal{H}^{d-1}(\partial_*\bigcup\{B(x,r)\in\mathcal{B}:\sphericalangle(e,x)\le\pi/4\}\cap C)\lesssim\mathcal{H}^{d-1}(\partial C).$$

If $K<N$ then we cover C by $\lesssim\left(\frac{N}{K}\right)^d$ many balls $C_1,C_2,\dots$ so that for each i we have $\operatorname{diam}(B)\ge N\operatorname{diam}(C_i)$. Then

$$\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B}\cap C)\le\sum_i\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B}\cap C_i)\lesssim\sum_i\mathcal{H}^{d-1}(\partial C_i)\lesssim\left(\frac{N}{K}\right)^d\mathcal{H}^{d-1}(\partial C).$$

□

For a set of balls $\mathcal{B}$ we denote by $\mathcal{B}_n$ the set of those $B\in\mathcal{B}$ with $\operatorname{diam}(B)\in[\frac{1}{2},1)2^n$ and $\mathcal{B}_{>n}=\bigcup_{k>n}\mathcal{B}_k$ and $\mathcal{B}_{\ge n},\mathcal{B}_{<n},\dots$ accordingly.

Lemma 4.3. Let $E\subset\mathbb{R}^d$ be measurable and $\mathcal{B}$ be a finite set of balls B with $\mathcal{L}(E\cap B)>\lambda\mathcal{L}(B)$. Then there is a set of balls $\mathcal{C}$ such that for each $n\in\mathbb{Z}$ the following holds.

(i) The balls in $\mathcal{C}_n$ are disjoint.

(ii) $\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_{n-1}\setminus\overline{E}^*$ is covered by $5\mathcal{C}_{\leq n}$.

(iii) Each $C\in\mathcal{C}_n$ has distance at most $2\operatorname{diam}(C)$ to $\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*$

(iv) and $\mathcal{H}^{d-1}(\partial_*E\cap C\cap\{x:\operatorname{dist}(x,\bigcup\mathcal{B}^{\complement})\geq\lambda d^{1-\frac{d}{2}}2^{n-3}\})\gtrsim\lambda^{\frac{d-1}{d}}\mathcal{H}^{d-1}(\partial C)$.

Proof. Apply Lemma 2.5 to each ball in $\mathcal{B}$ and denote by $\tilde{\mathcal{C}}$ the union of all these balls. They cover $\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*$. In particular $\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_{n-1}\setminus\overline{E}^*$ is covered by $\tilde{\mathcal{C}}_{\leq n}$. Let $n\in\mathbb{Z}$. By Lemma 1.9 there is a subcollection $\mathcal{C}_n$ of $\tilde{\mathcal{C}}_n$ of disjoint balls with $\bigcup\tilde{\mathcal{C}}_n\subset\bigcup 5\mathcal{C}_n$. That means (i) and (ii) are satisfied. Now remove those balls C from $\mathcal{C}_n$ such that $5C$ does not touch $\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*$. Then (ii) still holds and we also get (iii).

Let $C\in\mathcal{C}_n$. Let $B\in\mathcal{B}$ be the ball which gave rise to C. Since $B\subset\bigcup\mathcal{B}$ we have

$$\Big\{x:\operatorname{dist}(x,\bigcup\mathcal{B}^{\complement})>\frac{\lambda\operatorname{diam}C}{4d^{\frac{d}{2}-1}}\Big\}\supset\Big\{x:\operatorname{dist}(x,B^{\complement})>\frac{\lambda\operatorname{diam}C}{4d^{\frac{d}{2}-1}}\Big\}.$$

Then we invoke Lemma 2.5 to conclude (iv). □

Proposition 4.4. Let $\lambda\in(0,1)$. Let $E\subset\mathbb{R}^d$ be a set of locally finite perimeter and let $\mathcal{B}$ be a finite set of balls such that for each $B\in\mathcal{B}$ we have $\mathcal{L}(E\cap B)>\lambda\mathcal{L}(B)$. Then

$$\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*)\lesssim\lambda^{-\frac{d-1}{d}}(1-\log\lambda)\mathcal{H}^{d-1}\Big(\partial_*E\cap\overset{\circ}{\bigcup\mathcal{B}}^{*}\Big).$$

The idea of the proof of Proposition 4.4 is that we want to split $\partial_*\bigcup\mathcal{B}$ into pieces according to how far away $\partial_*\bigcup\mathcal{B}$ is from ∂_*E, and then identify for each such piece of $\partial_*\bigcup\mathcal{B}$ a corresponding piece of ∂_*E with comparable size.

Proposition 4.4 is the most crucial result in the paper. Since

$$\{\mathrm{M}1_E>\lambda\}=\bigcup\{B:\mathcal{L}(E\cap B)>\lambda\mathcal{L}(B)\}$$

it implies Proposition 1.5 due to an approximation scheme.

Proof of Proposition 4.4. We use Lemma 4.3. We first rearrange $\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*$ and divide it according to the $(\mathcal{C}_n)_n$ in Lemma 4.3, (ii) so that afterwards we can apply Lemma 4.1. We obtain

$$\begin{aligned}\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*)&=\mathcal{H}^{d-1}\Big(\bigcup_k\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_k\setminus\overline{E}^*\Big)\\&=\mathcal{H}^{d-1}\Big(\bigcup_k\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_k\cap\bigcup_{n\leq k+1}\bigcup 5\mathcal{C}_n\Big)\\&=\mathcal{H}^{d-1}\Big(\bigcup_n\bigcup_{k\geq n-1}\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_k\cap\bigcup 5\mathcal{C}_n\Big)\\&=\mathcal{H}^{d-1}\Big(\bigcup_n\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_{\geq n-1}\cap\bigcup 5\mathcal{C}_n\Big)\\&\leq\sum_n\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_{\geq n-1}\cap\bigcup 5\mathcal{C}_n)\end{aligned}$$

$$
\begin{aligned}
&= \sum_n \sum_{C\in\mathcal{C}_n} \mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B} \cap \partial_*\bigcup\mathcal{B}_{\geq n-1} \cap 5C). \\
&\lesssim \sum_n \sum_{C\in\mathcal{C}_n} \mathcal{H}^{d-1}(\partial C).
\end{aligned}
$$

In what follows we apply first (i), then (iv) and (iii). We obtain

$$
\begin{aligned}
\sum_{C\in\mathcal{C}_n} \mathcal{H}^{d-1}(\partial C) &\lesssim \lambda^{-\frac{d-1}{d}} \sum_{C\in\mathcal{C}_n} \mathcal{H}^{d-1}(\partial_* E \cap C \cap \{x : \operatorname{dist}(x, \bigcup\mathcal{B}^{\complement}) \geq \lambda d^{1-\frac{d}{2}} 2^{n-3}\}) \\
&= \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \bigcup\mathcal{C}_n \cap \{x : \operatorname{dist}(x, \bigcup\mathcal{B}^{\complement}) \geq \lambda d^{1-\frac{d}{2}} 2^{n-3}\}) \\
&\leq \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \{x : \lambda d^{1-\frac{d}{2}} 2^{n-3} \leq \operatorname{dist}(x, \bigcup\mathcal{B}^{\complement}) \leq 2^{n+1}\}).
\end{aligned}
$$

Now we sum over n. Since for a fixed number $r \in \mathbb{R}$ $\lambda d^{1-\frac{d}{2}} 2^{n-3} \leq r \leq 5 \cdot 2^n$ can only occur for $4 + (\frac{d}{2} - 1)\log_2 d - \log_2 \lambda$ many $n \in \mathbb{Z}$ we can bound

$$
\begin{aligned}
\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B} \setminus \overline{E}^*) &\leq \lambda^{-\frac{d-1}{d}} \sum_n \mathcal{H}^{d-1}(\partial_* E \cap \{x : \lambda d^{1-\frac{d}{2}} 2^{n-3} \leq \operatorname{dist}(x, \bigcup\mathcal{B}^{\complement}) \leq 2^{n+1}\}) \\
&\lesssim \lambda^{-\frac{d-1}{d}} (1 - \log\lambda) \mathcal{H}^{d-1}(\partial_* E \cap \bigcup\mathcal{B}).
\end{aligned}
$$

□

Remark 4.5. If the balls in $\bigcup_n \mathcal{C}_n$ were disjoint then we could get rid of the factor $1 - \log\lambda|$ by using Remark 3.2 instead of (iv).

