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A nonlinear Stein theorem

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Abstract. For vector valued solutions u to the p-Laplacian system $-\triangle_p u = F$ in a domain of $\mathbb{R}^n$, $p > 1$, $n \geq 2$, we prove that if F belongs to the limiting Lorentz space $L(n, 1)$, then Du is continuous.

To Bernard Dacorogna on his 60th birthday

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1. The result

A by now classical result of Stein [22] asserts that if $v \in W^{1,1}$ is a Sobolev function defined in $\mathbb{R}^n$ with $n \geq 2$, then

$$Dv \in L(n, 1) \implies v \text{ is continuous}. \tag{1}$$

The Lorentz space $L(n, 1)$ appearing in the above display consists of those measurable maps g satisfying the condition

$$\int_0^\infty |\{x : |g(x)| > t\}|^{1/n}\, dt < \infty$$

and (1) can be regarded as the limiting case of Sobolev-Morrey embedding theorem that asserts

$$Dv \in L^{n+\varepsilon} \implies v \in C^{0,\varepsilon/(n+\varepsilon)} \tag{2}$$

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whenever $\varepsilon > 0$. Note indeed that $L^{n+\varepsilon} \subset L(n,1) \subset L^n$ for every $\varepsilon > 0$, with all the inclusions being actually strict. Another version of Stein's theorem concerns the regularity of solutions to the non-homogeneous Laplacian system. Indeed, using (1) together with the standard Calderón-Zygmund theory allows to conclude with

$$\triangle u \in L(n,1) \implies Du \text{ is continuous}\,.$$

The aim of this paper is to prove the same result for solutions to the p-Laplacian system (with coefficients), as indeed established in the following:

Theorem 1. *Let $u \in W^{1,p}_{\rm loc}(\Omega,\mathbb{R}^N)$ with $p > 1$ and $N \geq 1$ be a local weak solution to the system*

$$-{\rm div}\,(\gamma(x)|Du|^{p-2}Du) = F \tag{3}$$

in an open subset $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Assume that

- *the vector field $F\colon \Omega \to \mathbb{R}^N$ satisfies $F \in L(n,1)$ locally in Ω*
- *the function $\gamma\colon \Omega \to [\nu, L]$ is Dini-continuous, where $0 < \nu \leq L < \infty$.*

Then Du is continuous in Ω.

The optimal character of the space $L(n,1)$ in the p-Laplacian setting also stems from the well-known regularity result

$$F \in L^{n+\varepsilon} \implies Du \text{ is locally Hölder continuous}\,,$$

which is the p-Laplacian counterpart of (2). Anyway, counterexamples working already in the linear case, show that Du can unbounded when $\triangle u \not\in L(n,1)$; see [2]. Moreover, we notice the relevant fact that the condition $F \in L(n,1)$ is independent of p and this is reflected in the approach we will actually take in the proof of Theorem 1. Indeed, the basic viewpoint adopted here is to look at (3) as a linear system in the nonlinear vector field $\gamma(x)|Du|^{p-2}Du$ rather than a nonlinear system in the gradient Du. This ultimately leads to write (3) as a decoupled system

$$\begin{cases} -{\rm div}\, H = F \\ H = \gamma(x)|Du|^{p-2}Du \end{cases}$$

so that the continuity of H eventually implies the one of Du. This viewpoint helps explaining why conditions for the continuity of Du in terms of optimal function spaces do not depend on p. The implementation of this heuristic argument is anyway not easy, and involves a considerable number of technicalities.

The space $L(n,1)$ already appears in the study of the p-Laplacian equations and systems [3,6,18] in connection to gradient L^∞-bounds, that in fact can be derived when $F \in L(n,1)$. In particular, in [18] we proved a scalar version of Theorem 1, but this result applies only to a single equation, and not to systems, and the gradient continuity has remained an open problem for systems. Moreover, only the case $p \geq 2$ has been considered in [18]. The novelty of Theorem 1 therefore lies not only in the fact that it covers the vectorial case $N > 1$, but also in that we are treating the full range $p > 1$. The approach we are developing here does not distinguish between the so-called singular case $1 < p < 2$ and the degenerate one $p \geq 2$. We also remark that, already in the case of the gradient L^∞-regularity, the two dimensional case $n = 2$ had remained an open issue in [3,6], essentially for technical reasons. This point is fixed in this paper by mean of a completely different approach. Indeed, in contrast to [18], where the analysis was based on the use of potential estimates (see [14,24,25]), here we pursue a different path using directly certain characterisations of Lorentz spaces (see for instance Section 2.3 below) and a careful linearisation approach.

Another feature of Theorem 1 is the presence of the coefficients - i.e. the function $\gamma(\cdot)$ - something that did not seem to be achievable with the known techniques. In this respect, the assumption of Dini-continuity of $\gamma(\cdot)$ is sharp. In fact, already in the case of linear and homogeneous elliptic equations

$$\operatorname{div}(A(x)Du) = 0 \tag{4}$$

the gradient of solutions is in general unbounded for continuos but not Dini-continuous matrices $A(\cdot)$, as shown in [13]. We just recall that the function $\gamma(\cdot)$ is Dini-continuous when there exists a concave, non-decreasing function $\omega\colon [0,\infty) \to [0,1]$ with $\omega(0) = 0$, satisfying

$$|\gamma(x) - \gamma(y)| \leq L\omega(|x-y|) \tag{5}$$

for every $x, y \in \Omega$ and such that

$$\int_0 \omega(\varrho)\, \frac{d\varrho}{\varrho} < \infty\,. \tag{6}$$

There has been recently a renewed interest in Dini-continuous coefficients and in related regularity issues, see for instance [20].

Although we preferred to concentrate on the model case in (3), the result of Theorem 1 continues to hold for systems with more general structures. For instance, with essentially minor modifications, we can treat general quasi-diagonal structures (sometime called “Uhlenbeck structure” as in [1]) of the type

$$-\operatorname{div}(g(|Du|)Du) = F\,, \qquad\qquad g(|Du|) \approx |Du|^{p-2}\,.$$

More general dependences on the coefficients can be considered as well. For instance we can treat models as

$$-\operatorname{div}(\langle A(x)Du, Du\rangle^{p-2} A(x)Du) = F\,,$$

where $A(x)$ is a bounded and strictly elliptic matrix with Dini-continuous entries (see for instance [15]). Yet minor modifications allow to deal with non-degenerate structures too, as for instance

$$-\operatorname{div}(\gamma(x)(\mu^2 + |Du|^2)^{(p-2)/2} Du) = F\,, \qquad\qquad \mu > 0\,.$$

We remark that a preliminary gradient continuity result for solutions to (3) has been obtained in [7] for the case $p \geq 2$ and under the suboptimal assumptions

$$F \in L\left(n, \frac{1}{p-1}\right) \qquad \text{and} \qquad \int_0 [\omega(\varrho)]^{2/p} \, \frac{d\varrho}{\varrho} < \infty\,. \tag{7}$$

In particular, the case of Dini continuity of coefficients was not covered. Both conditions in (7) worsen when p increases. Here sharp results are finally reached by a very careful linearisation argument that allows a better control of the degeneracy.

2. Preparatory material

2.1. General notation

In what follows we denote by c a general positive constant, possibly varying from line to line; special occurrences will be denoted by $c_1, c_2, \bar{c}_1, \bar{c}_2$ or the like. All such constants will always be *larger or equal than one*; moreover relevant dependencies on parameters will be emphasized using parentheses, i.e., $c_1 \equiv c_1(n, N, p, \nu, L)$ means that c_1 depends only on n, N, p, ν, L. We denote by

$$B(x_0, r) \equiv B_r(x_0) := \{x \in \mathbb{R}^n \,:\, |x - x_0| < r\}$$

the open ball with center x_0 and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_r \equiv B(x_0, r)$; moreover, with B being a generic ball with radius r we will denote by σB the ball concentric to B having radius σr, $\sigma > 0$. Unless otherwise stated, different balls in the same context will have the same center. We shall also denote $B \equiv B_1 = B(0, 1)$ if not differently specified. With $\mathcal{O} \subset \mathbb{R}^n$ being a measurable subset with positive measure, and with $g \colon \mathcal{O} \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, typically a gradient in the following, we shall denote by

$$(g)_{\mathcal{O}} \equiv \fint_{\mathcal{O}} g(x)\, dx := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} g(x)\, dx$$

its integral average; here $|\mathcal{O}|$ denotes the Lebesgue measure of $\mathcal{O}$. In the rest of the paper we shall use several times the following elementary property of integral averages:

$$\left(\fint_{\mathcal{O}} |g - (g)_{\mathcal{O}}|^{\gamma}\, dx\right)^{1/\gamma} \leq 2 \left(\fint_{\mathcal{O}} |g - A|^{\gamma}\, dx\right)^{1/\gamma}\,, \tag{8}$$

whenever $A \in \mathbb{R}^k$ and $\gamma \geq 1$. The oscillation of g on $\mathcal{O}$ is instead defined as

$$\operatorname*{osc}_{\mathcal{O}} g := \sup_{x, \tilde{x} \in \mathcal{O}} |g(x) - g(\tilde{x})|\,.$$

Finally, for $s \geq 1$, we shall denote its L^s-excess functional as

$$E(g, \mathcal{O}) \equiv E_s(g, \mathcal{O}) := \left(\fint_{\mathcal{O}} |g - (g)_{\mathcal{O}}|^s \, dx \right)^{1/s} .$$

Such a quantity is bound to provide an integral measure of the oscillations of g in $\mathcal{O}$.

