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POTENTIAL ESTIMATES AND GRADIENT BOUNDEDNESS FOR NONLINEAR PARABOLIC SYSTEMS

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Abstract. We consider a class of parabolic systems and equations in divergence form modeled by the evolutionary p-Laplacean system

$$u_t - \mathrm{div}(|Du|^{p-2}Du) = V(x,t)\,,$$

and provide L^∞-bounds for the spatial gradient of solutions Du via nonlinear potentials of the right hand side datum V. Such estimates are related to those obtained by Kilpeläinen & Malý [22] in the elliptic case. In turn, the potential estimates found imply optimal conditions for the boundedness of Du in terms of borderline rearrangement invariant function spaces of Lorentz type. In particular, we prove that if $V \in L(n+2,1)$ then $Du \in L^\infty_{\mathrm{loc}}$, where n is the space dimension, and this gives the borderline case of a result of DiBenedetto [5]; a significant point is that the condition $V \in L(n+2,1)$ is independent of p. Moreover, we find explicit forms of local a priori estimates extending those from [5] valid for the homogeneous case $V \equiv 0$.

1. Introduction and results

In this paper we study a class of nonlinear, possible degenerate parabolic systems and equations, whose main model is given by the evolutionary p-Laplacean system

$$u_t - \mathrm{div}(|Du|^{p-2}Du) = V(x,t)\,. \tag{1.1}$$

Our aim is to establish optimal local estimates for the L^∞-norm of the spatial gradient Du of local weak solutions in terms of suitable nonlinear potentials of the right hand side datum V. The results hereby presented actually cover a large class of parabolic problems, including for instance all scalar equations in divergence form of the type

$$u_t - \mathrm{div}\, a(Du) = V(x,t) \tag{1.2}$$

under suitable parabolicity and growth assumptions on the vector field $a(\cdot)$, and more general parabolic systems with so called quasidiagonal structure, that is, systems of the form

$$u_t - \mathrm{div}(g(|Du|^2)Du) = V(x,t)\,. \tag{1.3}$$

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Specifically, we shall consider parabolic problems of the type in (1.2) in a space-time cylinder $Q_* = \Omega \times (0, \tau_*)$, where Ω is an bounded domain in $\mathbb{R}^n$ and $\tau_* > 0$. The vector field V is assumed to belong to $L^2(Q_*, \mathbb{R}^N)$; by eventually letting $V \equiv 0$ outside Q_* we shall assume without loss of generality that $V \in L^2(\mathbb{R}^{n+1}, \mathbb{R}^N)$. The vector field $a\colon \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is required to be C^1-regular and to satisfy the following structural conditions:

$$\begin{cases} |a(z)| + |\partial a(z)| \left(|z|^2 + s^2\right)^{1/2} \le L \left(|z|^2 + s^2\right)^{(p-1)/2} \\ \nu \left(|z|^2 + s^2\right)^{(p-2)/2} |\lambda|^2 \le \langle \partial a(z)\lambda, \lambda \rangle \,, \end{cases} \tag{1.4}$$

for all $\lambda, z \in \mathbb{R}^{Nn}$, where

$$n \ge 2, \qquad N \ge 1, \qquad s \ge 0, \qquad 0 < \nu \le L < \infty, \qquad p > \frac{2n}{n+2} \,. \tag{1.5}$$

We immediately observe that the lower bound on p appearing in (1.5) is typical for the results that we are going to obtain, and essentially appears in all the gradient bounds for parabolic equations [5]. In fact, the estimates developed include L^∞-bounds for the gradient, which are known to fail if such assumption on p is not considered. We refer to [5, Chapter 8] for more results and techniques in this direction. We moreover recall that in (1.5) the number s stands for the parabolicity parameter; i.e. if $s = 0$ we are dealing with a degenerate parabolic equation/system, while $s > 0$ the problems is non-degenerate. As a matter of fact, after re-normalization it is easy to see that the cases one can confine himself to consider are $s = 0$ and $s = 1$.

Solutions $u\colon Q_* \to \mathbb{R}^N$ to (1.2) will be understood in the distributional sense, i.e.

$$\int_{Q_*} \left(-u\phi_t + \langle a(Du), D\phi \rangle\right) \, dx\, dt = \int_{Q_*} V\phi \, dx\, dt \tag{1.6}$$

is assumed to hold whenever $\phi \in C_0^\infty(Q_*)$; they will be moreover assumed to satisfy

$$u \in C^0(0, \tau_*; L^2_{\text{loc}}(\Omega, \mathbb{R}^N)) \cap L^p_{\text{loc}}(0, \tau_*; W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)) \,. \tag{1.7}$$

A well-established existence theory for solutions with such a regularity is available; we address the reader for instance to [5, 36] and related references. We remark that the available existence theory provides us with solutions satisfying (1.7).

When dealing with the scalar case, i.e. $N = 1$, that is when we are considering a single equation, the assumptions reported in (1.4)-(1.5) will be only ones considered on the field $a(\cdot)$. Instead, when dealing with the vectorial case $N > 1$, that is when dealing with a system and the solution is vector valued, we in addition assume the quasidiagonal structure

$$a(z) = g(|z|^2)z, \qquad z \in \mathbb{R}^{Nn}, \qquad \text{when } N > 1, \tag{1.8}$$

where $g(\cdot)$ is nonnegative function such that $g \in C^1((0,\infty))$. We shall thereby consider parabolic systems as in (1.3). The specific form of the equation in the vectorial case is sometimes referred to as Uhlenbeck structure [39], after the seminal work [39] in the elliptic setting. It is worth to remark here that when considering general parabolic systems as in (1.2), i.e. without assuming further structure properties as (1.8), solutions are known to exhibit singularities even when $V \equiv 0$ and therefore the estimates derived in the following cannot hold true; instead partial regularity holds and for this we for instance refer to [10, 30] and related references. The proof of the Hölder continuity of the spatial gradient of solutions to systems as (1.3) when the right hand side vector field V is smooth enough, is instead a fundamental result of DiBenedetto & Friedman [6, 7, 8]. For basic regularity properties of solutions in the elliptic case we refer for instance to [4, 9, 28, 29, 39].

In this paper, by means of a new estimate via nonlinear potentials, amongst the other things we catch a borderline case of the L^∞-results of DiBenedetto, finding optimal conditions on V to guarantee the local gradient boundedness of Du. See Remark 1.23.

1.1. **Potential estimates.** Let us start with the potential estimates. Since the breakthrough of Kilpeläinen & Malý [22] in the beginning of the 90s, it is known that solutions to nonlinear and possibly degenerate elliptic equations can be pointwise estimated in terms of nonlinear Wolff potentials of the type

$$\mathbf{W}^F_{\beta,p}(x^*,R) := \int_0^R \left(\frac{|F|(B_\varrho(x^*))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}\,, \qquad \beta \in (0,n/p]\,, \tag{1.9}$$

defined for general Borel measures F. Here $B(x,\varrho) \subset \mathbb{R}^n$ denotes the open ball centered at x^*, with radius ϱ. In turn, these are suitable nonlinear versions of standard Riesz potentials, that is

$$\mathbf{I}^F_\beta(x^*,R) = \mathbf{W}^F_{\beta/2,2}(x^*,R) = \int_0^R \frac{|F|(B_\varrho(x^*))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho}\,, \qquad \beta \in (0,n]\,. \tag{1.10}$$

Such estimates are the analog of those valid for the standard Poisson equation $-\triangle u = V$ obtainable via representation formulas involving the fundamental solution, and in turn allow to deduce all type of regularity results on solutions from the behavior of nonlinear potentials. Very recently, the possibility of extending such pointwise estimates to the gradient level has been observed starting from the paper [33] for nonlinear elliptic equations with linear growth ($p = 2$), and then extended to the general degenerate case in the papers [11, 14, 15]. In particular, such estimates allow to get sharp conditions for the gradient boundedness of solutions to elliptic problems as $-\operatorname{div} a(Du) = V$ in terms of the right hand side datum V.

The aim of this paper is to provide local L^∞-estimates for parabolic equations and systems of the type (1.2)-(1.3), in terms of a nonlinear potential of the right hand side datum V. More precisely, given a vector field

$F \in L^2(Q_*, \mathbb{R}^N)$, we define the nonlinear Riesz type potential $\mathbf{P}_r^F$ by

$$\mathbf{P}_r^F(x^*, t^*) := \int_0^r \left(\varrho^2 \fint_{Q_\varrho(x^*,t^*)} |F|^2 \, dx \, dt \right)^{1/2} \frac{d\varrho}{\varrho}\,, \tag{1.11}$$

where $Q_\varrho(x^*, t^*) = B_\varrho(x^*) \times (t^* - \varrho^2, t^*)$ denotes the standard parabolic cylinder with vertex at (x^*, t^*) and width $\varrho > 0$; see next section for the notation adopted in this paper. The reason why we are calling $\mathbf{P}_r^F$ a nonlinear Riesz potential is that $\mathbf{P}_r^F$ exhibits the same scaling properties of the usual Riesz-caloric potential - that is the obvious adaptation of the Riesz potential $\mathbf{I}_1^F$ to the parabolic setting:

$$\mathbf{I}_1^F(x^*, t^*; r) := \int_0^r \varrho \fint_{Q_\varrho(x^*,t^*)} |F| \, dx \, dt \, \frac{d\varrho}{\varrho}\,.$$

In particular, both $\mathbf{P}_r^F$ and $\mathbf{I}_1^F$ are independent of p - on the contrary to the Wolff potential in (1.9) - and Hölder's inequality gives $\mathbf{I}_1^F(x^*, t^*; r) \leq \mathbf{P}_r^F(x^*, t^*)$. The structure of $\mathbf{P}_r^F$ makes it an obviously good replacement for the standard Riesz potential in order to derive regularity results for gradient of solutions when the right hand side is at least of class L^2. Indeed, in this paper we obtain potential estimates for the spatial gradient of solutions in terms of $\mathbf{P}_r^F$ that in turn allow to obtain optimal gradient boundedness criteria in rearrangement invariant function spaces that are just as good as those implied by $\mathbf{I}_1^F$ for L^2-maps F. The matter is anyway much more complicated than in the elliptic case since when $p \neq 2$ the anisotropicity of the parabolic operators involved forces to use a delicate interaction between DiBenedetto's intrinsic geometry, and nonlinear potential techniques. We remark that since the work of DiBenedetto in the eighties - see [5] for an overview - the intrinsic geometry viewpoint has proved to be unavoidable in studying parabolic equations with p-growth; see for instance [25, 35].

The first result we are going to present is a nonlinear potential estimate in the degenerate case $p \geq 2$.

Theorem 1.1. *Under the assumptions* (1.4)*, let u be a distributional solution as in* (1.6)-(1.7) *to the equation* (1.2)*, or to the system* (1.3) *with $a(z) \equiv g(|z|^2)z$; assume also that $p \geq 2$. Then, for every $\ell \in (0, 1]$ there exists a constant c, depending only on n, N, p, L, ν, ℓ such that*

$$\begin{aligned} \|Du\|_{L^\infty(Q_{r/2})} \leq & c \left(\fint_{Q_r} (|Du| + s)^{p-2} |Du|^{2\ell} \, dx \, dt \right)^{1/2\ell} \\ & + c \|\mathbf{P}_r^V\|_{L^\infty(Q_r)}^{2/p} + c(s+1) \end{aligned} \tag{1.12}$$

holds for every parabolic cylinder $Q_r \subset Q_*$.

For the subquadratic case we instead have the following:

Theorem 1.2. *Under the assumptions* (1.4), *let u be a distributional solution as in* (1.6)-(1.7) *to the equation* (1.2), *or to the system* (1.3) *with $a(z) \equiv g(|z|^2)z$; assume also that*

$$2 \geq p > \frac{2n}{n+2}\,. \tag{1.13}$$

Then, for every ℓ satisfying

$$\frac{n(2-p)}{2} < 2\ell \leq p \tag{1.14}$$

there exists a constant c, depending only on n, N, p, L, ν, ℓ, such that

$$\begin{aligned} \|Du\|_{L^\infty(Q_{r/2})} \leq & c\left(\fint_{Q_r} |Du|^{2\ell}\,dx\,dt\right)^{2/[4\ell-n(2-p)]} \\ & + c\|\mathbf{P}_r^V\|_{L^\infty(Q_r)}^{4/[(n+2)p-2n]} + c(s+1) \end{aligned} \tag{1.15}$$

holds for every parabolic cylinder $Q_r \subset Q_*$.

Remark 1.16. We remark that the global outcome of Theorems 1.1-1.2 is that in any case $p > 2n/(n+2)$ it holds

$$\mathbf{P}_r^V \in L^\infty \Longrightarrow Du \in L^\infty_{\mathrm{loc}}(Q_*)\,. \tag{1.17}$$

A significant point here is that *condition* (1.17) *is independent of* p, in that the exponent p does not appear in the definition of the potential $\mathbf{P}_r^V$.

Remark 1.18. Condition (1.14) is of course non-void provided $n(2-p)/2 < p$, and this is exactly guaranteed by assuming (1.13). Observe that by taking $\ell = p/2$ in (1.15) we in particular obtain

$$\begin{aligned} \|Du\|_{L^\infty(Q_{r/2})} \leq & c\left(\fint_{Q_r} |Du|^{p}\,dx\,dt\right)^{2/[(n+2)p-2n]} \\ & + c\|\mathbf{P}_r^V\|_{L^\infty(Q_r)}^{4/[(n+2)p-2n]} + c(s+1)\,. \end{aligned} \tag{1.19}$$

Remark 1.20. The form of the estimates presented in Theorems 1.1 and 1.2 is in a certain sense optimal, as in the case $V = 0$ it allows to recover the sharp interpolated L^∞ bounds of DiBenedetto [5]; related bounds in L^q are available in [1]. Indeed, for solutions to the evolutionary p-Laplacean system

$$u_t - \mathrm{div}(|Du|^{p-2}Du) = 0$$

estimate [5, Chapter 8, Theorem 5.1] valid for the case $p \geq 2$ reads as

$$\|Du\|_{L^\infty(Q_{r/2})} \leq c\left(\fint_{Q_r} |Du|^{p-2+2\ell}\,dx\,dt\right)^{1/2\ell} + 1 \tag{1.21}$$

for every $\ell \in (0,1]$ where $c \equiv c(n,N,p,\nu,L,\ell)$, while in the case (1.13) estimate in [5, Chapter 8, Theorem 5.2] gives

$$\|Du\|_{L^\infty(Q_{r/2})} \leq c \left(\fint_{Q_r} |Du|^{2\ell}\, dx\, dt \right)^{1/[4\ell - n(2-p)]} + 1 \tag{1.22}$$

whenever $2\ell > n(2-p)/2$ holds. Estimates (1.21)-(1.22) are now a consequence of (1.12) and (1.15), respectively.

An interesting point in our approach to estimates (1.12) and (1.15) is that it allows for one-step proof of the gradient bounds. Indeed, estimates as (1.21)-(1.22) are usually obtained in three steps: i.e. first Moser's iteration technique to prove that $Du \in L^q$ for every $q < \infty$, then De Giorgi's to prove $Du \in L^\infty$, and finally interpolation to allow the variability of the exponent ℓ. On the contrary, we here achieve the general interpolated form in (1.12)-(1.15) in one step, via a suitable scheme of self-improving estimates and a variant of De Giorgi's iteration technique. Our strategy is to first achieve a very preliminary form of pointwise estimates via certain potentials that still involve the solution in a way that takes into account the intrinsic geometry of the problem. Then, in the second step, we switch from intrinsic potentials to non-intrinsic ones.

We finally remark that the exponent $4/[(n+2)p-2n]$ appearing in the right hand side of (1.15) is typical of subquadratic estimates in parabolic problems - see again [5] - and indicates the asymptotic failure of the estimate when $p \to 2n/(n+2)$. Indeed, when $p < 2n/(n+2)$ finite time blow-up maybe expected. See also Remark 1.27 below.