Now we extend Proposition 4.4 to the whole set $\{\mathrm{M}1_E > \lambda\}$.

Proof of Proposition 1.5. Note that

$$
\{\mathrm{M}1_E > \lambda\} = \bigcup\{B \subset \Omega : \mathcal{L}(B \cap E) > \lambda\mathcal{L}(B)\}.
$$

First we pass to a countable set of balls. By the Lindelöf property, for example Proposition 1.5 in [4], there is a sequence of balls with

$$
\{\mathrm{M}1_E > \lambda\} = B_1 \cup B_2 \cup \ldots
$$

such that for each i we have $\mathcal{L}(E \cap B_i) > \lambda\mathcal{L}(B_i)$. Denote $\mathcal{B}_n = \{B_1, \ldots, B_n\}$. Then $\bigcup\mathcal{B}_n$ converges to $\{\mathrm{M}1_E > \lambda\}$ in $L^1_{\mathrm{loc}}(\Omega)$. Furthermore by Lemma 2.6 we have

$$
\bigcup\mathcal{B}_n \subset \bigcup\mathcal{B}_n \cup \mathring{E}^* \subset \{\mathrm{M}1_E > \lambda\}
$$

which means that also $\bigcup\mathcal{B}_n \cup E$ converges to $\{\mathrm{M}1_E > \lambda\}$ in $L^1_{\mathrm{loc}}(\Omega)$. We apply this finite approximation using Lemma 1.10 and then divide the boundary using Lemma 1.6. Since E and $\mathring{E}^*$ agree up to a set of measure zero we have $\overline{(\mathring{E}^*)}^* = \overline{E}^*$ and $\partial_*(\mathring{E}^*) = \partial_* E$ so that we get

$$
\mathcal{H}^{d-1}(\partial_*\{\mathrm{M}1_E > \lambda\} \cap \Omega) \leq \limsup_{n\to\infty} \mathcal{H}^{d-1}(\partial_*(\bigcup\mathcal{B}_n \cup \mathring{E}^*) \cap \Omega)
$$

$$\leq \limsup_{n\to\infty} \mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B}_n \setminus \overline{E}^* \cap \Omega) + \mathcal{H}^{d-1}\Big(\partial_* E \setminus \Big(\bigcup^{\circ} \mathcal{B}_n\Big)^* \cap \Omega\Big). \quad (7)$$

By Lemma 2.6 the second summand is bounded by $\mathcal{H}^{d-1}(\partial_* E \cap \Omega \cap \{\mathrm{M}1_E > \lambda\})$. In fact, if $\mathcal{H}^{d-1}(\partial_* E \cap \Omega \cap \{\mathrm{M}1_E > \lambda\})$ is finite then the second summand in (7) even goes to 0 for $n \to \infty$. This is due to Lemma 2.6 for the uncentered maximal function, because

$$\Big(\bigcup^{\circ} \mathcal{B}_n\Big)^* \supset \bigcup \mathcal{B}_n$$

which is an increasing sequence that exhausts $\{\mathrm{M}1_E > \lambda\}$. In any case, it remains to estimate the first summand in (7). Note that all balls $B \in \mathcal{B}_n$ satisfy in particular $\mathcal{L}(E \cap B) > \lambda \mathcal{L}(B)$. Thus by Proposition 4.4 we have

$$\begin{aligned}\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B}_n \setminus \overline{E}^*) &\lesssim \lambda^{-\frac{d-1}{d}}(1-\log\lambda)\mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B}_n)\\ &\leq \lambda^{-\frac{d-1}{d}}(1-\log\lambda)\mathcal{H}^{d-1}(\partial_* E \cap \{\mathrm{M}1_E > \lambda\}).\end{aligned}$$

□

5 The optimal rate in λ

In this section we prove the following improvement of Proposition 1.5.

Proposition 5.1. Let M be the local uncentered maximal operator. Let $E \subset \mathbb{R}^d$ be a set with locally finite perimeter and $\lambda \in (0,1)$. Then

$$\mathcal{H}^{d-1}(\partial_* \{\mathrm{M}1_E > \lambda\} \cap \Omega) \lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \{\mathrm{M}1_E > \lambda\}).$$

More important than the statement of Proposition 5.1 is maybe the proof strategy. It may be helpful when attempting to generalize Theorem 1.3 to $\operatorname{var} \mathrm{M}f \lesssim \operatorname{var} f$ for general functions f of bounded variation.

Remark 5.2. From taking $\Omega = \mathbb{R}^d$ and $E = B(0,1)$ it follows that the rate $\lambda^{-\frac{d-1}{d}}$ in Proposition 5.1 is optimal.

In order to prove Proposition 5.1 it suffices to prove the following improvement of Proposition 4.4.

Proposition 5.3. Let $E \subset \mathbb{R}^d$ be a set of locally finite perimeter and let $\mathcal{B}$ be a finite set of balls such that for each $B \in \mathcal{B}$ we have $\lambda \mathcal{L}(B) < \mathcal{L}(E \cap B) \leq \frac{1}{2}\mathcal{L}(B)$. Then

$$\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B}) \lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B}).$$

Proof of Proposition 5.1. Let $\mathcal{B}$ be a finite set of balls B with $\mathcal{L}(B \cap E) \geq \lambda \mathcal{L}(B)$. Then

$$\begin{aligned}\mathcal{H}^{d-1}(\partial \bigcup \mathcal{B} \setminus \overline{E}^*) &\leq \mathcal{H}^{d-1}(\partial \{B \in \mathcal{B} : \mathcal{L}(B \cap E) > \frac{1}{2}\mathcal{L}(B)\} \setminus \overline{E}^*)\\ &\quad + \mathcal{H}^{d-1}(\partial \{B \in \mathcal{B} : \lambda \mathcal{L}(B) < \mathcal{L}(B \cap E) \leq \frac{1}{2}\mathcal{L}(B)\} \setminus \overline{E}^*)\end{aligned}$$

By Proposition 4.4 the first summand in the previous display is $\lesssim \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B})$ and by Proposition 5.3 the second summand is $\lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B})$. We conclude

$$\mathcal{H}^{d-1}(\partial \bigcup \mathcal{B} \setminus \overline{E}^*) \lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B}),$$

which is Proposition 4.4 without the factor $1-\log\lambda$. Thus we can repeat the proof of Proposition 1.5 verbatim without the factor $1-\log\lambda$. □

There is a weaker version of Proposition 5.3 which has a simpler proof, but already suffices to prove Proposition 5.1 for $\Omega = \mathbb{R}^d$.

Proposition 5.4. There is an $\varepsilon > 0$ depending only on the dimension such that for all $\lambda < \varepsilon$ the following holds. Let $E \subset \mathbb{R}^d$ be a set of locally finite perimeter and let $\mathcal{B}$ be a finite set of balls such that for each $B \in \mathcal{B}$ we have $\lambda\mathcal{L}(B) < \mathcal{L}(E \cap B) \le \varepsilon\mathcal{L}(B)$. Then there is a finite superset $\tilde{\mathcal{B}}$ of $\mathcal{B}$ consisting of balls B with $\mathcal{L}(E \cap B) > \lambda\mathcal{L}(B)$ that satisfies

$$\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}) \lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B}).$$

Proof of Proposition 5.1 for $\Omega = \mathbb{R}^d$. Take $\varepsilon > 0$ from Proposition 5.4. For $\lambda \ge \varepsilon$ Proposition 5.1 already follows from Proposition 1.5. It suffices to consider the case that there is an $x_0 \in \{\lambda < \mathrm{M}1_E \le \varepsilon\}$. Let $\mathrm{M}1_E(x) > \lambda$. Then there is a ball $C \ni x$ with $\mathcal{L}(E \cap C) > \lambda\mathcal{L}(C)$, while $\mathcal{L}(E \cap B(x_0, 2|x - x_0|)) \le \varepsilon\mathcal{L}(B(x_0, 2|x - x_0|))$. By continuity we can conclude that $\{\mathrm{M}1_E > \lambda\}$ is a union of balls B with $\lambda\mathcal{L}(B) < \mathcal{L}(E \cap B) < \varepsilon\mathcal{L}(B)$. Thus by the Lindelöf property there is a sequence of balls $(B_n)_n$ with $\lambda\mathcal{L}(B_n) < \mathcal{L}(E \cap B_n) < \varepsilon\mathcal{L}(B_n)$ such that $\{\mathrm{M}1_E > \lambda\} = B_1 \cup B_2 \cup \ldots$. Let $\tilde{\mathcal{B}}_n$ be the finite superset of $\mathcal{B}_n = \{B_1, \ldots, B_n\}$ from Proposition 5.4. Then

$$\bigcup \mathcal{B}_n \subset \bigcup \tilde{\mathcal{B}}_n \subset \{\mathrm{M}1_E > \lambda\}$$

which means that $\bigcup \tilde{\mathcal{B}}_n \cup \mathring{E}^*$ also converges to $\{\mathrm{M}1_E > \lambda\}$ in $L^1_{\mathrm{loc}}(\Omega)$. Thus we get as in the proof of Proposition 1.5 that

$$\mathcal{H}^{d-1}(\partial_* \{\mathrm{M}1_E > \lambda\}) \le \limsup_{n\to\infty} \mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}_n).$$

