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POINTWISE GRADIENT ESTIMATES

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Abstract. We survey a number of recent results concerning the possibility of proving pointwise gradient estimates via potentials for solutions to quasilinear, possibly degenerate, elliptic and parabolic equations.

1. Linear potentials and the classical approach

The aim of this paper is to survey a certain number of recent results pointing at the fact that *pointwise bounds for solutions to linear elliptic and parabolic equations actually hold also in the case of general, possibly degenerate elliptic equations.* Specifically, consider as a model case the Poisson equation

$$-\triangle u = \mu \,, \tag{1.1}$$

which, for simplicity, we shall initially consider in the whole $\mathbb{R}^n$ for $n \geq 2$; here μ is, again for simplicity, assumed to be smooth and compactly supported, while u is the unique solution, which decays to zero at infinity.

The classical representation formula involving the so called fundamental solution (Green's function) gives that

$$u(x) = \int G(x,y)\, d\mu(y) \,, \tag{1.2}$$

where

$$G(x,y) \approx \begin{cases} |x-y|^{2-n} & \text{if} \quad n \geq 3 \\ \log\ |x-y| & \text{if} \quad n = 2\,; \end{cases} \tag{1.3}$$

here the symbol $\approx$ denotes a relation of proportionality via a fixed constant, depending only on the dimension n. The value of the constant is in principle not relevant for our purposes. The representation formula in (1.2) allows to derive all the relevant integrability properties of u and its derivatives in terms of those of the right hand side datum. This goes via the analysis of Riesz potentials, which are defined as follows, with $\beta \in [0,n)$:

$$I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\beta}} \,. \tag{1.4}$$

This is called the β-Riesz potential of μ, where μ is a Borel measure defined on $\mathbb{R}^n$. By (1.2) we gain the following inequalities:

$$|u(x)| \lesssim |I_2(\mu)(x)| \qquad \text{and} \qquad |Du(x)| \lesssim I_1(|\mu|)(x)\,; \tag{1.5}$$

for simplicity we have only concentrated on the case $n > 2$. The second inequality has been actually obtained by differentiating (1.2). The previous estimates essentially encode all the basic integrability properties of solutions. In fact, using the regularizing property

$$I_\beta \colon L^\gamma \to L^{n\gamma/(n-\beta\gamma)} \,, \qquad \gamma > 1\,, \qquad \beta\gamma < n\,, \tag{1.6}$$

of the Riesz potentials, we infer, for instance, the following a priori estimate:

$$\|Du\|_{L^{n\gamma/(n-\gamma)}} \lesssim \|\mu\|_{L^\gamma} \tag{1.7}$$

whenever $\gamma < n$. Similarly, further estimates in rearrangement invariant functions spaces follow as well.

Although pointwise estimates (1.5) appear at the first sight too much linked to the linear case to leave hopes for a nonlinear analog, it is not so. The next sections are indeed dedicated to show in *what terms such an extension can be done.*

Remark 1.1. In the rest of the paper, unless otherwise specified, most of the results will be stated in the form of a priori estimates more regular solutions for solutions (for instance C^0, C^1-solutions, and in general $W^{1,p}$-solutions). In turn, such estimates allow, via the usual approximation arguments for instance described in [8, 9, 17, 25, 49], generalizations for equations with Borel measures as datum.

2. Nonlinear elliptic equations

As for the notation, in what follows, we denote by c a general positive constant, possibly varying from line to line; special occurrences will be denoted by c_1, c_2 etc, and relevant dependencies on parameters will be emphasized using parentheses. *All such constants will be larger or equal than one.* We also denote by $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ the open ball of $\mathbb{R}^n$ with center x and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_r \equiv B(x,r)$. Unless otherwise stated, different balls in the same context will have the same center.

Since we are going to deal with local results, given a Borel measure μ living in $\mathbb{R}^n$, we need a suitable, truncated version of the classical Riesz potentials defined in (1.4), that is

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho}\,, \tag{2.1}$$

and we note that the inequality $\mathbf{I}^{\mu}_{\beta}(x,R) \lesssim I_\beta(\mu)(x)$ holds whenever μ is a nonnegative measure. With A being a measurable subset with positive measure, and with $g\colon A \to \mathbb{R}^m$, $m \in \mathbb{N}$, being a measurable map, we shall denote by

$$\fint_A g(x)\,dx\,dt := \frac{1}{|A|}\int_A g(x)\,dx\,dt$$

its integral average; here $|A|$ denotes the Lebesgue measure of A. A similar notation is adopted if the integral is only in space or time.

The class of equations of interest here are those of quasilinear type, that is

$$-\mathrm{div}\ a(Du) = \mu \qquad \text{in } \Omega \tag{2.2}$$

whenever μ is a Borel measure with finite mass (that for the sake of simplicity we assume to be defined in $\mathbb{R}^n$) while $\Omega \subset \mathbb{R}^n$ is open and bounded subset and $n \geq 2$. The vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular and satisfying the following *growth and ellipticity assumptions*:

$$\begin{cases} |a(z)| + |a_z(z)|(|z|^2+s^2)^{1/2} \leq L(|z|^2+s^2)^{(p-1)/2} \\ \nu(|z|^2+s^2)^{(p-2)/2}|\xi|^2 \leq \langle a_z(z)\xi,\xi\rangle \end{cases} \tag{2.3}$$

whenever $z, \xi \in \mathbb{R}^n$, where $0 < \nu \leq L$ and $s \geq 0$ are fixed parameters. A model case for the previous situation is clearly given by considering the p-Laplacean equation

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu\,, \tag{2.4}$$

or by its nondegenerate version (when $s > 0$)

$$-\mathrm{div}\,[(|Du|^2+s^2)^{(p-2)/2}Du] = \mu\,.$$

Now, although estimates (1.5) could be still possible for nonlinear equations of the type (2.2) when $p = 2$, they certainly do not hold when $p \neq 2$. Indeed, if we consider a solution to $\operatorname{div}(|Du|^{p-2}Du) = \mu$ with $p \neq 2$, we see that $\tilde{u} = c^{\frac{1}{p-1}}u$ - and *not* cu - solves $\operatorname{div}(|D\tilde{u}|^{p-2}D\tilde{u}) = c\mu$ for $c \neq 0$. Therefore, in order to hope for a nonlinear analog of relations (1.5) we need to consider a suitable family of nonlinear potentials, capable to encode in their structure the scaling of equations of p-Laplacean type. The first example of such potentials is given by the so called nonlinear Wolff potentials.

Definition 1. Let μ be Borel measure with finite total mass on $\mathbb{R}^n$; the nonlinear Wolff potential is defined by

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}\,, \qquad \beta \in (0, n/p] \tag{2.5}$$

whenever $x \in \mathbb{R}^n$ and $0 < R \leq \infty$.

We immediately notice that for a suitable choice of the parameters β, p, Wolff potentials reduce to Riesz potentials, i.e. $\mathbf{I}^{|\mu|}_{\beta} \equiv \mathbf{W}^{\mu}_{\beta/2,2}$. Wolff potentials play a crucial role in nonlinear potential theory and in the description of the fine properties of solutions to nonlinear equations in divergence form. They first appear in the work of Havin & Maz'ya [23] and they have been popularized by the famous paper by Hedberg & Wolff [24].

