LOWER SEMICONTINUITY OF WEAK SUPERSOLUTIONS TO NONLINEAR PARABOLIC EQUATIONS

TUOMO KUUSI

ABSTRACT. We prove that weak supersolutions to equations similar to the evolutionary p-Laplace equation has lower semicontinuous representatives. The proof avoids the use of Harnack's inequality and, in particular, the use of parabolic BMO. Moreover, the result gives a new point of view to approach the continuity of the solutions to a second order partial differential equation in divergence form.

1. INTRODUCTION

We study weak supersolutions to the evolutionary p-Laplace equation

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) = 0 \tag{1.1}$$

in an open set $\mathcal{O}$ in $\mathbb{R}^{n+1}$ with $p > 2n/(n+2)$. They are defined with the aid of smooth nonnegative test functions under the integral sign. In their definition, weak supersolutions are merely assumed to be Sobolev functions and, in particular, are not assumed to be lower semicontinuous.

The motivation to study the lower semicontinuity is to obtain coherence between the definitions of weak supersolutions and so-called p-superparabolic functions. The latter are defined as lower semicontinuous functions obeying the comparison principle with respect to the weak continuous solutions of (1.1), see [6] by Kilpeläinen and Lindqvist. This class of functions has an essential role when studying potential theoretical aspects of parabolic partial differential equations. The lower semicontinuity yields often required topological information about the level sets of p-superparabolic functions.

Every weak supersolution obeys the comparison principle with respect to weak solutions in suitable Sobolev sense. A natural question is then whether a weak supersolution has a lower semicontinuous representative and, consequently, coincide with a p-superparabolic function. This question was raised in [8] by Kinnunen and Lindqvist. Indeed, our main result gives an affirmative answer. Our proof is based on a general principle and it generalizes for wider class of equations, see Remark 4.5. The obtained lower semicontinuity together with results in [8] and [5] about the equivalency of viscosity solutions and p-superparabolic functions by Juutinen, Lindqvist, and Manfredi,

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see also [10] and [9], implies that weak supersolutions, p-superparabolic functions, and viscosity supersolutions are equivalent concepts whenever they are locally bounded.

In the elliptic theory the semicontinuity of weak supersolutions is usually presented as a consequence of weak Harnack principle, see, for instance, Trudinger [12] and Heinonen, Kilpeläinen and Martio [4]. In [13] Ziemer shows the lower semicontinuity using parabolic Harnack inequalities for weak supersolutions to equations with general structure, but with linear growth.

Our proof does not rely on Harnack's inequality and, hence, it gives a new proof for the lower semicontinuity even in the elliptic case. In particular, our approach does not use parabolic BMO estimates for the logarithm of a weak supersolution. In fact, our proof is based on a reverse Hölder type estimate for weak subsolutions. The estimate is standard, see DiBenedetto [2], but we wish to give here an alternative proof using Moser's iteration method [11]. An advantage of our approach is that we need not to distinct the singular case from the degenerate case. Moreover, we find the proof rather straightforward and transparent.

2. Preliminaries

2.1. **Parabolic Sobolev Spaces.** Suppose that Ω is an open set in $\mathbb{R}^n$. The Sobolev space $W^{1,p}(\Omega)$ is defined to be the space of real-valued functions f such that $f \in L^p(\Omega)$ and the distributional first partial derivatives $\partial f/\partial x_i$, $i = 1, 2, \dots, n$, exist in Ω and belong to $L^p(\Omega)$. We use the norm

$$\|f\|_{1,p,\Omega} = \Big(\int_\Omega |f|^p \, dx \Big)^{1/p} + \Big(\int_\Omega |\nabla f|^p \, dx \Big)^{1/p}.$$

The Sobolev space with zero boundary values, $W_0^{1,p}(\Omega)$, is the closure of $C_0^\infty(\Omega)$ with respect to the Sobolev norm.

We denote by $L^p(t_1, t_2; W^{1,p}(\Omega))$, $t_1 < t_2$, the space of functions such that for almost every $t_1 < t < t_2$ the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$ and the norm

$$\|u\|_{L^p(t_1,t_2;W^{1,p}(\Omega))} = \Big(\int_{t_1}^{t_2} \int_\Omega \big(|u(x,t)|^p + |\nabla u(x,t)|^p \big) \, dx \, dt \Big)^{1/p}$$

is finite. Definition of the space $L^p(t_1, t_2; W_0^{1,p}(\Omega))$ is analogous.