1.2. **Function spaces criteria.** The estimates of Theorems 1.1 and 1.2 allow to get several boundedness criteria for Du in terms of V; here we shall present a few basic ones regarding borderline function spaces. The heuristic is now the following: as described before Theorem 1.1, the nonlinear potential $\mathbf{P}_r^F$ is strongly related to the Riesz caloric potential $\mathbf{I}_1^F$ in that they essentially exhibit the same scaling properties. In turn this implies that $\mathbf{P}_r^F$ allows to recover the same conditions for gradient boundedness implied by $\mathbf{I}_1^F$, as for instance the borderline ones using Lorentz spaces (we refer to Section 2.2 below for the relevant definitions). Specifically, we recall that in the elliptic case $-\triangle u = V$ we have that $\|Du\|_{L^\infty} \lesssim \|\mathbf{I}_1^V\|_{L^\infty}$ and this in turn implies Lorentz space regularity conditions: $V \in L(n,1)$ is sufficient to conclude with the gradient boundedness. Here we see that the potential $\mathbf{P}_r^F$ allows for completely similar conclusions in the parabolic case, where by elementary scaling consideration the right space with $L(n,1)$ has to be replaced is $L(n+2,1)$. Indeed, we have

Theorem 1.3. *Under the assumptions* (1.4)*, let u be a distributional solution as in* (1.6)-(1.7) *to the equation* (1.2)*, or to the system* (1.3) *with $a(z) \equiv g(|z|^2)z$. If $V \in L(n+2,1)$ locally in Q_*, then $Du \in L^\infty_{\mathrm{loc}}(Q_*)$.*

Remark 1.23. Theorem 1.3 provides us with an optimal borderline case of the L^∞-result in [5, Chapter VIII, Section 1-(ii)], where it is proved that

$$V \in L^{n+2+\varepsilon} \Longrightarrow Du \in L^\infty_{\mathrm{loc}}(Q_*)$$

whenever $\varepsilon > 0$. In fact we recall that $L^{n+2+\varepsilon} \subset L(n+2,1)$. We explicitly remark that a significant point of Theorem 1.3 is that the space $L(n+2,1)$ needed for the gradient boundedness is independent of p.

The next result tells us that when $V(\cdot)$ has a special structure, and in particular when it is time-independent, we get closer to the elliptic case; see also Remark 1.27 for more explanations and comments.

Theorem 1.4. *Under the assumptions* (1.4)*, let u be a distributional solution as in* (1.6)-(1.7) *to the equation* (1.2)*, or to the system* (1.3) *with $a(z) \equiv g(|z|^2)z$; assume also that $n > 2$. If*

$$V(x,t) = V_1(x)V_2(t) \tag{1.24}$$

where $V_1 \in L(n,1)$ and $V_2 \in L^\infty$ locally in Ω and $(0,\tau_*)$ respectively, then $Du \in L^\infty_{\mathrm{loc}}(Q_*)$.

As mentioned, we remark that the limiting role of the space $L(n,1)$ already appears when looking at Poisson $-\triangle u = V$: It happens that $V \in L(n,1)$ is a sharp condition for the gradient boundedness, while the weaker $V \in L^n$ is not sufficient. Moreover, when $V \in L^q$ for some $q > n$, then the gradient is Hölder continuous. We refer to [32, 33, 34] for a more detailed description of the setting and for further references.

Remark 1.25. We observe that by looking at the proof of Theorems 1.3-1.4 it is not difficult to see that it is also possible to have explicit local estimates of $\|Du\|_{L^\infty}$, in terms of the norms $\|V\|_{L(n+2,1)}$ and $\|V_1\|_{L(n,1)}\|V_2\|_{L^\infty}$, respectively.

1.3. **Refined bounds.** Considering the case of Theorem 1.4 we observe that when $V(\cdot)$ is time independent - or equivalently is of the form (1.24) with $V_2(t)$ being a locally bounded function - estimate (1.12) can be slightly improved in that a better exponent is allowed for the potential.

Theorem 1.5. *Under the assumptions* (1.4)*, let u be a distributional solution as in* (1.6)-(1.7) *to the equation* (1.2)*, or to the system* (1.3) *with $a(z) \equiv g(|z|^2)z$; assume also that $p \geq 2$ and that $V(x,t) \equiv V(x)$. Then, for every $\ell \in (0,1]$ there exists a constant c, depending only on n, N, p, L, ν, ℓ such that*

$$\begin{aligned} \|Du\|_{L^\infty(Q_{r/2})} \leq & c\left(\fint_{Q_r} (|Du|+s)^{p-2}\,|Du|^{2\ell}\,dx\,dt\right)^{1/2\ell} \\ & + c\|\mathbf{P}^V_r\|^{1/(p-1)}_{L^\infty(Q_r)} + c(s+1) \end{aligned} \tag{1.26}$$

holds for every parabolic cylinder $Q_r \subset Q_*$.

Remark 1.27. The occurrence of the smaller exponent $1/(p-1)$ instead of $2/p$ in (1.26), when $V(\cdot)$ is time independent, is not surprising. Indeed, this is exactly the parabolic version of the elliptic estimate in [14]. Thinking of the asymptotic profile - that is of course possible when $V(x,t)$ is independent of t - estimate (1.26) reduces to the elliptic one in [14]. We observe that apparently it is not possible to derive the same estimate in the case $p \leq 2$. This seems to be linked to the fact that when $p < 2$ the diffusivity of the equation weakens up to the stage that it becomes insufficient to prevent a blow-up when $p < 2n/(n+2)$. In turn a finite time blow-up of the gradient may already occur when $V(x,t)$, which acts as a source term, is time independent, so that it makes no sense to talk about stationary solutions. The exponent $4/[(n+2)p-2n]$ appearing in (1.19) therefore reflects the tendency of the estimate to deteriorate when p approaches the borderline exponent $2n/(n+2)$.

1.4. **Plan of the paper and technical novelties.** This is now as follows: in Section 2 we recall the basic notation and a number of basic properties of Lorentz spaces; in particular, we establish a few elementary but useful inequalities linking such spaces to the nonlinear potential introduced in (1.11).

In Section 3 we build up an approximation scheme aimed at reducing the proof of the regularity results to a priori estimates valid for a priori regular solutions. This involves a certain number of rather standard arguments and the proofs will be at this stage mostly sketched.

In Section 4 we shall prove a basic (weighted) Caccioppoli type estimate - see (4.2) below - valid for the function $v := |Du|^2$. As explained later, the validity of such an energy estimate stems from the fact that when the right hand side V is equal to zero, the quantity $v = |Du|^2$ turns out to be a subsolution of a suitable parabolic equation of porous medium type [24]. This fact, originally observed by Uhlenbeck in the elliptic case [39] and going back to Bernstein in the linear case, extends to parabolic systems as well, see [5]. A side effect of this property of $v \equiv |Du|^2$ is that, although the function v ceases to be a subsolution when $V \neq 0$, we can still prove an energy inequality for it, that in turn reduces to the one that v would automatically satisfy, when $V = 0$, as a subsolution of a linear parabolic equation. This fact holds in the scalar case $N = 1$ while in the general vectorial case it holds provided the structure assumption (1.8) is considered. It is worth to point out at this stage that, following a traditional path going back to the classical De Giorgi's paper [3], all the remaining proofs of the paper will use the fact that v satisfies (4.2), and the fact that u is a solution will not be anymore used.

In Section 5 we show how to apply the Caccioppoli's type inequality (4.2) to derive intrinsic potential estimates for v, namely (5.11) and (5.26). This means that the resulting estimates will not involve a potential of the right hand side $\mathbf{P}_r^V$, but rather a intrinsic potential of the type (2.2) below, that is $\mathbf{P}_{r,k}^{\widetilde{V}}$, where both k and $\widetilde{V}$ depend on v itself. The reason for this occurrence is that the energy estimate (4.2) turns out to be a weighted one when $p \neq 2$, with a weight of the type $(s^2+v)^{(p-2)/2}$ that is naturally linked to the structure of the p-Laplacean operator. In turn this is a typical fact when dealing with the evolutionary p-Laplacean operator, which clearly exhibits a space/time anisotropicity when $p \neq 2$.

In Section 6 we finally use properly scaling and iteration argument to set us free from the intrinsic potentials used, finally achieving the proof of the results. In Section 7 we outline some possible refinements concerning more general equations and systems of the type

$$u_t - \operatorname{div} a(x,t,u,Du) = b(x,t,u,Du)\,. \tag{1.28}$$

2. Preliminary results and notation

2.1. **Notation.** In this paper we follow the usual convention of denoting by c a general constant larger (or equal) than one, possibly varying from line to line; special occurrences will be denoted by c_1 etc; relevant dependence on parameters will be emphasized using parentheses. Again following a standard convention we shall denote by χ_A the indicator function of a set A. Moreover, given a real valued function g and a real number k, we shall denote

$$(g-k)_+ := \max\{g-k, 0\}\,.$$

With $A \subset \mathbb{R}^{n+1}$ being a measurable subset with positive measure, and with $g\colon A \to \mathbb{R}^k$ being a measurable map, we shall denote its average by

$$\fint_A g(x,t)\,dx\,dt := \frac{1}{|A|}\int_A g(x,t)\,dx\,dt\,,$$

$|A|$ being the Lebesgue measure of A. In the rest of the paper, when considering function spaces of vector valued i.e. taking values in $\mathbb{R}^k$ with $k > 1$, such as $W^{1,p}(\Omega,\mathbb{R}^k), C^0(\Omega,\mathbb{R}^k)$, etc, when not essential in the context we shall omit denoting $\mathbb{R}^k$ thereby simply denoting $W^{1,p}(\Omega), C^0(\Omega)$, etc.

We shall denote in a standard way

$$B_r(x^*) := \{x \in \mathbb{R}^n \,:\, |x-x^*| < r\}$$

the open ball with center x^* and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_r \equiv B_r(x^*)$. Moreover, when more than one ball will come into the play, they will always share the same center unless otherwise stated. We shall also denote $B \equiv B_1 = B_1(0)$; more in general, when no confusion will arise or when the specific radius or center will not be important we shall abbreviate by B any ball under consideration. When referring to a certain ball $B \subset \mathbb{R}^n$, we shall often indicate by $r(B)$ its radius. As usual, the standard parabolic cylinders with are defined as

$$Q_r(x^*,t^*) = B_r(x^*) \times (t^* - r^2, t^*)\,.$$

Beside the usual parabolic cylinder we shall it useful to define the intrinsic cylinders of the type

$$Q_{r,k}(x^*,t^*) = B_r(x^*) \times (t^* - k^{-(p-2)/2}r^2, t^*), \qquad k,r>0, \tag{2.1}$$

for $(x^*,t^*) \in \mathbb{R}^{n+1}$; obviously $Q_r(x^*,t^*) \equiv Q_{r,1}(x^*,t^*)$. The use of the word *intrinsic* stems from the fact that such cylinders will be used in a context where the number k depends on the behavior of the solutions on the same cylinder. This terminology has been introduced by DiBenedetto in [5]. As for the balls, when the vertex will not be important in the context, or when all the cylinders will share the same vertex, we shall simply denote $Q_r(x^*,t^*) \equiv Q_r$ or $Q_{r,k}(x^*,t^*) \equiv Q_{r,k}$.

Accordingly, beside the potential defined in (1.11), for an L^2-integrable map $F\colon Q_* \mapsto \mathbb{R}^N$ we define *the intrinsic nonlinear potential* $\mathbf{P}^F_{r,k}$ as

$$\mathbf{P}^F_{r,k}(x^*,t^*) = \int_0^r \left(\varrho^2 \fint_{Q_{\varrho,k}(x^*,t^*)} |F|^2\,dx\,dt\right)^{1/2} \frac{d\varrho}{\varrho}\,. \tag{2.2}$$

We obviously have $\mathbf{P}^F_{r,1} \equiv \mathbf{P}^F_r$.

We now recall a few basic consequences of the assumptions in (1.4). It is standard to check that the ellipticity property in $(1.4)_2$ implies the following strong form of monotonicity:

$$c^{-1}(|z_1|^2+|z_2|^2+s^2)^{(p-2)/2}|z_2-z_1|^2 \le \langle a(z_2)-a(z_1), z_2-z_1\rangle \tag{2.3}$$

where $c \equiv c(n,p,\nu) > 0$, and whenever $z_1, z_2 \in \mathbb{R}^{Nn}$. Finally, we recall a standard iteration lemma whose proof can be for instance found in [17].

Lemma 2.1. *Let $\xi > 0$ and $0 < \delta < 1$. Suppose that the function ϕ : $(0,1] \mapsto [0,\infty)$ is finite and satisfies*

$$\phi(\sigma') \le \delta\phi(\sigma) + \frac{A}{(\sigma-\sigma')^\xi} + B \tag{2.4}$$

whenever $0 < r/2 \le \sigma' < \sigma \le r$. Then there is a constant c depending only on ξ and δ such that

$$\phi(r/2) \le \frac{cA}{r^\xi} + cB\,.$$

The next elementary lemma is taken from [2].

Lemma 2.2. *For every $p \in [1,\infty)$ and $s \ge 0$ there is a constant c depending only on k and p such that the following inequality*

$$(|\xi|^2+s^2)^{p/2} \le c(|w|^2+s^2)^{p/2} + c(|w|^2+|\xi|^2+s^2)^{(p-2)/2}|\xi-w|^2$$

holds whenever $\xi, w \in \mathbb{R}^k$.

2.2. **Lorentz spaces and nonlinear potentials.** We start recalling a few basic definitions concerning Lorentz spaces. Let $F\colon Q_* \to \mathbb{R}^N$ be a measurable map, and let preliminary extend F to $\mathbb{R}^{n+1}$ by letting $F \equiv 0$ outside Q_*. We assume $|\{z \in Q_* : |F(z)| > t\}| < \infty$ for $t \geq 0$. The decreasing rearrangement $F^*\colon [0,\infty] \to [0,\infty]$ of F is defined as the (unique) non-increasing, right continuous function which is equi-distributed with $|F(\cdot)|$, that is

$$F^*(s) := \sup\left\{h \geq 0 \,:\, |\{z \in \mathbb{R}^{n+1} \,:\, |F(z)| > h\}| > s\right\}.$$

Now, the usual definition of the Lorentz space $L(\gamma,q)(Q_*) \equiv L(\gamma,q)$, for $\gamma \in (0,\infty)$ and $q \in (0,\infty)$ prescribes that

$$[F]_{\gamma,q} := \left(\frac{q}{\gamma}\int_0^\infty \left(F^*(\varrho)\varrho^{1/\gamma}\right)^q \frac{d\varrho}{\varrho}\right)^{1/q} < \infty\,. \tag{2.5}$$

The local version of Lorentz spaces is defined in the usual way by saying that $F \in L(\gamma,q)$ locally in Q_* iff $\chi_A F \in L(\gamma,q)$ for every open subset $A \Subset Q_*$.

Lorentz spaces refine the standard Lebesgue spaces and it follows from the definition that $L(\gamma,\gamma) \equiv L^\gamma$. For more on Lorentz spaces we refer for instance to [37]. A classical fact due to Hunt [19] states that when considering the maximal operator of F^* it is possible to obtain a quantity, in fact equivalent to the one in (2.5) when $\gamma > 1$, that defines a norm in $L(\gamma,q)$. More precisely, defining for $s > 0$ the following maximal operator:

$$F^{**}(s) := \frac{1}{s}\int_0^s F^*(t)\,dt \tag{2.6}$$

for $q < \infty$ the quantity

$$\|F\|_{\gamma,q} := \left(\frac{q}{\gamma}\int_0^\infty \left(F^{**}(\varrho)\varrho^{1/\gamma}\right)^q \frac{d\varrho}{\varrho}\right)^{1/q}$$

is such that

$$[F]_{\gamma,q} \leq \|F\|_{\gamma,q} \leq c(\gamma,q)[F]_{\gamma,q} \qquad\qquad \text{for } \gamma > 1\,, \tag{2.7}$$

see for instance [37, Theorem 3.21]. The following inequality, which is a straightforward corollary of the definition of rearrangement of a function, holds whenever $A \subseteq Q_*$ is a measurable set:

$$\int_A |F(x,t)|\,dx\,dt \leq \int_0^{|A|} F^*(s)\,ds\,. \tag{2.8}$$

In this paper we shall consider essentially two Lorentz spaces, that are $L(n+2,1)$ in $\mathbb{R}^{n+1}$ and $L(n,1)$ in $\mathbb{R}^n$. In this respect we briefly describe a connection between such spaces and the nonlinear potential defined in (1.11).