An important fact about Wolff potentials is that their behavior can be in several aspects recovered from that of Riesz potentials. Indeed, the following pointwise inequality holds:

$$\mathbf{W}^{\mu}_{\beta,p}(x,\infty) \leq cI_{\beta}\left\{[I_{\beta}(|\mu|)]^{1/(p-1)}\right\}(x) =: c\mathbf{V}_{\beta,p}(|\mu|)(x)\,. \tag{2.6}$$

The nonlinear potential $\mathbf{V}_{\beta,p}(\mu)(x)$ appearing in the right hand side of the previous inequality - often called the Havin-Maz'ya potential of μ - is a classical object in nonlinear potential theory, and together with the bound (2.6) comes from the pioneering work of Adams & Meyers [3] and Havin & Maz'ya [23]. Estimate (2.6) allows to derive several types of local estimates starting by the properties of the Riesz potential. As a matter of fact, although named after Wolff, Wolff potentials appear to have employed well before Wolff, see for instance [23].

Kilpeläinen & Malý [27, 28] were the first in proving that Wolff potentials locally control solutions to general quasi linear equations; later on, further approaches have been given by Trudinger & Wang [57, 58], Korte & Kuusi [30] and Duzaar & Mingione [17].

Theorem 2.1. *Let $u \in C^0(\Omega) \cap W^{1,p}(\Omega)$ be a weak solution to the equation* (2.2)*, under the assumptions* (2.3) *with $1 < p \leq n$, where μ is a Borel measure with finite total mass. Then there exists a constant $c \equiv c(n, p, \nu, L) > 0$ such that the following pointwise estimate holds whenever $B(x,R) \subseteq \Omega$:*

$$|u(x)| \leq c\mathbf{W}^{\mu}_{1,p}(x,R) + c\fint_{B(x,R)} (|u| + Rs)\,dy\,. \tag{2.7}$$

Note that for $p = 2$ we have that $\mathbf{W}^{\mu}_{1,p} \equiv \mathbf{I}^{|\mu|}_2$ and we retrieve a local analog of the first estimate in (1.5). The importance of Theorem 2.1 lies in several aspects: it locally allows to recover all the integrability results known for u via the properties of the Wolff potentials - see also (2.6). Moreover, Theorem 2.1 is the key tool for the proof of the sufficiency of the boundary Wiener criterium for nonlinear equations [28] (whose sufficiency had been previuosly established by Maz'ya [46]); further applications are given in the work of Phuc & Verbitsky [52, 53].[52, 53].

The possibility of extending pointwise potential estimates to the gradient remained an open issue discussed for a long while, and an answer came only recently.

The first case, when $p=2$ is contained in [49], and gives a componentwise estimate for the gradient which locally replicates the second one in (1.5).

Theorem 2.2 ([49])**.** *Let $u \in C^1(\Omega)$ be a solution to the equation* (2.2)*, under the assumptions* (2.3) *with $p=2$, with μ being a Borel measure with finite total mass. Then there exists a constant $c \equiv c(n,\nu,L)$ such that the pointwise estimate*

$$|D_\xi u(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c \fint_{B(x,R)} |D_\xi u| \, dy \tag{2.8}$$

holds whenever $\xi \in \{1,\ldots,n\}$ and whenever $B(x,R) \subseteq \Omega$.

The extension to the case $p \neq 2$ has entailed several steps. At the first sight, it appeared as Wolff potentials would play a major role in the estimation of the gradient too. In fact, the first gradient extension valid for solutions to equations with super-linear ($p \neq 2$) growth has been given in the following:

Theorem 2.3 ([17])**.** *Let $u \in C^1(\Omega)$ be a weak solution to the equation* (2.2) *under the assumptions* (2.3) *with $p \geq 2$, where μ is a Borel measure with finite total mass. Then there exists a constant $c \equiv c(n,p,\nu,L) > 0$ such that the pointwise estimate*

$$|Du(x)| \leq c\mathbf{W}^{\mu}_{1/p,p}(x,R) + c \fint_{B(x,R)} (|Du| + s) \, dy \tag{2.9}$$

holds whenever $B(x,R) \subseteq \Omega$.

Now, while estimate (2.9) is sufficiently precise to catch all those integrability properties described by spaces *that are not too close to L^∞* - for instance, for the model example in (2.4) it reproduces those in [8, 9, 14, 26, 47, 48] - it does not fully catch the known integrability results when one looks at, for instance, conditions guaranteeing the Lipschitz continuity of solutions. When looking at Lorentz spaces, estimate (2.9) (applied, via approximation, to $W^{1,p}$ solutions) gives

$$\mu \in L(n, 1/(p-1)) \Longrightarrow Du \in L^\infty \qquad \text{locally in } \Omega\,. \tag{2.10}$$

On the other hand, the result in [20] gives, for $n > 2$, that

$$\mu \in L(n,1) \Longrightarrow Du \in L^\infty \qquad \text{locally in } \Omega\,, \tag{2.11}$$

which is better than (2.10). Indeed, let us recall the definition of so-called Lorentz spaces $L(t,q)(\Omega)$, with $1 \leq t < \infty$ and $0 < q \leq \infty$. When $q < \infty$, a measurable map g belongs to $L(t,q)(\Omega)$ iff

$$\|g\|^q_{L(t,q)(\Omega)} := q \int_0^\infty \left(\lambda^t |\{x \in \Omega \ : \ |g(x)| > \lambda\}|\right)^{q/t} \frac{d\lambda}{\lambda} < \infty\,.$$

For $q = \infty$ Lorentz spaces are defined as Marcinkiewicz spaces $L(t,\infty)(\Omega) \equiv \mathcal{M}^t(\Omega)$; the local variant of such spaces is then obtained by saying $g \in L(t,q)(\Omega)$ locally iff $g \in L(t,q)(\Omega')$ whenever $\Omega' \Subset \Omega$ is a subset. Lorentz spaces "interpolate" Lebesgue spaces as the second parameter q "tunes" t in the following sense: whenever $0 < q < t < r \leq \infty$ we have, with continuous embeddings, that the following strict inclusions hold:

$$L^r \equiv L(r,r) \subset L(t,q) \subset L(t,t) \subset L(t,r) \subset L(q,q) \equiv L^q\,.$$

The gap between (2.10) and (2.11) leads to think that estimate (2.9) can be still improved. In particular, we observe that in (2.11) the imposed condition on μ to make Du locally bounded is independent of p. This hints the possibility of the existence of an estimate involving a potential being independent of p as well. Natural candidates at this stage are obviously the Riesz potentials, and, indeed, they reappear in the following:

Theorem 2.4 ([19, 41])**.** *Let $u \in C^1(\Omega)$ be a weak solution to the equation* (2.2) *under the assumptions* (2.3) *with $p \geq 2 - 1/n$, where μ is a Borel measure with finite total mass defined on Ω. Then there exists a constant c, depending only on n, p, ν, L, such that the pointwise estimate*

$$|Du(x)|^{p-1} \leq c\mathbf{I}_1^{|\mu|}(x,R) + c\left(\fint_{B(x,R)} (|Du| + s)\, dy\right)^{p-1} \tag{2.12}$$

holds whenever $B(x,R) \subseteq \Omega$.