2.2. **Weak super– and subsolutions.** Let $\mathcal{O}$ be an open set in $\mathbb{R}^{n+1}$. A function u is a weak local supersolution (subsolution) to (1.1) if for any $\Omega \times (\tau_1, \tau_2) \Subset \mathcal{O}$, Ω open in $\mathbb{R}^n$, the function u belongs to $L^p(\tau_1, \tau_2; W^{1,p}(\Omega))$ and it satisfies the integral inequality

$$\begin{aligned} &\int_{t_1}^{t_2} \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, dx \, dt - \int_{t_1}^{t_2} \int_\Omega u \frac{\partial \eta}{\partial t} \, dx \, dt \\ &\quad + \int_\Omega u(x,t_2)\eta(x,t_2) \, dx - \int_\Omega u(x,t_1)\eta(x,t_1) \, dx \geq (\leq) \; 0 \end{aligned} \tag{2.1}$$

for almost every $\tau_1 < t_1 < t_2 < \tau_2$ and for every nonnegative $\eta \in C_0^\infty(\Omega \times (\tau_1, \tau_2))$. The boundary terms above are taken in the sense of limits

$$\int_\Omega u(x,t_1)\eta(x,t_1)\,dx = \lim_{\sigma\to 0}\frac{1}{\sigma}\int_{t_1}^{t_1+\sigma}\int_\Omega u(x,t)\eta(x,t)\,dx\,dt$$

and

$$\int_\Omega u(x,t_2)\eta(x,t_2)\,dx = \lim_{\sigma\to 0}\frac{1}{\sigma}\int_{t_2-\sigma}^{t_2}\int_\Omega u(x,t)\eta(x,t)\,dx\,dt.$$

A function is a local weak solution if it is both local weak supersolution and subsolution.

Furthermore, few words about p-superparabolic functions are appropriate. First of all, the equation (1.1) has a self similar weak solution

$$\mathcal{B}(x,t) = t^{-n/\lambda}\Big(C - \frac{p-2}{p}\lambda^{1/(p-1)}\Big(\frac{|x|}{t^{1/\lambda}}\Big)^{p/(p-1)}\Big)^{(p-1)/(p-2)},$$

in $\mathbb{R}^n \times (0,\infty)$, where $\lambda = n(p-2)+p$. This so-called Barenblatt solution was discovered in [1]. It is easy see that the function

$$\mathcal{V}_T(x,t) = \begin{cases} \mathcal{B}(x,t), & (x,t) \in \mathbb{R}^n \times (T,+\infty), \\ 0, & \text{otherwise}, \end{cases}$$

is a weak supersolution to (1.1) in $\mathbb{R}^n \times (-\infty,+\infty)$ for every $T > 0$. However, when $T = 0$, the function $\mathcal{V}_0$ is not a weak supersolution anymore. It is the a priori integrability of the gradient that fails. However, $\mathcal{V}_0$ is a p-superparabolic function according to the following definition.

Definition 2.2. A function $u : \mathcal{O} \to \mathbb{R}$ is a p-superparabolic if

1. u is lower semicontinuous,
2. u is finite in a dense subset of $\mathcal{O}$, and
3. u satisfies the comparison principle on space time cylinder $D_{t_1,t_2} = \Omega \times (t_1,t_2)$, where Ω is open in $\mathbb{R}^n$, $t_1 < t_2$ and $D_{t_1,t_2} \Subset \mathcal{O}$: If $h \in C(\bar{D}_{t_1,t_2})$ is p-parabolic in D_{t_1,t_2} such that $h \le u$ on the parabolic boundary of D_{t_1,t_2} , then $h \le u$ in D_{t_1,t_2}.

Our main result, together with the comparison principle, implies that a weak supersolution has always a p-superparabolic representative. Partial converse is also true. First, one should pay attention to the fact that, in their definition, the p-superparabolic functions are not required to have any derivatives. Indeed, Theorem 1.4 in [8] shows that there are no other locally bounded p-superparabolic functions than weak supersolutions. See also [9]. More generally, if p-superparabolic function belongs to the parabolic Sobolev space, then it is a weak supersolution. Observe, however, that in view of the Barenblatt solution above, the additional assumption on the summability of the gradient is needed.