2.2. General setting for the proofs

In this section we build the basic set-up for the proof of Theorem 1. With $B(x_0, 2r) \subset \Omega$ being a fixed ball we then set for $j \geq 0$

$$B_j \equiv B(x_0, r_j)\,, \qquad r_j := \sigma^j r\,, \qquad \sigma \in (0, 1/4)\,. \tag{9}$$

We remark that the parameter σ is at the moment assumed to be just a number belonging to $(0, 1/4)$ and all the considerations through Section 2 will stay valid for any choice of it. Specific values of σ will be then used in Sections 3 and 4 below. Next, for $j \geq 0$, we define the maps $u = w_j + W_0^{1,p}(B_j, \mathbb{R}^N)$ as the unique solutions to the problems

$$\begin{cases} \operatorname{div}(\gamma(x)|Dw_j|^{p-2}Dw_j) = 0 & \text{in } B_j \\ \hfill w_j = u & \text{on } \partial B_j\,, \end{cases} \tag{10}$$

and, eventually, $v_j \in w_j + W_0^{1,p}(\frac{1}{2}B_j, \mathbb{R}^N)$ as the unique solutions to

$$\begin{cases} \operatorname{div}(\gamma(x_0)|Dv_j|^{p-2}Dv_j) = 0 & \text{in } \frac{1}{2}B_j \\ \hfill v_j = w_j & \text{on } \partial\left(\frac{1}{2}B_j\right). \end{cases} \tag{11}$$

Before going on, let us remark that since all the results in this papers are local in nature in the following we shall assume without loss of generality that $F\colon \mathbb{R}^n \to \mathbb{R}^N$ (this can be done for instance letting $F \equiv 0$ outside Ω) and that

$$u \in W^{1,p}(\Omega, \mathbb{R}^N) \qquad \text{and} \qquad F \in L(n,1)(\mathbb{R}^n, \mathbb{R}^N)\,. \tag{12}$$

2.3. A relevant series and Lorentz spaces

Let us recall a few basic facts about Lorentz spaces (for which we for instance refer to [23]). We recall that a map $g\colon \Omega \to \mathbb{R}^k$ belongs to the Lorentz space $L(s, \gamma) \equiv L(s, \gamma)(\Omega, \mathbb{R}^k)$ for $s \geq 1$ and $\gamma > 0$ iff

$$\|g\|_{L(s,\gamma)}^{\gamma} := \int_0^\infty (t^s|\{x \in \Omega \, : \, |g(x)| > t\}|)^{\gamma/s} \, \frac{dt}{t} < \infty\,. \tag{13}$$

The local variant of $L(s, \gamma)$ is then defined, as usual, by saying that $v \in L(s, \gamma)$ locally, iff $v \in L(s, \gamma)(\Omega', \mathbb{R}^k)$ for every open subset $\Omega' \Subset \Omega$. A useful characterization of Lorentz spaces can be given via rearrangements;

indeed, if $|g|^*\colon [0,|\Omega|] \to [0,\infty]$ denotes the non-increasing rearrangement of $|g|$ (see [10]) and if we consider the following maximal type operator

$$|g|^{**}(t) := \frac{1}{t}\int_0^t |g|^*(\varrho)\,d\varrho\,, \qquad t \in (0,|\Omega|]\,,$$

then we have

$$g \in L(s,\gamma) \Longleftrightarrow \int_0^{|\Omega|} [\varrho(|g|^{**}(\varrho))^s]^{\gamma/s}\,\frac{d\varrho}{\varrho} < \infty \qquad \text{if } s>1\,. \tag{14}$$

Moreover, it holds that

$$\int_0^{|\Omega|} [\varrho(|g|^{**}(\varrho))^s]^{\gamma/s}\,\frac{d\varrho}{\varrho} \le c(s,\gamma)\|g\|_{L(s,\gamma)}^{\gamma} \tag{15}$$

again for $s>1$ (see [9,23]).

With F being the vector field defined in Theorem 1 (but also recall (12)), and with the balls B_j being defined in (9), we consider the following quantity:

$$S_q(x_0,r,\sigma) := \sum_{j=0}^{\infty} r_j \left(\fint_{B_j} |F|^q\,dx\right)^{1/q}, \qquad q \in (1,n)\,. \tag{16}$$

The quantity $S_q(x_0,r,\sigma)$ defined in (16) plays a crucial role in the analysis of the fine pointwise properties of the gradient of solutions to systems as in (3). We are here interested in deriving an estimate for the quantity in the previous display in terms of the $L(n,1)$-norm of F, or, better saying, the $L(n/q,1/q)$ norm of $|F|^q$. More precisely, we have the following:

Lemma 1. *If $F \in L(n,1)$, $q \in (1,n)$ and $\sigma \in (0,1/4)$, then*

$$S_q(x_0,r,\sigma) \le c(n,q,\sigma)\|F\|_{L(n,1)} \tag{17}$$

holds for every $r>0$ and $x_0 \in \mathbb{R}^n$ and, more precisely, estimate

$$S_q(x_0,r,\sigma) \le \sigma^{1-n/q}\int_0^{2r} [\varrho^q(|F|^q)^{**}(\omega_n\varrho^n)]^{1/q}\frac{d\varrho}{\varrho} \le c\|F\|_{L(n,1)} \tag{18}$$

is true for a constant c depending only on n,q,σ.

Proof. By the classical Hardy-Littlewood inequality on rearrangements [10], and letting $r_{-1}=2r$, it follows that for $j\ge 0$ if $r_j \le \varrho \le r_{j-1}$

$$r_j^q \fint_{B_j} |F|^q\,dx \le \frac{r_j^q}{\omega_n r_j^n}\int_0^{\omega_n r_j^n} (|F|^q)^*(\xi)\,d\xi \le \sigma^{q-n}\varrho^q(|F|^q)^{**}(\omega_n\varrho^n)$$

holds whenever $j\ge 0$ and therefore we have

$$\left(r_j^q \fint_{B_j} |F|^q\,dx\right)^{1/q}$$

$$\leq \sigma^{1-n/q}\left[\log\left(\frac{1}{\sigma}\right)\right]^{-1}\int_{r_j}^{r_{j-1}}[\varrho^q(|F|^q)^{**}(\omega_n\varrho^n)]^{1/q}\frac{d\varrho}{\varrho}$$
$$\leq \sigma^{1-n/q}\int_{r_j}^{r_{j-1}}[\varrho^q(|F|^q)^{**}(\omega_n\varrho^n)]^{1/q}\frac{d\varrho}{\varrho}\,.$$

Summing up the previous inequalities gives (18). Here we just recall that ω_n standardly denotes the measure of the unit ball in $\mathbb{R}^n$. Next, notice that the right hand side of (18) is finite when $F \in L(n,1)$. Indeed, by the very definition of Lorentz spaces in (13) we have that $|F|^q \in L(n/q,1/q)$ and

$$\||F|^q\|_{L(n/q,1/q)} = q^q\|F\|^q_{L(n,1)}$$

then a change of variable and a repeated use of (15) give that

$$\begin{aligned}\int_0^\infty[\varrho^q(|F|^q)^{**}(\omega_n\varrho^n)]^{1/q}\frac{d\varrho}{\varrho} &= \frac{1}{n\omega_n^{1/n}}\int_0^\infty[\varrho((|F|^q)^{**}(\varrho))^{n/q}]^{1/n}\frac{d\varrho}{\varrho}\\ &\leq c(n,q)\||F|^q\|^{1/q}_{L(n/q,1/q)}\\ &= c(n,q)\|F\|_{L(n,1)}\,.\end{aligned}$$

Observe that we have used (14) with $g = |F|^q$, $s = n/q$ and $\gamma = 1/q$, since $n/q > 1$. The inequality in the last display together with (18) gives (17).

2.4. Facts of algebraic nature

We shall largely use the auxiliary vector field $V\colon \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ defined by

$$V(z) := |z|^{(p-2)/2}z\,, \tag{19}$$

and which is a locally Lipschitz bjiection from $\mathbb{R}^{Nn}$ into itself. This will be often employed in connection to the following inequality (see [9]):

$$\frac{|z_1-z_2|}{c_1} \leq \frac{|V(z_1)-V(z_2)|}{(|z_1|+|z_2|)^{(p-2)/2}} \leq c_1|z_1-z_2| \tag{20}$$

which is valid for all matrixes $z_1, z_2 \in \mathbb{R}^{Nn}$ that are not simultaneously null and for every $p > 1$, where $c_1 \equiv c_1(n,N,p)$. The previous inequality is relevant in manipulations involving the classical monotonicity estimate

$$(|z_1|+|z_2|)^{p-2}|z_1-z_2|^2 \leq c(n,N,p)\langle |z_1|^{p-2}z_1 - |z_2|^{p-2}z_2, z_1-z_2\rangle\,, \tag{21}$$

which again holds for all matrixes $z_1, z_2 \in \mathbb{R}^{Nn}$ and $p > 1$. We now give a lemma for the case $1 < p \leq 2$.

Lemma 2. *Let $p \in (1,2]$. There exists a constant c, depending only on n, N, p such that the following inequality holds whenever $z_1, z_2 \in \mathbb{R}^{Nn}$:*

$$|z_1-z_2| \leq c|V(z_1)-V(z_2)|^{2/p} + c|V(z_1)-V(z_2)||z_2|^{(2-p)/2}\,. \tag{22}$$

Proof. We assume $p \neq 2$, because otherwise the assertion is trivial since $V(z) = z$ when $p = 2$. We have

$$
\begin{aligned}
|z_1 - z_2| &= (|z_1| + |z_2|)^{(p-2)/2}|z_1 - z_2|(|z_1| + |z_2|)^{(2-p)/2} \\
&\overset{(20)}{\leq} c_1|V(z_1) - V(z_2)| \cdot (|z_1| + |z_2|)^{(2-p)/2} \\
&\leq c|V(z_1) - V(z_2)| \cdot \left(|z_1 - z_2|^{(2-p)/2} + |z_2|^{(2-p)/2}\right) .
\end{aligned}
$$

Then, using Young's inequality with conjugate exponents

$$
\left(\frac{2}{p}, \frac{2}{2-p}\right)
$$

we gain

$$
|z_1 - z_2| \leq (1/2)|z_1 - z_2| + c|V(z_1) - V(z_2)|^{2/p} + c|V(z_1) - V(z_2)||z_2|^{(2-p)/2}
$$

and (22) follows.