We recall that the lower bound $p > 2 - 1/n$ is linked to the fact that, in general, solutions to measure data problems do not belong to $W^{1,1}$ when $p < 2 - 1/n$. The involved part of (2.12) is when $p \geq 2$, and it has been obtained by the authors in [41] (see also [42] for a preliminary announcement); in fact, when $p < 2$, estimate (2.12) does not improve (2.9), which actually is not expected to hold in the subquadratic case. As a matter of fact, Theorem 2.4 implies Theorem 2.3 since

$$\mathbf{I}_1^{|\mu|}(x,R) \lesssim \left[\mathbf{W}^{\mu}_{1/p,p}(x,2R)\right]^{p-1} \qquad \text{holds when } p \geq 2\,.$$

The surprising character of Theorem 2.5 mainly relies on the fact that, although considering degenerate quasilinear equations, the gradient can be pointwise estimated via Riesz potentials exactly as it happens for solutions to the hyper-classical Poisson equation $-\triangle u = \mu$, for which estimate (2.8) is an immediate consequence of the classical representation formula via Green's functions. Indeed, we have

Corollary 2.1. *Let $u \in W^{1,p}(\mathbb{R}^n)$ be a local weak solution to the equation* (2.4) *with $p \geq 2 - 1/n$ and μ being a Borel measure with locally finite mass. Then there exists a constant c, depending only on n, p, such that the following estimate holds for every Lebesgue point $x \in \mathbb{R}^n$ of Du:*

$$|Du(x)|^{p-1} \leq c \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}}\,.$$

We notice that (2.11) now follows from (2.12) (when applied, via approximation, to $W^{1,p}$ solutions). Theorem 2.5 yields in turn the following, immediate

Corollary 2.2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.2) *under the assumptions* (2.3) *with $p \geq 2 - 1/n$, where μ is a Borel measure with finite total mass defined on Ω. Then*

$$\mathbf{I}_1^{|\mu|}(\cdot,R) \in L^\infty_{\rm loc}(\Omega) \text{ for some } R > 0 \quad \Longrightarrow \quad Du \in L^\infty_{\rm loc}(\Omega,\mathbb{R}^n)\,.$$

In particular, there exists a constant c, depending only on n, p, ν, L, such that the following estimate holds whenever $B_R \subseteq \Omega$:

$$\|Du\|_{L^\infty(B_{R/2})} \leq c\left\|\mathbf{I}_1^{|\mu|}(\cdot,R)\right\|^{1/(p-1)}_{L^\infty(B_R)} + c\fint_{B_R} (|Du| + s)\, dy\,. \tag{2.13}$$

The previous result is striking as it states that *the classical, sharp Riesz potential criterium implying the Lipschitz continuity of solutions to the Poisson equations remains valid when considering the p-Laplacean operator.* More analogies actually take place in the following:

Theorem 2.5 ([41])**.** *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.2) *under the assumptions* (2.3) *with $p \geq 2 - 1/n$, where μ is a Borel measure with finite total mass defined on Ω. If*

$$\lim_{R\to 0} \mathbf{I}_1^{|\mu|}(x,R) = 0 \quad \text{locally uniformly in } \Omega \text{ w.r.t. } x\,,$$

then Du is continuous in Ω.

Theorem 2.5 admits the following relevant corollary, providing gradient continuity when μ is a function belonging to a borderline Lorentz space:

Corollary 2.3. *Let $u \in W^{1,p}(\Omega)$ be as in Theorem 2.5. If $\mu \in L(n,1)$ locally in Ω, then Du is continuous in Ω.*

Corollary 2.3 extends (2.11), which is sharp already in the case of the Poisson equation; we remark that the two-dimensional case $n = 2$ of (2.11) oddly remained an open problem in [20], essentially for technical reasons, and this is settled by the previous corollary.

Finally, another immediate corollary of Theorem 2.5 is concerned with those measures satisfying special density properties.

Corollary 2.4. *Let $u \in W^{1,p}(\Omega)$ be as in Theorem 2.5. Assume that the measure μ satisfies the density condition*

$$|\mu|(B_R) \leq cR^{n-1}h(R)\,, \qquad \textit{where} \qquad \int_0^R h(\varrho)\,\frac{d\varrho}{\varrho} < \infty$$

for some $c \geq 0$ and for every ball $B_R \subset \mathbb{R}^n$. Then Du is continuous in Ω.

Remark 2.1 (Regular coefficients)**.** Theorem 2.4 extends to the case of more general equations of the type

$$-\operatorname{div} a(x, Du) = \mu \tag{2.14}$$

under the assumption that the dependence on x of the vector field is Dini-continuous. This condition is sharp in that estimate (2.12) ensures the local gradient boundedness when, for instance, $\mu = 0$, and in this situation it is known that Dini-continuity must be assumed [17, 18, 36].

Remark 2.2 (Measurable coefficients)**.** Theorem 2.4 cannot extend to the case of more general equations of the type (2.14) when the dependence on x is simply measurable. It is clear that an estimate as (2.9) cannot hold under assumptions (2.3), as in this case the maximal gradient regularity of solutions to equations as $\operatorname{div} a(x, Du) = 0$ is in general only given by

$$Du \in L^{p+\delta}_{\rm loc} \tag{2.15}$$

for some $\delta > 0$. This is essentially a consequence of Gehring's lemma and $\delta \equiv \delta(n, p, \nu, L)$ is a universal exponent depending only on the ellipticity properties of the operator. However, something remains; more precisely a non-local version of estimate (2.9) still holds yielding level sets information rather than pointwise. Moreover, such an estimate is bound to provide regularity results in accordance to the maximal gradient regularity in (2.15), in that it will provide in the best possible case gradient estimates in L^q with $q < p + \delta$, where δ is exactly the exponent in (2.15) given by Gehring's lemma. Before stating the result, some definitions: we recall the definition of the (restricted and noncentered) fractional maximal function operator relative to a cube $Q_0 \subseteq \mathbb{R}^n$ defined as

$$M^*_{\beta,Q_0}(g)(x) := \sup_{Q \subseteq Q_0,\; x \in Q} |Q|^{\beta/n} \fint_Q |g(y)|\, dy\,, \qquad \beta \in [0, n)\,,$$

where the sup is taken with respect all the cubes Q contained in Q_0; all the cubes here have sides parallel to the coordinate axes. It goes without saying that a similar definition can be given when g is replaced by a measure in an obvious way.