The study of the pointwise behavior p-superparabolic functions is relevant since they are defined at every point of their domain. If the value is changed even at a single point, then the obtained function may not be p-superparabolic anymore. By theorem 5.2 in [8] every p-superparabolic function v satisfies

$$v(x,t) = \operatorname*{ess\,lim\,inf}_{(y,s)\to(x,t)} v(y,s)$$

for all (x,t) in the domain of v. It is noteworthy that the same limit process defines the lower semicontinuous representative of a weak supersolution in our proof.

As the last remark we mention that in [7] the authors show that also any unbounded p-superparabolic function has spatial derivatives in Sobolev sense.

2.3. **Notation.** We let $B_R(x)$ stand for a ball of radius R and center x in $\mathbb{R}^n$. If K is a bounded measurable set in $\mathbb{R}^n$ and f a measurable function, we denote

$$\fint_K f\,dx = \frac{1}{|K|}\int_K f\,dx,$$

where $|K|$ is the Lebesgue measure of the set K. If Ω and Ω' are open sets, Ω' bounded and the closure of Ω' belongs to Ω we denote $\Omega' \Subset \Omega$. By the Steklov average of a measurable function we mean

$$u_h(x,t) = \frac{1}{h}\int_t^{t+h} u(x,s)\,ds.$$

3. Boundedness of weak subsolutions

In this section we prove, using Moser's iteration technique, that subsolutions are bounded. Similar results can be found in [2] which applies intrinsic De Giorgi's method. Moser's method is based on the Sobolev's inequality and the following Caccioppoli estimate. We sketch the proof of it for the sake of completeness.

Lemma 3.1. *Let Ω be an open bounded set in $\mathbb{R}^n$. Suppose that u is a nonnegative subsolution in $\Omega \times (\tau_1,\tau_2)$. Let $\varepsilon \geq 1$. Then there is a constant $C = C(p,\varepsilon)$ such that*

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_\Omega |\nabla u|^p u^{-1+\varepsilon}\varphi^p\,dx\,dt + \operatorname*{ess\,sup}_{\tau_1<t<\tau_2}\int_\Omega u^{1+\varepsilon}\varphi^p\,dx \\ \leq C\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1+\varepsilon}|\nabla\varphi|^p\,dx\,dt + C\int_{\tau_1}^{\tau_2}\int_\Omega u^{1+\varepsilon}\left|\frac{\partial\varphi}{\partial t}\right|\varphi^{p-1}\,dx\,dt,\end{aligned}$$

where $\varphi \in C_0^\infty(\Omega\times(\tau_1,\tau_2))$.

Proof. From the weak formulation we have that subsolutions satisfy the following regularized integral inequality

$$0 \geq \int_\Omega \frac{\partial u_h}{\partial t}(x,t)\eta_h(x,t)\,dx + \int_\Omega \big(|\nabla u|^{p-2}\nabla u\big)_h \cdot \nabla\eta_h(x,t)\,dx$$

for almost every $\tau_1 < t < \tau_2 - h$. We choose the test function

$$\eta_h = \min(u_h,k)^{\varepsilon-1}u_h\varphi^p,$$

where φ belongs to $C_0^\infty(\Omega\times(\tau_1,\tau_2))$. It is admissible after an approximation. We denote the limit as $\eta = \min(u,k)^{\varepsilon-1}u\varphi^p$. We then choose $\tau_1 < t_1 < t_2 < \tau_2 - h$ and integrate the regularized equation. It follows from the properties of Steklov averages (see [2]) that

$$\big(|\nabla u|^{p-2}\nabla u\big)_h \cdot \nabla\eta_h(x,t) \to |\nabla u|^{p-2}\nabla u \cdot \nabla\eta(x,t)$$

as $h \to 0$ in $L^1(\operatorname{spt}\varphi)$. When $u < k$ we have by Young's inequality that