2.5. Homogeneous systems with Dini-continuous coefficients

Our recent paper [17] deals with the spatial gradient continuity properties of solutions to nonlinear parabolic systems of p-Laplacian type

$$
u_t - \operatorname{div}(\gamma(x,t)|Du|^{p-2}Du) = 0
$$

and we have proved that if the coefficient function $\gamma(\cdot)$ is Dini-continuous with respect to the space variable x then Du is continuous. Related a priori estimates have been provided. The results obtained in [17] obviously apply to the elliptic case $\operatorname{div}(\gamma(x)|Du|^{p-2}Du) = 0$ and this section is devoted to restate the a priori estimates derived in [17] in a way tailored to the our needs here. The outcome is indeed the following:

Theorem 2. *Let w_j be as in* (10) *with $j \geq 0$, then Dw_j is continuous. Moreover*

- *There exists a constant $c_2 \equiv c_2(n, N, p, \nu, L) \geq 1$ and positive radius $R_1 \equiv R_1(n, N, p, \nu, L, \omega(\cdot))$ such that if $r \leq R_1$, then the following holds:*

$$
\sup_{\frac{1}{2}B_j} |Dw_j| \leq c_2 \fint_{B_j} |Dw_j| \, dx . \tag{23}
$$

- *Assume that the inequality*

$$
\sup_{\frac{1}{2}B_j} |Dw_j| \leq A\lambda \tag{24}
$$

holds for some $A \geq 1$ and $\lambda > 0$. Then, for any $\delta \in (0,1)$ there exists a positive constant $\sigma_1 \equiv \sigma_1(n, N, p, \nu, L, \omega(\cdot), A, \delta) \in (0, 1/4)$ such that

$$
\operatorname*{osc}_{\sigma_1 B_j} Dw_j \leq \delta\lambda . \tag{25}
$$

Proof. The proof is actually an adaptation of the relevant arguments from [17]. We start from the a priori estimate (23). Using [17, Theorem 1.1] together with a standard covering argument it follows that there exists a radius $R_1 \equiv R_1(n, N, p, \nu, L, \omega(\cdot)) > 0$, such that

$$\sup_{\frac{1}{2}B} |Dw_j| \leq c \qquad \text{where } c \equiv c\left(n, N, p, \nu, L, \fint_B |Dw_j|^p \, dx\right)$$

holds whenever $B \subset B_j$ is a ball (not necessarily concentric to B_j) with radius smaller of equal than R_1. By a standard rescaling procedure, that is considering the new solution to $\operatorname{div}(\gamma(x)|D\tilde{w}|^{p-2}D\tilde{w}) = 0$ defined by

$$\tilde{w} := \left(\fint_B |Dw_j|^p \, dx\right)^{-1/p} w_j \Longrightarrow \left(\fint_B |D\tilde{w}|^p \, dx\right)^{1/p} = 1\,,$$

we have that

$$\sup_{\frac{1}{2}B} |D\tilde{w}| \leq c \equiv c\,(n, N, p, \nu, L)\ .$$

Scaling back to w_j yields

$$\sup_{\frac{1}{2}B} |Dw_j| \leq c \left(\fint_B |Dw_j|^p \, dx\right)^{1/p}$$

with $c \equiv c(n, N, p, \nu, L)$, whenever B is a ball contained in B_j. In turn, a by now standard interpolation/covering argument leads to the fact that the exponent in the previous inequality can be lowered:

$$\sup_{\frac{1}{2}B} |Dw_j| \leq c(t) \left(\fint_B |Dw_j|^t \, dx\right)^{1/t} \qquad \text{holds for every } t > 0\,,$$

from which (23) follows by taking $B \equiv B_j$ and $t = 1$. The adaptation of (25) from the parabolic arguments of [17] needs more care. In particular the so-called "intrinsic geometry" needed in [17] and taken from the work of DiBenedetto [4], becomes immaterial in this context. In this respect, we recall that in [17] we worked with intrinsic cylinders with vertex (x_0, t_0) of the type $Q_\varrho^\lambda(x_0, t_0) = B(x, \varrho) \times (t_0 - \lambda^{2-p}\varrho^2, t_0)$, for a number $\lambda > 0$ which is suitably related to the size of the gradient on the same cylinder $Q_\varrho^\lambda(x_0, t_0)$. When dealing with elliptic problems such peculiar cylinders disappear and they are simply replaced by standard Euclidean balls $B_\varrho(x_0)$. We shall now prove (25) by showing that for a suitable choice of σ_1 as described in the statement, we have

$$|Dw_j(\tilde{x}) - Dw_j(\tilde{y})| \leq \delta\lambda \qquad \forall\ \tilde{x}, \tilde{y} \in \sigma_1 B_j\,. \tag{26}$$

Therefore following the arguments in [17, (5.15)] we find that for every $\varepsilon \in (0, 1)$ there exists a positive radius $R_\varepsilon \equiv R_\varepsilon(n, N, p, \nu, L, \omega(\cdot), \varepsilon) \in (0, 1/16)$ such that

$$|(V(Dw_j))_{B_\tau(\tilde{x})} - (V(Dw_j))_{B_\varrho(\tilde{x})}| \leq (A\lambda)^{p/2}\varepsilon \tag{27}$$

and

$$\left(\fint_{B_\varrho(\tilde{x})} |V(Dw_j) - (V(Dw_j))_{B_\varrho(\tilde{x})}|^2 \, dx \right)^{1/2} \leq (A\lambda)^{p/2}\varepsilon \qquad (28)$$

hold whenever $\tilde{x} \in \frac{1}{4}B_j$ and $0 < \tau \leq \varrho \leq R_\varepsilon r_j$; notice that R_ε is in particular independent of λ, A and the considered cylinder B_j. The vector field $V(\cdot)$ has been introduced in (19). Since Dw_j is continuous we let $\tau \to 0$ in (27) thereby obtaining

$$|V(Dw_j(\tilde{x})) - (V(Dw_j))_{B_\varrho(\tilde{x})}| \leq (A\lambda)^{p/2}\varepsilon \qquad \forall\ \varrho \in (0, R_\varepsilon r_j]\,. \qquad (29)$$

Now, observe that if $\tilde{y}, \tilde{x} \in \sigma_1 B_j$ and $\sigma_1 \leq R_\varepsilon/32$ it obviously follows that

$$B_{R_\varepsilon r_j/8}(\tilde{y}) \subset B_{R_\varepsilon r_j}(\tilde{x}) \subset \frac{1}{4}B_j\,. \qquad (30)$$

Then, using also Jensen's inequality we have

$$\begin{aligned}
&|(V(Dw_j))_{B_{R_\varepsilon r_j/8}(\tilde{y})} - (V(Dw_j))_{B_{R_\varepsilon r_j/8}(\tilde{x})}| \\
&\quad \leq \fint_{B_{R_\varepsilon r_j/8}(\tilde{y})} |V(Dw_j) - (V(Dw_j))_{B_{R_\varepsilon r_j/8}(\tilde{x})}| \, dx \\
&\quad \overset{(8)}{\leq} 2 \fint_{B_{R_\varepsilon r_j/8}(\tilde{y})} |V(Dw_j) - (V(Dw_j))_{B_{R_\varepsilon r_j}(\tilde{x})}| \, dx \\
&\quad \overset{(30)}{\leq} 16^n \fint_{B_{R_\varepsilon r_j}(\tilde{x})} |V(Dw_j) - (V(Dw_j))_{B_{R_\varepsilon r_j}(\tilde{x})}| \, dx \\
&\quad \leq 16^n \left(\fint_{B_{R_\varepsilon r_j}(\tilde{x})} |V(Dw_j) - (V(Dw_j))_{B_{R_\varepsilon r_j}(\tilde{x})}|^2 \, dx \right)^{1/2} \\
&\quad \overset{(28)}{\leq} 16^n (A\lambda)^{p/2}\varepsilon\,.
\end{aligned}$$

By using the previous estimate and (29) twice (centered in $\tilde{y}$ and $\tilde{x}$) together with triangle inequality we easily gain

$$|V(Dw_j(\tilde{x})) - V(Dw_j(\tilde{y}))| \leq 48^n (A\lambda)^{p/2}\varepsilon\,. \qquad (31)$$

In order to show the validity of (26) we now distinguish between two cases. The first is when $p \geq 2$. Let us observe that we can assume that

$$\max\{|Dw_j(\tilde{x})|, |Dw_j(\tilde{y})|\} \geq \delta\lambda/2 \qquad (32)$$

holds otherwise we are done. We then have

$$\begin{aligned}
|Dw_j(\tilde{x}) - Dw_j(\tilde{y})| &\overset{(20)}{\leq} c_1 \frac{|V(Dw_j(\tilde{x})) - V(Dw_j(\tilde{y}))|}{(|Dw_j(\tilde{x})| + |Dw_j(\tilde{y})|)^{(p-2)/2}} \\
&\overset{(32)}{\leq} c_1 2^{p/2-1} \delta^{1-p/2} \lambda^{1-p/2} |V(Dw_j(\tilde{x})) - V(Dw_j(\tilde{y}))|
\end{aligned}$$

$$\stackrel{(31)}{\leq} c_1 48^n 2^{p/2-1} \delta^{1-p/2} A^{p/2} \lambda \varepsilon \,.$$

In the case $1 < p \leq 2$ we instead have

$$\begin{aligned} |Dw_j(\tilde{x}) - Dw_j(\tilde{y})| &\stackrel{(20)}{\leq} c_1 \frac{|V(Dw_j(\tilde{x})) - V(Dw_j(\tilde{y}))|}{(|Dw_j(\tilde{x})| + |Dw_j(\tilde{y})|)^{(p-2)/2}} \\ &\stackrel{(24)}{\leq} c_1 2^{1-p/2} A^{1-p/2} \lambda^{1-p/2} |V(Dw_j(\tilde{x})) - V(Dw_j(\tilde{y}))| \\ &\stackrel{(31)}{\leq} c_1 48^n 2^{1-p/2} A \lambda \varepsilon \,. \end{aligned}$$

Then, in the case $p \geq 2$ we choose

$$\varepsilon := \frac{\delta^{p/2}}{c_1 48^n 2^{p/2-1} A^{p/2}} \,.$$

This determines the radius $R_\varepsilon \equiv R_\varepsilon(n, N, p, \nu, L, \omega(\cdot), A, \delta) > 0$ for which (27) and (28) work and therefore we conclude with (26) with the choice $\sigma_1 := R_\varepsilon/32$. In the case $1 < p < 2$ we instead choose

$$\varepsilon := \frac{\delta}{c_1 48^n 2^{1-p/2} A}$$

and conclude similarly.

2.6. Regularity for the p-Laplacian system

We here recall some basic properties of solutions to the p-Laplacian system; our basic reference here are [5,8,9]. We again restate all the results for the maps v_j defined in (11).

Theorem 3. *Let v_j be as in (11) with $j \geq 0$, then Dv_j is continuous. Moreover*

- *There exists a constant $c_3 \geq 1$ depending only on n, N, p such that*

$$\sup_{\frac{1}{4}B_j} |Dv_j| \leq c_3 \fint_{\frac{1}{2}B_j} |Dv_j| \, dx \tag{33}$$

- *For every $A \geq 1$ there exist constants*

$$c_4 \equiv c_4(n, N, p, A) \equiv \tilde{c}_4(n, N, p) A$$

and $\alpha \equiv \alpha(n, N, p) \in (0, 1)$ such that

$$\sup_{\frac{1}{4}B_j} |Dv_j| \leq A\lambda \quad \Longrightarrow \quad \operatorname*{osc}_{\tau B_j} Dv_j \leq c_4 \tau^\alpha \lambda \qquad \forall\, \tau \in (0, 1/4) \,. \tag{34}$$

Proof. Estimate (33) is a slightly stronger version of a classical estimate going back to work of Uhlenbeck [26]. In the form presented here it can be retrieved using (23) as v_j solves a systems with constants - and hence Dini-continuous - coefficients. As for (34) we just point out that this an immediate consequence of a classical estimate that again goes back to the work of Uhlenbeck [26]. Indeed, in [26] the following inequality is proved to hold for solutions to the homogenous p-Laplacian system:

$$|Dv_j(\tilde{x}) - Dv_j(\tilde{y})| \leq c(n, N, p) \left(\fint_{B_R} |Dv_j|^p \, dx \right)^{1/p} \left(\frac{|\tilde{x} - \tilde{y}|}{R} \right)^{\alpha},$$

whenever $B_R \subset B_j$ and $\tilde{x}, \tilde{y} \in B_{R/2}$. See also [5,8,9,19] for more genertal cases covering the full range $p > 1$.