Lemma 2.3. *Let $F \in L^2(\mathbb{R}^{n+1}, \mathbb{R}^N)$; for every $r > 0$ it holds that*

$$\|\mathbf{P}_r^F\|_{L^\infty} \leq c_1 \int_0^{\omega_n r^{n+2}} \left(\left(|F|^2\right)^{**}(\varrho)\varrho^{2/(n+2)}\right)^{1/2} \frac{d\varrho}{\varrho}\,, \tag{2.9}$$

where the constant c_1 depends only on n and ω_n denotes the measure of the unit ball in $\mathbb{R}^n$. In particular

$$F \in L(n+2,1) \Longrightarrow \mathbf{P}_r^F \in L^\infty\,. \tag{2.10}$$

Proof. Observe that (2.8) implies

$$\varrho^2 \fint_{Q_\varrho(x^*,t^*)} |F|^2\,dx\,dt \leq \frac{c\varrho^2}{\omega_n \varrho^{n+2}} \int_0^{\omega_n \varrho^{n+2}} (|F|^2)^*(s)ds \leq \varrho^2 (|F|^2)^{**}(\omega_n \varrho^{n+2})\,.$$

Therefore we also have

$$\mathbf{P}_r^F(x^*,t^*) \leq \int_0^r \left[(|F|^2)^{**}(\omega_n \varrho^{n+2})\right]^{1/2} d\varrho$$

and (2.9) follows changing variables in the previous inequality since the point (x^*,t^*) is arbitrary. Finally, the implication in (2.10) is a consequence of the fact that $F \in L(n+2,1)$ iff $|F|^2 \in L((n+2)/2, 1/2)$, together with (2.7). $\square$

A completely similar proof gives the following:

Lemma 2.4. *Let $F \in L^2(\mathbb{R}^{n+1}, \mathbb{R}^N)$ be time independent, and let $n > 2$. If $F \in L(n,1)$ then $\mathbf{P}_r^F \in L^\infty$. The same holds if $F(x,t) = F_1(x)F_2(t)$ with $F_1 \in L(n,1)$ and $F_2 \in L^\infty$.*

3. Approximation and preliminary reduction to a priori estimates

In this section we show how to reduce the proof of the theorems stated in the Introduction to the case the solutions considered are supposed to enjoy additional regularity properties. Specifically after the results in this section we shall show how to reduce the case that the solution u considered is such that

$$u \in L^2_{\rm loc}(0,\tau_*; W^{2,2}_{\rm loc}(\Omega,\mathbb{R}^N) \cap W^{1,\infty}_{\rm loc}(\Omega,\mathbb{R}^N))\,, \quad Du \in C^{0,\alpha}_{\rm loc}(Q_*, \mathbb{R}^{Nn}) \tag{3.1}$$

for some $\alpha \in (0,1]$. The procedure is rather standard and similar approximation methods useful in the present regularity setting are described both in the elliptic and in the parabolic setting for instance in [14, 16]. For this reason we shall sketch several parts of the proofs, describing in detail only those needing additional explanations.

3.1. **Approximation in the scalar case.** We start explaining the approximation procedure in the scalar case $N = 1$. First of all let us remark that, since the estimates of Theorems 1.1-1.5 are local in nature, up to consider smaller cylinders of the type $\Omega' \times (\sigma, \tau_* - \sigma)$, with $\Omega' \Subset \Omega$ being a Lipschitz subdomain and $\sigma \in (0, \tau_*)$, we can in the following assume that

$$u \in C^0(0, \tau_*; L^2(\Omega)) \cap L^p(0, \tau_*; W^{1,p}(\Omega))$$

and that Ω is itself Lipschitz regular. Yet, we recall that up to letting $V \equiv 0$ outside Q_*, we shall assume that $V \in L^2(\mathbb{R}^{n+1})$.

We start mollifying the vector field $a(\cdot)$ and truncate the source term $V(\cdot)$ as follows. Let $\varepsilon > 0$ and let $\theta_\varepsilon \in C_0^\infty(B_\varepsilon(0))$ be a standard mollifier with $B_\varepsilon(0) \subset \mathbb{R}^n$, such that $\int_{\mathbb{R}^n} \theta_\varepsilon(z)\, dz = 1$. Define

$$a_\varepsilon(z) = \int_{\mathbb{R}^n} \theta_\varepsilon(z-y) a(y)\, dy.$$

Then $a_\varepsilon(\cdot)$ is clearly a smooth vector field and, moreover, the following structural conditions hold:

$$\begin{cases} |a_\varepsilon(z)| + |\partial a_\varepsilon(z)|\,(|z|^2 + s_\varepsilon^2)^{1/2} \leq c\,(|z|^2 + s_\varepsilon^2)^{(p-1)/2} \\ c^{-1}\,(|z|^2 + s_\varepsilon^2)^{(p-2)/2}\,|\lambda|^2 \leq \langle \partial a_\varepsilon(z)\lambda, \lambda\rangle \end{cases}$$

whenever $z, \lambda \in \mathbb{R}^n$, with a constant $c \geq 1$ depending only n, p, ν, L but not on ε; for this see also [14, 16]. Here

$$s_\varepsilon = s + \varepsilon > 0\,. \tag{3.2}$$

By using $(1.4)_1$ and mean value theorem

$$\begin{aligned} |a(y) - a(z)| \leq |y - z| \sup_{\xi \in B_\varepsilon(z)} |\partial a(\xi)| &\leq \varepsilon c |z| \sup_{\xi \in B_\varepsilon(z)} \left(|\xi|^2 + s^2\right)^{(p-2)/2} \\ &\leq \varepsilon c \left(|z|^2 + s_\varepsilon^2\right)^{(p-1)/2} \end{aligned}$$

for all $y \in B_\varepsilon(z)$. It thus follows that

$$|a(z) - a_\varepsilon(z)| \leq \varepsilon c \left(|z|^2 + s_\varepsilon^2\right)^{(p-1)/2}. \tag{3.3}$$

Moreover, we truncate V as

$$V_\varepsilon = \min\left\{\frac{1}{\varepsilon}, \max\left\{-\frac{1}{\varepsilon}, V\right\}\right\} \in L^\infty(Q_*).$$

We now define $u_\varepsilon \in C^0(0, \tau_*; L^2(\Omega)) \cap L^p(0, \tau_*; W^{1,p}(\Omega))$ as the unique solution to the following Cauchy-Dirichlet problem:

$$\begin{cases} (u_\varepsilon)_t - \operatorname{div}\, a_\varepsilon(Du_\varepsilon) = V_\varepsilon & \text{in } Q_* \\ u_\varepsilon = u \quad \text{on } \partial_{\mathrm{p}} Q_*\,. \end{cases} \tag{3.4}$$

Here $\partial_{\mathrm{p}} Q_*$ stands for the parabolic boundary of Q_*, i.e.,

$$\partial_{\mathrm{p}} Q_* = (\Omega \times \{0\}) \cup (\partial\Omega \times [0, \tau_*])\,.$$

The lateral boundary values are taken in sense of traces i.e. it holds that $u_\varepsilon - u \in L^p(0,\tau_*;W^{1,p}_0(\Omega))$ and initial values continuously in L^2, i.e., $\|(u_\varepsilon - u)(\cdot,t)\|_{L^2(\Omega)} \to 0$ as $t \downarrow 0$. Such a solution u_ε exists by standard existence theory, see [36]. The nondegeneracy/nonsingularity of the equation of (3.4) and boundedness of V_ε guarantee, by standard regularity theory [5], that

$$u_\varepsilon \in L^2_{\rm loc}(0,\tau_*;W^{2,2}_{\rm loc}(\Omega)) \cap C^\alpha(Q_*), \qquad Du_\varepsilon \in C^{0,\alpha}(Q_*)$$

holds for some $\alpha \in (0,1)$.

In the following, by $\{u_\varepsilon\}$ we shall actually mean a sequence $\{u_{\varepsilon_n}\}$, where $\varepsilon_n \to 0$; moreover, at the end of the proof of the next lemma, we shall pass to subsequences, that will be still denoted, for the sake of simplicity, by $\{u_\varepsilon\}$.

Lemma 3.1. *Under the assumptions of Theorems 1.1-1.5, up to a non-relabeled subsequence, we have that*

$$u_\varepsilon \to u \qquad \text{strongly in} \ \ L^p_{\rm loc}(0,\tau_*;W^{1,p}(\Omega))\,. \tag{3.5}$$

In particular, if the solutions u_ε satisfy a priori estimates in (1.12), (1.15) *and* (1.26) *uniformly with respect to $\varepsilon > 0$, then so does u.*

Proof. Extend first $u_\varepsilon - u$ to be zero outside the set $\Omega \times (0,\tau_*)$ and let $\varrho_h \in C^\infty_0(-h,h)$, $h > 0$, be a standard symmetric mollifier, such that $\int_{\mathbb{R}} \varrho_h \, dt = 1$. Denote in short

$$[w]_h(t) = \int_{\mathbb{R}} \varrho_h(t-s)w(s)\,ds$$

for any integrable function w. Let us subtract the equations (1.2) and $(3.4)_1$ and eventually test with $\phi_h = [[u-u_\varepsilon]_h\theta_h]_h$, and $\phi = \phi_0 = (u-u_\varepsilon)\theta$, where $\theta_h = (\tau_* - t - 2h)_+/(\tau_* - 2h)$, $0 < h < \tau_*/4$, and

$$\theta = \theta_0 = (\tau_* - t)_+/\tau_*\,. \tag{3.6}$$

We then have $\phi_h \in W^{1,p}_0(Q_*)$ and

$$\begin{aligned}\int_{Q_*} &\langle a(Du) - a_\varepsilon(Du_\varepsilon), D\phi_h\rangle \; dx\,dt \\ &= \int_{Q_*} (V - V_\varepsilon)\phi_h \, dx\,dt + \int_{Q_*} (u-u_\varepsilon)(\phi_h)_t \, dx\,dt.\end{aligned} \tag{3.7}$$

Let us estimate the terms appearing on the right hand side of the previous inequality. First, Fubini's theorem and integration by parts implies

$$\begin{aligned}\int_{Q_*} (u-u_\varepsilon)(\phi_h)_t \, dx\,dt &= \frac{1}{2}\int_{Q_*} [u-u_\varepsilon]_h^2(\theta_h)_t \, dx\,dt \\ &= -\frac{1}{2(\tau_*-2h)}\|[u-u_\varepsilon]_h\|^2_{L^2(Q_*)}\,.\end{aligned}$$

Moreover,

$$\left| \int_{Q_*} (V - V_\varepsilon)\phi_h \, dx \, dt \right| \le \|[V - V_\varepsilon]_h\|_{L^2(Q_*)} \|[u - u_\varepsilon]_h\|_{L^2(Q_*)}$$

holds again by Fubini's theorem and then by Hölder's inequality. Thus Young's inequality yields

$$\begin{aligned} &\left| \int_{Q_*} (V - V_\varepsilon)\phi_h \, dx \, dt \right| \\ &\qquad \le \frac{(\tau_* - 2h)}{2} \|[V - V_\varepsilon]_h\|^2_{L^2(Q_*)} + \frac{1}{2(\tau_* - 2h)} \|[u - u_\varepsilon]_h\|^2_{L^2(Q_*)}. \end{aligned}$$

Combining these, we obtain

$$\limsup_{h \to 0} \left(\int_{Q_*} (V - V_\varepsilon)\phi_h \, dx \, dt + \int_{Q_*} (u - u_\varepsilon)(\phi_h)_t \, dx \, dt \right) \le \frac{\tau_*}{2} \|V - V_\varepsilon\|^2_{L^2(Q_*)}.$$

Furthermore, since $u_\varepsilon, u \in L^p(0, \tau_*; W^{1,p}(\Omega))$, we have by the L^p-convergence of mollifiers that

$$\langle a(Du) - a_\varepsilon(Du_\varepsilon), D\phi_h \rangle \to \langle a(Du) - a_\varepsilon(Du_\varepsilon), D\phi \rangle$$

in $L^1(Q_*)$ as $h \to 0$. Thus we conclude with

$$\limsup_{\varepsilon \to 0} \int_{Q_*} \langle a(Du) - a_\varepsilon(Du_\varepsilon), D\phi \rangle \; dx \, dt \le \limsup_{\varepsilon \to 0} \frac{\tau_*}{2} \|V - V_\varepsilon\|^2_{L^2(Q_*)} = 0\,. \tag{3.8}$$

We now rewrite the integrand in (3.8) as

$$\begin{aligned} &\langle a(Du) - a_\varepsilon(Du_\varepsilon), D\phi \rangle \\ &\quad = \langle a(Du) - a(Du_\varepsilon), D\phi \rangle + \langle a(Du_\varepsilon) - a_\varepsilon(Du_\varepsilon), D\phi \rangle \;. \end{aligned} \tag{3.9}$$

Using (3.3) we conclude

$$\langle a(Du_\varepsilon) - a_\varepsilon(Du_\varepsilon), Du - Du_\varepsilon \rangle \le \varepsilon c \left(|Du_\varepsilon|^2 + s_\varepsilon^2 \right)^{(p-1)/2} |D(u - u_\varepsilon)|. \tag{3.10}$$

On the other hand, (2.3) implies

$$c \, \langle a(Du) - a(Du_\varepsilon), Du - Du_\varepsilon \rangle \ge \left(|Du|^2 + |Du_\varepsilon|^2 + s_\varepsilon^2 \right)^{(p-2)/2} |D(u - u_\varepsilon)|^2 \tag{3.11}$$

for all $p > 1$. The last inequality together with Lemma 2.2 further gives

$$|Du_\varepsilon|^p \le c(s_\varepsilon^p + |Du|^p + \langle a(Du) - a(Du_\varepsilon), Du - Du_\varepsilon \rangle)$$

and in view of (3.9) and (3.10), by using Young's inequality, we also have

$$|Du_\varepsilon|^p \le c(s_\varepsilon^p + |Du|^p + \langle a(Du) - a_\varepsilon(Du_\varepsilon), Du - Du_\varepsilon \rangle) \tag{3.12}$$

for sufficiently small ε. Thus, by (3.8) and the previous estimate, we conclude with

$$\int_{Q_*} |Du_\varepsilon|^p\theta \, dx\, dt \le c\left(s_\varepsilon^p + \int_{Q_*} |Du|^p \, dx\, dt + \|V - V_\varepsilon\|^2_{L^2(Q_*)}\right).$$

This, on the other hand, together with (3.10), readily implies

$$\int_{Q_*} \langle a(Du_\varepsilon) - a_\varepsilon(Du_\varepsilon), D\phi\rangle \; dx\, dt \to 0 \tag{3.13}$$

as $\varepsilon \to 0$. Therefore (3.8), (3.9), and (3.11) yield

$$\lim_{\varepsilon\to 0}\int_{Q_*} \left(|Du|^2 + |Du_\varepsilon|^2 + s_\varepsilon^2\right)^{(p-2)/2} |D(u-u_\varepsilon)|^2\theta \, dx\, dt = 0.$$

In the case $p \ge 2$ we immediately obtain that

$$Du_\varepsilon \to Du \qquad \text{in } L^p(Q_*)\,. \tag{3.14}$$

In the case $p < 2$ we obtain the same by Hölder's inequality

$$\begin{aligned}\int_{Q_*} |D(u-u_\varepsilon)|^p\theta \, dx\, dt &\le \left(\int_{Q_*} \left(|Du|^2 + |Du_\varepsilon|^2 + s_\varepsilon^2\right)^{p/2} \theta \, dx\, dt\right)^{(2-p)/p} \\ &\quad \times \left(\int_{Q_*} \left(|Du|^2 + |Du_\varepsilon|^2 + s_\varepsilon^2\right)^{(p-2)/2} |D(u-u_\varepsilon)|^2\theta \, dx\, dt\right)^{2/p} \to 0\end{aligned}$$

as $\varepsilon \to 0$ since the first term on right is bounded by (3.11) and the second one converges to zero by (3.13). Now (3.14) follows again taking into account the definition of θ in (3.6), and using a standard diagonal argument. In turn, Sobolev's embedding theorem implies the convergence of u_ε to u in L^p and we conclude with (3.5). We finally conclude showing how to pass to the limit in estimate (1.12), applied to u_ε. The proof for the a priori estimates of Theorems 1.2 and 1.5 is completely similar. Assuming the uniform validity of (1.12) for u_ε, noting that $\mathbf{P}_r^{V_\varepsilon} \le \mathbf{P}_r^V$, using lower semicontinuity in order to deal with the left hand side and the strong convergence for the gradients in L^p to treat the right hand one, we have

$$\begin{aligned}\|Du\|_{L^\infty(Q_{r/2})} &\le \liminf_{\varepsilon\to 0} \|Du_\varepsilon\|_{L^\infty(Q_{r/2})} \\ &\le c\limsup_{\varepsilon\to 0}\left(\fint_{Q_r} (|Du_\varepsilon| + s)^{p-2}\,|Du_\varepsilon|^{2\ell}\, dx\, dt\right)^{1/2\ell} \\ &\qquad +c\|\mathbf{P}_r^V\|^{2/p}_{L^\infty(Q_r)} + c(s+1) \\ &\le c\left(\fint_{Q_r} (|Du| + s)^{p-2}\,|Du|^{2\ell}\, dx\, dt\right)^{1/2\ell} \\ &\qquad +c\|\mathbf{P}_r^V\|^{2/p}_{L^\infty(Q_r)} + c(s+1)\,.\end{aligned}$$

Note that to conclude the converge of the right hand side integrals it is essential to have that $p-2+2\ell \leq p$. The proof for estimates (1.15) and (1.26) is completely similar. In the case of (1.15) we have of course to assume that $2\ell \leq p$. □

3.2. **Approximation in the vectorial case.** Here we pass to examine the vectorial case $N > 1$. The only different point with respect to the scalar case is that the regularized vector fields $a_\varepsilon(\cdot)$ have to be constructed in such a way that the additional structure condition (1.8) is also satisfied. To this aim we recall a technique already used in [16, Lemma 3.2]. For $\varepsilon > 0$ we define

$$a_\varepsilon(z) := g_\varepsilon(|z|^2)z\,, \qquad \text{where} \;\; g_\varepsilon(t) := g(\varepsilon^2 + t)\,. \tag{3.15}$$

Using (1.4) an (1.8) it is now easy to see that the same (1.4) are satisfied by $a_\varepsilon(\cdot)$ with s_ε defined as in (3.2). The rest of the proof proceeds exactly as in the scalar case.