$$\begin{aligned}|\nabla u|^{p-2}\nabla u\cdot\nabla\eta \geq &\varepsilon u^{\varepsilon-1}\varphi^p|\nabla u|^p - pu^\varepsilon|\nabla u|^{p-1}\varphi^{p-1}\nabla\varphi\\ \geq &\frac{\varepsilon}{p}u^{\varepsilon-1}\varphi^p|\nabla u|^p - \Big(\frac{p}{\varepsilon}\Big)^{p-1}u^{p-1+\varepsilon}|\nabla\varphi|^p\end{aligned}$$

for almost every (x,t) in the support of φ. Similarly, when $u \geq k$, we have

$$|\nabla u|^{p-2}\nabla u\cdot\nabla\eta \geq \frac{1}{p}k^{\varepsilon-1}\varphi^p|\nabla u|^p - p^{p-1}u^p k^{\varepsilon-1}|\nabla\varphi|^p$$

for almost every (x,t) in the support of φ. Furthermore, we integrate by parts and obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \frac{\partial u_h}{\partial t}\min(u_h,k)^{\varepsilon-1}u_h\varphi^p\,dx\,dt &= \int_{t_1}^{t_2}\int_\Omega \frac{\partial g(u_h)^{1+\varepsilon}}{\partial t}\varphi^p\,dx\,dt\\ \to -\int_{t_1}^{t_2}\int_\Omega g(u)\frac{\partial\varphi^p}{\partial t}\,dx\,dt &+ \int_\Omega g(u(x,t))\varphi^p(x,t)\,dx\Big|_{t=t_1}^{t_2}\Big).\end{aligned}$$

for almost every $\tau_1 < t_1 < t_2 < \tau_2$ as $h \to 0$. Here we have denoted $g(s) = \int_0^s \min(k,r)^{\varepsilon-1}r\,dr$. We now choose $\tau_1 < t_2 < \tau_2$ such that

$$\int_\Omega u^{1+\varepsilon}(x,t_2)\varphi^p(x,t_2)\,dx \geq \frac{1}{2}\operatorname*{ess\,sup}_{\tau_1<t<\tau_2}\int_\Omega u^{1+\varepsilon}\varphi^p\,dx,$$

and let $t_1 \to \tau_1$. We collect results, apply monotone convergence theorem and conlude the result of the lemma. □

3.1. **Estimates for the essential supremum of a subsolution.** For the first lemma we assume that the nonnegative subsolution is bounded below away from zero i.e.

$$u \geq \Big(\frac{R^p}{\rho^p T}\Big)^{1/(p-2)} > 0$$

with some $\rho > 0$. We remark that for the heat equation ($p = 2$) this condition reduces to $T \simeq R^2$.

Lemma 3.2. *Let $\mathcal{O}$ be an open bounded set in $\mathbb{R}^{n+1}$. Suppose that u is a subsolution in $\mathcal{O}$ and $B_R(x_0)\times(t_0-T,t_0) \Subset \mathcal{O}$ and*

$$u \geq \Big(\frac{R^p}{\rho^p T}\Big)^{1/(p-2)} > 0, \quad \rho > 0.$$

Then there exists a constant $C = C(n,p)$ such that for $p \geq 2$ we have

$$\operatorname*{ess\,sup}_{B_{\sigma R}(x_0)\times(t_0-\sigma^p T,t_0)} u \leq \Big(\frac{T}{R^p}\frac{C(1+\rho)^{n+p}}{(1-\sigma)^{n+p}}\fint_{t_0-T}^{t_0}\fint_{B_R(x_0)} u^{p-2+\delta}\,dx\,dt\Big)^{1/\delta}$$

and for $2n/(n+2) < p < 2$

$$\begin{aligned}\operatorname*{ess\,sup}_{B_{\sigma R}(x_0)\times(t_0-\sigma^p T,t_0)} u \leq \Big(&\frac{R^n}{T^{n/p}}\frac{C(1+1/\rho)^{n+p}}{(1-\sigma)^{n+p}}\\ &\times\fint_{t_0-T}^{t_0}\fint_{B_R(x_0)} u^{n(2-p)/p+\delta}\,dx\,dt\Big)^{1/\delta}\end{aligned}$$

for every $1/2 < \sigma < 1$ and $\delta > 0$.