Theorem 2.6 ([48, 51])**.** *Let $u \in W^{1,p}(\Omega)$ be a weak solution to* (2.14) *under the assumptions* (2.3) *(more precisely $z \mapsto a(x, z)$ satisfies* (2.3) *a.e. $x \in \Omega$ and the vector field $a(\cdot)$ is Carathéodory regular), where μ is a Borel measure with finite total mass and $p > 2 - 1/n$. Let $Q_{2R} \Subset \Omega$ be a cube and let $M^* \equiv M^*_{Q_{2R}}$*

denote the restricted maximal operator with respect to Q_{2R}. *There exist constants* $\delta \equiv \delta(n,p,\nu,L) > 0$ *and* $A \equiv A(n,p,\nu,L) > 1$ *such that: For every* $T > 1$ *there exists* $\varepsilon \equiv \varepsilon(n,p,\nu,L,T) \in (0,1)$ *such that*

$$\begin{aligned}\Big|\Big\{x \in Q_R : M^*((|Du|+s))(x) > AT\lambda\Big\}\Big| \\ \leq T^{-(p+\delta)}\Big|\Big\{x \in Q_R : M^*((|Du|+s))(x) > \lambda\Big\}\Big| \\ +\Big|\Big\{x \in Q_R :\ [M^*_{1,Q_{2R}}(\mu)]^{1/(p-1)} > \varepsilon\lambda\Big\}\Big|\end{aligned}$$

holds whenever

$$\lambda \geq c(n)T^{p+\delta} \fint_{Q_{2R}} (|Du|+s)\, dx\,.$$

The connection to Theorem 2.4 is that the fractional maximal operator considered above is pointwise estimated by the Riesz potential considered in Theorem 2.4. The result above has been obtained in [48] when $p \geq 2$, while the case $2-1/n < p < 2$ can be obtained following [51]. We refer to [36, 37] for more results on equations with coefficients.

3. Nonlinear potential estimates for parabolic problems

Here we switch to the pointwise gradient potential estimates available in the case of parabolic equations of the type

$$u_t - \operatorname{div} a(Du) = \mu \tag{3.1}$$

considered in cylindrical domains $\Omega_T = \Omega \times (-T,0)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 2$, and $T > 0$. In the most general case μ is a Borel measure with finite total mass, for simplicity defined on $\mathbb{R}^{n+1}$; therefore we shall assume that $|\mu|(\mathbb{R}^{n+1}) < \infty$. The C^1-vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy (2.3) whenever $z,\xi \in \mathbb{R}^n$, where $0 < \nu \leq L$ and $s \geq 0$. The model example for the equations treated here is given by the familiar evolutionary p-Laplacean equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu\,. \tag{3.2}$$

As solutions to (3.1) are usually obtained via approximation with solutions to equations with more regular data and solutions - see [7] for the necessary explanations and basic existence and regularity results - we shall always assume to deal with energy solutions, i.e., we say that u is a solution to (3.1) if

$$u \in C^0(-T,0;L^2(\Omega)) \cap L^p(-T,0;W^{1,p}(\Omega)) \tag{3.3}$$

and u solves (3.1) in the distributional sense

$$-\int_{\Omega_T} u\varphi_t\, dx\, dt + \int_{\Omega_T} \langle a(Du), D\varphi\rangle\, dx\, dt = \int_{\Omega_T} \varphi\, d\mu$$

whenever $\varphi \in C^\infty_c(\Omega_T)$. Moreover, in the following we shall always deal with the case the measure is an integrable function $\mu \in L^1(\mathbb{R}^{n+1})$. We shall in other words confine ourselves to describe a priori estimates valid for *a priori more regular solutions and data.*

We shall start from the case $p = 2$, dealt within [17], when the treatment is similar to that for elliptic equations. For this, we need some additional terminology. Let us recall that given points $(x,t),(x_0,t_0) \in \mathbb{R}^{n+1}$ the standard parabolic metric is defined by

$$d_{\rm par}((x,t),(x_0,t_0)) := \max\{|x-x_0|, \sqrt{|t-t_0|}\} \approx \sqrt{|x-x_0|^2+|t-t_0|}$$

and the related metric balls with radius R with respect to this metric are given by the cylinders of the type $B(x_0,R)\times(t_0-R^2,t_0+R^2)$. The "caloric" Riesz potential is now built starting from

$$I_\beta(\mu)((x,t)) := \int_{\mathbb{R}^{n+1}} \frac{d\mu((\tilde{x},\tilde{t}))}{d_{\mathrm{par}}((\tilde{x},\tilde{t}),(x,t))^{N-\beta}}\,, \qquad 0<\beta\leq N:=n+2\,,$$

whenever $(x,t)\in\mathbb{R}^{n+1}$. In order to be used in estimates for parabolic equations, it is convenient to introduce its local version via the usual backward parabolic cylinders - with "vertex" at (x_0,t_0) - that is

$$Q(x_0,t_0;R) := B(x_0,R)\times(t_0-R^2,t_0)\,. \tag{3.4}$$

The Riesz potential is now given by

$$\mathbf{I}^\mu_\beta(x_0,t_0;R) := \int_0^R \frac{|\mu|(Q(x_0,t_0;\varrho))}{\varrho^{N-\beta}}\,\frac{d\varrho}{\varrho}\,, \qquad 0<\beta\leq N\,.$$

The main result in the parabolic case is

Theorem 3.1 (Caloric potential gradient bound)**.** *Under the assumptions* (2.3) *with $p=2$, let u be a weak solution to* (3.1)*, and such that such that Du is continuous in Ω_T. Then there exists a constant $c\equiv c(n,\nu,L)$ such that the estimate*

$$|Du(x_0,t_0)| \leq c\mathbf{I}^\mu_1(x_0,t_0;R) + c\,⨍_{Q(x_0,t_0;R)} (|Du|+s)\,dx\,dt \tag{3.5}$$

holds whenever $Q(x_0,t_0;R)\subseteq\Omega$.

3.1. **Degenerate/singular problem: the case $p>2$.** The degenerate/singular case $p\neq 2$ *is a different story*, and the pointwise gradient estimates involve substantial new ingredients as the operators involved do not have a universal scaling. Therefore, the concept of *intrinsic geometry*, developed by DiBenedetto [12], comes into the play. This is linked to the fact that multiplying a solution to (3.2) by a constant does not yield a solution to a similar equation. The intrinsic geometry prescribes that - although the equations considered are anisotropic - they behave as isotropic ones when considered in space/time cylinders *whose sizes depend on the solution itself.* To outline how the intrinsic approach works, let us consider a domain, actually a cylinder Q, where, roughly speaking, the size of the gradient is approximately λ – possibly in some integral averaged sense – that is

$$|Du|\approx\lambda>0\,. \tag{3.6}$$

In this case we shall consider *intrinsic cylinders*, i.e. cylinders of the type

$$Q=Q^\lambda_r(x_0,t_0)\equiv B(x_0,r)\times(t_0-\lambda^{2-p}r^2,t_0)\,, \tag{3.7}$$

where $B(x_0,r)\subset\mathbb{R}^n$ is the usual Euclidean ball centered at x_0 and with radius $r>0$. Note that when $\lambda\equiv 1$ or when $p=2$, the cylinder in (3.7) reduces to the standard parabolic cylinder in (3.4). Indeed, the case $p=2$ is the only one admitting a non-intrinsic scaling and local estimates have a homogeneous character. The *heuristics of the intrinsic scaling method* can now be easily described as follows: assuming that, in a cylinder Q as in (3.7), the size of the gradient is approximately λ as in (3.6). We then have that the equation $u_t-\operatorname{div}(|Du|^{p-2}Du)=0$ looks like $u_t=\operatorname{div}(\lambda^{p-2}Du)=\lambda^{p-2}\triangle u$, which, after a scaling and considering $v(x,t):=u(x_0+\varrho x,t_0+\lambda^{2-p}\varrho^2 t)$ in $B(0,1)\times(-1,0)$, reduces to the heat equation $v_t=\triangle v$ in $B(0,1)\times(-1,0)$. This equation, in fact, admits favorable and homogeneous a priori estimates for solutions. The success of this strategy is therefore linked to a rigorous construction of such cylinders in the context of intrinsic definitions. Indeed, the way to express a condition as (3.6) is typically in an averaged sense like for instance

$$\left(\frac{1}{|Q_r^\lambda|}\int_{Q_r^\lambda}|Du|^{p-1}\,dx\,dt\right)^{1/(p-1)} = \left(\fint_{Q_r^\lambda}|Du|^{p-1}\,dx\,dt\right)^{1/(p-1)} \approx \lambda\,. \tag{3.8}$$

A problematic aspect in (3.8) occurs as the value of the integral average must be comparable to a constant which is in turn involved in the construction of its support $Q_r^\lambda \equiv Q_r^\lambda(x_0,t_0)$, exactly according to (3.7). As a consequence of the use of such intrinsic geometry, all the a priori estimates for solutions to evolutionary equations of p-Laplacean type admit formulations becoming homogeneous only when formulated in terms of intrinsic parameters and cylinders as λ and Q_r^λ.