4. A Caccioppoli inequality of porous medium type

In this section we prove a weighted Caccioppoli estimate which is the starting point of all the later results. More precisely we shall exploit the well-known fact that in the case of a zero right-hand side $V = 0$, the quantity $v = |Du|^2$ turn out to be a subsolution of a parabolic equation of porous medium type. The idea is now that, although the presence of a non-zero right hand side V does not allow to draw a similar conclusion, the same principle can be nevertheless used to prove directly that v satisfies a weighted Caccioppoli type estimate. Such an inequality in turn coincides with the usual one that v would automatically satisfy as a subsolution in the case $V = 0$.

We shall give actually two proofs: the first is for general parabolic equations; the second one for systems satisfying the quasidiagonal structure condition in (1.3). By the remarks and the approximation procedure in Section 3 we shall assume to work with a nondegenerate equation/system, that is the parameter s in (1.4) is assumed to be larger than zero - and to deal with a smoother solution, more precisely

$$u \in L^2(0, \tau_*; W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega))\,. \tag{4.1}$$

In the rest of the section we shall denote by Q a general cylinder of the type $Q \equiv B \times (\tau_1, \tau_2) \subset Q_*$. In this case we shall denote $T := (\tau_1, \tau_2)$.

Proposition 4.1. *Let u be a distributional solution to* (1.2) *with $s > 0$, and satisfying the regularity condition in* (4.1). *Let $v = |Du|^2$ and $k > 0$; let $\phi \in C^\infty_0(Q)$ be nonnegative. Then there is a constant c_1, depending only on*

n,p,ν,L, such that

$$
\begin{aligned}
\operatorname*{ess\,sup}_{t\in T}\int_B (v-k)_+^2\phi^2\,dx &+ \int_Q (s^2+v)^{(p-2)/2}|D(v-k)_+|^2\phi^2\,dx\,dt\\
&\leq c_1\int_Q \Big((s^2+v)^{(p-2)/2}(v-k)_+^2|D\phi|^2+(v-k)_+^2\left((\phi^2)_t\right)_+\Big)\,dx\,dt \qquad (4.2)\\
&\quad + c_1\int_Q \widetilde{V}_k^2\phi^2\,dx\,dt
\end{aligned}
$$

where

$$
\widetilde{V}_k := \left(s^2+v\right)^{1-p/4}|V|\chi_{\{v>k\}}\,. \tag{4.3}
$$

Proof. In the following we shall repeatedly use Einstein's summation convention on repeated indexes; moreover, we shall denote by $\phi\in C_0^\infty(Q)$ a smooth test function that we will eventually change and choose several times. Actually the choice we shall make of the functions ϕ will not lead to a C^∞ choice; on the other hand, by the assumed regularity of u in (4.1) and usual density arguments, will always lead to an admissible choice in integral identities where the choice of ϕ will be made.

Finally, note that the usual problem - typical when dealing with parabolic equations - that consists of considering solutions that are not differentiable with respect to time can be bypassed as usual by using a smoothing procedure via mollifiers or via Steklov averages; for this we refer to [5]. It is anyway important to remark here that, when testing with functions like $v\equiv|Du|^2$ or the like, this is allowed since we are assuming more regularity on the solution, namely (4.1).

Indeed, recalling (4.1), in the weak formulation (1.6) instead of ϕ we insert $D_m\phi$, $m\in\{1,\dots,n\}$, and integrate by parts to get

$$
\int_Q \left(-D_m u\phi_t+(D_m(a_i(Du)))D_i\phi\right)\,dx\,dt = -\int_Q VD_m\phi\,dx\,dt.
$$

Let us introduce the matrix

$$
\widetilde{a}_{i,j}(z)=\frac{(a_i)_{z_j}(z)}{(s^2+|z|^2)^{(p-2)/2}}
$$

for $z\in\mathbb{R}^n$, which is uniformly elliptic in the sense that

$$
|\widetilde{a}(z)|\leq L, \qquad \langle\widetilde{a}(z)\lambda,\lambda\rangle\geq\nu|\lambda|^2 \tag{4.4}
$$

hold for all $z,\lambda\in\mathbb{R}^n$. By using that $D_m a_i(Du)=(a_i)_{z_j}(Du)(D_{mj}u)$ we therefore arrive at

$$
\int_Q \left(-D_m u\phi_t+(s^2+|Du|^2)^{(p-2)/2}\widetilde{a}_{i,j}D_{mj}uD_i\phi\right)\,dx\,dt = -\int_Q VD_m\phi\,dx\,dt\,. \tag{4.5}
$$

Next, instead of ϕ, insert $\phi D_m u$ in (4.5); note that this is one of the points that has to be justified via the use of mollifiers or Steklov averages. Since

$$D_{mj}u(D_i(\phi D_m u)) = D_{mj}uD_{mi}u\phi + D_{mj}uD_m uD_i\phi$$

and

$$-D_m u(\phi D_m u)_t = -\frac{1}{2}((D_m u)^2)_t\phi - (D_m u)^2\phi_t$$

hold, integration by parts first yields

$$-\int_Q D_m u(\phi D_m u)_t\,dx\,dt = -\frac{1}{2}\int_Q (D_m u)^2\phi_t\,dx\,dt$$

and thus we gain

$$\begin{aligned}
&\int_Q \frac{1}{2}(D_m u)^2\phi_t\,dx\,dt \\
&\qquad + \int_Q (s^2+|Du|^2)^{(p-2)/2}\widetilde{a}_{i,j}\left(D_{mj}uD_{mi}u\phi + D_{mj}uD_m uD_i\phi\right)\,dx\,dt \\
&\qquad\qquad = -\int_Q V\left(\phi D_{mm}u + D_m uD_m\phi\right)\,dx\,dt\,,
\end{aligned}$$

which in turn holds whenever $\phi \in C_0^\infty(Q)$. Summing up the previous identity over $m \in \{1,\ldots,n\}$, recalling that since $v = |Du|^2$ it then follows $D_j v = 2D_{jm}uD_m u$, we obtain

$$\begin{aligned}
I + II :=& \int_Q (s^2+v)^{(p-2)/2}\widetilde{a}_{i,j}D_j vD_i\phi\,dx\,dt \\
&+ \sum_m \int_Q (s^2+v)^{(p-2)/2}\phi\widetilde{a}_{i,j}D_{mj}uD_{mi}u\,dx\,dt \\
=& -\sum_m \left(\int_Q V\left(\phi D_{mm}u + D_m uD_m\phi\right)\,dx\,dt\right) + \frac{1}{2}\int_Q v\phi_t\,dx\,dt \\
:=& \, III + IV\,.
\end{aligned} \tag{4.6}$$

Now we replace ϕ with $(v-k)_+\phi^2$, $k > 0$, and with $\phi \geq 0$; note that this again requires justification via use of Steklov averages since v does not have time derivatives. We estimate the resulting terms in (4.6), starting by IV. We get

$$IV = \frac{1}{2}\int_Q v((v-k)_+\phi^2)_t\,dx\,dt = \frac{1}{4}\int_Q (v-k)_+^2(\phi^2)_t\,dx\,dt\,.$$

We then pass to II; Young's inequality in turn allows to establish

$$\widetilde{a}_{i,j}D_j vD_i((v-k)_+\phi^2) \geq \frac{1}{c}|D(v-k)_+|^2\phi^2 - c(v-k)_+^2|D\phi|^2$$

so that

$$\begin{aligned}\int_Q &(s^2{+}v)^{(p-2)/2}|D(v{-}k)_+|^2\phi^2\,dx\,dt\\ &\le cI + c\int_Q (s^2{+}v)^{(p-2)/2}(v{-}k)_+^2|D\phi|^2\,dx\,dt\,.\end{aligned}$$

Again we have

$$\begin{aligned}\int_Q &(v{-}k)_+(s^2{+}v)^{(p-2)/2}|D^2u|^2\phi^2\,dx\,dt\\ &\le c\int_Q (v{-}k)_+(s^2{+}v)^{(p-2)/2}\widetilde{a}_{i,j}D_{mj}uD_{mi}u\phi^2\,dx\,dt = cII\,.\end{aligned}$$

Combining the last four estimates yields

$$\begin{aligned}&\int_Q (v{-}k)_+(s^2{+}v)^{(p-2)/2}|D^2u|^2\phi^2\,dx\,dt\\ &-\frac{1}{4}\int_Q (v{-}k)^2(\phi^2)_t\,dx\,dt + \int_Q (s^2{+}v)^{(p-2)/2}|D(v{-}k)_+|^2\phi^2\,dx\,dt\\ &\quad\le c\int_Q \left((s^2{+}v)^{(p-2)/2}(v{-}k)_+^2|D\phi|^2\right)\,dx\,dt + c|III|\,.\end{aligned}\tag{4.7}$$

To estimate $|III|$ we observe that

$$\begin{aligned}&|D_muD_m((v{-}k)_+\phi^2)|\\ &\quad\le cv^{1/2}\phi\left(|D(v{-}k)_+|\phi + (v{-}k)_+|D\phi|\right)\\ &\quad\le c(s^2{+}v)^{(1-p/4)}\phi\left((s^2{+}v)^{(p-2)/4}|D(v{-}k)_+|\phi\right.\\ &\qquad\qquad\left.+(s^2{+}v)^{(p-2)/4}(v{-}k)_+|D\phi|\right)\,,\end{aligned}$$

and

$$\begin{aligned}|(v{-}k)_+\phi^2D_{mm}u| &\le c\sqrt{(v{-}k)_+}(s^2{+}v)^{-(p-2)/4}\sqrt{(v{-}k)_+}(s^2{+}v)^{(p-2)/4}|D^2u|\phi^2\\ &\le c(s^2{+}v)^{(4-p)/4}\sqrt{(v{-}k)_+}(s^2{+}v)^{(p-2)/4}|D^2u|\phi^2\,.\end{aligned}$$

Using the last two inequalities and again applying Young's inequality, we obtain, with $\varepsilon \in (0,1)$,

$$\begin{aligned}|III| &\le \int_Q |V\left((v{-}k)_+\phi^2D_{mm}u{+}D_muD_m((v{-}k)_+\phi^2)\right)|\,dx\,dt\\ &\le c(\varepsilon)\int_Q V^2\phi^2(s^2{+}v)^{2-p/2}\chi_{\{v>k\}}\,dx\,dt\\ &\qquad+\varepsilon\int_Q (s^2{+}v)^{(p-2)/2}\left((v{-}k)_+|D^2u|^2\phi^2 + |D(v{-}k)_+|^2\phi^2\right)\,dx\,dt\\ &\qquad+c\int_Q (s^2{+}v)^{(p-2)/2}(v{-}k)_+^2|D\phi|^2\,dx\,dt.\end{aligned}$$

Connecting the above estimates for the terms $I, \ldots, IV$ to (4.7), and choosing $\varepsilon \equiv \varepsilon(n,p,\nu,L) > 0$ small enough in order to reabsorb terms we have:

$$
\begin{aligned}
-\frac{1}{4}\int_Q (v-k)_+^2(\phi^2)_t\,dx\,dt + \int_Q (s^2+v)^{(p-2)/2}|D(v-k)_+|^2\phi^2\,dx\,dt \\
\leq c\int_Q (s^2+v)^{(p-2)/2}(v-k)_+^2|D\phi|^2\,dx\,dt + c\int_Q \widetilde{V}_k^2\phi^2\,dx\,dt\,.
\end{aligned}
\tag{4.8}
$$

We now take now $\tau \in T$, and in (4.8) replace by ϕ^2 with $\phi^2\theta_j$, where $\theta_j := (\chi_{(-\infty,\tau)} * \Phi_j)$, where $\{\Phi_j\}$ is a sequence of standard smooth and unidimensional mollifiers. We obviously have that $\theta_j \to \chi_{(-\infty,\tau)}$ as $j \to \infty$, and θ_j are smooth; letting $j \to \infty$ in the corresponding version of (4.8) we gain

$$
\begin{aligned}
&\int_{B\times\{t\leq\tau\}} (s^2+v)^{(p-2)/2}|D(v-k)_+|^2\phi^2\,dx\,dt + \int_B [(v-k)_+^2\phi^2](x,\tau)\,dx \\
&\quad\leq c\int_Q \Big((s^2+v)^{(p-2)/2}(v-k)_+^2|D\phi|^2 + (v-k)_+^2\left((\phi^2)_t\right)_+\Big)\,dx\,dt \\
&\qquad + c\int_Q \widetilde{V}_k^2\phi^2\,dx\,dt\,.
\end{aligned}
\tag{4.9}
$$

Being $\tau \in T$ arbitrary in the previous inequality (4.2) follows. □

The previous result has a full analog in the case we are dealing with solutions for parabolic systems under the additional structure conditions reported in (1.8).

Proposition 4.2. *Let u be a vector valued solution to* (1.3) *with $s > 0$, under the assumptions* (1.4) *with $a(z) \equiv g(|z|^2)z$, and enjoying the regularity in* (4.1). *Let $v = |Du|^2$, $k > 0$ and $\phi \in C_0^\infty(Q)$. Then there is a constant c_1, depending only on n, N, p, ν, L, such that inequality* (4.2) *holds with $\widetilde{V}_k$ defined as* (4.3).