Proof. Let $\sigma R \le s < S < R$. We set

$$R_0 = S, \quad R_j = (S - (S-s)(1-2^{-j})), \quad j = 0,1,2,\dots,$$

and denote

$$U_j = B_j \times \Gamma_j = B_{R_j}(x_0) \times (t_0 - (R_j/R_0)^p T, t_0).$$

We choose test functions $\varphi_j \in C^\infty(U_j) \cap C(\bar{U}_j)$, $j = 0,1,2,\dots$, such that

$$0 \le \varphi_j \le 1, \qquad \varphi_j = 0 \quad \text{on} \quad \partial_p U_j, \qquad \varphi_j = 1 \quad \text{in} \quad U_{j+1}$$

and

$$|\nabla \varphi_j| \le \frac{C}{S-s} 2^j, \quad \Big|\frac{\partial \varphi_j}{\partial t}\Big| \le \frac{R^p}{T}\frac{C}{(S-s)^p} 2^{pj}.$$

The first step in the proof is to apply parabolic Sobolev's inequality, See [2], Proposition 3.1, p. 7. It implies

$$\begin{aligned}\int_{U_{j+1}} u^{\kappa\alpha}\,dx\,dt \le& \int_{U_j} (u^{\alpha/p}\varphi_j^{\beta/p})^{\kappa p}\,dx\,dt\\ \le& C\int_{U_j} |\nabla(u^{\alpha/p}\varphi_j^{\beta/p})|^p\,dx\,dt \Big(\operatorname*{ess\,sup}_{\Gamma_j}\int_{B_j}(u^{\alpha/p}\varphi_j^{\beta/p})^{(\kappa-1)n}\,dx\Big)^{p/n}\end{aligned}$$

for some $\alpha \in \mathbb{R}$, $\beta \ge 1$ and $\kappa > 1$. We choose

$$\alpha = p-1+\varepsilon, \quad \kappa = 1 + \frac{p(1+\varepsilon)}{n(p-1+\varepsilon)}, \quad \beta = \frac{p(p-1+\varepsilon)}{1+\varepsilon},$$

where $\varepsilon \ge 1$. We use Lemma 3.1 to estimate terms on the right-hand side. First, we have

$$\begin{aligned}\operatorname*{ess\,sup}_{\Gamma_j}\int_{B_j}(u^{\alpha/p}\varphi_j^{\beta/p})^{(\kappa-1)n}\,dx =& \operatorname*{ess\,sup}_{\Gamma_j}\int_{B_j} u^{1+\varepsilon}\varphi_j^p\,dx\\ \le& C\Big(\int_{U_j} u^{p-1+\varepsilon}|\nabla\varphi_j|^p\,dx\,dt + \int_{U_j} u^{1+\varepsilon}\Big|\frac{\partial\varphi_j}{\partial t}\Big|\varphi_j^{p-1}\,dx\,dt\Big).\end{aligned}$$

A similar estimate gives

$$\begin{aligned}&\int_{U_j} |\nabla(u^{\alpha/p}\varphi_j^{\beta/p})|^p\,dx\,dt\\ &\le C\varepsilon^p\Big(\int_{U_j}|\nabla\varphi_j|^p u^{p-1+\varepsilon}\,dx\,dt + \int_{U_j} u^{1+\varepsilon}\Big|\frac{\partial\varphi_j}{\partial t}\Big|\varphi_j^{p-1}\,dx\,dt\Big).\end{aligned}$$

Moreover, when $p > 2$, by the assumption $u^{1+\varepsilon} \le (\rho^p T/R^p)u^{p-1+\varepsilon}$, we obtain

$$\int_{U_{j+1}} u^{p-1+p/n+(1+p/n)\varepsilon}\,dx\,dt \le \Big(\frac{C(1+\rho)^p 2^{jp}\varepsilon^p}{(S-s)^p}\int_{U_j} u^{p-1+\varepsilon}\,dx\,dt\Big)^{1+p/n}.$$

If $p < 2$, the assumption implies $u^{p-1+\varepsilon} \le (R^p/\rho^p T)u^{1+\varepsilon}$ and hence

$$\int_{U_{j+1}} u^{p-1+p/n+(1+p/n)\varepsilon}\,dx\,dt \le \Big(\frac{R^p(1+1/\rho)^p}{T}\frac{C2^{jp}\varepsilon^p}{(S-s)^p}\int_{U_j} u^{1+\varepsilon}\,dx\,dt\Big)^{1+p/n}.$$