The approach of [38, 39, 40] proposes *to adopt the intrinsic geometry approach in the context of nonlinear potential estimates.* This provides a class of *intrinsic Wolff potentials* that reveal to be the natural objects to be considered, as their structure allows to recast the behavior of the Barenblatt solution - the so-called nonlinear fundamental solution. For this reason reason we introduce the following *intrinsic Wolff potential*:

$$\mathbf{W}_\lambda^\mu(x_0,t_0;R) := \int_0^R\left(\frac{|\mu|(Q_\varrho^\lambda(x_0,t_0))}{\lambda^{2-p}\varrho^{N-1}}\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\,, \qquad N := n+2\,,$$

defined starting by intrinsic cylinders $Q_\varrho^\lambda(x_0,t_0)$ as in (3.7). We have

Theorem 3.2 ([39])**.** *Let u be a solution to* (3.1)*; assume that* (2.3) *hold with $p\geq 2$, and that Du is continuous in Ω_T and $\mu\in L^1$. There exists a constant $c\geq 1$, depending only on n,p,ν,L, such that whenever $Q_R^\lambda \equiv Q_R^\lambda(x_0,t_0) \equiv B(x_0,R)\times(t_0-\lambda^{2-p}R^2,t_0)\subset\Omega_T$ is an intrinsic cylinder with vertex at (x_0,t_0), then*

$$c\mathbf{W}_\lambda^\mu(x_0,t_0;R) + c\left(\fint_{Q_R^\lambda}(|Du|+s)^{p-1}\,dx\,dt\right)^{1/(p-1)} \leq \lambda \tag{3.9}$$

implies

$$|Du(x_0,t_0)|\leq\lambda\,.$$

The nonlinear potential $\mathbf{W}_\lambda^\mu$ appearing in (3.9) is the natural intrinsic counterpart of the Wolff potential $\mathbf{W}_{1/p,p}^\mu$ intervening in (2.9). In fact, when considering the associated elliptic stationary problem, μ being time independent, Theorem 3.2 gives back (2.9). Estimate (3.9), in turn, essentially gives back (2.8), when applied in the stationary setting, as well the classical L^∞-bound of DiBenedetto [12] who indeed proved that

$$c\left(\fint_{Q_R^\lambda}(|Du|+s)^{p-1}\,dx\,dt\right)^{1/(p-1)} \leq \lambda \quad\Longrightarrow\quad |Du(x_0,t_0)|\leq\lambda\,.$$

The formulation of Theorem 3.2 involves intrinsic quantities and conditions, that, as such, appear at a first sight to be problematic to verify. Nevertheless, as shown in the next theorem, which in fact follows as a corollary, Theorem 3.2 always implies local a priori estimates *via parabolic Wolff potentials*, on arbitrary parabolic cylinders $Q_R\subset\Omega_T$:

$$\tilde{\mathbf{W}}_{\beta,p}^\mu(x_0,t_0;R) := \int_0^R\left(\frac{|\mu|(Q(x_0,t_0;\varrho))}{\varrho^{N-\beta p}}\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\,, \qquad \beta\in(0,N/p]\,;$$

notice that

$$\tilde{\mathbf{W}}_{\beta/2,2}^\mu(x_0,t_0;R) = \mathbf{I}_\beta^\mu(x_0,t_0;R)\,.$$

We then have

Corollary 3.1 ([39]). *Let u be a solution to* (3.1)*; assume that* (2.3) *hold with $p \geq 2$, and that Du is continuous in Ω_T and $\mu \in L^1$. There exists a constant c, depending only on n, p, ν, L, such that*

$$|Du(x_0,t_0)| \leq c\left[\tilde{\mathbf{W}}^{\mu}_{1/p,p}(x_0,t_0;R)\right]^{p-1} + c \fint_{Q_R} (|Du|+s+1)^{p-1}\,dx\,dt \tag{3.10}$$

holds whenever $Q(x_0,t_0;R) \equiv B(x_0,R)\times(t_0-R^2,t_0) \subset \Omega_T$ is a standard parabolic cylinder with vertex at (x_0,t_0).

In particular, in the case $p = 2$ we recover (3.5). To check the consistency of estimate (3.10) with the ones already present in the literature we observe that when $\mu \equiv 0$, estimate (3.10) reduces the classical L^∞-gradient bound available for solutions to the evolutionary p-Laplacean equation; see [12, Chapter 8, Theorem 5.1']; see also Remark 4.3 below. It is interesting to see that when switching to a non-intrisic formulation, local estimates immediately show an anisotropicity under the form of a deficit scaling exponent, which in this case is $p-1$, precisely reflecting the lack of homogeneity of the equations considered. This is typical when considering anisotropic problems, and similar deficit scaling exponents typically appear in the a priori estimates from [1, 2, 35].

Finally, when μ is time independent, or admits a favorable decomposition, it is possible to avoid the intrinsic geometry effect in the potential term and we go back to the elliptic regime.

Corollary 3.2 ([39]). *Let u be a solution to* (3.1)*; assume that* (2.3) *hold with $p \geq 2$, and that Du is continuous in Ω_T and $\mu \in L^1$. Assume that the measure μ satisfies $|\mu| \leq \mu_0 \otimes f$, where $f \in L^\infty(-T,0)$ and μ_0 is a Borel measure on Ω with finite total mass. Then there exists a constant c, depending only on n,p,ν,L, such that*

$$|Du(x_0,t_0)| \leq c\|f\|_{L^\infty}^{1/(p-1)}\mathbf{W}^{\mu_0}_{1/p,p}(x_0,R) + c\fint_{Q_R}(|Du|+s+1)^{p-1}\,dx\,dt$$

whenever $Q_R(x_0,t_0) \equiv B(x_0,R)\times(t_0-R^2,t_0) \subset \Omega_T$ is a standard parabolic cylinder having (x_0,t_0) as vertex. The (elliptic) Wolff potential $\mathbf{W}^{\mu_0}_{1/p,p}$ is defined in (2.5)*.*

3.2. **The subquadratic case.** Here we go back to Riesz type potentials. Also this time, we have to find a suitable intrinsic formulation; it is convenient to consider intrinsic cylinders of the type

$$\tilde{Q}^{\lambda}_{R}(x_0,t_0) := B(x_0,\lambda^{(p-2)/2}R)\times(t_0-R^2,t_0)\,.$$