Proof. The proof is in several respects similar to that of the previous proposition valid for general equations, nevertheless we give the full details for the sake of readability. The weak formulation (1.6) component-wise now reads as

$$
\int_Q (-u^\alpha\phi_t^\alpha + a_i^\alpha(Du)D_i\phi^\alpha)\,dx\,dt = \int_Q V^\alpha\phi^\alpha\,dx\,dt,
$$

where $\alpha \in \{1,\ldots,N\}$. Instead of ϕ^α, we insert $D_m\phi^\alpha$, $m \in \{1,\ldots,n\}$, and integrate by parts to get

$$
\int_Q (-D_m u^\alpha\phi_t + (D_m(a_i^\alpha(Du)))D_i\phi^\alpha)\,dx\,dt = -\int_Q V^\alpha D_m\phi^\alpha\,dx\,dt\,. \tag{4.10}
$$

Note that (1.8) implies

$$\partial_{z_j^\beta} a_i^\alpha(z) = g(|z|^2)\delta_{ij}\delta_{\alpha\beta} + 2g'(|z|^2) z_i^\alpha z_j^\beta \,,$$

where δ_{ij} denotes the usual Kronecker's symbol; therefore

$$D_m(a_i^\alpha(Du)) = g(|Du|^2) D_{im}u^\alpha + 2g'(|Du|^2)\sum_{j,\beta} D_{jm}u^\beta D_j u^\beta D_i u^\alpha \,. \tag{4.11}$$

For later convenience we again introduce the following matrix, which turns out to be uniformly elliptic in the sense of (4.4):

$$\widetilde{a}_{i,j}^{\alpha,\beta}(x,t) = \frac{\partial_{z_j^\beta} a_i^\alpha(Du(x,t))}{(s^2+|Du(x,t)|^2)^{(p-2)/2}} \,.$$

Next, by (1.4) we notice that

$$\begin{aligned} &\sum_{i,j,\alpha,\beta} \int_Q \left(g(|Du|^2)\delta_{ij}\delta_{\alpha\beta} + 2g'(|Du|^2) D_i u^\alpha D_j u^\beta \right) D_i F^\alpha D_j F^\beta \, dx\, dt \\ &\quad \geq \frac{1}{c} \int_Q \left(s^2 + |Du|^2\right)^{(p-2)/2} |DF|^2 \, dx\, dt \end{aligned} \tag{4.12}$$

holds for all vector fields $F \in L^p(0,\tau_*; W^{1,2}(\Omega;\mathbb{R}^n))$. Similarly, on the diagonal $\alpha = \beta$, the inequality

$$\begin{aligned} &\sum_{i,j,\alpha} \int_Q \left(g(|Du|^2)\delta_{ij} + 2g'(|Du|^2) D_i u^\alpha D_j u^\alpha \right) D_i F^\alpha D_j F^\alpha \, dx\, dt \\ &\quad \geq \frac{1}{c} \int_Q \left(s^2 + |Du|^2\right)^{(p-2)/2} |DF|^2 \, dx\, dt \end{aligned} \tag{4.13}$$

holds. Using (4.11) in (4.10) yields

$$\begin{aligned} I^\alpha &:= -\int_Q V^\alpha D_m \phi^\alpha \, dx\, dt \\ &= \int_Q \left(-D_m u^\alpha \phi_t^\alpha + (D_m(a_i^\alpha(Du))) D_i \phi^\alpha\right) \, dx\, dt. \\ &= -\int_Q D_m u^\alpha \phi_t^\alpha \, dx\, dt \\ &\quad + \sum_{i,j,\beta} \int_Q 2g'(|Du|^2) D_{jm}u^\beta D_j u^\beta D_i u^\alpha D_i \phi^\alpha \, dx\, dt \\ &\quad + \sum_i \int_Q g(|Du|^2) D_{im} u^\alpha D_i \phi^\alpha \, dx\, dt \\ &=: II_1^\alpha + II_2^\alpha + II_3^\alpha \,. \end{aligned}$$

Replacing then ϕ^α with $D_m u^\alpha (v-k)_+\phi^2$, $\phi \in C_0^\infty(Q)$, $k>0$, after summation we have

$$\begin{aligned} I &:= -\sum_{m,\alpha}\int_Q V^\alpha D_m\phi^\alpha\,dx\,dt = -\sum_{m,\alpha}\int_Q V^\alpha D_{mm}u^\alpha(v-k)_+\phi^2\,dx\,dt \\ &\quad -\sum_{m,\alpha}\int_Q V^\alpha D_m u^\alpha D_m(v-k)_+\phi^2\,dx\,dt \\ &\quad -\sum_{m,\alpha}\int_Q 2V^\alpha\phi D_m u^\alpha(v-k)_+D_m\phi\,dx\,dt \\ &=: I_1+I_2+I_3 \end{aligned}$$

with $\phi^\alpha = D_m u^\alpha(v-k)_+\phi^2$. Therefore we ultimately obtain

$$I_1+I_2+I_3 = \sum_\alpha \left(II_1^\alpha + II_2^\alpha + II_3^\alpha\right)\,. \tag{4.14}$$

We now treat the terms in the right hand side of (4.14). First, recalling that $v=|Du|^2$, integration by parts yields

$$II_1 := -\sum_{m,\alpha}\int_Q D_m u^\alpha (D_m u^\alpha(v-k)_+\phi^2)_t\,dx\,dt = -\frac{1}{4}\int_Q (v-k)_+^2(\phi^2)_t\,dx\,dt. \tag{4.15}$$

Next, we have

$$\begin{aligned} II_2 &:= \sum_{i,j,m,\alpha,\beta}\int_Q 2g'(|Du|^2)D_{jm}u^\beta D_j u^\beta D_i u^\alpha D_i(D_m u^\alpha(v-k)_+\phi^2)\,dx\,dt \\ &= \sum_{i,j,m,\alpha,\beta}\int_Q 2g'(|Du|^2)D_{jm}u^\beta D_j u^\beta D_i u^\alpha D_{im}u^\alpha(v-k)_+\phi^2\,dx\,dt \\ &\quad + \sum_{i,j,m,\alpha,\beta}\int_Q 2g'(|Du|^2)D_{jm}u^\beta D_j u^\beta D_i u^\alpha D_m u^\alpha D_i(v-k)_+\phi^2\,dx\,dt \\ &\quad + \sum_{i,j,m,\alpha,\beta}\int_Q 4g'(|Du|^2)D_{jm}u^\beta D_j u^\beta D_i u^\alpha D_m u^\alpha(v-k)_+\phi D_i\phi\,dx\,dt \\ &=: II_{2,1}+II_{2,2}+II_{2,3}\,, \end{aligned}$$

and

$$\begin{aligned} II_3 &:= \sum_{i,m,\alpha}\int_Q g(|Du|^2)D_{im}u^\alpha D_i(D_m u^\alpha(v-k)_+\phi^2)\,dx\,dt \\ &= \sum_{i,m,\alpha}\int_Q g(|Du|^2)D_{im}u^\alpha D_{im}u^\alpha(v-k)_+\phi^2\,dx\,dt \\ &\quad + \sum_{i,m,\alpha}\int_Q g(|Du|^2)D_{im}u^\alpha D_m u^\alpha D_i(v-k)_+\phi^2\,dx\,dt \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,m,\alpha}\int_Q 2g(|Du|^2)D_{im}u^\alpha D_m u^\alpha (v-k)_+\phi D_i\phi\,dx\,dt\\
=:\ & II_{3,1}+II_{3,2}+II_{3,3}\,.
\end{aligned}
$$

After rewriting the sum of integrands of $II_{2,1}$ and $II_{3,1}$ as

$$
(v-k)_+\phi^2\sum_{i,j,m,\alpha,\beta}\left(g(|Du|^2)\delta_{ij}\delta_{\alpha\beta}+2g'(|Du|^2)D_ju^\beta D_iu^\alpha\right)D_{jm}u^\beta D_{im}u^\alpha,
$$

we appeal to the ellipticity condition (4.12) and conclude with

$$
II_{2,1}+II_{3,1}\geq \frac{1}{c}\int_Q (v-k)_+(s^2+v)^{(p-2)/2}|D^2u|^2\phi^2\,dx\,dt. \tag{4.16}
$$

Furthermore, since

$$
D_m(v-k)_+ = D_m(|Du|^2-k)_+ = 2\chi_{\{v>k\}}\sum_{j,\beta}D_{jm}u^\beta D_ju^\beta, \tag{4.17}
$$

we obtain

$$
II_{2,2}=\sum_{i,m,\alpha}\int_Q 2g'(|Du|^2)D_iu^\alpha D_mu^\alpha D_m(v-k)_+D_i(v-k)_+\phi^2\,dx\,dt
$$

and

$$
II_{3,2}=\sum_{i,m}\delta_{im}\int_Q g(|Du|^2)D_m(v-k)_+D_i(v-k)_+\phi^2\,dx\,dt.
$$

Thus (4.13) implies

$$
II_{2,2}+II_{3,2}\geq\frac{1}{c}\int_Q (s^2+v)^{(p-2)/2}|D(v-k)_+|^2\phi^2\,dx\,dt\,. \tag{4.18}
$$

Using again (4.17) and Young's inequality, we also obtain

$$
\begin{aligned}
|II_{2,3}|+|II_{3,3}| &\leq c\int_Q (s^2+v)^{(p-2)/2}|D(v-k)_+|(v-k)_+|D\phi|\phi\,dx\,dt\\
&\leq \varepsilon\int_Q (s^2+v)^{(p-2)/2}|D(v-k)_+|^2\phi^2\,dx\,dt\\
&\qquad + c(\varepsilon)\int_Q (s^2+v)^{(p-2)/2}(v-k)_+^2|D\phi|^2\,dx\,dt\,,
\end{aligned}
$$

where $\varepsilon\in(0,1)$. Combining the last inequality with (4.16) and (4.18), recalling (4.15), and choosing $\varepsilon\equiv\varepsilon(n,N,p,\nu,L)$ small enough in order to reabsorb terms, we arrive at

$$
\begin{aligned}
II_1+&II_2+II_3\\
&\geq -\frac{1}{4}\int_Q (v-k)^2(\phi^2)_t\,dx\,dt+\int_Q (s^2+v)^{(p-2)/2}|D(v-k)_+|^2\phi^2\,dx\,dt\\
&\quad+\int_Q (v-k)_+(s^2+v)^{(p-2)/2}|D^2u|^2\phi^2\,dx\,dt\\
&\quad-c\int_Q (s^2+v)^{(p-2)/2}(v-k)_+^2|D\phi|^2\,dx\,dt\,,
\end{aligned}
\tag{4.19}
$$

for $c\equiv c(n,N,p,\nu,L)$. Furthermore, again by Young's inequality, we obtain

$$
\begin{aligned}
I_1:=\sum_\alpha I_1^\alpha \leq& c\int_Q |V||D^2u|(v-k)_+\phi^2\,dx\,dt\\
\leq& \varepsilon\int_Q (v-k)_+(s^2+v)^{(p-2)/2}|D^2u|^2\phi^2\,dx\,dt+c(\varepsilon)\int_Q \widetilde{V}_k^2\phi^2\,dx\,dt.
\end{aligned}
$$

Similarly, we find

$$
I_2:=\sum_\alpha I_2^\alpha \leq \varepsilon\int_Q (s^2+v)^{(p-2)/2}|D(v-k)_+|^2\phi^2\,dx\,dt+c(\varepsilon)\int_Q \widetilde{V}_k^2\phi^2\,dx\,dt
$$

and

$$
I_3:=\sum_\alpha I_3^\alpha \leq c\int_Q (s^2+v)^{(p-2)/2}(v-k)_+^2|D\phi|^2\,dx\,dt+c\int_Q \widetilde{V}_k^2\phi^2\,dx\,dt\,,
$$

where we recall that, as in (4.3), it is $\widetilde{V}_k=(s^2+v)^{1-p/4}\,|V|\chi_{\{v>k\}}$.

Merging the last three inequalities with (4.19) and (4.14), finally choosing ε small in order to reabsorb terms, we conclude with

$$
\begin{aligned}
-\frac{1}{4}\int_Q (v-k)^2(\phi^2)_t\,dx\,dt&+\int_Q (s^2+v)^{(p-2)/2}|D(v-k)_+|^2\phi^2\,dx\,dt\\
&\leq c\int_Q (s^2+v)^{(p-2)/2}(v-k)_+^2|D\phi|^2\,dx\,dt+c\int_Q \widetilde{V}_k^2\phi^2\,dx\,dt.
\end{aligned}
$$

The statement now follows as after (4.8). □

5. Intrinsic potential estimates

In this section, following a strategy that goes back to the fundamental paper of De Giorgi [3], all the a priori estimates for solutions will be obtained as a consequence of the sole Caccioppoli inequality in Propositions 4.1 and 4.2. For this reason, the proofs below do not distinguish between the scalar ($N=1$) and the vectorial case ($N>1$). What we are actually going to do is to give statements valid for general functions v satisfying an immediate corollary of inequality (4.2), that is inequality (5.5) in Proposition 5.1 below. Then, in the next section we shall apply the results obtained with the peculiar choice $v = |Du|^2$, thereby proving the results for solutions to general equations and systems as (1.2).

5.1. **Further preliminary estimates.** A useful feature of reverse Hölder's inequalities is their self-improving property. We shall in the following need to use this fact when the reverse integral inequality will be considered - in the context of inequalities as (5.5) below - with respect to a measure involving the solution itself, and more precisely

$$d\mu(x,t) := \frac{1}{|Q|}\left(\frac{s^2+|Du|^2}{k}\right)^{(p-2)/2} dx\,dt = \frac{1}{|Q|}\left(\frac{s^2+v}{k}\right)^{(p-2)/2} dx\,dt\,,$$

for suitable parabolic cylinders Q and numbers $k > 0$. For this reason the following result is stated in full generality with respect to a general measure; the proof is an adaptation of the one in [18], and we report it in full extent for the sake of completeness.

Lemma 5.1. *Let μ be a nonnegative Borel measure with finite total mass. Let $0 < q < p < s < +\infty$ and $\xi, M \geq 0$, and let $(\sigma U)_\sigma$ be a family of open sets with the property*

$$\sigma' U \subset \sigma U \subset 1U = U$$

whenever $0 < \sigma' < \sigma \leq 1$. Suppose that $w \in L^p(U;\mu)$ is a non-negative function satisfying

$$\left(\int_{\sigma' U} w^s\,d\mu\right)^{1/s} \leq \frac{c_0}{(\sigma-\sigma')^{\xi}}\left(\int_{\sigma U} w^p\,d\mu\right)^{1/p} + M \tag{5.1}$$

for all $1/2 \leq \sigma' < \sigma \leq 1$. Then there is a positive constant c depending only on c_0, ξ, s, p, and q such that

$$\left(\int_{\sigma U} w^s\,d\mu\right)^{1/s} \leq \frac{c}{(1-\sigma)^{\xi'}}\left[\left(\int_{U} w^q\,d\mu\right)^{1/q} + M\right]. \tag{5.2}$$

for all $0 < \sigma < 1$, where

$$\xi' := \frac{\xi p(s-q)}{q(s-p)}\,.$$

Proof. Define

$$\Psi := \sup_{1/2<\sigma<1}\left\{(1-\sigma)^{\xi(1/\theta-1)}\left(\int_{\sigma U} w^p\,d\mu\right)^{1/p}\right\},$$

where $0 < \theta < 1$ satisfies

$$1 = \frac{p\theta}{q} + \frac{p(1-\theta)}{s}. \tag{5.3}$$

By (5.1) we have

$$
\begin{aligned}
(1-\sigma)^{\xi/\theta} &\left(\int_{\sigma U} w^s\, d\mu\right)^{1/s} \\
&\leq 2^{\xi/\theta} c_0 \left(1-\frac{1+\sigma}{2}\right)^{\xi(1/\theta-1)} \left(\int_{\frac{1+\sigma}{2}U} w^p\, d\mu\right)^{1/p} + (1-\sigma)^{\xi/\theta} M \\
&\leq 2^{\xi/\theta} c_0 \Psi + M
\end{aligned}
$$

for all $1/2 \leq \sigma < 1$. For every $\varepsilon > 0$, we find $\widetilde{\sigma} \equiv \widetilde{\sigma}(\varepsilon) \in (1/2, 1)$ such that

$$
\Psi \leq (1-\widetilde{\sigma})^{\xi(1-\theta)/\theta} \left(\int_{\widetilde{\sigma} U} w^p\, d\mu\right)^{1/p} + \varepsilon.
$$

Hölder's and Young's inequalities, and (5.3), then give

$$
\begin{aligned}
\Psi &\leq (1-\widetilde{\sigma})^{\xi(1-\theta)/\theta} \left(\int_{\widetilde{\sigma} U} w^{p(1-\theta)} w^{p\theta}\, d\mu\right)^{1/p} + \varepsilon \\
&\leq (1-\widetilde{\sigma})^{\xi(1-\theta)/\theta} \left(\int_{\widetilde{\sigma} U} w^s\, d\mu\right)^{(1-\theta)/s} \left(\int_{\widetilde{\sigma} U} w^q\, d\mu\right)^{\theta/q} + \varepsilon \\
&\leq \left(2^{\xi/\theta} c_0 \Psi + M\right)^{1-\theta} \left(\int_{\widetilde{\sigma} U} w^q\, d\mu\right)^{\theta/q} + \varepsilon \\
&\leq \frac{1}{2}\Psi + \frac{1}{2} M + c \left(\int_{\widetilde{\sigma} U} w^q\, d\mu\right)^{1/q} + \varepsilon\,.
\end{aligned}
$$

Therefore,

$$
\Psi \leq M + c \left(\int_{\widetilde{\sigma} U} w^q\, d\mu\right)^{1/q} + 2\varepsilon \leq M + c \left(\int_{U} w^q\, d\mu\right)^{1/q} + 2\varepsilon
$$

follows. But since

$$
\left(\int_{\sigma U} w^s\, d\mu\right)^{1/s} \leq \frac{c_0 \Psi}{(1-\sigma)^{\xi/\theta}} + M \leq \frac{c}{(1-\sigma)^{\xi/\theta}} \left[\left(\int_{U} w^q\, d\mu\right)^{1/q} + M + 2\varepsilon\right],
$$

we obtain the result after letting $\varepsilon \to 0$. $\square$

For the rest of the paper we shall fix the following numbers

$$
\gamma = 2 - 2/\kappa > 1 \qquad\qquad \kappa := \begin{cases} \dfrac{2n}{n-2} & \text{if } n > 2 \\ 4 & \text{if } n = 2. \end{cases} \tag{5.4}
$$

The next lemma exploits a basic consequence of the Caccioppoli inequalities found in Propositions 4.1 and 4.2.