The next lemma presents estimates involving excess functionals defined for exponents smaller that the natural one i.e. p; estimates of such kind are usually called estimates below the natural growth exponent.

Lemma 3. *Let v_j be as in* (11) *for $j \geq 0$. For every choice of $\bar{\varepsilon} > 0$ and $A \geq 1$ there exists a constant $\sigma_2 \in (0, 1/4)$, depending only on $n, N, p, \bar{\varepsilon}, A$, such that if $\sigma \in (0, \sigma_2]$ and*

$$\frac{\lambda}{A} \leq \sup_{\sigma B_j} |Dv_j| \leq \sup_{\frac{1}{4} B_j} |Dv_j| \leq A\lambda \tag{35}$$

holds, then

$$\left(\fint_{\sigma B_j} |Dv_j - (Dv_j)_{\sigma B_j}|^t \, dx \right)^{1/t} \leq \bar{\varepsilon} \left(\fint_{\frac{1}{2} B_j} |Dv_j - (Dv_j)_{\frac{1}{2} B_j}|^t \, dx \right)^{1/t} \tag{36}$$

also holds, whenever $t \in [1, 2]$.

Proof. The proof goes in two steps. In the first one we recall some basic facts from classical regularity theory for degenerate elliptic systems, asserting a decay estimate of the type in (36) for an excess functional of, let's say, traditional type, that is involving the vector field $V(\cdot)$ introduced in (19). In the second step we show how to use the bounds in (35) to commute the result of the first step in the decay estimate we want, that is (36).

Step 1: A preliminary decay estimate. The vector field $V(\cdot)$ is useful in reformulating the regularity properties of solutions to the p-Laplacian system, indeed a standard result (see for instance [8,9]) is the following decay estimate:

$$\begin{aligned} &\left(\fint_{B_\varrho} |V(Dv_j) - (V(Dv_j))_{B_\varrho}|^2 \, dx \right)^{1/2} \\ &\qquad \leq c \left(\frac{\varrho}{R} \right)^{\beta} \left(\fint_{B_R} |V(Dv_j) - (V(Dv_j))_{B_R}|^2 \, dx \right)^{1/2} \end{aligned} \tag{37}$$

that holds for a constant $c \equiv c(n,N,p)$, whenever $B_\varrho \subset B_R$ are concentric balls contained in $\frac{1}{2}B_j$ (but not necessarily concentric to B_j). See for instance [5,8,9]. The first remark to do is that the previous inequality can be actually reformulated in terms of lower exponents, that is

$$\left(\fint_{B_\varrho} |V(Dv_j) - (V(Dv_j))_{B_\varrho}|^t\,dx\right)^{1/t} \leq c\left(\frac{\varrho}{R}\right)^\beta \left(\fint_{B_R} |V(Dv_j) - (V(Dv_j))_{B_R}|^t\,dx\right)^{1/t} \tag{38}$$

for every $t \in [1,2]$ and again for $c \equiv c(n,N,p)$. For the sake of completeness, let us briefly recall the proof of (38) since it is a consequence of a few facts from the regularity theory of general elliptic systems that are not commonly used in the literature. We start by the following reverse Hölder type inequality:

$$\left(\fint_{B_{R/2}} |V(Dv_j) - z_0|^{2\chi}\,dx\right)^{1/(2\chi)} \leq c\left(\fint_{B_R} |V(Dv_j) - z_0|^2\,dx\right)^{1/2}, \tag{39}$$

which holds whenever $B_R \subset \frac{1}{2}B_j$ and $z_0 \in \mathbb{R}^{Nn}$, for a constant $c \equiv c(n,N,p)$, where $\chi = n/(n-1)$; see for instance [8,16]. The well-known self improving property of reverse Hölder type inequalities also reflects in the fact that the range of exponents for which the reverse property holds can be actually extended in terms of the exponents involved. Indeed, proceeding as in [21, Lemmas 3.1 and 3.2] we have that (39) implies that the following inequality holds uniformly in $t \in [1,2]$ for every ball $B_R \subset \frac{1}{2}B_j$:

$$\left(\fint_{B_{R/2}} |V(Dv_j) - z_0|^2\,dx\right)^{1/2} \leq c\left(\fint_{B_R} |V(Dv_j) - z_0|^t\,dx\right)^{1/t} \tag{40}$$

with $c \equiv (n,N,p)$. Using estimates (37) and (40) with the specific choice $z_0 = (V(Dv_j))_{B_R}$, and yet Hölder's inequality we indeed deduce, for $0 < \varrho \leq R/2$

$$\begin{aligned}
&\left(\fint_{B_\varrho} |V(Dv_j) - (V(Dv_j))_{B_\varrho}|^t\,dx\right)^{1/t} \\
&\qquad \leq \left(\fint_{B_\varrho} |V(Dv_j) - (V(Dv_j))_{B_\varrho}|^2\,dx\right)^{1/2} \\
&\qquad \leq c\left(\frac{\varrho}{R}\right)^\beta \left(\fint_{B_{R/2}} |V(Dv_j) - (V(Dv_j))_{B_R}|^2\,dx\right)^{1/2} \\
&\qquad \leq c\left(\frac{\varrho}{R}\right)^\beta \left(\fint_{B_R} |V(Dv_j) - (V(Dv_j))_{B_R}|^t\,dx\right)^{1/t}.
\end{aligned}$$

We have also used (8) in the second-last estimate and note that the constant c depends, as usual, only on n, N, p. On the other hand, (38) follows by trivial means when $R/2 \leq \varrho \leq R$ so that (38) is completely proved.

Step 2: Proof of (36). The value of σ_2 will be determined in due course of the proof, while we shall argue under the assumption (35) for some $\sigma \leq \sigma_2$. Towards the determination of the number σ_2 we start looking at (34) and define the radius $r > 0$ as

$$r \equiv r(n, N, p, A) := \left(\frac{1}{16\tilde{c}_4 A^2}\right)^{1/\alpha} \Longrightarrow \qquad \operatorname*{osc}_{rB_j} Dv_j \leq \frac{\lambda}{4A}\,. \tag{41}$$

With $\sigma \leq \sigma_2 \leq r$, and σ_2 yet to be determined, if $x \in rB_j$, then we have

$$\begin{aligned} |Dv_j(x)| &= \sup_{\sigma B_j} |Dv_j| + |Dv_j(x)| - \sup_{\sigma B_j} |Dv_j| \\ &\stackrel{(35)}{\geq} \frac{\lambda}{A} - \operatorname*{osc}_{rB_j} Dv_j \stackrel{(41)}{\geq} \frac{\lambda}{4A} \end{aligned}$$

so that, again taking (35) into account we conclude with

$$\frac{\lambda}{4A} \leq \inf_{rB_j} |Dv_j| \leq \sup_{\frac{1}{4}B_j} |Dv_j| \leq A\lambda\,. \tag{42}$$

Let us now first consider the case $p \geq 2$. As $V(\cdot)$ is a bjiection of $\mathbb{R}^{Nn}$ we can define the matrix $z(\sigma) \in \mathbb{R}^{Nn}$ such that

$$z(\sigma) := V^{-1}((V(Dv_j))_{\sigma B_j}) \iff V(z(\sigma)) = (V(Dv_j))_{\sigma B_j}\,. \tag{43}$$

As a straightforward consequence of (42) and of the definition of $V(\cdot)$, we have that there exists a constant $c \equiv c(n, N, p) \geq 4$ such that

$$\frac{\lambda}{cA} \leq |z(\sigma)| \leq cA\lambda\,. \tag{44}$$

Keeping in mind the content of the last two displays we now have that

$$\begin{aligned} &\left(\fint_{\sigma B_j} |Dv_j - (Dv_j)_{\sigma B_j}|^t \,dx\right)^{1/t} \\ &\quad \stackrel{(8)}{\leq} 2\left(\fint_{\sigma B_j} |Dv_j - z(\sigma)|^t \,dx\right)^{1/t} \\ &\quad \stackrel{(20)}{\leq} c\left(\fint_{\sigma B_j} (|Dv_j| + |z(\sigma)|)^{(2-p)t/2} |V(Dv_j) - V(z(\sigma))|^t \,dx\right)^{1/t} \\ &\stackrel{(43),(44)}{\leq} c\left(\frac{A}{\lambda}\right)^{(p-2)/2} \left(\fint_{\sigma B_j} |V(Dv_j) - V((Dv_j))_{\sigma B_j}|^t \,dx\right)^{1/t} \end{aligned}$$

$$\overset{(38)}{\le} c\left(\frac{A}{\lambda}\right)^{(p-2)/2}\left(\frac{\sigma}{r}\right)^{\beta}\left(\fint_{rB_j}|V(Dv_j)-(V(Dv_j))_{rB_j}|^t\,dx\right)^{1/t}$$

$$\overset{(8)}{\le} c\left(\frac{A}{\lambda}\right)^{(p-2)/2}\left(\frac{\sigma}{r}\right)^{\beta}\left(\fint_{rB_j}|V(Dv_j)-V((Dv_j)_{\frac14 B_j})|^t\,dx\right)^{1/t}$$

$$\overset{(20)}{\le} c\left(\frac{A}{\lambda}\right)^{(p-2)/2}\left(\frac{\sigma}{r}\right)^{\beta}$$

$$\cdot\left(\fint_{rB_j}(|Dv_j|+|(Dv_j)_{\frac14 B_j}|)^{(p-2)t/2}|Dv_j-(Dv_j)_{\frac14 B_j}|^t\,dx\right)^{1/t}$$

$$\overset{(42)}{\le} cA^{p-2}\left(\frac{\sigma}{r}\right)^{\beta}\left(\fint_{rB_j}|Dv_j-(Dv_j)_{\frac14 B_j}|^t\,dx\right)^{1/t}$$

$$\overset{(8)}{\le} cA^{p-2}r^{-n/t}\left(\frac{\sigma}{r}\right)^{\beta}\left(\fint_{\frac12 B_j}|Dv_j-(Dv_j)_{\frac12 B_j}|^t\,dx\right)^{1/t}$$