Note that the ratio between the space and the time scales remains invariant when considering such cylinders instead of those considered in (3.7). This time we have

Theorem 3.3 ([40]). *Let u be a solution to* (3.1)*; assume that* (2.3) *hold with $2-1/(n+1) < p \leq 2$, and that Du is continuous in Ω_T and $\mu \in L^1$. There exists a constant $c \geq 1$, depending only on n,p,ν,L, such that whenever $\tilde{Q}^{\lambda}_{R} \equiv \tilde{Q}^{\lambda}_{R}(x_0,t_0) \equiv B(x_0,\lambda^{(p-2)/2}R)\times(t_0-R^2,t_0) \subset \Omega_T$ is an intrinsic cylinder with vertex at (x_0,t_0), such that if*

$$c\mathbf{I}^{\mu}_{1}(x_0,t_0;R_\lambda) + c\fint_{Q^{\lambda}_{R}}(|Du|+s)\,dx\,dt \leq \lambda \tag{3.11}$$

holds with $R_\lambda = \lambda^{(p-2)/2}R$, then

$$|Du(x_0,t_0)| \leq \lambda\,.$$

Note that in the previous theorem the intrinsic nature of the potential comes from the fact that we are considering the usual parabolic Riesz potential $\mathbf{I}^{\mu}_{1}$, but with a radius that depends on λ. In a way, the formulation in (3.11) is *less intrinsic* *than the one in* (3.9). As for the case $p \geq 2$, the previous theorem implies a priori estimates on standard parabolic cylinders. Indeed we have

Corollary 3.3 ([40])**.** *Let u be a solution to* (3.1)*; assume that* (2.3) *hold with $2 - 1/(n+1) < p \leq 2$, and that Du is continuous in Ω_T and $\mu \in L^1$. Then there exists a constant c, depending only on n, p, ν, L, but not on (x_0, t_0), the solution u, or the vector field $a(\cdot)$, such that*

$$|Du(x_0,t_0)| \leq c\,[\mathbf{I}_1^{\mu}(x_0,t_0;R)]^{2/[(n+1)p-2n]} + c\left(\fint_{Q_R} (|Du|+s+1)\,dx\,dt\right)^{2/[2-n(2-p)]} \tag{3.12}$$

holds whenever $Q_R \equiv Q_R(x_0,t_0) \equiv B(x_0,R)\times(t_0-R^2,t_0) \subset \Omega_T$ is a standard parabolic cylinder with vertex at (x_0,t_0).

Again, as for the case $p \geq 2$, when μ is time independent or admits a favorable decomposition, elliptic Riesz potentials reappear.

Corollary 3.4 ([40])**.** *Let u be a solution to* (3.1)*; assume that* (2.3) *hold with $2 - 1/(n+1) < p \leq 2$, and that Du is continuous in Ω_T and $\mu \in L^1$. Moreover, assume that the decomposition $\mu = \mu_0 \otimes f$ holds, where μ_0 is a finite mass Borel measure on $\mathbb{R}^n$ and $f \in L^\infty(-T,0)$. The following holds for a.e. $(x_0,t_0) \in \Omega_T$: There exists a constant c, depending only on n, p, ν, L, but not on (x_0,t_0), the solution u, or the vector field $a(\cdot)$, such that*

$$|Du(x_0,t_0)| \leq c\|f\|_{L^\infty}^{1/(p-1)}\,[\mathbf{I}_1^{\mu_0}(x_0,R)]^{1/(p-1)} + c\left(\fint_{Q_R} (|Du|+s+1)\,dx\,dt\right)^{2/[2-n(2-p)]} \tag{3.13}$$

whenever $Q_R(x_0,t_0) \equiv B(x_0,R)\times(t_0-R^2,t_0) \subset \Omega_T$ is a standard parabolic cylinder having (x_0,t_0) as vertex. The (elliptic) Riesz potential $\mathbf{I}_1^{\mu_0}$ is defined in (2.1).

Remark 3.1 (Structure of the exponents)**.** It is worthwhile to analyze the exponents appearing in (3.12), and in particular to make a comparison with the ones appearing in (3.13), as they precisely reflect the structure properties of the equation, and in particular of the Barenblatt (fundamental) solutions. The number $2/[2-n(2-p)]$ is the same one appearing in the typical gradient estimates for homogeneous equations ($\mu = 0$) and reflects the gradient nature of the estimate in question. Indeed, when $\mu \equiv 0$ estimate (3.12) reduces to the classical one obtained in [12, Chapter 8, Theorem 5.2']. The exponent $2/[(n+1)p-2n]$ instead blows up as $p \to 2n/(n+1)$ and reflects the non-homogeneity of the equation studied, as well as the structure of the Barenblatt solution; see Section 3.3 below and [40] for the subquadratic case of interest here. Such exponent indeed intervenes in those estimates related to the Barenblatt solution, as for instance the Harnack inequalities in [13, 34]. For the very same reason the exponent $2/[(n+1)p-2n]$ relates to the fact that the right hand side measure μ in general depends on time, and it disappears when μ is time-independent. This is completely natural as in this case it is possible to consider stationary solutions. Yet, it is interesting to compare estimate (3.12) with the main result in [2], where a completely similar dependence on the exponents appears.

3.3. **Comparison with the Barenblatt solution.** For the sake of brevity, we shall concentrate here on the case $p > 2$. The standard quality test for potential estimates, as for instance those in (2.7) and (2.9), consists of measuring the extent they allow to reproduce the behavior of fundamental solutions, i.e. the behavior of those special solutions obtained by taking $\mu \equiv \delta$, where δ is the Dirac measure charging one point. In the case of the evolutionary p-Laplacean equation $u_t - \operatorname{div}(|Du|^{p-2}Du) = \delta$ in $\mathbb{R}^{n+1}$ with Dirac datum δ charging the origin, an explicit (very weak) solution - so-called Barenblatt solution - is given by

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-n/\theta}\left(c_b - \theta^{1/(1-p)}\dfrac{p-2}{p}\left(\dfrac{|x|}{t^{1/\theta}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)} & t>0 \\ 0 & t \leq 0\,, \end{cases}$$

when $p > 2$. Here $\theta := n(p-2)+p$ and c_b is a constant normalizing the solution so that

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x,t)\,dx = 1$$

for all $t > 0$. In a similar way, for the range $2 - 1/(n+1) < p < 2$, we have

$$\mathcal{B}_p(x,t) := \begin{cases} t^{-n/\theta}\left[\dfrac{2-p}{p}\theta^{1/(1-p)}\left(c_b + \left(\dfrac{|x|}{t^{1/\theta}}\right)^{p/(p-1)}\right)\right]^{(p-1)/(p-2)} & t>0 \\ 0 & t \leq 0\,. \end{cases}$$

A direct computation reveals that the gradient of $\mathcal{B}_p(x,t)$ satisfies the estimate

$$|D\mathcal{B}_p(x_0,t_0)| \leq ct_0^{-(n+1)/\theta} \tag{3.14}$$

whenever $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$; in turn this prescribes the blow-up behavior of the fundamental solution at the origin as well as the decay behavior at infinity. Both are typical for a situation, where a Dirac measure appears. *The crucial point is now that the bound appearing in* (3.14) *is directly implied by Theorems* 3.2 *and* 3.3. Moreover, as Theorems 3.2-3.3 hold for general equations, the same bound also holds for solutions to general equations of the type

$$u_t - \operatorname{div} a(Du) = \delta \qquad \text{in } \mathbb{R}^{n+1}\,, \tag{3.15}$$

under assumptions (2.3); see also [44]. This result should be anyway compared to the one in [12, Chapter 11, Theorem 2.1, (2.4)]. Of course, when considering equations with genuine measure data as (3.15), we have to consider those solutions considered in [8, 7], and obtained by approximation processes, as limits of solutions with more regular data. As the estimate of Theorem 3.2 is stable under such approximation methods, Theorem 3.2 applies to solutions of (3.15) as well, modulo considering Lebesgue points of Du rather than any points.