Proposition 5.1. *Let $v \in C^0(Q_*)$ be a nonnegative function satisfying* (4.2) *with $s>0$ in the cylinder $Q = B \times T \subset Q_*$. Then there is a new constant c depending only on n, p and the constant c_1 appearing in* (4.2)*, such that*

$$\left(\fint_Q \left(s^2+v\right)^{(p-2)/2} (v-k)_+^{2\gamma} \phi^{2\gamma}\, dx\, dt\right)^{1/\gamma}$$
$$\leq cr(B)^{2/\gamma}|T|^{1-1/\gamma}$$
$$\times \left(\fint_Q \left[(v-k)_+^2 \left(\left(s^2+v\right)^{(p-2)/2}|D\phi|^2 + \left((\phi^2)_t\right)_+\right) + \widetilde{V}_k^2\right] dx\, dt\right) \tag{5.5}$$

holds for all $\phi \in C_0^\infty(Q)$ and $k>0$, where γ has been defined in (5.4).

Proof. To begin with, we denote $m = (p-2)/2$, and then use the identity

$$(s^2+v)^m (v-k)_+^{2(2-2/\kappa)} \phi^{2(2-2/\kappa)}$$
$$= \left((s^2+v)^{m/2}(v-k)_+\phi\right)^{\kappa 2/\kappa} \left((v-k)_+^2\phi^2\right)^{1-2/\kappa},$$

together with Hölder's inequality, as follows:

$$\fint_Q (s^2+v)^m (v-k)_+^{2(2-2/\kappa)} \phi^{2(2-2/\kappa)}\, dx\, dt$$
$$\leq \fint_T \left(\left(\fint_B \left((s^2+v)^{m/2}(v-k)_+\phi\right)^\kappa dx\right)^{2/\kappa} \left(\fint_B (v-k)_+^2\phi^2\, dx\right)^{1-2/\kappa}\right) dt. \tag{5.6}$$

Applying Sobolev's inequality, we further have

$$\left(\fint_B \left((s^2+v)^{m/2}(v-k)_+\phi\right)^\kappa dx\right)^{2/\kappa}$$
$$\leq cr(B)^2 \fint_B |D((s^2+v)^{m/2}(v-k)_+\phi)|^2\, dx. \tag{5.7}$$

The straightforward calculation

$$|D((s^2+v)^{m/2}(v-k)_+\phi)|^2$$
$$\leq 2(s^2+v)^m |D(v-k)_+|^2\phi^2 \left(1+\left(\frac{m}{2}\right)^2 \frac{(v-k)_+^2}{(s^2+v)^2}\right)$$
$$+ 2(s^2+v)^m(v-k)_+^2|D\phi|^2$$
$$\leq c(s^2+v)^m|D(v-k)_+|^2\phi^2 + c(s^2+v)^m(v-k)_+^2|D\phi|^2,$$

together with the Caccioppoli estimate (4.2), implies

$$
\begin{aligned}
&\fint_Q |D((s^2+v)^{m/2}(v-k)_+\phi)|^2\,dx\,dt \\
&\qquad \le c\fint_Q \left[(v-k)_+^2\left((s^2+v)^m|D\phi|^2+\left((\phi^2)_t\right)_+\right)+\widetilde{V}_k^2\right]dx\,dt\,,
\end{aligned}
\tag{5.8}
$$

where $\widetilde{V}_k$ has been defined in (4.3). Moreover, again (4.2) gives

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in T}\fint_B (v-k)_+^2\phi^2\,dx \\
&\qquad \le c|T|\fint_Q \left[(v-k)_+^2\left((s^2+v)^m|D\phi|^2+\left((\phi^2)_t\right)_+\right)+\widetilde{V}_k^2\right]dx\,dt.
\end{aligned}
\tag{5.9}
$$

Inserting (5.8)-(5.9) in (5.6) and (5.7) completes the proof. □

5.2. **Estimates in the case $p \geq 2$.** The result we are proving here is a general pointwise estimate valid for any function v satisfying (5.5), in terms of a nonlinear potential of the functions V. More precisely, by eventually taking $v \equiv |Du|^2$, in this first step we obtain a bound which still involves the solution u in the right hand side with a dependence that becomes dangerous in the degenerate case $s = 0$ (of course here we are assuming that $s > 0$, but the estimate worsens when s approaches zero). Moreover, the estimate involves a free parameter $k > 0$, used to achieve a sort of intrinsic scaling in the estimate. In this respect we shall use intrinsic cylinders - we are here adopting the terminology introduced by DiBenedetto [11] - of the type $Q_{r,k}(x^*,t^*)$ defined in (2.1). Needless to say, every such cylinder is of course assumed to be contained in the starting cylinder Q_*. In the following we shall consider intrinsic potentials of the type $\mathbf{P}^F_{r,k}$ introduced in (2.2).

Theorem 5.1. *Let*

$$
0<\ell<2\gamma = 4-4/\kappa, \qquad \kappa := \begin{cases} \dfrac{2n}{n-2} & \text{if } n>2 \\ 4 & \text{if } n=2\,, \end{cases}
\tag{5.10}
$$

and let $v \in C^0(Q_*)$ be a non-negative function satisfying (4.2) for every cylinder $Q \subset Q_*$; assume also that $p \geq 2$ and $s > 0$. Then there exists a constant c, depending only on ℓ and n, p, c_1, such that

$$
\begin{aligned}
v(x^*,t^*) \le{}& k + c\left(\fint_{Q_{r,k}(x^*,t^*)}\left(\frac{s^2+v}{k}\right)^{(p-2)/2}(v-k)_+^\ell\,dx\,dt\right)^{1/\ell} \\
&+ ck^{-(p-2)/4}\mathbf{P}^{\widetilde{V}_k}_{r,k}(x^*,t^*)
\end{aligned}
\tag{5.11}
$$

holds for all $k > 0$ such that $Q_{r,k}(x^*,t^*) \subset Q_*$, where

$$
\widetilde{V}_k = |V|(s^2+v)^{1-p/4}\chi_{\{v>k\}}. \tag{5.12}
$$

Proof. In the following, when not differently specified, the symbol c will denote a constant depending only on n, p, c_1.

Step 1: Decay estimate. With $Q_{r,k}(x^*,t^*)$ being the cylinder in question, denote in short

$$\sigma Q_j = \sigma B_j \times \sigma T_j = B_{\sigma r_j}(x^*) \times (t^* - k^{-m}(\sigma r_j)^2, t^*), \tag{5.13}$$

where

$$\begin{cases} m = \frac{p-2}{2}, \qquad k_0 \equiv k > 0, \qquad 0 < \sigma \leq 1 \\ \qquad r_j = \frac{r}{2^{j+1}}, \qquad j = -1, 0, 1, \ldots, \end{cases} \tag{5.14}$$

and, for $1/2 \leq \sigma' < \sigma \leq 1$, we choose cutoff functions $\phi_j \in C_0^\infty(\sigma Q_j)$, $0 \leq \phi_j \leq 1$, such that $\phi_j = 1$ in $\sigma' Q_j$ and

$$|D\phi_j| \leq \frac{4}{r_j(\sigma - \sigma')} \qquad \text{and} \qquad (\phi_t)_+ \leq \frac{4k^m}{r_j^2(\sigma - \sigma')}\,. \tag{5.15}$$

Let $\{k_j\}$ be a nondecreasing sequence to be chosen later, and such that $k_j \geq k$ holds for every $j \in \mathbb{N}$. Then, obviously,

$$(v - k_j)_+^2((\phi_j^2)_t)_+ \leq \frac{c}{r_j^2(\sigma - \sigma')}(s^2+v)^m(v-k_j)_+^2\,.$$

Note that the assumption $p \geq 2$ is needed here. Inequality (5.5) then yields

$$\begin{aligned} &\fint_{\sigma' Q_j} (s^2+v)^m (v-k_j)_+^{2\gamma}\,dx\,dt \\ &\quad\leq c\frac{r_j^{2(1-\gamma)}}{|T_j|^{1-\gamma}}\left(\fint_{\sigma Q_j}\left[(s^2+v)^m\frac{(v-k_j)_+^2}{(\sigma-\sigma')^2} + r_j^2\widetilde{V}_k^2\right]dx\,dt\right)^\gamma \\ &\quad= ck^{m(1-\gamma)}\left(\fint_{\sigma Q_j}\left[(s^2+v)^m\frac{(v-k_j)_+^2}{(\sigma-\sigma')^2} + r_j^2\widetilde{V}_k^2\right]dx\,dt\right)^\gamma \end{aligned} \tag{5.16}$$

for all k_j. Thus we arrive at

$$\begin{aligned} &\fint_{\sigma' Q_j}\left(\frac{s^2+v}{k}\right)^m (v-k_j)_+^{2\gamma}\,dx\,dt \\ &\quad\leq c\left(\fint_{\sigma Q_j}\left[\left(\frac{s^2+v}{k}\right)^m\frac{(v-k_j)_+^2}{(\sigma-\sigma')^2} + \frac{r_j^2}{k^m}\widetilde{V}_k^2\right]dx\,dt\right)^\gamma, \end{aligned} \tag{5.17}$$

which holds for a constant c depending only on n, p, c_1, and whenever $1/2 \leq \sigma' < \sigma \leq 1$. By defining the Borel measure

$$d\mu(x,t) = \frac{1}{|Q_j|}\left(\frac{s^2+v}{k}\right)^m dx\,dt$$

and denoting

$$M = \frac{r_j^2}{k^m} \fint_{Q_j} \widetilde{V}_k^2 \, dx \, dt, \qquad w = (v - k_j)_+^2, \qquad Q = Q_j \tag{5.18}$$

inequality (5.17) takes the form

$$\left(\int_{\sigma' Q} w^\gamma \, d\mu \right)^{1/\gamma} \leq \frac{c}{(\sigma - \sigma')^2} \int_{\sigma Q} w \, d\mu + cM$$

for $1/2 \leq \sigma' < \sigma \leq 1$. Since $\gamma > 1$ by (5.4), we can apply Lemma 5.1 and therefore it follows that

$$\left(\int_{\frac{1}{2} Q} w^\gamma \, d\mu \right)^{1/\gamma} \leq c(\ell) \left[\left(\int_Q w^\ell \, d\mu \right)^{1/\ell} + M \right]$$

for all $0 < \ell < \gamma$. Rewriting this in the original notation, that is (5.18), gives

$$\begin{aligned} &\left(\fint_{Q_{j+1}} \left(\frac{s^2 + v}{k} \right)^m (v - k_j)_+^{2\gamma} \, dx \, dt \right)^{1/\gamma} \\ &\qquad \leq c \left(\fint_{Q_j} \left(\frac{s^2 + v}{k} \right)^m (v - k_j)_+^{\ell} \, dx \, dt \right)^{2/\ell} + \frac{c r_j^2}{k^m} \fint_{Q_j} \widetilde{V}_k^2 \, dx \, dt, \end{aligned} \tag{5.19}$$

which holds whenever $0 < \ell < 2\gamma$; the constant c depends on n, p, c_1 and ℓ. Next, we observe that

$$(v - k_j)_+^{2\gamma} \geq (v - k_{j+1})_+^\ell (k_{j+1} - k_j)^{2\gamma - \ell}$$

and by denoting

$$Y_j := \left(\fint_{Q_j} \left(\frac{s^2 + v}{k} \right)^m (v - k_j)_+^\ell \, dx \, dt \right)^{1/\ell}, \quad M_j := c \left(\frac{r_j^2}{k^m} \fint_{Q_j} \widetilde{V}_k^2 \, dx \, dt \right)^{1/2},$$

inequality (5.19) becomes, after taking the square root,

$$(k_{j+1} - k_j)^{1 - \ell/(2\gamma)} Y_{j+1}^{\ell/(2\gamma)} \leq c Y_j + M_j. \tag{5.20}$$

Step 2: Iteration. We here use a variant of a method introduced by Kilpeläinen & Malý in [22]. Define k_j's inductively as

$$k_{j+1} = k_j + \frac{Y_j}{\delta}, \qquad k_0 = k, \qquad j = 0, 1, \ldots \qquad \delta > 0. \tag{5.21}$$

This obviously defines a nondecreasing sequence. Assume first that

$$k_{j+2} - k_{j+1} \geq \frac{1}{2} (k_{j+1} - k_j) \tag{5.22}$$

holds for a certain index $j \geq 0$. It then follows that

$$\begin{aligned}(k_{j+1}-k_j)^{1-\ell/(2\gamma)}Y_{j+1}^{\ell/(2\gamma)} =&\delta^{\ell/(2\gamma)}(k_{j+1}-k_j)^{1-\ell/(2\gamma)}(k_{j+2}-k_{j+1})^{\ell/(2\gamma)}\\ \geq&(\delta/2)^{\ell/(2\gamma)}(k_{j+1}-k_j).\end{aligned}$$

Inserting this into (5.20) gives

$$(\delta/2)^{\ell/(2\gamma)}(k_{j+1}-k_j) \leq c\delta(k_{j+1}-k_j) + M_j$$

with $c \equiv c(n, p, c_1, \ell)$. First, since $\ell < 2\gamma$, there is small enough $\delta > 0$ depending only on n, p, ν, L, and ℓ such that

$$c\delta \leq \frac{1}{2}\left(\frac{\delta}{2}\right)^{\ell/(2\gamma)},$$

leading to

$$k_{j+1}-k_j \leq cM_j \tag{5.23}$$

with $c \equiv c(n, p, c_1, \ell)$. Second, we always have

$$\begin{aligned}k_{j+2} - k_{j+1} =&\frac{Y_{j+1}}{\delta}\\ =&\frac{1}{\delta}\left(\fint_{Q_{j+1}}\left(\frac{s^2+v}{k}\right)^m(v-k_{j+1})_+^\ell\,dx\,dt\right)^{1/\ell}\\ \leq&\frac{c(n,\ell)}{\delta}\left(\fint_{Q_j}\left(\frac{s^2+v}{k}\right)^m(v-k_j)_+^\ell\,dx\,dt\right)^{1/\ell}\\ =&c\,(k_{j+1}-k_j)\,.\end{aligned} \tag{5.24}$$

Therefore, under the assumption (5.22) we have (5.23) and yet

$$k_{j+2} - k_{j+1} \leq c\,(k_{j+1} - k_j) \leq cM_j.$$

In all cases we conclude

$$k_{j+2} - k_{j+1} \leq \frac{1}{2}\,(k_{j+1} - k_j) + cM_j\,,$$

whenever $j \geq 0$. Telescoping the previous inequality gives

$$\sum_{j=0}^{\infty}(k_{j+2} - k_{j+1}) \leq (k_1 - k_0) + c\sum_{j=0}^{\infty}M_j\,.$$

Therefore, using also (5.24) repeatedly, we get

$$\begin{aligned}\lim_{j\to\infty} k_j = \sum_{j=0}^{\infty}(k_{j+2}-k_{j+1})+k_1 &\le k_1+(k_1-k_0)+c\sum_{j=0}^{\infty} M_j\\ &= k+(2/\delta)Y_0+c\sum_{j=0}^{\infty} M_j\\ &\equiv k+cY_0+c\sum_{j=0}^{\infty} M_j\,.\end{aligned}$$