$$\overset{(41)}{\le} cA^{p-2+2(n+\beta)/\alpha}\sigma^{\beta}\left(\fint_{\frac12 B_j}|Dv_j-(Dv_j)_{\frac12 B_j}|^t\,dx\right)^{1/t}$$

$$\le cA^{p-2+2(n+\beta)/\alpha}\sigma_2^{\beta}\left(\fint_{\frac12 B_j}|Dv_j-(Dv_j)_{\frac12 B_j}|^t\,dx\right)^{1/t}$$

for a constant $c \equiv c(n, N, p)$. Therefore is it now sufficient to take σ_2 such that

$$\sigma_2 := \left\{\left(\frac{\bar{\varepsilon}}{cA^{p-2+2(n+\beta)/\alpha}}\right)^{1/\beta}, \left(\frac{1}{16\tilde{c}_4 A^2}\right)^{1/\alpha}\right\} \le r$$

and (36) follows in the case $p \ge 2$. The proof in the case $1 < p \le 2$ is completely similar, and in some sense dual, as it just requires to use the inequalities in (42) and (44) in an order which is reversed with respect to the one used in the case $p \ge 2$. The final outcome, again for a constant $c \equiv c(n, N, p)$, is the inequality

$$\left(\fint_{\sigma B_j}|Dv_j-(Dv_j)_{\sigma B_j}|^t\,dx\right)^{1/t}$$

$$\le cA^{2-p+2(n+\beta)/\alpha}\sigma_2^{\beta}\left(\fint_{\frac12 B_j}|Dv_j-(Dv_j)_{\frac12 B_j}|^t\,dx\right)^{1/t}$$

and we again conclude with (36) by taking σ_2 as

$$\sigma_2 := \left\{\left(\frac{\bar{\varepsilon}}{cA^{2-p+2(n+\beta)/\alpha}}\right)^{1/\beta}, \left(\frac{1}{16\tilde{c}_4 A^2}\right)^{1/\alpha}\right\} \le r\,.$$

The proof is complete.

2.7. Preliminary estimates

With $p > 1$ given, in this paper we denote by p^* the usual Sobolev conjugate exponent in the sense of

$$p^* := \begin{cases} \dfrac{np}{n-p} & \text{if } n > p \\ \text{any number larger than } p & \text{if } p \geq n\,. \end{cases} \tag{45}$$

Moreover, the exponent $p' = p/(p-1)$ is the usual conjugate exponent of p. We start with a first comparison estimate.

Lemma 4. *Let u be as in Theorem 1 with $p \geq 2$ and w_j, v_j as in* (10)-(11), *respectively, with $j \geq 0$. There exists a constant $c_5 \equiv c_5(n, N, p, \nu, q)$, such that*

$$\left(\fint_{B_j} |Du - Dw_j|^p \, dx\right)^{1/p} \leq c_5 \left(r_j^q \fint_{B_j} |F|^q \, dx\right)^{1/[q(p-1)]} \tag{46}$$

holds for every $q \geq (p^*)'$ when $p < n$, and for every $q > 1$ when $p \geq n$. Moreover, when $j \geq 1$ and for another constant $c_6 \equiv c_6(n, N, p, \nu, q, \sigma)$, it holds that

$$\left(\fint_{B_j} |Dw_{j-1} - Dw_j|^p \, dx\right)^{1/p} \leq c_6 \left(r_{j-1}^q \fint_{B_{j-1}} |F|^q \, dx\right)^{1/[q(p-1)]} . \tag{47}$$

Proof. Using the weak formulations of (3) and (10) we have

$$\fint_{B_j} \gamma(x)\langle |Du|^{p-2}Du - |Dw_j|^{p-2}Dw_j, D\varphi\rangle \, dx = \fint_{B_j} \langle F, \varphi\rangle \, dx \,,$$

which is valid for every $\varphi \in W^{1,p}_0(B_j, \mathbb{R}^N)$; there we choose $\varphi = u - w_j$. Using (21) we find that the inequality

$$\fint_{B_j} (|Dw_j| + |Du|)^{p-2}|Du - Dw_j|^2 \, dx \leq c \fint_{B_j} |F||u - w_j| \, dx \tag{48}$$

holds for $p > 1$ and $c \equiv c(n, N, p, \nu)$. Since $p \geq 2$, we have

$$\begin{aligned} \fint_{B_j} |Du - Dw_j|^p \, dx &\leq \fint_{B_j} |F||u - w_j| \, dx \\ &\leq \left(\fint_{B_j} |u - w_j|^{p^*}\right)^{1/p^*} \left(\fint_{B_j} |F|^{(p^*)'} \, dx\right)^{1/(p^*)'} \\ &\leq c r_j \left(\fint_{B_j} |Du - Dw_j|^p\right)^{1/p} \left(\fint_{B_j} |F|^q \, dx\right)^{1/q} \end{aligned}$$

so that (57) follows. Notice that here we are implicitly using the fact that, with $q > 1$ and $p \geq n$, by (45) we can choose p^* such that $1 < (p^*)' \leq q$;

the dependence of the constant c_5 on q stems from this fact. As for (47), this follows applying (46) on B_{j-1} and B_j

$$\begin{aligned}
&\fint_{B_j} |Dw_{j-1} - Dw_j|^p \, dx \\
&\leq \frac{c}{\sigma^n} \fint_{B_{j-1}} |Du - Dw_{j-1}|^p \, dx + c \fint_{B_j} |Du - Dw_j|^p \, dx \\
&\leq c\sigma^{-n-\frac{p(n-q)}{q(p-1)}} \left(r_{j-1}^q \fint_{B_{j-1}} |F|^q \, dx \right)^{p/[q(p-1)]} .
\end{aligned}$$

The proof is complete.

Lemma 5. *Let w_j, v_j as in* (10)-(11)*, respectively, with $j \geq 0$. There exists a constant $c \equiv c(n, N, p, \nu, L)$ such that*

$$\fint_{\frac{1}{2}B_j} |V(Dv_j) - V(Dw_j)|^2 \, dx \leq c[\omega(r_j)]^2 \fint_{\frac{1}{2}B_j} |Dw_j|^p \, dx \tag{49}$$

holds. Moreover, when $p \geq 2$ there exists a constant $c_7 \equiv c_7(n, N, p, \nu, L)$ such that the following holds too:

$$\left(\fint_{\frac{1}{2}B_j} |Dv_j - Dw_j|^p \, dx \right)^{1/p} \leq c_7[\omega(r_j)]^{2/p} \left(\fint_{\frac{1}{2}B_j} |Dw_j|^p \, dx \right)^{1/p} . \tag{50}$$

Proof. The weak formulations of (10) and (11) give

$$\fint_{\frac{1}{2}B_j} \langle \gamma(x_0)|Dv_j|^{p-2}Dv_j - \gamma(x)|Dw_j|^{p-2}Dw_j, D\varphi \rangle \, dx = 0 \, ,$$

which holds whenever $\varphi \in W_0^{1,p}(\frac{1}{2}B_j, \mathbb{R}^N)$ and can be re-written as

$$\begin{aligned}
I_1 &:= \fint_{\frac{1}{2}B_j} \gamma(x_0)\langle |Dv_j|^{p-2}Dv_j - |Dw_j|^{p-2}Dw_j, D\varphi \rangle \, dx \\
&= \fint_{\frac{1}{2}B_j} (\gamma(x) - \gamma(x_0))\langle |Dw_j|^{p-2}Dw_j, D\varphi \rangle \, dx =: I_2 \, ,
\end{aligned}$$

where we have chosen $\varphi = v_j - w_j$. By using (20)-(21) we have

$$\fint_{\frac{1}{2}B_j} |V(Dv_j) - V(Dw_j)|^2 \, dx \leq cI_1 \leq c|I_2| \tag{51}$$

with $c \equiv c(n, N, p, \nu)$. We estimate I_2 as follows, using also the concavity of $\omega(\cdot)$ and the fact that $\omega(\cdot) \leq 1$:

$$|I_2| \overset{(5)}{\leq} c\omega(r_j) \fint_{\frac{1}{2}B_j} |Dw_j|^{p-1}|Dv_j - Dw_j| \, dx$$

$$
\begin{aligned}
&\le\ c\omega(r_j)\fint_{\frac12 B_j}(|Dw_j|+|Dv_j|)^{p-1}|Dv_j-Dw_j|\,dx\\
&=\ c\omega(r_j)\fint_{\frac12 B_j}(|Dw_j|+|Dv_j|)^{p/2}(|Dw_j|+|Dv_j|)^{(p-2)/2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot|Dv_j-Dw_j|\,dx\\
&\overset{(20)}{\le}\ c\omega(r_j)\fint_{\frac12 B_j}(|Dw_j|+|Dv_j|)^{p/2}|V(Dv_j)-V(Dw_j)|\,dx\\
&\le\ \varepsilon\fint_{\frac12 B_j}|V(Dv_j)-V(Dw_j)|^2\,dx\\
&\qquad\qquad\qquad+\frac{c[\omega(r_j)]^2}{\varepsilon}\fint_{\frac12 B_j}(|Dw_j|+|Dv_j|)^p\,dx\\
&\le\ \varepsilon\fint_{\frac12 B_j}|V(Dv_j)-V(Dw_j)|^2\,dx+\frac{c[\omega(r_j)]^2}{\varepsilon}\fint_{\frac12 B_j}|Dw_j|^p\,dx\,,
\end{aligned}
$$

where $c\equiv c(n,N,p,L)$. Notice that in the last estimate we have used the inequality

$$
\fint_{\frac12 B_j}|Dv_j|^p\,dx\le\fint_{\frac12 B_j}|Dw_j|^p\,dx\,, \tag{52}
$$

which is a consequence of the fact that v_j minimizes the p-energy

$$
z\mapsto\fint_{\frac12 B_j}|Dz|^p\,dx
$$

in its Dirichlet class $w_j+W^{1,p}_0(\frac12 B_j,\mathbb{R}^N)$. Connecting the above estimate found for I_2 to (51) yields (49), by standardly choosing $\varepsilon\equiv\varepsilon(n,N,p,\nu,L)$ suitably small and reabsorbing terms. When $p\ge 2$, again applying (20) we obtain

$$
\left(\fint_{\frac12 B_j}|Dv_j-Dw_j|^p\,dx\right)^{1/p}\le\left(\fint_{\frac12 B_j}|V(Dv_j)-V(Dw_j)|^2\,dx\right)^{1/p}
$$

so that (50) readily follows from (49).