Remark 3.2. Again, as in the elliptic case, Theorems 3.2 and 3.3 open the way to the proof of criteria to establish the gradient continuity. For this we refer to [40].

4. The case of systems and modified potentials

In this section we would like to point out some extension of the result of Corollary 2.2 to the case of certain classes of elliptic and parabolic systems. For the sake of simplicity we shall consider the basic p-Laplacean system

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu \qquad u\colon \Omega \to \mathbb{R}^m\,, \quad m \geq 2 \tag{4.1}$$

together with its evolutionary version

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu \qquad u\colon \Omega_T \to \mathbb{R}^m\,, \quad m \geq 2\,. \tag{4.2}$$

Let us start with the elliptic case. We shall assume that $\mu \in L^2$. In this situation we have

$$\mathbf{I}_1^{|\mu|}(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-1}}\,\frac{d\varrho}{\varrho} = \omega_n \int_0^R ⨍_{B(x,\varrho)} |\mu|\,dy\,d\varrho\,,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. Therefore Hölder's inequality yields

$$\mathbf{I}_1^{|\mu|}(x,R) \le c\int_0^R \left(\fint_{B(x,\varrho)} |\mu|^2\, dy \right)^{1/2} d\varrho \le c\mathbf{P}^\mu(x,R)\,,$$

where we have defined the new potential

$$\mathbf{P}^\mu(x,R) := \int_0^R \left(\frac{|\mu|^2(B(x,\varrho))}{\varrho^{n-2}} \right)^{1/2} \frac{d\varrho}{\varrho}\,.$$

Dimensionally speaking, it scales as the Riesz potential $\mathbf{I}_1^{|\mu|}$. Here we are using the more compact notation

$$|\mu|^2(B(x,\varrho)) := \int_{B(x,\varrho)} |\mu|^2\, dy\,.$$

The first results we present are about general local weak solutions to (4.1) and it is the analog of the L^∞-estimate (2.13) using the nonlinear potential $\mathbf{P}^\mu$.

Theorem 4.1 ([20]). *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ be a weak solution to the system* (4.1) *for $p > 1$; there exists a constant c, depending only on n and p, such that*

$$\|Du\|_{L^\infty(B_{R/2})} \le c\|\mathbf{P}^\mu(\cdot,R)\|^{1/(p-1)}_{L^\infty(B_R)} + c\left(\fint_{B_R} |Du|^p\, dy \right)^{1/p}$$

holds whenever $B_R \subset \Omega$.

Again a sharp characterization of the Lipschitz continuity of solutions follows via use of Lorentz spaces.

Corollary 4.1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ be a weak solution to the system* (4.1) *for $p > 1, n > 2$. If $\mu \in L(n,1)$ holds, then Du is locally bounded in Ω.*

The parabolic version of the previous results involves similar potentials based on standard parabolic cylinders. More precisely, we define

$$\mathbf{P}^\mu(x_0,t_0;R) := \int_0^R \left(\frac{|\mu|^2(Q_\varrho(x_0,t_0;\varrho))}{\varrho^{N-2}} \right)^{1/2} \frac{d\varrho}{\varrho}\,, \qquad N = n+2\,.$$

In the parabolic case, as it is standard, the shape of the estimates drastically change in the two cases $p \ge 2$ and $p < 2$. The first result we are going to present is a nonlinear potential estimate in the degenerate case $p \ge 2$.

Theorem 4.2 ([35]). *Let u be a distributional solution to* (4.2) *as in* (3.3)*, with $p \ge 2$. Then, for every $\ell \in (0,1]$ there exists a constant c, depending only on n, m, p, ℓ such that*

$$\|Du\|_{L^\infty(Q_{R/2})} \le c\|\mathbf{P}^\mu(\cdot;R)\|^{2/p}_{L^\infty(Q_R)} + c\left(\fint_{Q_R} |Du|^{p-2+2\ell}\, dx\, dt \right)^{1/2\ell} + c$$

holds for every standard parabolic cylinder $Q_R \subset Q_T$.

For the subquadratic case we instead have the following:

Theorem 4.3 ([35]). *Let u be a distributional solution to* (4.2) *as in* (3.3)*. Assume also that*

$$2 \ge p > \frac{2n}{n+2}\,. \tag{4.3}$$

Then, for every ℓ satisfying

$$\frac{n(2-p)}{2} < 2\ell \le p\,, \tag{4.4}$$

there exists a constant c, depending only on n, m, p, ℓ, such that

$$\|Du\|_{L^\infty(Q_{R/2})} \leq c\|\mathbf{P}^\mu(\cdot;R)\|_{L^\infty(Q_R)}^{4/[(n+2)p-2n]} + c\left(\fint_{Q_R} |Du|^{2\ell}\,dx\,dt\right)^{2/[4\ell-n(2-p)]} + c$$

holds for every standard parabolic cylinder $Q_R \subset Q_T$.

Remark 4.1. We remark that the global outcome of Theorems 4.2-4.3 is that in any case $p > 2n/(n+2)$ it holds

$$\mathbf{P}^\mu(\cdot;R) \in L^\infty \Longrightarrow Du \in L^\infty_{\rm loc}(Q_T, \mathbb{R}^{mn})\,.$$

A significant point here is that *the previous condition is independent of* p, in that the exponent p does not appear in the definition of the potential $\mathbf{P}^\mu(\cdot;R)$.

Remark 4.2. Condition (4.4) is of course non-void provided $n(2-p)/2 < p$, and this is exactly guaranteed by assuming (4.3). Observe that by taking $\ell = p/2$ in Theorem 4.3 we, in particular, obtain

$$\|Du\|_{L^\infty(Q_{R/2})} \leq c\|\mathbf{P}^\mu(\cdot;R)\|_{L^\infty(Q_R)}^{4/[(n+2)p-2n]} + c\left(\fint_{Q_R} |Du|^{p}\,dx\,dt\right)^{2/[(n+2)p-2n]} + c\,.$$

Remark 4.3. Forms of the estimates presented in Theorems 4.2 and 4.3 are in a certain sense optimal as in the case $\mu = 0$ they allow to recover the sharp interpolated L^∞ bounds of DiBenedetto [12]; related bounds in L^q are available in [2]. Indeed, for solutions to the evolutionary p-Laplacean system

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0\,, \tag{4.5}$$

estimate [12, Chapter 8, Theorem 5.1] valid for the case $p \geq 2$ reads as

$$\|Du\|_{L^\infty(Q_{R/2})} \leq c\left(\fint_{Q_R} |Du|^{p-2+2\ell}\,dx\,dt\right)^{1/2\ell} + c \tag{4.6}$$

for every $\ell \in (0,1]$, where $c \equiv c(n,m,p,\ell)$, while in the case (4.3) estimate in [12, Chapter 8, Theorem 5.2] gives

$$\|Du\|_{L^\infty(Q_{R/2})} \leq c\left(\fint_{Q_R} |Du|^{2\ell}\,dx\,dt\right)^{1/[4\ell-n(2-p)]} + c \tag{4.7}$$

whenever $2\ell > n(2-p)/2$ holds. Estimates (4.6)-(4.7) are now a consequence of Theorems 4.2 and 4.3, respectively.