Since the sequence $\{k_j\}$ is bounded and nondecreasing, the limit above exists and hence Y_j converges to zero and we obtain

$$v(x^*,t^*)\le \lim_{j\to\infty} k_j \le k+cY_0+c\sum_{j=0}^{\infty} M_j.$$

Observe that at this point we are using that v is continuous. Estimate (5.11) now follows recalling the definition of to estimate

$$\begin{aligned}Y_0 =&\frac{c}{\delta}\left(\fint_{Q_{r/2,k}(x^*,t^*)}\left(\frac{s^2+v}{k}\right)^{(p-2)/2}(v-k)_+^{\ell}\,dx\,dt\right)^{1/\ell}\\ \le& c\left(\fint_{Q_{r,k}(x^*,t^*)}\left(\frac{s^2+v}{k}\right)^{(p-2)/2}(v-k)_+^{\ell}\,dx\,dt\right)^{1/\ell}\end{aligned}$$

and observing that, with $r_{-1}\equiv r$, it is possible to estimate

$$\begin{aligned}\sum_{j=0}^{\infty} M_j &= c\sum_{j=0}^{\infty}\frac{r_j}{k^{m/2}}\left(\fint_{Q_j}\widetilde{V}_k^2\,dx\,dt\right)^{1/2}\\ &\le ck^{-m/2}\sum_{j=0}^{\infty}\left(\fint_{Q_j}\widetilde{V}_k^2\,dx\,dt\right)^{1/2}\int_{r_j}^{r_{j-1}}\varrho\frac{d\varrho}{\varrho}\\ &\le ck^{-m/2}\sum_{j=0}^{\infty}\int_{r_j}^{r_{j-1}}\left(\varrho^2\fint_{Q_{\varrho,k}(x^*,t^*)}\widetilde{V}_k^2\,dx\,dt\right)^{1/2}\frac{d\varrho}{\varrho}\\ &= ck^{-m/2}\int_0^{r}\left(\varrho^2\fint_{Q_{\varrho,k}(x^*,t^*)}\widetilde{V}_k^2\,dx\,dt\right)^{1/2}\frac{d\varrho}{\varrho}\\ &= ck^{-(p-2)/4}\mathbf{P}_{r,k}^{\widetilde{V}_k}(x^*,t^*).\end{aligned}$$

The proof is complete. □

5.3. **Estimates in the case $p \leq 2$.** The singular case $p \leq 2$ forces us to use a posteriori information about the gradient. The scale between the time and space variables will now depend on the supremum of the gradient, while in turn it is exactly the gradient L^∞-norm we are trying to estimate; this procedure provides a closer linkage between our techniques and DiBenedetto's intrinsic geometry approach.

Theorem 5.2. *Let ℓ, γ and κ as in (5.10) and and let $v \in C^0(Q_*)$ be a non-negative function satisfying (4.2) for every cylinder $Q \subset Q_*$; assume also that $1 < p \leq 2$ and $s > 0$. Finally, assume that for $(x_*, t_*) \in Q_*$ it holds that*

$$Q_{r,k}(x^*, t^*) = B_r(x^*) \times (t^* - k^{-(p-2)/2} r^2, t^*) \subset Q_*$$

where k is a positive number such that the intrinsic inequality is satisfied

$$4k \geq \operatorname*{ess\,sup}_{Q_{r,k}(x^*,t^*)} \left(s^2 + v\right) . \tag{5.25}$$

Then there exists a constant c depending only on n, p, c_1 and ℓ such that

$$v(x^*, t^*) \leq k + c \left(\fint_{Q_{r,k}(x^*,t^*)} (v-k)_+^\ell \, dx \, dt \right)^{1/\ell} + c k^{-(p-2)/4} \mathbf{P}_{r,k}^{\widetilde{V}_k}(x^*, t^*) \tag{5.26}$$

where $\mathbf{P}_{r,k}^{\widetilde{V}_k}(x^*, t^*)$ and $\widetilde{V}_k$ are as in (2.2) and (5.12), respectively.

Proof. The proof is a variant of the one developed for Theorem 5.1, and we shall adopt the notation introduced there; in particular the one in (5.13) and (5.14). The definitions of Q_j and k imply

$$r_j^{2/\gamma} |T_j|^{1-1/\gamma} = k^{-m(1-1/\gamma)} r_j^2$$

while on the other hand we notice that

$$4^{-|m|} \chi_{\{v > k_j\}} \leq \chi_{\{v > k_j\}} \left(\frac{s^2 + v}{k} \right)^m \leq \left(\frac{s^2 + k_j}{k} \right)^m = \left(\frac{s^2 + k_j}{k} \right)^{\frac{p-2}{2}} \leq 1 \tag{5.27}$$

for all $k_j \geq k$, the first inequality in the previous line being actually a consequence of (5.25). Furthermore, we choose cut-off functions $\phi_j \in C_0^\infty(\sigma Q_j)$ as in the proof of Theorem 5.1, and in particular such that (5.15) are satisfied; again as for Theorem 5.1 we take $\{k_j\}$ to be a nondecreasing sequence such that $k_j \geq k$ holds for every $j \in \mathbb{N}$. Substituting estimates into (5.5) and using (5.27) repeatedly, gives

$$
\begin{aligned}
&\frac{k^m}{4^{|m|}} \fint_{Q_{j+1}} (v-k_j)_+^{2\gamma}\, dx\, dt \\
&\le \fint_{Q_{j+1}} (s^2+v)^m (v-k_j)_+^{2\gamma}\, dx\, dt \\
&\qquad \le c k^{m(1-\gamma)} \left(\fint_{\sigma Q_j} \left[\frac{(v-k_j)_+^2}{(\sigma-\sigma')^2} \left((s^2+v)^m + k^m\right) + r_j^2 \widetilde{V}_k^2 \right] dx\, dt \right)^{\gamma} \\
&\qquad \le c k^{m} \left(\fint_{\sigma Q_j} \left[\frac{(v-k_j)_+^2}{(\sigma-\sigma')^2} + \frac{r_j^2}{k^m} \widetilde{V}_k^2 \right] dx\, dt \right)^{\gamma} .
\end{aligned}
\tag{5.28}
$$

Thus we obtain

$$
\left(\fint_{\sigma' Q_j} (v-k_j)_+^{2\gamma}\, dx\, dt \right)^{1/\gamma} \le c \fint_{\sigma Q_j} \left[\frac{(v-k_j)_+^2}{(\sigma-\sigma')^2} + \frac{r_j^2}{k^m} \widetilde{V}_k^2 \right] dx\, dt \tag{5.29}
$$

whenever $1/2 \le \sigma' < \sigma \le 1$. proceeding as for Theorem 5.1, after (5.17), by Lemma 5.1 we conclude

$$
\left(\fint_{Q_{j+1}} (v-k_j)_+^{2\gamma}\, dx\, dt \right)^{1/\gamma} \le c \left(\fint_{Q_j} (v-k_j)_+^{2\ell}\, dx\, dt \right)^{1/\ell} + c \frac{r_j^2}{k^m} \fint_{Q_j} \widetilde{V}_k^2\, dx\, dt \tag{5.30}
$$

for all $0 < \ell < \gamma$, and c also depends on ℓ. The assertion now follows as in the degenerate case, i.e. as in the proof of Theorem 5.1, but this time without the weight $(s^2+v)^m$ inside the integrals. Specifically, this time we define

$$
Y_j := \left(\fint_{Q_j} (v-k_j)_+^{\ell}\, dx\, dt \right)^{1/\ell}, \qquad M_j := c \left(\frac{r_j^2}{k^m} \fint_{Q_j} \widetilde{V}_k^2\, dx\, dt \right)^{1/2},
$$

where $\{k_j\}$ is defined as in (5.21) with $k_0 := k$ and proceed as in the Step 2 of the proof of Theorem 5.1. □

Remark 5.31. In the proof of the previous result no lower bound on p other than $p > 1$ is needed. This is essentially due to two facts: first we are assuming that v is already bounded, in such a way that the integral in the right hand side of (5.26) is always finite; second, the right hand side potential in (5.26) still depends on v in various ways. Eventually, when setting the estimate free from such a dependence in the right hand side we shall need (1.5). See the proof of Theorem 1.2 below.

6. Non-intrinsic estimates and proof of the results

In this section we prove the main results concerning solutions of parabolic equations of the type (1.2) and systems of the type (1.3). The strategy is the following: thanks to (4.2) and eventually to Proposition 5.5, we shall use estimates (5.11) and (5.26) with the choice $v \equiv |Du|^2$. Such inequalities are intrinsic in that a dependence on the solution itself shows up in the potential considered on the right hand side; we shall then make them non-intrinsic via suitable choices of the parameter k and some iteration/covering procedures. At this stage the importance of the lower bound on p in (1.5) appears evident in a rather subtle way. For this reason, in the applications below of estimates (5.11) and (5.26), all the constants depending on c_1 appearing in (4.2), will then exhibit the general dependence upon the data n, N, p, ν, L and the interpolation parameter ℓ. We also recall that, due to the preliminary reduction made in Section 3, we shall confine to prove the results under the form of a priori estimates, and in particular, for those solutions satisfying the conditions of Proposition 5.1 with $v = |Du|^2$. In particular, we shall assume that Du is continuous, and this is legal thanks to the available regularity theory for solution to smoother equations, see (3.1).

Remark 6.1. Before going on with the proofs, let us point out an important fact. Estimates (5.11) and (5.26) have been derived as a priori estimates, i.e. estimates for a priori continuous and in particular locally bounded functions v. As mentioned just a few lines above, in order to achieve the proof of the theorems we shall use them with the choice $v \equiv |Du|^2 \equiv |Du_\varepsilon|^2$ and then match the resulting a priori estimates with the result of Lemma 3.1. At this stage we will need that the right hand sides in the corresponding estimates stay uniformly bounded with respect to ε; for this reason we shall always assume to take ℓ in (5.11) and (5.26) such that $p - 2 + 2\ell \leq p$ when $p \geq 2$, and in particular in this case we shall take $\ell \leq 1$. When instead $p \leq 2$ then we shall take ℓ such that $2\ell \leq p$; this condition has to be matched with $\ell > n(2-p)/4$ and altogether we have to assume (1.14). In turn this is possible exactly when (1.13) is in force.

Proof of Theorem 1.1. We recall we are assuming without loss of generality that $V \in L^2(\mathbb{R}^{n+1}, \mathbb{R}^N)$, so that we can consider the quantity $\|\mathbf{P}_r^V\|_{L^\infty(Q)}$ on arbitrary cylinders Q. We distinguish two cases; the first is when $2 \leq p < 4$. We appeal to estimate (5.11) with $v \equiv |Du|^2$, taking $k = 1$; this gives, after a few elementary manipulations

$$
\begin{aligned}
|Du(x^*, t^*)| \leq & c \left(\fint_{Q_\rho(x^*, t^*)} (|Du| + s)^{p-2} |Du|^{2\ell} \, dx \, dt \right)^{1/2\ell} \\
& + c[\mathbf{P}_\rho^{\widetilde{V}_1}(x^*, t^*)]^{1/2} + 1 \,.
\end{aligned}
\tag{6.2}
$$

Estimate (6.2) is obviously valid whenever $Q_\rho(x^*,t^*) \subset Q_r$. We start observing the following inequality:

$$\mathbf{P}_\rho^{\widetilde{V}_1}(x^*,t^*) \leq c\Big(s + \|Du\|_{L^\infty(Q_\rho(x^*,t^*))}\Big)^{\frac{4-p}{2}} \mathbf{P}_\rho^V(x^*,t^*)$$

that matched with (6.2) gives

$$\begin{aligned} |Du(x^*,t^*)| \leq & c \left(\fint_{Q_\rho(x^*,t^*)} (|Du|+s)^{p-2} |Du|^{2\ell} \, dx\, dt \right)^{1/2\ell} \\ & + c\Big(s + \|Du\|_{L^\infty(Q_\rho(x^*,t^*))}\Big)^{\frac{4-p}{4}} [\mathbf{P}_\rho^V(x^*,t^*)]^{1/2} + 1\,. \end{aligned} \tag{6.3}$$

In turn, by applying Young's inequality with conjugate exponents $4/(4-p)$ and $4/p$, we conclude with

$$\begin{aligned} |Du(x^*,t^*)| \leq & c \left(\fint_{Q_\rho(x^*,t^*)} (|Du|+s)^{p-2} |Du|^{2\ell} \, dx\, dt \right)^{1/2\ell} \\ & + (1/2)\|Du\|_{L^\infty(Q_\rho(x^*,t^*))} + c[\mathbf{P}_\rho^V(x^*,t^*)]^{2/p} + cs + 1\,. \end{aligned} \tag{6.4}$$

We now select

$$r/2 \leq \sigma' < \sigma \leq r \tag{6.5}$$

and consequently determine cylinders with the same vertex

$$Q_{r/2} \subset Q_{\sigma'} \subset Q_\sigma \subset Q_r \tag{6.6}$$

and eventually apply (6.4) with $\rho = (\sigma'-\sigma)/2$ in place of r, and to every point $(x^*,t^*) \in Q_{\sigma'}$. This easily leads to

$$\begin{aligned} \|Du\|_{L^\infty(Q_{\sigma'})} \leq & \frac{c}{(\sigma-\sigma')^{(n+2)/2\ell}} \left(\int_{Q_r} (|Du|+s)^{p-2} |Du|^{2\ell} \, dx\, dt \right)^{1/2\ell} \\ & + (1/2)\|Du\|_{L^\infty(Q_\sigma)} + c\|\mathbf{P}_r^V\|_{L^\infty(Q_r)}^{2/p} + cs + 1\,. \end{aligned} \tag{6.7}$$

We then apply Lemma 2.1 with the choice

$$\phi(\sigma) = \|Du\|_{L^\infty(Q_\sigma)}, \qquad B = c\|\mathbf{P}_r^V\|_{L^\infty(Q_r)}^{2/p} + cs + 1$$

and

$$A = \left(\int_{Q_r} (|Du|+s)^{p-2} |Du|^{2\ell} \, dx\, dt \right)^{1/2\ell},$$

thereby getting the desired estimate, that is (1.12).

We now treat the case $p \geq 4$. In this case we proceed with a different choice of the number k when applying (5.11). Specifically, we fix

$$k = 1 + \|\mathbf{P}_r^V\|_{L^\infty(Q_r)}^{4/p}\,, \tag{6.8}$$

so that, with $v = |Du|^2$, we have, in particular,

$$(s^2{+}v)^{1-p/4}\chi_{\{v>k\}} \le (s^2{+}k)^{(4-p)/4} \qquad \text{in } \ Q_r\,.$$

Keeping (5.11) in mind, as $k \geq 1$, also by mean of the previous inequality we may estimate

$$\begin{aligned} k^{-(p-2)/4}\mathbf{P}^{\widetilde{V}_k}_{r,k}(x^*,t^*) &\le \mathbf{P}^{\widetilde{V}_k}_{r}(x^*,t^*) \\ &\le (s^2{+}k)^{(4-p)/4}\mathbf{P}^{V}_{r}(x^*,t^*) \\ &\le \|\mathbf{P}^{V}_{r}\|^{(4-p)/p}_{L^\infty(Q_r)}\mathbf{P}^{V}_{r}(x^*,t^*) \\ &\le \|\mathbf{P}^{V}_{r}\|^{4/p}_{L^\infty(Q_r)}\,. \end{aligned} \tag{6.9}$$

Moreover, observe that with $v = |Du|^2$ we have

$$\begin{aligned} &\fint_{Q_{\rho,k}(x^*,t^*)} \left(\frac{s^2{+}v}{k}\right)^{(p-2)/2} (v-k)_+^{\ell}\,dx\,dt \\ &\qquad \le c\fint_{Q_\rho(x^*,t^*)} (|Du|+s)^{p-2}\,|Du|^{2\ell}\,dx\,dt\,. \end{aligned} \tag{6.10}$$

Writing (5.11) with the choice in (6.8), and using (6.9)-(6.10), we obtain

$$\begin{aligned} |Du(x^*,t^*)| \le{}& c\left(\fint_{Q_\rho(x^*,t^*)} (|Du|+s)^{p-2}\,|Du|^{2\ell}\,dx\,dt\right)^{1/2\ell} \\ &+ c\|\mathbf{P}^{V}_{r}\|^{2/p}_{L^\infty(Q_r)} + 1\,. \end{aligned} \tag{6.11}$$

Now, with the choices made in (6.5)-(6.6) - we take a standard cylinder $Q_\rho(x^*,t^*) \subset Q_\sigma$ such that $(x^*,t^*) \in Q_\sigma$ and $\rho = (\sigma-\sigma')/2$ - estimate (6.11) implies

$$\begin{aligned} \|Du\|_{L^\infty(Q_{\sigma'})} \le{}& \frac{c}{(\sigma-\sigma')^{(n+2)/2\ell}}\left(\int_{Q_r} (|Du|+s)^{p-2}\,|Du|^{2\ell}\,dx\,dt\right)^{1/2\ell} \\ &+ c\|\mathbf{P}^{V}_{r}\|^{2/p}_{L^\infty(Q_r)} + 1\,, \end{aligned} \tag{6.12}$$

which is the analog of (6.7). Now we conclude taking $\sigma' = r/2$ and $\sigma = 3r/4$; no iteration is necessary in this case. □

In order to prove Theorem 1.5 the basic idea is that when the right hand side potential $V(\cdot)$ is independent of time, then an additional scaling argument allows to get a better power for the potential, which is exactly the one showing up in the elliptic case [14].