By Proposition 5.4 we have

$$\begin{aligned}\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}_n) &\lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B}_n) \\ &\le \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \{\mathrm{M}1_E > \lambda\}).\end{aligned}$$

□

5.1 The global case $\Omega = \mathbb{R}^d$

In this subsection we present a proof of Proposition 5.4. It already contains some of the ideas for the general local case Proposition 5.3.

Proof of Proposition 5.4. Restrict $\varepsilon \le \frac{1}{2}$. Let $\mathcal{F}'$ be the collection of balls from Lemma 2.4 applied to $E \cap \bigcup \mathcal{B}$ and $\mathcal{B}$. Then let $\tilde{\mathcal{F}}$ be the countable disjoint subcollection from Lemma 1.9. Extract from that a finite subcollection $\mathcal{F}$ so that for every $B \in \mathcal{B}$ we have

$$\mathcal{L}(E \cap \bigcup 5\mathcal{F} \cap B) \ge \frac{\lambda}{2}\mathcal{L}(B). \tag{8}$$

This is possible since $\mathcal{B}$ is finite. For every $F \in \mathcal{F}$ the ball $B = (2\lambda)^{-\frac{1}{d}}F$ satisfies

$$\mathcal{L}(E \cap B) \ge \mathcal{L}(E \cap F) = \frac{\mathcal{L}(F)}{2} = \lambda\mathcal{L}(B).$$

Add all those balls B to $\mathcal{B}$. Then $\mathcal{B}$ is still finite.

Here $\mathcal{F}$ serves as a decomposition of E into pieces $F \cap E$ where each piece has a substantial amount of boundary. Recall that $\mathcal{H}^{d-1}(\partial_* E \cap F) \gtrsim \mathcal{H}^{d-1}(\partial F)$. The overall goal now is to collect for each F its contribution to $\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B})$ and show that it is bounded by ∂F.

Let $r > 0$ and $F \in \mathcal{F}$ with $\operatorname{diam} F \ge 8r\lambda^{\frac{1}{d}}$. Then $(2\lambda)^{-\frac{1}{d}}F$ has diameter at least $4r$. Restrict further $\varepsilon \le 20^{-d}$, i.e. $5 \cdot 4r\varepsilon^{\frac{1}{d}} + r \le 2r$. That means any ball $B \in \mathcal{B}$ with diameter at most r that intersects $5F$ is entirely contained in $(2\lambda)^{-\frac{1}{d}}F \in \mathcal{B}$, which means we may remove B from $\mathcal{B}$ without changing $\partial_* \bigcup \mathcal{B} \setminus \overline{E}^*$. Or conversely, if $B \in \mathcal{B}$ has diameter r and $F \in \mathcal{F}$ such that $5F$ intersects B then $\operatorname{diam} F < 8r\lambda^{\frac{1}{d}}$. Thus if we further restrict $\varepsilon \le \frac{1}{2}40^{-d}$ then

$$\frac{\mathcal{L}(5F)}{\mathcal{L}(B)} < \frac{5^d 8^d r^d \varepsilon}{r^d} \le \frac{1}{2}. \tag{9}$$

Now denote by $\mathcal{B}_n$ the set of balls in $\mathcal{B}$ with $\operatorname{diam} B \in [\frac{1}{2}, 1)2^n$ and let $B \in \mathcal{B}_n$. Denote by $\mathcal{F}_n$ the set of those balls with $\operatorname{diam} F \in 2^n\lambda^{\frac{1}{d}}[4, 8)$. Let $F \in \mathcal{F}$ such that $5F$ intersects B. Then

$$F \in \mathcal{F}_k \qquad \text{for some } k \le n. \tag{10}$$

By (9) we can apply Corollary 2.3 with $X = B$ and $E = 5F$ and get

$$\begin{aligned} \mathcal{H}^{d-1}(\partial B) &\lesssim \Big(\frac{\mathcal{L}(5F \cap B)}{\mathcal{L}(B)}\Big)^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial 5F \cap B) \\ &\lesssim \Big(\frac{\mathcal{L}(5F \cap B)}{\mathcal{L}(B)}\Big)^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F). \end{aligned} \tag{11}$$

Since any F such that $5F$ intersects B is contained in $2B$, we get from (8) that

$$\frac{\lambda}{2}\mathcal{L}(B) \le \sum_{F \subset 2B} \mathcal{L}(5F \cap B).$$

We rewrite the last display as

$$\mathcal{H}^{d-1}(\partial B) \le 2 \sum_{F \subset 2B} \frac{\mathcal{L}(5F \cap B)}{\lambda\mathcal{L}(B)} \mathcal{H}^{d-1}(\partial B).$$

We apply (11) on the right-hand side and remember (10), i.e. that $F \subset \bigcup_{k\leq n} \mathcal{F}_k$. So we get

$$\begin{aligned}
\mathcal{H}^{d-1}(\partial B) &\lesssim \sum_{F\subset 2B} \Big(\frac{\mathcal{L}(5F\cap B)}{\lambda\mathcal{L}(B)}\Big)^{\frac{1}{d}} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F) \\
&\lesssim \sum_{k\leq n} \sum_{F\in\mathcal{F}_k, F\subset 2B} \Big(\frac{\mathcal{L}(5F\cap B)}{\lambda\mathcal{L}(B)}\Big)^{\frac{1}{d}} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F) \\
&\leq \sum_{k\leq n} \sum_{F\in\mathcal{F}_k, F\subset 2B} \Big(\frac{\mathcal{L}(5F)}{\lambda\mathcal{L}(B)}\Big)^{\frac{1}{d}} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F) \\
&\lesssim \sum_{k\leq n} \sum_{F\in\mathcal{F}_k, F\subset 2B} 2^{k-n} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F). \qquad (12)
\end{aligned}$$

This estimate can be seen as a way to distribute $\mathcal{H}^{d-1}(\partial B)$ over the balls F that it contains. The next step will be to turn the dependence around, and see for a fixed F, for how much variation of $\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B})$ it is responsible.

Since $\mathcal{B}_n$ is finite we have

$$\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B}_n) = \sum_{B\in\mathcal{B}_n} \mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \mathcal{B}_n).$$

We again multiply each summand by a number bounded from below according to (12).

$$\begin{aligned}
&\sum_{B\in\mathcal{B}_n} \mathcal{H}^{d-1}(\partial B\cap \partial_* \bigcup \mathcal{B}_n) \\
&\lesssim \sum_{B\in\mathcal{B}_n} \frac{\mathcal{H}^{d-1}(\partial B\cap\partial_*\bigcup\mathcal{B}_n)}{\mathcal{H}^{d-1}(\partial B)} \sum_{k\leq n} \sum_{F\in\mathcal{F}_k, F\subset 2B} 2^{k-n}\lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F) \\
&= \lambda^{-\frac{d-1}{d}} \sum_{k\leq n} 2^{k-n} \sum_{F\in\mathcal{F}_k} \mathcal{H}^{d-1}(\partial F) \sum_{B\in\mathcal{B}_n, 2B\supset F} \frac{\mathcal{H}^{d-1}(\partial B\cap\partial_*\bigcup\mathcal{B}_n)}{\mathcal{H}^{d-1}(\partial B)}.
\end{aligned}$$