2.8. Basic comparison estimate for $p\ge 2$

For the case $p\ge 2$ we now fix the number q as

$$
q:=\begin{cases}\dfrac{np}{n+p} & \text{if } n>2 \text{ or } p>2\\[2mm] 3/2 & \text{if}\quad n=p=2\,.\end{cases} \tag{53}
$$

We notice that with this definition we have that

$$
\begin{cases}\dfrac{np}{np-n+p}=(p^*)'\le q<n & \text{if } p<n\\[2mm] \qquad 1<q<n & \text{if } p\ge n\end{cases} \tag{54}
$$

and

$$q' := \begin{cases} \dfrac{np'}{n-p'} & \text{if } n > 2 \text{ or } p > 2 \\ 3 & \text{if} \quad n = p = 2\,. \end{cases} \tag{55}$$

The value of the exponent q will remain fixed as above for the rest of the paper as long as we shall be considering the case $p \geq 2$. The value of q will be defined in a different way when $1 < p < 2$ (see (68) below).

Lemma 6. *Let u be as in Theorem 1 with $p \geq 2$ and w_j, v_j as in* (10)-(11)*, respectively, for $j \geq 1$, and finally let the number q be as in* (53)*. Assume that for $\lambda > 0$ it holds that*

$$r_{j-1}\left(\fint_{B_{j-1}} |F|^q\,dx\right)^{1/q} \leq \lambda^{p-1} \tag{56}$$

and that for a constant $A \geq 1$ the inequalities

$$\sup_{\frac{1}{2}B_j} |Dw_j| \leq A\lambda \qquad \text{and} \qquad \frac{\lambda}{A} \leq |Dw_{j-1}| \leq A\lambda \qquad \text{in } B_j \tag{57}$$

also hold. Then there exists a constant $c_8 \equiv c_8(n, N, p, \nu, L, A, \sigma)$ such that

$$\left(\fint_{\frac{1}{2}B_j} |Du - Dv_j|^{p'}\,dx\right)^{1/p'} \\ \leq c_8\omega(r_j)\lambda + c_8\lambda^{2-p} r_{j-1}\left(\fint_{B_{j-1}} |F|^q\,dx\right)^{1/q}. \tag{58}$$

Proof. We will first prove that there exists a constant $c \equiv c(n, p, \nu, A, \sigma)$ such that

$$\left(\fint_{B_j} |Du - Dw_j|^{p'}\,dx\right)^{1/p'} \leq c\lambda^{2-p} r_{j-1}\left(\fint_{B_{j-1}} |F|^q\,dx\right)^{1/q}. \tag{59}$$

Then we shall prove that there exists $c \equiv c(n, N, p, \nu, L, A, \sigma)$ such that

$$\left(\fint_{\frac{1}{2}B_j} |Dv_j - Dw_j|^{p'}\,dx\right)^{1/p'} \\ \leq c\omega(r_j)\lambda + c\lambda^{2-p} r_{j-1}\left(\fint_{B_{j-1}} |F|^q\,dx\right)^{1/q}. \tag{60}$$

At this point (58) follows combining (59) and (60). The rest of the proof accordingly splits into two steps.

Step 1: Proof of (59). We will use the following rescaled maps:

$$\bar{w}_{j-1} := \frac{w_{j-1}}{\lambda} \qquad \text{and} \qquad \bar{w}_j := \frac{w_j}{\lambda}\,. \tag{61}$$

Then we estimate, by mean of (57)

$$\begin{aligned}
&\left(\fint_{B_j} |Du - Dw_j|^{p'}\,dx\right)^{1/p'} \\
&\quad \leq A^{p-2}\left(\fint_{B_j} |D\bar{w}_{j-1}|^{p'(p-2)}|Du - Dw_j|^{p'}\,dx\right)^{1/p'} \\
&\quad \leq c\left(\fint_{B_j} |D\bar{w}_j - D\bar{w}_{j-1}|^{p'(p-2)}|Du - Dw_j|^{p'}\,dx\right)^{1/p'} \\
&\qquad + c\left(\fint_{B_j} |D\bar{w}_j|^{p'(p-2)}|Du - Dw_j|^{p'}\,dx\right)^{1/p'} =: cI_3 + cI_4 \qquad (62)
\end{aligned}$$

where the constant c depends on p, A. Then we find

$$\begin{aligned}
I_3 &\overset{(61)}{=} \lambda^{2-p}\left(\fint_{B_j} |Dw_j - Dw_{j-1}|^{p'(p-2)}|Du - Dw_j|^{p'}\,dx\right)^{1/p'} \\
&\leq \lambda^{2-p}\left(\fint_{B_j} |Dw_j - Dw_{j-1}|^{p}\,dx\right)^{(p-2)/p} \\
&\qquad \cdot\left(\fint_{B_j} |Du - Dw_j|^{p}\,dx\right)^{1/p} \qquad (63) \\
&\overset{(46),(47)}{\leq} c\lambda^{2-p}\left(r_{j-1}^q \fint_{B_{j-1}} |F|^q\,dx\right)^{1/q},
\end{aligned}$$

where c depends on n, N, p, ν, σ. We proceed with the estimation of I_4; we write

$$I_4 = \left(\fint_{B_j} |D\bar{w}_j|^{(p-2)/(p-1)}|Du - Dw_j|^{p'}|D\bar{w}_j|^{p-2}\,dx\right)^{1/p'}$$

and apply Hölder's inequality with respect to the measure $\mu = |D\bar{w}_j|^{p-2}\,dx$ with conjugate exponents $2/p'$ and $2/(2-p')$ (when $p > 2$), to get

$$I_4 \leq \left(\fint_{B_j} |D\bar{w}_j|^{p-2}|Du - Dw_j|^{2}\,dx\right)^{1/2}\left(\fint_{B_j} |D\bar{w}_j|^{p}\,dx\right)^{(p-2)/(2p)}. \qquad (64)$$

Now we observe that

$$\fint_{B_j} |D\bar{w}_j|^{p}\,dx \overset{(61)}{\leq} \frac{c}{\lambda^p}\fint_{B_j} |Dw_j - Dw_{j-1}|^{p}\,dx + c\fint_{B_j} |D\bar{w}_{j-1}|^{p}\,dx$$

$$\overset{(47),(57)}{\leq} \frac{c}{\lambda^p}\left(r_{j-1}^q \fint_{B_{j-1}} |F|^q\, dx\right)^{\frac{p}{q(p-1)}} + c \overset{(56)}{\leq} c$$

and the constant c depends on n, N, p, ν, A, σ. Merging the last estimate with (64) and recalling (61) yields

$$\begin{aligned} I_4 &\leq c\lambda^{(2-p)/2}\left(\fint_{B_j} |Dw_j|^{p-2}|Du - Dw_j|^2\, dx\right)^{1/2} \\ &\leq c\lambda^{(2-p)/2}\left(\fint_{B_j} (|Dw_j| + |Du|)^{p-2}|Du - Dw_j|^2\, dx\right)^{1/2} \\ &\overset{(48)}{\leq} c\lambda^{(2-p)/2}\left(\fint_{B_j} |F||u - w_j|\, dx\right)^{1/2} \end{aligned} \tag{65}$$

again for $c \equiv c(n, N, p, \nu, A, \sigma)$. We estimate the last integral appearing in the above display as follows:

$$\begin{aligned} \fint_{B_j} |F||u - w_j|\, dx &\leq \left(\fint_{B_j} |u - w_j|^{q'}\, dx\right)^{1/q'} \left(\fint_{B_j} |F|^q\, dx\right)^{1/q} \\ &\overset{(55)}{\leq} c\left(\fint_{B_j} |Du - Dw_j|^{p'}\, dx\right)^{1/p'} \left(r_j^q \fint_{B_j} |F|^q\, dx\right)^{1/q}. \end{aligned}$$

Of course in the last inequality we have used Sobolev embedding theorem. Connecting the last estimate with (65) and in turn with (62), (63) yields

$$\begin{aligned} \left(\fint_{B_j} |Du - Dw_j|^{p'}\, dx\right)^{1/p'} &\leq c\lambda^{2-p}\left(r_{j-1}^q \fint_{B_{j-1}} |F|^q\, dx\right)^{1/q} \\ &+c\left[\left(\fint_{B_j} |Du - Dw_j|^{p'}\, dx\right)^{1/p'} \lambda^{2-p}\left(r_{j-1}^q \fint_{B_{j-1}} |F|^q\, dx\right)^{1/q}\right]^{1/2} \\ &\leq \frac{1}{2}\left(\fint_{B_j} |Du - Dw_j|^{p'}\, dx\right)^{1/p'} + c\lambda^{2-p}\left(r_{j-1}^q \fint_{B_{j-1}} |F|^q\, dx\right)^{1/q} \end{aligned}$$

for a constant c depending only on n, N, p, ν, A, σ and (59) follows.

Step 2: Proof of (60). Using the first inequality in (57) in (49) and (50) yields

$$\fint_{\frac{1}{2}B_j} |V(Dv_j) - V(Dw_j)|^2\, dx \leq c[\omega(r_j)]^2(A\lambda)^p \tag{66}$$

and

$$\left(\fint_{\frac{1}{2}B_j} |Dv_j - Dw_j|^p\, dx\right)^{1/p} \leq c[\omega(r_j)]^{2/p}A\lambda\,, \tag{67}$$

respectively. Recalling the definitions in (61) we write

$$
\begin{aligned}
&\left(\fint_{\frac{1}{2}B_j} |Dv_j - Dw_j|^{p'}\, dx\right)^{1/p'}\\
&\stackrel{(57)}{\leq} A^{p-2}\left(\fint_{\frac{1}{2}B_j} |D\bar{w}_{j-1}|^{p'(p-2)}|Dv_j - Dw_j|^{p'}\, dx\right)^{1/p'}\\
&\leq c\left(\fint_{\frac{1}{2}B_j} |D\bar{w}_j - D\bar{w}_{j-1}|^{p'(p-2)}|Dv_j - Dw_j|^{p'}\, dx\right)^{1/p'}\\
&\qquad +c\left(\fint_{\frac{1}{2}B_j} |D\bar{w}_j|^{p'(p-2)}|Dv_j - Dw_j|^{p'}\, dx\right)^{1/p'} =: cI_5 + cI_6\,.
\end{aligned}
$$