For $p \ge 2$ we then choose

$$\varepsilon_j = 2(1+p/n)^j - 1, \quad \alpha_j = p-1+\varepsilon^j,$$

$j = 0, 1, 2, \dots$, so that $p - 1 + p/n + (1 + p/n)\varepsilon_j = p - 1 + \varepsilon_{j+1}$. For $p < 2$ we set

$$\varepsilon_j = (2 + n(p-2)/p)(1 + p/n)^j - (n(p-2) + p)/p, \quad \alpha_j = 1 + \varepsilon_j,$$

$j = 0, 1, 2, \dots$, and $p - 1 + p/n + (1 + p/n)\varepsilon_j = 1 + \varepsilon_{j+1}$. Thus, we have

$$I_{j+1} \leq \Big(C^j Y^p I_j\Big)^{1+p/n}, \tag{3.3}$$

where

$$I_j = \int_{U_j} u^{\alpha_j}\, dx\, dt, \quad Y = \frac{1}{S-s} \times \begin{cases} (1+\rho), & p \geq 2, \\ R(1 + 1/\rho)/T^{1/p}, & p < 2. \end{cases}$$

Next, a direct calculation gives

$$\prod_{k=0}^{j} C^{(j-k)(1+p/n)^k} = \Big(\prod_{k=0}^{j} C^{k(1+p/n)^{-k}}\Big)^{(1+p/n)^j} \leq C^{(1+p/n)^{j+1}}$$

and

$$\sum_{k=1}^{j+1} (1 + p/n)^k = \frac{n+p}{p}\big((1+p/n)^{j+1} - 1\big).$$

The calculation shows that the constant will stay bounded in the iteration below. We repeatedly use (3.3) and get

$$I_{j+1}^{1/\alpha_{j+1}} \leq \Big(CY^{n+p} \int_{U_0} u^{\max(p,2)}\, dx\, dt\Big)^{(1+p/n)^{j+1}/\alpha_{j+1}}.$$

Since $(1 + p/n)^j/\alpha_j \to 1/2$, $p \geq 2$, and $(1 + p/n)^j/\alpha_j \to 1/(2 + n(p-2)/p)$, $p < 2$, as $j \to \infty$, we conclude

$$\operatorname*{ess\,sup}_{Q(s)} u \leq \Big(Y^{n+p} C \int_{Q(S)} u^{\max(p,2)}\, dx\, dt\Big)^{1/\min(2, 2+n(p-2)/p)},$$

where we have denoted $Q(s) = B(x_0, s) \times (t_0, t_0 - (s/S)^p T)$. When $p \geq 2$, we obtain by Young's inequality for every $2 > \delta \geq \min\{\delta_0, 1\}$ that

$$\begin{aligned} \operatorname*{ess\,sup}_{Q(s)} u \leq& \Big(\operatorname*{ess\,sup}_{Q(S)} u^{2-\delta} \frac{C(1+\rho)^{n+p}}{(S-s)^{n+p}} \int_{Q(S)} u^{p-2+\delta}\, dx\, dt\Big)^{1/2} \\ \leq& \frac{1}{2} \operatorname*{ess\,sup}_{Q(S)} u + \Big(\frac{C(1+\rho)^{n+p}}{(S-s)^{n+p}} \int_{Q(S)} u^{p-2+\delta}\, dx\, dt\Big)^{1/\delta}. \end{aligned}$$

Similarly, when $p < 2$, we have

$$\operatorname*{ess\,sup}_{Q(s)} u \leq \frac{1}{2} \operatorname*{ess\,sup}_{Q(S)} u + \Big(\frac{CR^{n+p}(1 + 1/\rho)^{n+p}}{T^{(n+p)/p}(S-s)^{n+p}} \int_{Q(S)} u^{n(2-p)/p+\delta}\, dx\, dt\Big)^{1/\delta}.$$

A standard iteration argument (see e.g. [3], Lemma 5.1) proves the claim of the lemma. □

Suppose that u is a nonnegative subsolution. We may then apply the previous lemma for the subsolution $v = u + (R^p/\rho^p T)^{1/(p-2)}$ and obtain the following theorem.