Now we have reorganized $\partial_* \bigcup \mathcal{B}_n$ according to the $F \in \mathcal{F}$. We want to bound the contribution of each F uniformly. For each $F \in \mathcal{F}_k$ for which there is a $B \in \mathcal{B}_n$ with $F \subset 2B$, denote by B_F a largest such B. Then for all $B \in \mathcal{B}_n$ with $F \subset 2B$ have $B \subset 3B_F$. Thus

$$\sum_{B\in\mathcal{B}_n, 2B\supset F} \frac{\mathcal{H}^{d-1}(\partial B\cap\partial_*\bigcup\mathcal{B}_n)}{\mathcal{H}^{d-1}(\partial B)} \lesssim \sum_{B\in\mathcal{B}_n, B\subset 3B_F} \frac{\mathcal{H}^{d-1}(\partial B\cap\partial_*\bigcup\mathcal{B}_n)}{\mathcal{H}^{d-1}(\partial B_F)},$$

which is uniformly bounded according to Lemma 4.1. Therefore we can conclude

$$\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B}_n) \lesssim \lambda^{-\frac{d-1}{d}} \sum_{k\leq n} 2^{k-n} \sum_{F\in\mathcal{F}_k} \mathcal{H}^{d-1}(\partial F).$$

The interaction between the scales is small enough so that we can just sum over all scales and obtain

$$\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B}) \leq \sum_n \mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B}_n)$$

$$\begin{aligned}
&\lesssim \lambda^{-\frac{d-1}{d}} \sum_k \sum_{n\geq k} 2^{k-n} \sum_{F\in\mathcal{F}_k} \mathcal{H}^{d-1}(\partial F)\\
&\lesssim \lambda^{-\frac{d-1}{d}} \sum_k \sum_{F\in\mathcal{F}_k} \mathcal{H}^{d-1}(\partial F)\\
&= \lambda^{-\frac{d-1}{d}} \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial F).
\end{aligned}$$

Now we use Lemma 2.4 and the isoperimetric inequality (1) to get back from $\mathcal{F}$ to E. Recall that $\mathcal{H}^{d-1}(\partial F) \lesssim \mathcal{H}^{d-1}(\partial_* E \cap F)$ and that the balls in $\mathcal{F}$ are disjoint. Hence we can conclude

$$\begin{aligned}
\mathcal{H}^{d-1}(\partial_* \bigcup \mathcal{B}) &\lesssim \lambda^{-\frac{d-1}{d}} \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial_* E \cap F)\\
&\leq \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B}).
\end{aligned}$$

□

5.2 The general local case $\Omega \subset \mathbb{R}^d$

In this subsection we present a proof of Proposition 5.3. It requires a few more steps than the proof of Proposition 5.4.

Lemma 5.5. Let $\lambda \leq 2^{-\frac{d+1}{2}} d^{-\frac{3}{2}}$ and B, C be balls with $\operatorname{diam} C \geq \operatorname{diam} B$ and $\mathcal{L}(B \cap C) \leq \lambda \mathcal{L}(B)$. Then $(1 - 2d^{\frac{3}{d+1}} \lambda^{\frac{2}{d+1}})B$ and C are disjoint.

For the application we only need that for λ small enough B and $(3/4)^{\frac{1}{d}} C$ are disjoint. Since $\operatorname{diam} C \geq \operatorname{diam} B$ this follows if $(3/4)^{\frac{1}{d}} B$ and C are disjoint. The rate in λ also plays no role.

Proof. We first do some calculations. Let σ_d be the measure of the d dimensional unit ball. We have

$$\frac{\sigma_d}{\sigma_{d-1}} = \pi^{\frac{1}{2}} \frac{\Gamma(d/2+1)}{\Gamma(d/2+1/2)}.$$

By Stirling's formula it holds for all $x \geq 1$ that

$$\Gamma(1+x) \in [\sqrt{2\pi}, e] x^{x+\frac{1}{2}} e^{-x}.$$

Thus for $d \geq 3$ we have

$$\begin{aligned}
\frac{\sigma_d}{\sigma_{d-1}} &\leq \frac{e}{\sqrt{2\pi}} \frac{(d/2)^{(1+d)/2}}{((d-1)/2)^{d/2}} e^{-\frac{1}{2}}\\
&= \frac{e^{\frac{1}{2}}}{\sqrt{2\pi}} \left(1 + \frac{1}{d-1}\right)^{d/2} \left(\frac{d}{2}\right)^{\frac{1}{2}}\\
&\leq \frac{e^{\frac{1}{2}}}{\sqrt{2\pi}} e^{\frac{1}{2}} \left(1 + \frac{1}{d-1}\right)^{1/2} \left(\frac{d}{2}\right)^{\frac{1}{2}}\\
&= \frac{e}{2\sqrt{\pi}} \left(d + 1 + \frac{1}{d-1}\right)^{1/2}
\end{aligned}$$

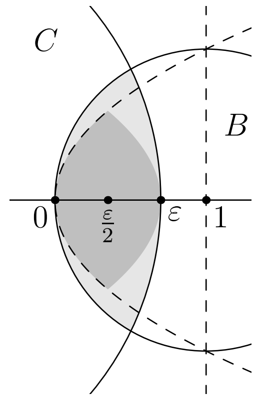

Figure 4: The lower bound for $\mathcal{L}(B \cap C)$ in the proof of Lemma 5.5.

$$\begin{aligned} &\le \frac{e}{2\sqrt{\pi}}\left(1+\frac{1}{3}+\frac{1}{6}\right)^{1/2} d^{\frac{1}{2}} \\ &\le d^{\frac{1}{2}}. \end{aligned} \tag{13}$$

After rescaling, rotation and translation it suffices to consider the case that there are $r \ge 1, 0 < \varepsilon \le 2$ such that $B = B(e_1, 1)$ and $C = B((\varepsilon - r)e_1, r)$. We bound $\mathcal{L}(B \cap C)$ from below by the marked area in Figure 4. For $x \in \mathbb{R}^d$ denote $\bar{x}_1 = (x_2, \dots, x_d)$. The two spheres ∂B and ∂C intersect in a plane orthogonal to e_1 that is between $\frac{\varepsilon}{2}e_1$ and εe_1. Thus

$$\left\{x : \bar{x}_1^2 < x_1 < \frac{\varepsilon}{2}\right\} \subset \left\{x \in B : x_1 < \frac{\varepsilon}{2}\right\} \subset B \cap C$$

and by symmetry and $r \ge 1$ also the image of the first set mirrored at $x_1 = \frac{\varepsilon}{2}$ is containd in $B \cap C$, so that

$$\begin{aligned} \mathcal{L}(B \cap C) &> 2\mathcal{L}\left(\left\{x : \bar{x}_1^2 < x_1 < \frac{\varepsilon}{2}\right\}\right) \\ &= 2\int_0^{\frac{\varepsilon}{2}} \sigma_{d-1} h^{\frac{d-1}{2}} \,\mathrm{d}h \\ &= 2^{-\frac{d-3}{2}} \frac{\sigma_{d-1}}{d+1} \varepsilon^{\frac{d+1}{2}}, \end{aligned}$$

Therefore since $\mathcal{L}(B \cap C) \le \lambda \mathcal{L}(B) = \lambda \sigma_d$ we can conclude the following upper bound for ε using (13).

$$\begin{aligned} \varepsilon^{\frac{d+1}{2}} &\le \frac{\lambda(d+1)\sigma_d}{\sigma_{d-1}} 2^{\frac{d-3}{2}} \\ &\le 2^{\frac{d+1}{2}} \lambda \frac{(d+1)d^{\frac{1}{2}}}{4} \\ &\le 2^{\frac{d+1}{2}} \lambda d^{\frac{3}{2}}, \\ \varepsilon &\le 2d^{\frac{3}{d+1}} \lambda^{\frac{2}{d+1}}. \end{aligned}$$

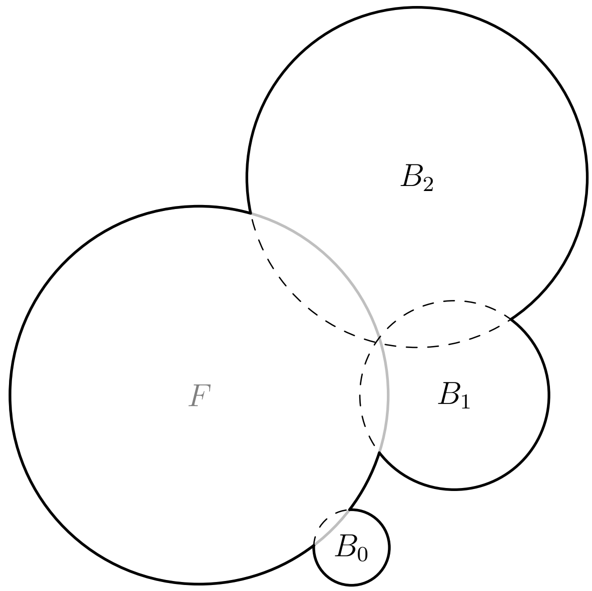

Figure 5: The objects in Lemma 5.7.