To estimate I_5 we argue as follows:

$$
\begin{aligned}
I_5 &\leq \left(\fint_{\frac{1}{2}B_j} |D\bar{w}_j - D\bar{w}_{j-1}|^{p}\, dx\right)^{(p-2)/p}\left(\fint_{\frac{1}{2}B_j} |Dv_j - Dw_j|^{p}\, dx\right)^{1/p}\\
&\stackrel{(67)}{\leq} \left(\fint_{\frac{1}{2}B_j} |D\bar{w}_j - D\bar{w}_{j-1}|^{p}\, dx\right)^{(p-2)/p}[\omega(r_j)]^{2/p}\lambda\\
&\leq c\omega(r_j)\lambda + c\lambda\fint_{\frac{1}{2}B_j} |D\bar{w}_j - D\bar{w}_{j-1}|^{p}\, dx\\
&\stackrel{(61)}{=} c\omega(r_j)\lambda + c\lambda^{1-p}\fint_{\frac{1}{2}B_j} |Dw_j - Dw_{j-1}|^{p}\, dx\\
&\stackrel{(47)}{\leq} c\omega(r_j)\lambda + c\lambda^{1-p}\left(r_{j-1}^q\fint_{B_{j-1}} |F|^{q}\, dx\right)^{p/[q(p-1)]}\\
&\stackrel{(56)}{\leq} c\omega(r_j)\lambda + c\lambda^{2-p}r_{j-1}\left(\fint_{B_{j-1}} |F|^{q}\, dx\right)^{1/q}.
\end{aligned}
$$

As for I_6, observe that, since $|D\bar{w}_j| \leq A$ in $\frac{1}{2}B_j$, then

$$
\begin{aligned}
I_6 &\leq A^{(p-2)/2}\left(\fint_{\frac{1}{2}B_j} |D\bar{w}_j|^{p'(p-2)/2}|Dv_j - Dw_j|^{p'}\, dx\right)^{1/p'}\\
&\leq c\left(\fint_{\frac{1}{2}B_j} |D\bar{w}_j|^{p-2}|Dv_j - Dw_j|^{2}\, dx\right)^{1/2}\\
&\stackrel{(61)}{=} c\lambda^{(2-p)/2}\left(\fint_{\frac{1}{2}B_j} |Dw_j|^{p-2}|Dv_j - Dw_j|^{2}\, dx\right)^{1/2}\\
&\leq c\lambda^{(2-p)/2}\left(\fint_{\frac{1}{2}B_j} (|Dv_j| + |Dw_j|)^{p-2}|Dv_j - Dw_j|^{2}\, dx\right)^{1/2}
\end{aligned}
$$

$$\overset{(20)}{\leq} c\lambda^{(2-p)/2}\left(\fint_{\frac{1}{2}B_j}|V(Dv_j)-V(Dw_j)|^2\,dx\right)^{1/2}$$

$$\overset{(66)}{\leq} c\omega(r_j)\lambda\,.$$

Merging the estimates found for I_5 and I_6 with (62) finally yields (60).

2.9. Basic comparison estimate for $1<p\leq 2$

We fix the exponent q for the case $1<p\leq 2$ as follows:

$$q:=\begin{cases}\dfrac{np}{np-n+p}=(p^*)' & \text{if } 1<p<2\\ \quad 3/2 & \text{if } n=p=2\,.\end{cases} \tag{68}$$

With this definition condition (54) is still satisfied. Moreover, note that the above definition of q coincides with the one in (53) when $p=2$. In this case we are giving the definition twice since we want to show that the different exponents used in the cases $p\geq 2$ and $1<p\leq 2$ actually coincide for $p=2$, and that all the estimates we have for the p-Laplacian operator are stable when p approaches 2. For the same reason, the constants c_5 and c_8 appearing in the next lemma have the same name of the similar constants introduced in (46) and (58), respectively; this is due to the fact that the constants below will play a similar role in the case $1<p\leq 2$ and the of those in (46) and (58) when $p\geq 2$.

Lemma 7. *Let u be as in Theorem 1 with $1<p\leq 2$ and w_j, v_j as in* (10)-(11)*, respectively, for $j\geq 1$, and finally let the number q be as in* (68)*. Assume that for $\lambda>0$ the inequalities*

$$r_j\left(\fint_{B_j}|F|^q\,dx\right)^{1/q}\leq\lambda^{p-1} \tag{69}$$

and

$$\left(\fint_{B_j}|Du|^p\,dx\right)^{1/p}\leq\lambda \tag{70}$$

hold. Then there exists a constant $c_8\equiv c_8(n,N,p,\nu,L,A,\sigma)$ such that

$$\left(\fint_{\frac{1}{2}B_j}|Du-Dv_j|^p\,dx\right)^{1/p}$$

$$\leq c_8\omega(r_j)\lambda+c_8\lambda^{2-p}r_{j-1}\left(\fint_{B_{j-1}}|F|^q\,dx\right)^{1/q}. \tag{71}$$

Moreover, there exists a constant $c_5\equiv c_5(n,N,p,\nu)$ such that

$$\left(\fint_{B_j}|Du-Dw_j|^p\,dx\right)^{1/p}\leq c_5\lambda^{2-p}r_j\left(\fint_{B_j}|F|^q\,dx\right)^{1/q}. \tag{72}$$

Proof. We will first prove (72) and then we will prove

$$\left(\fint_{\frac{1}{2}B_j} |Dv_j - Dw_j|^p \, dx\right)^{1/p} \leq c\lambda\omega(r_j) + c\lambda^{2-p} r_j \left(\fint_{B_j} |F|^q \, dx\right)^{1/q} \tag{73}$$

for a constant $c \equiv c(n, N, p, \nu, L, A, \sigma)$, so that (71) will follow via elementary manipulations and triangle inequality. We start applying (22) in order to have

$$|Du - Dw_j|^p \leq c|V(Du) - V(Dw_j)|^2 + c|V(Du) - V(Dw_j)|^p |Du|^{(2-p)p/2}$$

for $c \equiv c(n, N, p)$, so that, by means of Hölder's inequality we have

$$\begin{aligned}\left(\fint_{B_j} |Du - Dw_j|^p \, dx\right)^{1/p} &\leq c \left(\fint_{B_j} |V(Du) - V(Dw_j)|^2 \, dx\right)^{1/p} \\ &\quad + c \left(\fint_{B_j} |V(Du) - V(Dw_j)|^2 \, dx\right)^{1/2} \left(\fint_{B_j} |Du|^p \, dx\right)^{(2-p)/2p} \\ &=: cI_7 + cI_8 \, .\end{aligned}$$

For I_7 we instead get

$$\begin{aligned}I_7 &\overset{(20),(48)}{\leq} c\left(\fint_{B_j} |F||u - w_j| \, dx\right)^{1/p} \\ &\leq c\left(\fint_{B_j} |u - w_j|^{q'} \, dx\right)^{1/pq'} \left(\fint_{B_j} |F|^q \, dx\right)^{1/pq} \\ &\overset{(68)}{\leq} c\left(\fint_{B_j} |Du - Dw_j|^p \, dx\right)^{1/p^2} \left(r_j^q \fint_{B_j} |F|^q \, dx\right)^{1/pq} \\ &\leq \varepsilon\left(\fint_{B_j} |Du - Dw_j|^p \, dx\right)^{1/p} + c\left(r_j^q \fint_{B_j} |F|^q \, dx\right)^{1/[q(p-1)]} \\ &\overset{(69)}{\leq} \varepsilon\left(\fint_{B_j} |Du - Dw_j|^p \, dx\right)^{1/p} + c\lambda^{2-p} r_j \left(\fint_{B_j} |F|^q \, dx\right)^{1/q} ,\end{aligned}$$

where $c \equiv c(n, N, p, \nu)$. Proceeding similarly as in the last display, for I_8 we obtain

$$\begin{aligned}I_8 &\overset{(70)}{\leq} \lambda^{(2-p)/2}\left(\fint_{B_j} |V(Du) - V(Dw_j)|^2 \, dx\right)^{1/2} \\ &\overset{(20),(48)}{\leq} c\lambda^{(2-p)/2}\left(\fint_{B_j} |F||u - w_j| \, dx\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\le \quad c\lambda^{(2-p)/2}\left(\fint_{B_j}|Du-Dw_j|^p\,dx\right)^{1/2p}\left(r_j^q\fint_{B_j}|F|^q\,dx\right)^{1/2q}\\
&\le \quad \varepsilon\left(\fint_{B_j}|Du-Dw_j|^p\,dx\right)^{1/p}+c\lambda^{2-p}r_j\left(\fint_{B_j}|F|^q\,dx\right)^{1/q}.
\end{aligned}
$$

Connecting the estimates in the last three displays, choosing$\varepsilon \equiv \varepsilon(n, N, p, \nu)$ small enough and reabsorbing terms yields (72). Next we turn to the proof of (73). From the proof of Lemma 6 we recall that the content of display (51) still holds in the case $1 < p \le 2$. Then we use (20) to have

$$
|Dv_j-Dw_j|^p \le c|V(Dv_j)-V(Dw_j)|^p(|Dv_j|+|Dw_j|)^{p(2-p)/2}
$$

so that, using Hölder's inequality we conclude with (73) as follows and therefore the proof of the lemma is complete:

$$
\begin{aligned}
&\left(\fint_{\frac{1}{2}B_j}|Dv_j-Dw_j|^p\,dx\right)^{1/p}\\
&\le c\left(\fint_{\frac{1}{2}B_j}|V(Dv_j)-V(Dw_j)|^2\,dx\right)^{1/2}\\
&\qquad\qquad\cdot\left(\fint_{\frac{1}{2}B_j}(|Dv_j|+|Dw_j|)^p\,dx\right)^{(2-p)/2p}\\
&\overset{(49),(52)}{\le} c\omega(r_j)\left(\fint_{\frac{1}{2}B_j}|Dw_j|^p\,dx\right)^{1/p}\\
&\le c\omega(r_j)\left(\fint_{\frac{1}{2}B_j}|Du-Dw_j|^p\,dx\right)^{1/p}+c\omega(r_j)\left(\fint_{\frac{1}{2}B_j}|Du|^p\,dx\right)^{1/p}\\
&\overset{(72),(70)}{\le} c\lambda^{2-p}r_j\left(\fint_{B_j}|F|^q\,dx\right)^{1/q}+c\omega(r_j)\lambda\,.
\end{aligned}
$$

3. A pointwise gradient bound

We start with a gradient L^∞-bound which has its own interest for essentially three reasons. First, L^∞-bounds for gradients of solutions are available for p-Laplacian systems without coefficients (i.e. $\gamma(\cdot)$ is a constant function) as derived in [3, 6], but the known techniques do not apparently extend to treat the case when coefficients are involved, at least in the case they are not differentiable, as here. Second, already in the case when no coefficients are present the two dimensional case $n = 2$ remained an open issue, essentially for technical reasons. Third, and most importantly for us, many of the arguments developed here are at the origin of those used in the proof of the main Theorem 1 in the final Section 4. To formulate the result we introduce the exponents s and q as follows:

$$s := \begin{cases} p' \text{ if } & p \geq 2 \\ p \text{ if } 1 < p \leq 2 \end{cases} \tag{74}$$

and

$$q := \begin{cases} \text{the number in (53) if } & p \geq 2 \\ \text{the number in (68) if } 1 < p \leq 2\,. \end{cases} \tag{75}$$