Theorem 3.4. *Let $\mathcal{O}$ be an open set in $\mathbb{R}^{n+1}$. Suppose that u is a nonnegative subsolution in $\mathcal{O}$ and $B_R(x_0) \times (t_0 - T, t_0) \Subset \mathcal{O}$. Then there exists a constant $C = C(n, p, \rho)$ such that for $p \geq 2$*

$$\operatorname*{ess\,sup}_{B_R(x_0)\times(t_0,t_0-T/2^p)} u \leq C\Big(\frac{R^p}{\rho^p T}\Big)^{1/(p-2)} + \frac{CT}{R^p} \fint_{t_0-T}^{t_0} \fint_{B_{2R}(x_0)} u^{p-1}\,dx\,dt$$

and for $2n/(n+2) < p < 2$

$$\operatorname*{ess\,sup}_{B_R(x_0)\times(t_0,t_0-T/2^p)} u \leq C\Big(\frac{R^p}{\rho^p T}\Big)^{1/(p-2)} + \frac{CR^n}{T^{n/p}} \fint_{t_0-T}^{t_0} \fint_{B_{2R}(x_0)} u^{n(2-p)/p+1}\,dx\,dt.$$

Remark 3.5. When $p = 2$ we choose $\rho = 2$ and $T = R^2$. Then the result reduces to

$$\operatorname*{ess\,sup}_{B_R(x_0)\times(t_0,t_0-R^2/4)} u \leq C \fint_{t_0-R^2}^{t_0} \fint_{B_{2R}(x_0)} u\,dx\,dt.$$

Remark 3.6. It is clear that sets $B_{\sigma R}(x_0) \times (t_0 - \sigma^p T, t_0)$, $0 < \sigma \leq 1$, above may be replaced with $B_{\sigma R}(x_0) \times (t_0 - \sigma^p T, t_0 + \sigma^p T)$, $0 < \sigma \leq 1$.

If u is a supersolution, then $\max(-u, 0)$ is a nonnegative subsolution. Hence we may apply Theorem 3.4 and obtain

Corollary 3.7. *A weak supersolution in an open set $\mathcal{O}$ in $\mathbb{R}^{n+1}$ is locally essentially bounded below.*

4. Lower semicontinuity of weak supersolutions

To prove the lower semicontinuity of weak supersolutions, we apply Theorem 3.4 together with Lebesgue's differentiation theorem. To this end, let u be a measurable function in an open set $\mathcal{O}$ in $\mathbb{R}^{n+1}$. We define the essential limes inferior as

$$\operatorname*{ess\,lim\,inf}_{(y,s)\to(x,t)} u(y,s) = \lim_{R\to 0} \operatorname*{ess\,inf}_{B_R(x)\times(t-R^p,t+R^p)} u.$$

The following theorem is our main result. It states that every weak supersolution has a lower semicontinuous representative.

Theorem 4.1. *Suppose that u is a weak supersolution in an open set $\mathcal{O}$ in $\mathbb{R}^{n+1}$. Then*

$$\widetilde{u}(x,t) = \operatorname*{ess\,lim\,inf}_{(y,s)\to(x,t)} u(y,s)$$

is a supersolution in $\mathcal{O}$. The function $\widetilde{u}$ is lower semicontinuous and $\widetilde{u} = u$ almost everywhere in $\mathcal{O}$.

Proof. We denote

$$U_R^M = B_R(0) \times (-M^{p-2}R^p, M^{p-2}R^p), \quad M \in \mathbb{N}.$$

Let E_M be the set of Lebesgue points with respect to the basis $\{U_R^M\}$ i.e.

$$E_M = \{(x,t) \in \mathcal{O} : \lim_{R\to 0} \fint_{(x,t)+U_R^M} |u(x,t) - u(y,s)|^\beta\,dy\,ds = 0\},$$

where $\beta = \max(n(2-p)/p+1, p-1)$. It is a straightforward calculation to show that $E_{M+1} \subset E_M$ for every $M \in \mathbb{N}$. Therefore, we obtain

$$E \equiv \bigcap_{M\in\mathbb{N}} E_M = E_1.$$

We also have $|E| = |\mathcal{O}|$. We define the set

$$V = \{(x,t) \in E : |u(x,t)| < \infty\}.$$

Clearly $\big|V\big| = \big|\mathcal{O}\big|$ by the summability of u. Let now $(x_0,t_0) \in V$ and $Q_R^M = (x_0,t_0) + U_R^M$, $M \in \mathbb{N}$. We first claim that

$$u(x_0,t_0) \le \operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)} u(x,t). \tag{4.2}$$