One can check this bound also for $d = 1, 2$. This finishes the proof because $(1 - \varepsilon)B$ and C are disjoint. ☐

Lemma 5.6. Let B be a ball and $\mathcal{F}$ be a set of balls F with $\mathcal{L}(\bigcup \mathcal{F} \cap B) \geq \lambda \mathcal{L}(B)$. Then there is a ball $F \in \mathcal{F}$ that intersects $(1 - \lambda/d)B$.

Proof. Since

$$\begin{aligned}\mathcal{L}(B \setminus (1 - \lambda/d)B) &= d\mathcal{L}(B) \int_{1-\lambda/d}^{1} r^{d-1} \, \mathrm{d}r \\ &< \lambda \mathcal{L}(B)\end{aligned}$$

$\bigcup \mathcal{F}$ cannot lie outside of $(1 - \lambda/d)B$. ☐

Lemma 5.7. Let $\lambda > 0$ and let F be a ball and $\mathcal{B}$ a finite set of balls B with $\mathcal{L}(B \cap F) \geq \lambda \mathcal{L}(B)$. Then

$$\mathcal{H}^{d-1}(\partial(F \cup \bigcup \mathcal{B})) \lesssim (1 - \log \lambda)\lambda^{-2+\frac{3}{d+1}} \mathcal{H}^{d-1}(\partial F).$$

The rate in λ plays no role for the application and is probably also not optimal.

Proof. After translation and scaling it suffices to consider $F = B(0, 1)$. We split $\mathcal{B}$ into

$$\begin{aligned}\mathcal{B}^{0,-} &= \Big\{B(x, r) \in \mathcal{B} : r \leq \frac{1}{2},\ |x| \leq 1 - \frac{r}{2}\Big\}, \\ \mathcal{B}^{0,+} &= \Big\{B(x, r) \in \mathcal{B} : r \leq \frac{1}{2},\ |x| > 1 - \frac{r}{2}\Big\},\end{aligned}$$

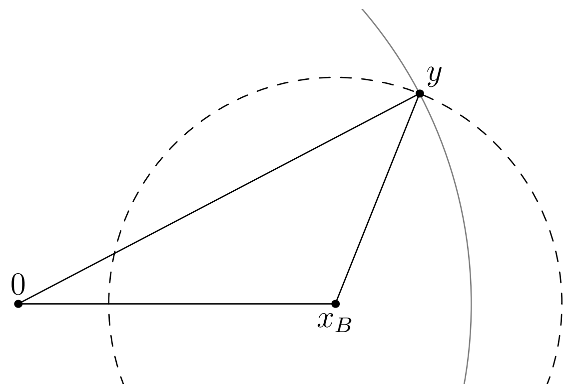

Figure 6: The case $B \in \mathcal{B}^{0,-}$.

$$\mathcal{B}^1 = \Big\{ B(x,r) \in \mathcal{B} : r > \frac{1}{2} \Big\}.$$

It suffices to bound the perimeter of each component separately.

First consider $\mathcal{B}^1$. For each $n \geq 1$ take a ball $B_n \in \mathcal{B}^1$ with $\operatorname{diam} B_n \in [\frac{1}{2}, 1)2^n$, if one exists. The largest such n is bounded by $\lceil 1 - \log_2 \lambda/d \rceil$. For each such $n \geq 1$ all balls $B \in \mathcal{B}^1$ with $\operatorname{diam} B \in [\frac{1}{2}, 1)2^n$ are contained in $8B_n$. Thus by Lemma 4.1 we have

$$\begin{aligned}
\mathcal{H}^{d-1}(\partial \bigcup \mathcal{B}^1) &\leq \sum_{n=1}^{\lceil 1-\log_2 \lambda/d \rceil} \mathcal{H}^{d-1}\Big(\partial \bigcup \Big\{ B \in \mathcal{B}^1 : \operatorname{diam} B \in \Big[\frac{1}{2}, 1\Big) 2^n \Big\}\Big) \\
&\lesssim \sum_{n=1}^{\lceil 1-\log_2 \lambda/d \rceil} \mathcal{H}^{d-1}(\partial 8B_n) \\
&\lesssim \sum_{n=1}^{\lceil 1-\log_2 \lambda/d \rceil} 2^{(d-1)n} \\
&\lesssim \lambda^{-\frac{d-1}{d}} \\
&\lesssim (1 - \log \lambda) \lambda^{-2+\frac{3}{d+1}},
\end{aligned}$$

and we are done with $\mathcal{B}^1$.

For $B \in \mathcal{B}^{0,-} \cup \mathcal{B}^{0,+}$ denote by x_B the center of B.

Claim. Let $B \in \mathcal{B}^{0,-}$ and $y \in \partial B \setminus B(0,1)$. Then $\sphericalangle(y, y - x_B) \leq \frac{\pi}{3}$.

Proof. Denote $B = B(x_B, r)$. Clearly $\sphericalangle(y, y - x_B)$ increases the closer y is to $\partial B(0,1)$. Thus it suffices to consider $y \in \partial B \cap \partial B(0,1)$, and $\sphericalangle(y, y - x_B)$ does not depend on the choice of y, but only on $|x_B|$ and r. Consider the triangle with endpoints $0, y, x_B$. It has sidelengths $1, r, |x_B|$ with $r \leq \frac{1}{2}$ and $1 - r \leq |x_B| \leq 1 - \frac{r}{2}$. So by the law of cosines

$$\cos \sphericalangle(y, y - x_B) = \frac{1 + r^2 - |x_B|^2}{2r}$$

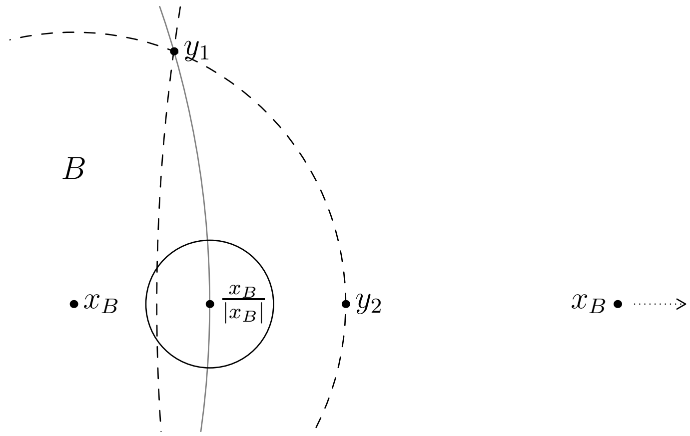

Figure 7: The two extremal cases of $B \in \mathcal{B}^{0,+}$.

$$\begin{aligned}
&\geq \frac{1 + r^2 - (1 - r/2)^2}{2r} \\
&= \frac{\frac{3}{4}r^2 + r}{2r} \\
&= \frac{\frac{3}{4}r + 1}{2} \\
&\geq \frac{1}{2}.
\end{aligned}$$

□

By the claim at each point $y \in \partial B$ the angle between y and ∂B is at least $\frac{\pi}{6}$. That means for each $z \in B(0,1)$ there is exactly one $y_z \in \partial(F \cup \bigcup \mathcal{B}^{0,-})$ with $y_z/|y_z| = z$. Furthermore the mapping $z \mapsto |y_z|$ is Lipschitz with constant $\tan(\pi/2 - \pi/6) = \sqrt{3}$ and bounded by $\frac{5}{4}$. Thus

$$\mathcal{H}^{d-1}(\partial \bigcup \mathcal{B}^{0,-}) \leq 2(5/4)^{d-1}\mathcal{H}^{d-1}(\partial B(0,1)).$$

For $B \in \mathcal{B}^{0,+}$ denote $r_B = \operatorname{dist}(x_B/|x_B|, \partial B(0,1) \cap \partial B)$.