Remark 4.4. Although being stated for the model cases (4.1) and (4.2), the results in this sections hold for certain classes of elliptic and parabolic systems. More precisely, we can consider quasidiagonal systems, i.e. of the type

$$-\operatorname{div}(g(|Du|^2)Du) = \mu \qquad \text{and} \qquad u_t - \operatorname{div}(g(|Du|^2)Du) = \mu\,.$$

The specific form above in the vectorial case is sometimes referred to as Uhlenbeck structure, after the seminal work [59] in the elliptic setting. It is relevant as it allows to rule out singularities when considering the homogeneous case (4.5).

5. Asymptotic regularity

The techniques developed for proving the parabolic potential estimates of Section 3 allows for proving the analog of a few results that, being rather classical in the elliptic case, were still an open issue in the parabolic one. The first one deals with general systems which are parabolic only in a very weak sense. Specifically, we take a model problem of the type

$$u_t - \operatorname{div} a(Du) = 0\,, \tag{5.1}$$

where the solution $u\colon \Omega_T \to \mathbb{R}^N$ is in general a vector valued map as in (3.3) and solves (5.1) in the distributional sense. The system is considered in the cylindrical domain $\Omega_T = \Omega \times (-T, 0)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $T > 0$, $n \geq 2$. At this stage, we make no other assumption on the C^1-vector field $a(\cdot)$ than

$$|a(z)| + |a_z(z)|(|z| + 1) \leq L(|z| + 1)^{p-1}\,, \tag{5.2}$$

which has to hold whenever $z \in \mathbb{R}^{Nn}$, and the following C^1-asymptotic closeness condition to the field $|z|^{p-2}z$:

$$\lim_{|z|\to\infty} \frac{|a_z(z) - b_z(z)|}{|z|^{p-2}} = 0\,, \qquad \text{where} \ \ b(z) := |z|^{p-2}z\,. \tag{5.3}$$

In particular, we are not assuming that the system considered is parabolic in that parabolicity only holds at infinity, in an asymptotic sense. Here, as in the rest of the paper, we shall always assume the lower bound

$$\frac{2n}{n+2} < p \tag{5.4}$$

that is in fact necessary to obtain the following regularity result:

Theorem 5.1 ([43])**.** *Let u be a solution to* (5.1) *under the assumptions* (5.2)-(5.3)*; then $Du \in L^\infty_{\rm loc}(\Omega_T)$. Moreover, there exists a constant c depending only on n, N, p, ν, L and the rate of convergence in* (5.3)*, such that*

$$|Du(x_0, t_0)| \leq c \left[\fint_{Q_r(x_0,t_0)} (|Du|^p + 1)\, dx\, dt \right]^{d/p} \tag{5.5}$$

holds whenever $Q_r(x_0, t_0) \subset \Omega_T$ is a standard parabolic cylinder with vertex (x_0, t_0), where (x_0, t_0) is a Lebesgue point for Du. Here

$$d := \begin{cases} \frac{p}{2} & \text{if} \quad p \geq 2 \\ \frac{2p}{p(n+2)-2n} & \text{if} \quad \frac{2n}{n+2} < p < 2 \end{cases}$$

is the scaling deficit exponent of the p-Laplacean system.

Estimate (5.5) is in a sense optimal, compare with Remark 4.3. Also compare the above estimate with the ones in [2] showing the occurrence of a scaling deficit exponent d precisely reflecting the anisotropicity of the operator considered; see [1] for a comparison with a related anisotropic elliptic situation. Asymptotic regularity results of the type just described are often crucial in establishing dimension estimates for singular sets of solutions to elliptic system (see for instance [31, 32, 33]) and in several problems coming from mathematical materials science. For the elliptic versions we refer to the starting work of Chipot & Evans [11] for linear problems, eventually extended to nonlinear settings for instance in [11, 22, 15, 21, 22, 55, 56, 54].

The second related result is instead concerned with a borderline case of the standard gradient Hölder continuity results. When dealing with truly parabolic systems, as for instance

$$u_t - \operatorname{div}(\gamma(x,t)|Du|^{p-2}Du) = 0\,, \tag{5.6}$$

Dini continuity of coefficients actually implies the continuity of the (spatial) gradient. This fact, being classical and sharp in the elliptic case, was still an open issue in the parabolic one and it has been established in [43] for general equations and for systems with quasi-diagonal structure. For this, let us set

$$\omega(\varrho) := \sup_{\substack{t\in(-T,0),x,y\in B_\varrho \\ B_\varrho\subset\Omega}} |\gamma(x,t) - \gamma(y,t)|\,,$$

and assume the Dini-continuity of $\gamma(\cdot)$ with respect to the space variables, i.e.

$$\int_0^1 \omega(\varrho)\,\frac{d\varrho}{\varrho} < \infty\,. \tag{5.7}$$

Theorem 5.2 ([43])**.** *Let u be a solution to* (5.6) *under the asumptions* (5.4) *and* (5.7)*. Then Du is continuous in Ω_T.*

The previous theorem extends to general classes of parabolic equations of the type

$$u_t - \operatorname{div} a(x,t,Du) = 0\,, \tag{5.8}$$

where the vector field $a\colon \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies assumptions

$$\begin{cases} |a(x,t,z)| + |a_z(x,t,z)|(|z|^2+s^2)^{1/2} \leq L(|z|^2+s^2)^{(p-1)/2} \\ \nu(|z|^2+s^2)^{(p-2)/2}|\xi|^2 \leq \langle a_z(x,t,z)\xi,\xi\rangle \\ |a(x,t,z)-a(x_0,t,z)| \leq L\omega(|x-x_0|)L(|z|^2+s^2)^{(p-1)/2} \end{cases} \tag{5.9}$$

whenever $z,\xi \in \mathbb{R}^n$ and $(x,t),(x_0,t) \in \Omega_T$. Numbers s,ν,L are assumed to satisfy $0<\nu\leq L$ and $s\geq 0$. Here $\omega(\cdot)$ is assumed to satisfy (5.7).

Theorem 5.3 ([43])**.** *Let u be a solution to* (5.8) *under the assumptions* (5.9) *and* (5.7)*. Then Du is continuous in Ω_T.*

The results above can be generalized to more general systems and equations of the type

$$u_t - \operatorname{div}(\gamma(x,t)|Du|^{p-2}Du) = \operatorname{div} G(x,t)\,,$$

provided suitable regularity assumptions are made on the right hand side vector field $G(\cdot)$. For instance, a suitable form of Dini continuity of $x \mapsto G(x,\cdot)$ suffices to conclude with the continuity of Du.

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