Proof of Theorem 1.5. We start taking a standard parabolic cylinder $Q_\varrho \equiv Q_\varrho(x^*,t^*) \subset Q_*$. Then we take $k \geq 0$ such that

$$k \ge \left(\|\mathbf{P}^{V}_{\varrho}\|_{L^\infty(Q_\varrho)} + 1\right)^{1/(p-1)}\,. \tag{6.13}$$

Since $p \geq 2$ and $k \geq 1$, and therefore $Q_{\varrho,k^2} \subset Q_r$, we have that

$$\|\mathbf{P}_{\varrho}^{V}\|_{L^\infty(Q_{\varrho,k^2})} \leq k^{p-1}\,. \tag{6.14}$$

We recall that, denoting by (x^*,t^*) the vertex of the cylinder Q_ϱ, it is $Q_{\varrho,k^2}(x^*,t^*) = B_\varrho(x^*)\times(t^*-k^{2-p}\varrho^2,t^*)$. We now rescale the equation/system in question (1.2) defining

$$\tilde{u}(\tilde{x},\tilde{t}) := \frac{u(x^*+\varrho\tilde{x}, t^*+k^{2-p}\varrho^2\tilde{t})}{k\varrho}\,, \tag{6.15}$$

whenever $(\tilde{x},\tilde{t}) \in B_1(0)\times(-1,0) \equiv Q_1$, and

$$\tilde{V}(\tilde{x}) := \frac{\varrho V(x^*+\varrho\tilde{x})}{k^{p-1}}$$

whenever $\tilde{x} \in \mathbb{R}^n$. Accordingly, we define the new vector field

$$\tilde{a}(z) := \frac{a(kz)}{k^{p-1}}\,. \tag{6.16}$$

It is now not difficult to see that $\tilde{u}$ weakly solves the equation/system

$$\tilde{u}_{\tilde{t}} - \operatorname{div}\tilde{a}(D\tilde{u}) = \tilde{V}(\tilde{x}) \tag{6.17}$$

in Q_1, and moreover, the new vector field $\tilde{a}(\cdot)$ satisfies (1.4) with s replaced by s/k. Most importantly - and here we need that $V(\cdot)$ is independent of t - we have, by a simple change of variable argument, that

$$\|\mathbf{P}_{1}^{\tilde{V}}\|_{L^\infty(Q_1)} \leq 1\,. \tag{6.18}$$

We are therefore able to apply Theorem 1.1 that, together with (6.18), gives

$$\|D\tilde{u}\|_{L^\infty(Q_{1/2})} \leq c\left(\fint_{Q_1}(|D\tilde{u}|+(s/k))^{p-2}\,|D\tilde{u}|^{2\ell}\,d\tilde{x}\,d\tilde{t}\right)^{1/2\ell} + c(s/k+1)\,. \tag{6.19}$$

By scaling back - i.e. going back to u and V - in inequality (6.19), we obtain

$$\begin{aligned}\|Du\|_{L^\infty(Q_{\varrho/2,k^2})} \leq{}& ck^{(2-p)/2\ell}\left(\fint_{Q_{\varrho,k^2}}(|Du|+s)^{p-2}\,|Du|^{2\ell}\,dx\,dt\right)^{1/2\ell}\\ &+c(k+s)\,.\end{aligned} \tag{6.20}$$

On the other hand, we also have

$$\begin{aligned}k^{(2-p)/2\ell}&\left(\fint_{Q_{\varrho,k^2}}(|Du|+s)^{p-2}\,|Du|^{2\ell}\,dx\,dt\right)^{1/2\ell}\\ &\leq\left(\fint_{Q_{\varrho}}(|Du|+s)^{p-2}\,|Du|^{2\ell}\,dx\,dt\right)^{1/2\ell}.\end{aligned} \tag{6.21}$$

Therefore, matching (6.20) and (6.21) yields

$$\|Du\|_{L^\infty(Q_{\varrho/2,k^2})} \le c\left(\fint_{Q_\varrho} (|Du|+s)^{p-2}\,|Du|^{2\ell}\,dx\,dt\right)^{1/2\ell} + c(k+s)\,, \tag{6.22}$$

whenever k satisfies (6.13). To conclude the proof we take a cylinder of the type $Q_{r/8,k} \equiv Q_{r/8,k}(x^*,t^*)$ such that $Q_{r/8,k} \subset Q_r$ and

$$\|Du\|_{L^\infty(Q_{r/8,k})} = \|Du\|_{L^\infty(Q_{r/2})}$$

and then apply estimate (6.22) with $\varrho = r/4$ and with the choice

$$k = \left(\|\mathbf{P}_r^V\|_{L^\infty(Q_r)} + 1\right)^{1/(p-1)}\,,$$

which obviously satisfies (6.13). $\square$

We now focus on the case $p \le 2$.

Proof of Theorem 1.2. The proof is more delicate than the one for Theorem 1.1, since the choice of k involves the intrinsic condition in (5.25); needless to say this time the starting point is formula (5.26) that we shall apply with the choice $v = |Du|^2$. Observe that, with $Q_\rho(x^*,t^*) \subset Q_*$ and $k \ge 1$ we have

$$Q_{k^{(p-2)/4}\rho,k}(x^*,t^*) \equiv B_{k^{(p-2)/4}\rho}(x^*) \times (t^*-\rho^2,t^*) \subset Q_\rho(x^*,t^*) \subset Q_*\,. \tag{6.23}$$

Next we take

$$k = 1 + (1/4)\Big(s^2 + \|Du\|^2_{L^\infty(Q_\rho(x^*,t^*))}\Big) \tag{6.24}$$

and we notice that by (6.23) inequality (5.25) is satisfied if we consider the cylinder $Q_{r,k} \equiv Q_{k^{(p-2)/4}\rho,k}(x^*,t^*)$, that is we take $r = k^{(p-2)/4}\rho$. We therefore apply (5.26) getting

$$\begin{aligned} v(x^*,t^*) &\le k + c\left(\fint_{Q_{k^{(p-2)/4}\rho,k}(x^*,t^*)} (v-k)_+^\ell\,dx\,dt\right)^{1/\ell} \\ &\qquad + ck^{(2-p)/4}\mathbf{P}^{\widetilde{V}_k}_{k^{(p-2)/4}\rho,k}(x^*,t^*)\,. \end{aligned} \tag{6.25}$$

With ω_n denoting the measure of the unit ball in $\mathbb{R}^n$ we let

$$f(\varrho) := \left(\frac{k^{(p-2)/2}}{\omega_n \varrho^n}\int_{Q_{\varrho,k}(x^*,t^*)} |\widetilde{V}_k|^2\,dx\,dt\right)^{1/2}\,,$$

and then, changing variables and keeping (6.23) in mind, obtain

$$\begin{aligned} \mathbf{P}^{\widetilde{V}_k}_{k^{(p-2)/4}\rho,k}(x^*,t^*) &= \int_0^{k^{(p-2)/4}\rho}\left(\varrho^2\fint_{Q_{\varrho,k}(x^*,t^*)} |\widetilde{V}_k|^2\,dx\,dt\right)^{1/2}\frac{d\varrho}{\varrho} \\ &= \int_0^{k^{(p-2)/4}\rho} f(\varrho)\frac{d\varrho}{\varrho} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\rho f(k^{(p-2)/4}\varrho)\frac{d\varrho}{\varrho} \\
&= k^{(p-2)/4}\int_0^\rho \left(\varrho^2 \fint_{Q_{k^{(p-2)/4}\varrho,k}(x^*,t^*)} |\widetilde{V}_k|^2\,dx\,dt\right)^{1/2}\frac{d\varrho}{\varrho} \\
&\leq k^{(n-2)(2-p)/8}\int_0^\rho \left(\varrho^2 \fint_{Q_\varrho(x^*,t^*)} |\widetilde{V}_k|^2\,dx\,dt\right)^{1/2}\frac{d\varrho}{\varrho} \\
&= k^{(n-2)(2-p)/8}\mathbf{P}_\rho^{\widetilde{V}_k}(x^*,t^*)\,.
\end{aligned} \tag{6.26}
$$

Moreover, again appealing to (6.23), we have

$$
\fint_{Q_{k^{(p-2)/4}\rho,k}(x^*,t^*)} (v-k)_+^\ell\,dx\,dt \leq k^{n(2-p)/4}\fint_{Q_\rho(x^*,t^*)} (v-k)_+^\ell\,dx\,dt\,. \tag{6.27}
$$

Combining (6.26) and (6.27) with (6.25) gives

$$
\begin{aligned}
v(x^*,t^*) \leq k + ck^{n(2-p)/4l}&\left(\fint_{Q_\rho(x^*,t^*)} (v-k)_+^\ell\,dx\,dt\right)^{1/\ell} \\
&+ ck^{n(2-p)/8}\mathbf{P}_\rho^{\widetilde{V}_k}(x^*,t^*)\,.
\end{aligned} \tag{6.28}
$$

Then we have, again recalling the choice in (6.24)

$$
k^{n(2-p)/8}\mathbf{P}_\rho^{\widetilde{V}_k}(x^*,t^*) \leq c(s^2+k)^{n(2-p)/8+(4-p)/4}\mathbf{P}_\rho^{V}(x^*,t^*)\,. \tag{6.29}
$$

Using (6.28), together with (6.29), yields

$$
\begin{aligned}
|Du(x^*,t^*)| \leq c\Big(s+\|Du\|_{L^\infty(Q_\rho(x^*,t^*))}\Big)^{n(2-p)/4\ell}&\left(\fint_{Q_\rho(x^*,t^*)} |Du|^{2\ell}\,dx\,dt\right)^{1/2\ell} \\
+ c\Big(s+\|Du\|_{L^\infty(Q_\rho(x^*,t^*))}\Big)^{n(2-p)/8+(4-p)/4}&[\mathbf{P}_\rho^{V}(x^*,t^*)]^{1/2} \\
+ 1 + (1/4)\Big(s+\|Du\|_{L^\infty(Q_\rho(x^*,t^*))}\Big)& \\
+ c\left(\fint_{Q_\rho(x^*,t^*)} |Du|^{2\ell}\,dx\,dt\right)^{1/2\ell} &+ c[\mathbf{P}_\rho^{V}(x^*,t^*)]^{1/2}\,.
\end{aligned} \tag{6.30}
$$

We now notice that

$$
\frac{n(2-p)}{8}+\frac{(4-p)}{4} < 1 \Longleftrightarrow p > \frac{2n}{n+2}
$$

and therefore by Young's inequality we have

$$
\begin{aligned}
&\Big(s+\|Du\|_{L^\infty(Q_\rho(x^*,t^*))}\Big)^{n(2-p)/8+(4-p)/4}[\mathbf{P}_\rho^V(x^*,t^*)]^{1/2}\\
&\qquad \le (1/4)\|Du\|_{L^\infty(Q_\rho)}+c[\mathbf{P}_\rho^V(x^*,t^*)]^{4/[(n+2)p-2n)]}+cs\,.
\end{aligned}
\tag{6.31}
$$

Moreover, since we are assuming $\ell > n(2-p)/4$, again by Young's inequality we obtain

$$
\begin{aligned}
&\Big(s+\|Du\|_{L^\infty(Q_\rho(x^*,t^*))}\Big)^{n(2-p)/4\ell}\left(\fint_{Q_\rho(x^*,t^*)}|Du|^{2\ell}\,dx\,dt\right)^{1/2\ell}\\
&\le (1/4)\|Du\|_{L^\infty(Q_\rho(x^*,t^*))}+c\left(\fint_{Q_\rho(x^*,t^*)}|Du|^{2\ell}\,dx\,dt\right)^{2/[4\ell-n(2-p)]}+cs
\end{aligned}
\tag{6.32}
$$

and matching (6.31)-(6.32) to (6.30) yields

$$
\begin{aligned}
|Du(x^*,t^*)| \le{}& c\left(\fint_{Q_\rho(x^*,t^*)}|Du|^{2\ell}\,dx\,dt\right)^{2/[4\ell-n(2-p)]}\\
&+c[\mathbf{P}_\rho^V(x^*,t^*)]^{4/[(n+2)p-2n)]}+c(s+1)\\
&+(3/4)\Big(s+\|Du\|_{L^\infty(Q_\rho(x^*,t^*))}\Big)\,.
\end{aligned}
$$

This last estimate is similar to (6.4) and therefore, proceeding as after (6.4) via Lemma 2.1, the desired inequality (1.15) finally follows. □

Proof of Theorems 1.3 and 1.4. The proof is now a simple consequence of Theorems 1.1-1.2 and Lemmas 2.3-2.4. □

7. Possible extensions and refinements

The techniques demonstrated in the previous sections allow for a rather large number of refinements and extensions, especially when keeping in mind the results valid in the elliptic case [14]. We shall confine ourselves to outline the results obtainable in terms of a priori estimates, that is estimates for a priori regular solutions, as done in Sections 4-6. The relative approximation and regularization arguments necessary to obtain relevant existence and regularity results can be obtained by combining the methods of Section 3 with those from [14].

- All the results of this paper extend mutatis mutandis to problems with time dependent coefficients of the type

  $$u_t - \operatorname{div} a(t, Du) = V(x,t)$$

  where the vector field $t \mapsto a(t,z)$ is just measurable. This follows by the fact that all the a priori estimates are basically a consequence of the Caccioppoli's inequality (4.2); in turn the derivation of this estimate involves a differentiation of the equation considered only with respect to the space variable x. Therefore a measurable dependence with respect to t does not affect the proofs.

- When considering more general operators of the type
$$u_t - \operatorname{div} a(x,t,u,Du) = V(x,t)$$
the situation changes and estimates can be obtained provided the partial map $(x,u) \mapsto a(x,\cdot,u,\cdot)$ is assumed to be differentiable and suitable growth conditions are assumed as usual of the derivatives a_x and a_u.
- We finally examine the case of right hand sides with more general growth, that is equations and systems of the type
$$u_t - \operatorname{div} a(Du) = b(x,t,u,Du)\,. \tag{7.1}$$
Here we shall consider growth conditions of the type:
$$|b(x,t,u,Du)| \leq L(V(x,t)|Du|^q + 1)\,,$$
while additional inessential refinements can be made adding further lower order terms. We shall start by the case $p \geq 2$. A priori L^∞-bounds follow considering (7.1), combining estimates of Theorems 1.1, 1.2 and 1.5 with the iteration methods developed in [14]. Indeed, as shown in [14], a priori L^∞-bounds for solutions to equations as in (7.1) are a consequence of estimates as (1.12) and a suitable localization and iteration lemmas that at this stage apply to the parabolic case as well. In particular, using estimate (1.12), and assuming that $q \leq p/2$, is possible to prove that Du is locally bounded provided so is $\mathbf{P}_r^V$. Instead, when considering the case $V(x,t) \equiv V(x)$, then using estimate (1.26) it is then possible to prove the same result assuming only that $q \leq p-1$. In the case $2 - 2/n < p \leq 2$ we instead have to assume that $q < 4/[(n+2)p - 2n]$, by using this time estimate (1.19).

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