Claim 5.8. Let $B \in \mathcal{B}^{0,+}$. Then $\operatorname{dist}(B(\frac{x_B}{|x_B|}, \frac{r_B}{4}), \mathbb{R}^d \setminus (B(0,1) \cup B)) \geq r_B/12$.

Proof. It suffices to check the distance at parallel points of $\partial B(\frac{x_B}{|x_B|}, \frac{r_B}{4})$ and $\partial(B(0,1) \cup B)$ and where $\partial(B(0,1) \cup B)$ has a corner. Denote $r = \operatorname{diam} B/2$. If y_1 is a corner of $\partial(B(0,1) \cup B)$ then $y_1 \in \partial B(0,1) \cap \partial B$ and there $\operatorname{dist}(B(\frac{x_B}{|x_B|}, \frac{r_B}{4}), y_1) = \frac{3}{4}r_B$.

The only parallel points lie on the line that goes through the origin and x_B. Of those points the only interesting one in $\partial(B(0,1) \cup B)$ is $y_2 = x_B + r\frac{x_B}{|x_B|}$. If $|x_B| \geq 1$ then $r_B \leq r$ which implies $\operatorname{dist}(B(\frac{x_B}{|x_B|}, \frac{r_B}{4}), y_2) \geq \frac{3}{4}r \geq \frac{3}{4}r_B$. Thus it suffices to consider $|x_B| \leq 1$. In this case since $|x_B| \geq 1 - \frac{r}{2}$ we have $|\frac{x_B}{|x_B|} - x_B| \leq \frac{r}{2}$ and thus $r_B \leq \frac{3}{2}r$. Therefore $\operatorname{dist}(B(\frac{x_B}{|x_B|}, \frac{r_B}{4}), y_2) \geq \frac{r}{2} - \frac{r_B}{4} \geq \frac{r}{2} - \frac{3}{8}r = \frac{r}{8} \geq \frac{r_B}{12}$. □

For each $n \in \mathbb{Z}$ denote by $\mathcal{B}_n$ the set of $B \in \mathcal{B}^{0,+}$ with $r_B \in [\frac{1}{2}, 1)2^n$. Let $B \in \mathcal{B}^{0,+}$ and $C \in \mathcal{B}^{0,+}$ with $C \neq B$. If $C \subset B(0,1) \cup B$ then C does not contribute to $\partial(\bigcup \mathcal{B}^{0,+} \cup B(0,1))$. Thus it suffices to consider the case that each $C \in \mathcal{B}^{0,+}$ intersects $\mathbb{R}^d \setminus (B(0,1) \cup B)$.

Claim 5.9. There is a $c \in \mathbb{Z} + \frac{\log_2 \lambda}{d+1}$ that depends only on the dimension such that for all n, k with $k - n \leq \frac{\log_2 \lambda}{d+1} - c$ and $B \in \mathcal{B}_n$ no $C \in \mathcal{B}_k$ intersects $B(x_B/|x_B|, r_B/4)$.

Proof. Note that $\lambda 2^{-d} \operatorname{diam}(B)^d \leq \mathcal{L}(B \cap B(0,1)) \sim r_B^{d+1}/\operatorname{diam}(B)$. Thus $r_B \leq \operatorname{diam} B \lesssim \lambda^{-\frac{1}{d+1}} r_B$. This means $2^{n-1} \leq \operatorname{diam} B \lesssim \lambda^{-\frac{1}{d+1}} 2^n$ and similarly $\operatorname{diam} C \lesssim \lambda^{-\frac{1}{d+1}} 2^k$. Thus for c small enough and $k \leq n + \frac{\log_2 \lambda}{d+1} - c$ we have

$$\operatorname{diam} C \leq \lambda^{-\frac{1}{d+1}} 2^{n+\log_2 \lambda/(d+1)} 2^{-1}/12 = 2^{n-1}/12 \leq r_B/12.$$

So since we assumed that C intersects $\mathbb{R}^d \setminus (B(0,1) \cup B)$ it follows from Claim 5.8 that C cannot intersect $B(x_B/|x_B|, r_B/4)$. □

Take a disjoint subcollection $\tilde{\mathcal{B}}_n$ of $\mathcal{B}_n$ such that $5\tilde{\mathcal{B}}_n$ covers $\mathcal{B}_n$. Then by Lemma 4.1 we have

$$\begin{aligned}
\mathcal{H}^{d-1}(\partial \bigcup \mathcal{B}_n) &\leq \sum_{B \in \tilde{\mathcal{B}}_n} \mathcal{H}^{d-1}(5B \cap \partial \bigcup \mathcal{B}_n) \\
&\lesssim \lambda^{-\frac{d}{d+1}} \sum_{B \in \tilde{\mathcal{B}}_n} \mathcal{H}^{d-1}(\partial 5B) \\
&\lesssim \lambda^{-2+\frac{3}{d+1}} \sum_{B \in \tilde{\mathcal{B}}_n} r_B^{d-1} \\
&\lesssim \lambda^{-2+\frac{3}{d+1}} \sum_{B \in \tilde{\mathcal{B}}_n} \mathcal{H}^{d-1}(\partial B(0,1) \cap B(x_B/|x_B|, r_B/4)) \\
&\leq \lambda^{-2+\frac{3}{d+1}} \mathcal{H}^{d-1}(\partial B(0,1) \cap \bigcup \{B(x_B/|x_B|, r_B/4) : B \in \mathcal{B}_n\}).
\end{aligned}$$

So we can conclude from Claim 5.9

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{B}_{n+k(c-\frac{\log_2 \lambda}{d+1})}) &\lesssim \lambda^{-2+\frac{3}{d+1}} \sum_{k \in \mathbb{Z}} \mathcal{H}^{d-1}\Big(\partial B(0,1) \cap \bigcup \Big\{B(x_B/|x_B|, r_B/4) : B \in \mathcal{B}_{n+k(c-\frac{\log_2 \lambda}{d+1})}\Big\}\Big) \\
&= \lambda^{-2+\frac{3}{d+1}} \mathcal{H}^{d-1}\Big(\partial B(0,1) \cap \bigcup \Big\{B(x_B/|x_B|, r_B/4) : B \in \bigcup_{k \in \mathbb{Z}} \mathcal{B}_{n+k(c-\frac{\log_2 \lambda}{d+1})}\Big\}\Big) \\
&\leq \lambda^{-2+\frac{3}{d+1}} \mathcal{H}^{d-1}(\partial B(0,1))
\end{aligned}$$

and thus

$$\begin{aligned}
\mathcal{H}^{d-1}(\partial \mathcal{B}^{0,+}) &\leq \sum_{n=0}^{c-\frac{\log_2 \lambda}{d+1}} \sum_{k \in \mathbb{Z}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{B}_{n+k(c-\frac{\log_2 \lambda}{d+1})}) \\
&\lesssim \Big(c - \frac{\log_2 \lambda}{d+1}\Big) \lambda^{-2+\frac{3}{d+1}}.
\end{aligned}$$

□

Proof of Proposition 5.3. According to Lemma 2.4, for every $B \in \mathcal{B}$, almost every point in $E \cap B$ is contained in a ball $F \subset B$ with

$$\mathcal{L}(F \cap E) = \frac{1}{2}\mathcal{L}(F).$$

Denote by $\mathcal{G}$ the set of all such balls F. By scaling it suffices to consider the case that all balls in $\mathcal{G}$ and $\mathcal{B}$ have diameter at most 1. We build inductively sequences $(\mathcal{F}_n)_{n=0}^{-\infty}, (\mathcal{G}_n)_{n=0}^{-\infty}$ of subsets of $\mathcal{G}$. We denote $\mathcal{F}_{>n} = \bigcup_{n<k\leq 0} \mathcal{F}_n$ and $\mathcal{G}_{>n}$ and $\mathcal{F}_{n<\cdot\leq k}$ accordingly. Assume we are at scale $n \leq 0$. Denote by $\mathcal{B}_n$ the set of balls in $\mathcal{B}$ with $\operatorname{diam} B \in [\frac{1}{2}, 1)2^n$. Decompose $\mathcal{B}_n$ into

$$\mathcal{B}_n^0 = \Big\{B \in \mathcal{B}_n : \mathcal{L}(\bigcup 5\mathcal{F}_{>n} \cap B) \leq \frac{\lambda}{2}\mathcal{L}(B)\Big\},$$
$$\mathcal{B}_n^1 = \Big\{B \in \mathcal{B}_n : \mathcal{L}(\bigcup 5\mathcal{F}_{>n} \cap B) > \frac{\lambda}{2}\mathcal{L}(B)\Big\}$$

and $\mathcal{B}_n^1$ into

$$\mathcal{B}_n^{1,0} = \Big\{B \in \mathcal{B}_n^1 : \mathcal{L}(\bigcup \mathcal{F}_{>n} \cap B) \leq \frac{1}{8^{\frac{d+1}{2}} d^{\frac{d+7}{2}}}\mathcal{L}(B)\Big\},$$
$$\mathcal{B}_n^{1,1} = \Big\{B \in \mathcal{B}_n^1 : \mathcal{L}(\bigcup \mathcal{F}_{>n} \cap B) > \frac{1}{8^{\frac{d+1}{2}} d^{\frac{d+7}{2}}}\mathcal{L}(B)\Big\}.$$