We make the counter assumption

$$u(x_0,t_0) - \operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)} u(x,t) = \varepsilon > 0.$$

Let R_0 be a radius such that

$$|\operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)} u(x,t) - \operatorname*{ess\,inf}_{Q_R^1} u| \le \varepsilon/2$$

for every $0 < R \le R_0$. Consequently, we have

$$u(x_0,t_0) - \operatorname*{ess\,inf}_{Q_R^1} u \ge \varepsilon/2 \tag{4.3}$$

for such R. We then define the subsolution $v = (u(x_0,t_0) - u)_+$. Since $(x_0,t_0) \in V$, we find for any $M \in \mathbb{N}$ a radius $R_1 = R_1(M)$ such that

$$\fint_{Q_{R_1}^M} v^\beta \,dx\,dt \le \fint_{Q_{R_1}^M} |u(x_0,t_0) - u(x,t)|^\beta \,dx\,dt \le M^{-\alpha-1},$$

where $\alpha = \max(p-2, n(2-p)/p)$. Furthermore, we obtain from Theorem 3.4 and Remark 3.6 (for $p = 2$ we use Remark 3.5) that

$$\operatorname*{ess\,sup}_{Q_{R_1/2}^M} v \le \frac{C}{M} + CM^\alpha \fint_{Q_{R_1}^M} v^\beta \,dx\,dt \le \frac{C}{M}.$$

We first fix $M = \lceil 4C/\varepsilon \rceil$ and then take $0 < R \le R_0$ so small that $Q_R^1 \subset Q_{R_1/2}^M$. It follows by (4.3) that

$$\varepsilon/4 \ge \operatorname*{ess\,sup}_{Q_{R_1/2}^M} v \ge \operatorname*{ess\,sup}_{Q_R^1} v \ge u(x_0,t_0) - \operatorname*{ess\,inf}_{Q_R^1} u \ge \varepsilon/2,$$

which gives the contradiction. We have thus proved (4.2). Moreover, by the definition of a Lebesgue point, we have

$$u(x_0,t_0) \le \operatorname*{ess\,lim\,inf}_{(y,s)\to(x_0,t_0)} u(y,s) \le \lim_{R\to 0} \fint_{Q_R^1} u(y,s)\,dy\,ds = u(x_0,t_0).$$

Consequently $\widetilde{u} = u$ almost everywhere. Finally, it is an easy exercise to show that since

$$\widetilde{u}(x,t) = \operatorname*{ess\,lim\,inf}_{(y,s)\to(x,t)} \widetilde{u}(y,s)$$

for every $(x,t) \in \mathcal{O}$, then $\widetilde{u}$ is a lower semicontinuous function. The result follows. $\square$

Remark 4.4. The functions $\widetilde{u}$ and u coincide at least in the set of Lebesgue points of u in which u is finite.

Remark 4.5. The proof of Theorem 4.1 is based on a general principle. The result also holds for weak supersolutions of

$$\nabla \cdot \big(\mathcal{A}(x,t,u,\nabla u)\big) = \frac{\partial u}{\partial t},$$

where $\mathcal{A}$ satisfies growth bounds

$$\begin{aligned} \mathcal{A}(x,t,u,\xi)\cdot\xi &\geq \nu|\xi|^p \\ |\mathcal{A}(x,t,u,\xi)| &\leq \gamma|\xi|^{p-1} \end{aligned} \tag{4.6}$$

for every $(u,\xi) \in \mathbb{R}^n$ and for almost every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Here $\nu \leq \gamma$ are positive constants. In addition, $\mathcal{A} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is assumed to be a Carathéodory function, that is, $(x,t) \mapsto \mathcal{A}(x,t,u,\xi)$ is measurable for every (u,ξ) in $\mathbb{R} \times \mathbb{R}^n$ and $(u,\xi) \mapsto \mathcal{A}(x,t,u,\xi)$ is continuous for almost every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

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Helsinki University of Technology

E-mail address: `tuomo.kuusi@tkk.fi`