Denote by $\mathcal{G}_n$ the set of balls $G \in \mathcal{G}$ with $\operatorname{diam} G \in [\frac{1}{2}, 1)2^n$ which intersect $E \setminus \bigcup 5\mathcal{F}_{>n}$ or are for some $k \geq n$ and some $B \in \mathcal{B}_k^{1,0}$ contained in $B \setminus \bigcup \mathcal{F}_{n<\cdot\leq k}$. Set $\mathcal{F}_n$ to be a maximal disjoint subcollection of $\mathcal{G}_n$.

Denote $\mathcal{F} = \bigcup_n \mathcal{F}_n$, $\mathcal{B}_0 = \bigcup_n \mathcal{B}_n^0$ and $\mathcal{B}^{1,0}$ and $\mathcal{B}^{1,1}$ accordingly. Here are a few properties of those ball collections.

(i) $5\mathcal{F}_n$ covers $\mathcal{G}_n$.

(ii) $5\mathcal{F}$ covers almost all of E.

(iii) $(3/4)^{\frac{1}{d}}\mathcal{F}$ is disjoint.

(iv) If $B \in \mathcal{B}_n^0$ then $5\mathcal{F}_{\leq n}$ covers at least $\frac{\lambda}{2}$ of B.

(v) If $B \in \mathcal{B}_n^{1,0}$ then $5\mathcal{F}_{\leq n}$ covers at least λ of B.

Proof. (i) By maximality of $\mathcal{F}_n$ every $G \in \mathcal{G}_n$ intersects an $F \in \mathcal{F}_n$. Since $\operatorname{diam} G \leq 2 \operatorname{diam} F$ this means $G \subset 5F$.

(ii) Let $G \in \mathcal{G}$ with $\operatorname{diam} G \in [\frac{1}{2}, 1)2^n$. If $G \in \mathcal{G}_n$ then by (i) G is contained in $\bigcup 5\mathcal{F}_n$. If $G \notin \mathcal{G}_n$ then G is contained in $5\mathcal{F}_{>n}$ because G intersects E. And since $\mathcal{G}$ covers almost all of E this means so does $5\mathcal{F}$.

(iii) For each n $\mathcal{F}_n$ is disjoint. It remains to show that it is disjoint from $(3/4)^{\frac{1}{d}}\mathcal{F}_{>n}$. So assume $F \in \mathcal{F}_n$. If F was chosen because it intersects $E \setminus 5\mathcal{F}_{>n}$ then it doesn't intersect $\mathcal{F}_{>n}$. It remains to consider the case that there is a $k \geq n$ and a $B \in \mathcal{B}_k^{1,0}$ such that $F \subset B$ and F does not intersect $\mathcal{F}_{n<\cdot\leq k}$. Since $B \in \mathcal{B}_k^{1,0}$ for every $G \in \mathcal{F}_{>k}$ we have

$$\mathcal{L}(B \cap G) \leq \frac{1}{8^{\frac{d+1}{2}} d^{\frac{d+7}{2}}}\mathcal{L}(B),$$

so that by Lemma 5.5 $(1-1/(4d))B$ and G are disjoint. Since $(3/4)^{\frac{1}{d}} \le 1-1/(4d)$ and $\operatorname{diam} G \ge \operatorname{diam} B$ this means that $(3/4)^{\frac{1}{d}}G$ and B are disjoint, too. Hence also F and $(3/4)^{\frac{1}{d}} \bigcup \mathcal{F}_{>k}$ are disjoint.

(iv) Let $B \in \mathcal{B}_n^0$. Then because $5\mathcal{F}$ covers almost all of E and

$$\mathcal{L}(\bigcup 5\mathcal{F}_{>n} \cap B) \le \frac{\lambda}{2}\mathcal{L}(B)$$

we must have

$$\mathcal{L}(\bigcup 5\mathcal{F}_{\le n} \cap B) \ge \frac{\lambda}{2}\mathcal{L}(B).$$

(v) Since $\mathcal{L}(B \cap E) \ge \lambda \mathcal{L}(B)$ it suffices to show that $5\mathcal{F}_{\le n}$ covers $E \cap B$. Recall that almost all of $B \cap E$ is covered by the union of all $G \in \mathcal{G}$ with $G \subset B$ and $\operatorname{diam} G < 2^n$, so it suffices to show that each such G is contained in $5\mathcal{F}_{\le n}$. Let $k \le n$ and $G \subset B$ with $\operatorname{diam} G \in [\frac{1}{2}, 1)2^k$. If G intersects $\bigcup \mathcal{F}_{k<\cdot\le n}$ then $G \subset \bigcup 5\mathcal{F}_{k<\cdot\le n}$. If G does not intersect $\bigcup \mathcal{F}_{k<\cdot\le n}$ then $G \in \mathcal{G}_k$ and thus by (i) we have $G \subset \bigcup 5\mathcal{F}_k$.

□

Denote $\tilde{\mathcal{B}} = \mathcal{B}^0 \cup \mathcal{B}^{1,0}$ so that $\mathcal{B} = \tilde{\mathcal{B}} \cup \mathcal{B}^{1,1}$. Then by Lemma 1.6 we have

$$\partial_* \bigcup \mathcal{B} \subset \partial_* \bigcup \tilde{\mathcal{B}} \cup \big(\partial_* \bigcup \mathcal{B}^{1,1} \setminus \overline{\bigcup \tilde{\mathcal{B}}}^*\big).$$

Note that for finite sets of balls the topological and measure theoretical notions agree up to $d-1$ dimensional measure zero. By Lemma 5.6 for every $B \in \mathcal{B}^{1,1}$ there is an $F \in \mathcal{F}$ with $\operatorname{diam} F > \operatorname{diam} B$ that intersects $(1 - 8^{-\frac{d+1}{2}} d^{-\frac{d+7}{2}}/d)B$. By Lemma 2.4 F is further contained in a $B_F \in \mathcal{B}$. Since $\operatorname{diam} B < \operatorname{diam} B_F$ we have $B \ne B_F$. For each $F \in \mathcal{F}$ denote by $\mathcal{B}(F)$ the set of $B \in \mathcal{B}$ with $\operatorname{diam} B < \operatorname{diam} F$ such that F intersects $(1 - 8^{-\frac{d+1}{2}} d^{-\frac{d+7}{2}}/d)B$. Then

$$\begin{aligned}
\partial \bigcup \mathcal{B}^{1,1} \setminus \overline{\bigcup \tilde{\mathcal{B}}} &\subset \partial \bigcup \mathcal{B}^{1,1} \setminus \bigcup \mathcal{B} \\
&\subset \bigcup_{F \in \mathcal{F}} \partial \bigcup (\mathcal{B}^{1,1} \cap \mathcal{B}(F)) \setminus \bigcup \mathcal{B} \\
&\subset \bigcup_{F \in \mathcal{F}} \partial \bigcup (\mathcal{B}^{1,1} \cap \mathcal{B}(F)) \setminus (\bigcup \mathcal{B}(F) \cup B_F) \\
&\subset \bigcup_{F \in \mathcal{F}} \partial \bigcup \mathcal{B}(F) \setminus (\bigcup \mathcal{B}(F) \cup B_F) \\
&= \bigcup_{F \in \mathcal{F}} \partial \bigcup \mathcal{B}(F) \setminus B_F \\
&\subset \bigcup_{F \in \mathcal{F}} \partial (F \cup \bigcup \mathcal{B}(F)).
\end{aligned}$$

Thus Lemma 5.7 implies

$$\mathcal{H}^{d-1}(\partial \bigcup \mathcal{B}^{1,1} \setminus \overline{\bigcup \tilde{\mathcal{B}}}) \lesssim \sum_{F \in \mathcal{F}} \mathcal{H}^{d-1}(\partial F)$$

Recall that we made $(3/4)^{\frac{1}{d}}\mathcal{F}$ disjoint and that by Lemma 2.4 we for each $F \in \mathcal{F}$ we have $F \subset \bigcup \mathcal{B}$ and $\mathcal{L}(F \cap E) = \mathcal{L}(F)/2$. Thus $\mathcal{L}((\frac{3}{4})^{\frac{1}{d}} F \cap E) \in [\frac{1}{4}, \frac{3}{4}]\mathcal{L}(F)$ and so by the relative isoperimetric inequality (1) we can conclude

$$\begin{aligned}\sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial F) &\lesssim \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial (3/4)^{\frac{1}{d}} F) \\ &\lesssim \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial_* E \cap (3/4)^{\frac{1}{d}} F) \\ &\leq \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B}). \end{aligned} \tag{14}$$

It remains to prove

$$\mathcal{H}^{d-1}(\partial \bigcup \tilde{\mathcal{B}}) \lesssim \lambda^{-\frac{d-1}{d}} \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial F). \tag{15}$$

For $n \in \mathbb{Z}$ denote by $\tilde{\mathcal{B}}_n$ the set of balls $B \in \tilde{\mathcal{B}}$ with $\operatorname{diam} B \in [\frac{1}{2}, 1)2^n$. Let $B \in \tilde{\mathcal{B}}_n$ and $F \in \mathcal{F}_{\leq n}$ such that $5F$ intersects B. Then $F \subset 9B$. By (iv) and (v) this means

$$\frac{\lambda}{2}\mathcal{L}(B) \leq \mathcal{L}(B \cap \bigcup 5\mathcal{F}_{\leq n}) \leq \sum_{F\subset 9B} \mathcal{L}(5F \cap B). \tag{16}$$

For each $k \in \mathbb{Z}$ denote by $\tilde{\mathcal{F}}_k$ the set of balls $F \in \mathcal{F}$ with $\operatorname{diam} F \in [\frac{1}{2}, 1)2^k \lambda^{\frac{1}{d}}$. We make a case distinction. If there is a $k \geq n$ and an $F \subset 9B$ with $F \in \tilde{\mathcal{F}}_k$ we have

$$\begin{aligned}\mathcal{H}^{d-1}(\partial B) &= \frac{\mathcal{H}^{d-1}(\partial B)}{\mathcal{H}^{d-1}(\partial F)} \mathcal{H}^{d-1}(\partial F) \\ &\leq 2^d \frac{2^{n(d-1)}}{\lambda^{\frac{d-1}{d}} 2^{k(d-1)}} \mathcal{H}^{d-1}(\partial F) \\ &\sim 2^{(n-k)(d-1)} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F), \end{aligned} \tag{17}$$

and we are done with this case for the moment.

Assume all $F \subset 9B$ are contained in $\tilde{\mathcal{F}}_{<n}$. Then for each $F \subset 9B$ we can apply Corollary 2.3 with $X = B$ and $E = 5F$ and get

$$\begin{aligned}\mathcal{H}^{d-1}(\partial B) &\lesssim \Big(\frac{\mathcal{L}(5F \cap B)}{\mathcal{L}(B)}\Big)^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial 5F \cap B) \\ &\lesssim \Big(\frac{\mathcal{L}(5F \cap B)}{\mathcal{L}(B)}\Big)^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F). \end{aligned} \tag{18}$$

We rewrite (16) as

$$\mathcal{H}^{d-1}(\partial B) \leq 2 \sum_{F\subset 9B} \frac{\mathcal{L}(5F \cap B)}{\lambda \mathcal{L}(B)} \mathcal{H}^{d-1}(\partial B).$$

We apply (18) on the right-hand side and get

$$\mathcal{H}^{d-1}(\partial B) \lesssim \sum_{F\subset 9B} \Big(\frac{\mathcal{L}(5F \cap B)}{\lambda \mathcal{L}(B)}\Big)^{\frac{1}{d}} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F)$$

$$\lesssim \sum_{k\le n} \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 9B} \Big(\frac{\mathcal{L}(5F\cap B)}{\lambda\mathcal{L}(B)}\Big)^{\frac{1}{d}} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F)$$
$$\le \sum_{k\le n} \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 9B} \Big(\frac{\mathcal{L}(5F)}{\lambda\mathcal{L}(B)}\Big)^{\frac{1}{d}} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F)$$
$$\lesssim \sum_{k\le n} \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 9B} 2^{k-n} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F). \tag{19}$$

If $d = 1$ then Proposition 5.4 is straightforward to prove directly, so it suffices to consider $d \ge 2$. There we can combine (17) and (19) into

$$\mathcal{H}^{d-1}(\partial B) \lesssim \lambda^{-\frac{d-1}{d}} \sum_k 2^{-|k-n|} \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 9B} \mathcal{H}^{d-1}(\partial F)$$

for simplicity. This estimate can be seen as a way to distribute $\mathcal{H}^{d-1}(\partial B)$ over the balls F that it contains. The next step will be to turn the dependence around, and see for a fixed F, for how much variation of $\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}})$ it is responsible.

Since $\tilde{\mathcal{B}}_n$ is finite we have

$$\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}_n) = \sum_{B\in\tilde{\mathcal{B}}_n} \mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)$$

and we multiply each summand by a number bounded from below according to (19)

$$\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}_n) \lesssim \sum_{B\in\tilde{\mathcal{B}}_n} \frac{\mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)}{\mathcal{H}^{d-1}(\partial B)} \sum_k \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 9B} 2^{-|k-n|} \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial F)$$
$$= \lambda^{-\frac{d-1}{d}} \sum_k 2^{-|k-n|} \sum_{F\in\tilde{\mathcal{F}}_k} \mathcal{H}^{d-1}(\partial F) \sum_{B\in\tilde{\mathcal{B}}_n, 9B\supset F} \frac{\mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)}{\mathcal{H}^{d-1}(\partial B)}.$$

Now we have reorganized $\partial_* \bigcup \tilde{\mathcal{B}}_n$ according to the $F \in \mathcal{F}$. We want to bound the contribution of each F uniformly. For each $F \in \tilde{\mathcal{F}}_k$ for which there is a $B \in \tilde{\mathcal{B}}_n$ with $F \subset 9B$, denote by B_F a largest such B. Then for all $B \in \tilde{\mathcal{B}}_n$ with $F \subset 9B$ have $B \subset 3B_F$. Thus

$$\sum_{B\in\tilde{\mathcal{B}}_n, 9B\supset F} \frac{\mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)}{\mathcal{H}^{d-1}(\partial B)} \lesssim \sum_{B\in\tilde{\mathcal{B}}_n, B\subset 99B_F} \frac{\mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)}{\mathcal{H}^{d-1}(\partial B_F)}$$

which is uniformly bounded according to Lemma 4.1. Therefore we can conclude

$$\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}_n) \lesssim \lambda^{-\frac{d-1}{d}} \sum_k 2^{-|k-n|} \sum_{F\in\tilde{\mathcal{F}}_k} \mathcal{H}^{d-1}(\partial F).$$

The interaction between the scales is small enough so that we can just sum over all scales and obtain

$$\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}) \le \sum_n \mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}_n)$$

$$
\begin{aligned}
&\lesssim \lambda^{-\frac{d-1}{d}} \sum_k \sum_n 2^{-|k-n|} \sum_{F\in\tilde{\mathcal{F}}_k} \mathcal{H}^{d-1}(\partial F) \\
&\lesssim \lambda^{-\frac{d-1}{d}} \sum_k \sum_{F\in\tilde{\mathcal{F}}_k} \mathcal{H}^{d-1}(\partial F) \\
&= \lambda^{-\frac{d-1}{d}} \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial F),
\end{aligned}
$$

and we have proven (15). Now from (14) we can conclude $\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}}) \lesssim \lambda^{-\frac{d-1}{d}} \mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B})$ and finish the proof. □

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