We again notice an overlapping in the two definitions above when $p = 2$; this is not by chance, but it is done with the purpose to show that whenever we shall distinguish between the cases $p \geq 2$ and $1 < p \leq 2$, all the methods will coincide for the case $p = 2$. The gradient bound is now featured in the following:

Theorem 4. *Let u be as in Theorem 1; then Du is locally bounded in Ω. Moreover, there exists a constant $c \geq 1$ and a positive radius R_0, both depending only on $n, N, p, \nu, L, \omega(\cdot)$, such that the pointwise estimate*

$$|Du(x_0)| \leq c \left(\fint_{B(x_0,r)} |Du|^s \, dx \right)^{1/s} + c\|F\|_{L(n,1)}^{1/(p-1)} \tag{76}$$

holds whenever $B(x_0, 2r) \subset \Omega$, x_0 is a Lebesgue point of Du and $2r \leq R_0$. Estimate (76) *holds with no restriction on r when the coefficient function $\gamma(\cdot)$ is constant.*

Remark 1. Theorem 4 is the first step towards the proof of the gradient continuity stated in Theorem 1; on the other hand, once Theorem 1 is gained, it is clear that estimate (76) holds for every point x_0. Moreover, a standard covering argument based on (76) gives that the following estimate:

$$\|Du\|_{L^\infty(B_{r/2})} \leq c \left(\fint_{B_r} |Du|^p \, dx \right)^{1/p} + c\|F\|_{L(n,1)}^{1/(p-1)}$$

holds for every $B_r \subset \Omega$.

Proof (of Theorem 4). Let us first briefly unveil the skeleton of the proof. We shall use the quantity $S(x_0, r, \sigma) \equiv S_q(x_0, r, \sigma)$ defined in (16), the number q introduced in (75), and the chain of shrinking balls B_i considered in (9). This chain will have as starting ball the one considered in the statement of the Theorem, that is, $B(x_0, r)$ with $r \leq R_0$. Both the parameter $\sigma \in (0, 1/4)$ appearing in (9) and the radius R_0 will be chosen in a few lines as a functions depending only on $n, N, p, \nu, L, \omega(\cdot)$, but not on the solution u. More precisely, we shall prove that

$$|Du(x_0)| \leq \lambda := H_1 \left(\fint_{B(x_0,r)} |Du|^s \, dx \right)^{1/s} + H_2[S(x_0, r, \sigma)]^{1/(p-1)}\,, \tag{77}$$

where the large constants H_1, H_2 are again going to be chosen in a few lines as functions depending only on $n, N, p, \nu, L, \omega(\cdot)$. The proof of (76) then follows by using (17) while the local boundedness of Du follows via a standard covering argument. In order to prove (77) we show that the inequality $|(Du)_{B_i}| \leq \lambda$ holds at least for a subsequence of indexes i. Since x_0 is assumed to be a Lebesgue point of Du, (77) therefore follows. We tell in advance that in what follows certain auxiliary constants will be deliberately chosen larger/smaller than necessary both for the sake of readability and to recall the reader that their role is exactly to be large/small. The rest of the proof proceeds now in three steps. Needless to say, in (77) we may assume that $\lambda > 0$, otherwise the statement of the Theorem 4 follows trivially.

Step 1: Choice of the constants and exit time. Keeping Theorems 2 and 3 in mind, we set

$$A := 10^{5n} \max\{c_2, c_3\}\,, \quad \delta := 10^{-5}\,, \quad \bar{\varepsilon} := 4^{-(n+4)}\,, \tag{78}$$

where $c_2 \equiv c_2(n, N, p, \nu, L)$ and $c_3 \equiv c_3(n, N, p)$ are as in Theorem 2 and Theorem 3, respectively; hence, A depends only on n, N, p, ν, L. The choice in (78) allows to determine the constant $\sigma_1 \equiv \sigma_1(n, N, p, \nu, L, \omega(\cdot))$ in Theorem 2, the constants $c_4 \equiv c_4(n, N, p, A)$ and $\alpha \equiv \alpha(n, N, p) > 0$ in Theorem 3 and $\sigma_2 \equiv \sigma_2(n, N, p, \nu, L)$ in Lemma 3. We now define the quantity

$$\sigma := \min\{\sigma_1, \sigma_2\} \in (0, 1/4) \tag{79}$$

that again depends only on $n, N, p, \nu, L, \omega(\cdot)$ by taking into account all the previous dependencies. Finally, with such a choice of σ and q we determine the constants c_5, c_6, c_7, c_8 in Lemmas 4-7, as once again functions depending on $n, N, p, \nu, L, \omega(\cdot)$. We only notice that the constants c_5 and c_8 have been defined twice (Lemmas 4-6 and Lemma 7, respectively), since they play a similar role according to which of the cases $p \geq 2$ and $1 < p \leq 2$ is actually occurring. We then define $k \equiv k(n, N, p, \nu, L, \omega(\cdot))$ as the smallest integer satisfying

$$c_4 \sigma^{k\alpha} \leq \frac{\sigma^n}{10^6} \quad \text{and} \quad k \geq 2\,. \tag{80}$$

Finally, to put order in all the constant whose name starts by "c", we introduce

$$c_f = \prod_{i=2}^{8} c_i\,, \tag{81}$$

which is of course larger than all the constants contributing to the above product as all of them are larger than one. Again, with the choice made above c_f only depends on $n, N, \nu, L, \omega(\cdot)$. We now proceed with the choice of H_1, H_2 and R_0. We set

$$\begin{cases} H_1 := 10^{5n} \sigma^{-2n} \\ H_2 := \left[6^n 10^6 \sigma^{-n(k+5)} c_f\right]^{\max\{1, 1/(p-1)\}} \end{cases}. \tag{82}$$

In this way also the constants H_1 and H_2 ultimately depend only on n, N, p, ν, L and $\omega(\cdot)$. As for R_0, we recall the definition of $R_1 \equiv R_1(n, N, p, \nu, L, \omega(\cdot))$ from Theorem 2 and then we fix a radius R_2 to be small enough to verify

$$\sigma^{-n(k+4)}[\omega(R_2)]^{\min\{1,2/p\}} + \sigma^{-1} \int_0^{R_2} \omega(\varrho) \, \frac{d\varrho}{\varrho} \leq \frac{\sigma^{2n}}{6^n 10^6 c_f} \tag{83}$$

is satisfied. Notice that this is possible by (6). Finally we set

$$R_0 := \min\{R_1, R_2\}/4$$

and again we have a quantity that depends on $n, N, p, \nu, L, \omega(\cdot)$. This means that, in the rest of the proof, when considering the ball in the statement of Theorem 4, and when adopting the setting of Section 2.2 with the maps w_j, v_j in (10)-(11), we will always have $2r \leq R_0$. In particular, for the shrinking balls B_i defined in (9) we use the constant σ that has been determined in (79). Accordingly, we define the following relevant quantities for $i \geq 0$:

$$\begin{cases} a_i := |(Du)_{B_i}| \\ E_i := E_s(Du, B_i) = \left(\fint_{B_i} |Du - (Du)_{B_i}|^s \, dx \right)^{1/s} \end{cases} \tag{84}$$

$$\mu_i := r_i \left(\fint_{B_i} |F|^q \, dx \right)^{1/q}, \qquad \text{so that } S(x_0, r, \sigma) = \sum_{i=0}^{\infty} \mu_i \, . \tag{85}$$

Finally, for integers $i \geq 1$, we define

$$C_i := \left(\fint_{B_{i-1}} |Du|^s \, dx \right)^{1/s} + \left(\fint_{B_i} |Du|^s \, dx \right)^{1/s} + 2\sigma^{-n} E_i \, .$$

Now we notice that the choice of H_1 in (82) allows to conclude that

$$C_1 \leq 4\sigma^{-2n} \left(\fint_{B(x_0,r)} |Du|^s \, dx \right)^{1/s} \leq \frac{4\sigma^{-2n}\lambda}{H_1} \leq \frac{\lambda}{1000} \, .$$

We further observe that, thanks to the inequality in the above display, we may proceed under the additional assumption that there exists an exit time index $i_e \geq 1$ such that

$$C_{i_e} \leq \frac{\lambda}{1000} \qquad \text{and} \qquad C_i > \frac{\lambda}{1000} \qquad \forall\, i > i_e \, . \tag{86}$$

Indeed, were this not the case, there would exist a subsequence $\{i_j\}_j$ of indexes such that $C_{i_j} \leq \lambda/1000$, that would in turn imply that

$$|Du(x_0)| = \lim_j a_{i_j} \leq \limsup_j C_{i_j} \leq \lambda \, ,$$

because x_0 is a Lebesgue point of Du. Hence (76) would follow. Therefore the rest of the proof proceeds assuming (86). Before going on let us record another couple of estimates that are going to be used in the following. The first one is

$$\sigma^{-n(k+4)}[\omega(r_0)]^{\min\{1,2/p\}} + \sum_{i=0}^{\infty}\omega(r_i) \le \frac{\sigma^{2n}}{6^n 10^6 c_f}\,. \tag{87}$$

Estimate (87) is a consequence of (83) and of the next computation

$$\begin{aligned}\int_0^{R_2}\omega(\varrho)\,\frac{d\varrho}{\varrho} &= \sum_{i=0}^{\infty}\int_{r_{i+1}}^{r_i}\omega(\varrho)\,\frac{d\varrho}{\varrho} + \int_{r_0}^{R_2}\omega(\varrho)\,\frac{d\varrho}{\varrho}\\ &\ge \log\left(\frac{1}{\sigma}\right)\sum_{i=0}^{\infty}\omega(r_{i+1}) + \log 4\omega(r_0) \ge \sigma\sum_{i=0}^{\infty}\omega(r_i)\end{aligned} \tag{88}$$

where we used the fact that $r \equiv r_0 \in (0, R_2/4]$ and that $\sigma \le 1/4$. The second estimate follows directly from the definition of $S(x_0, r, \sigma)$, (77) and (82), and is the following:

$$\mu_j = r_j\left(\fint_{B_j}|F|^q\,dx\right)^{1/q} \le \lambda^{p-1} \qquad \text{for every } j \ge 0\,. \tag{89}$$

This inequality will be used to apply Lemmas 6 and 7. Finally we remark that due to the exit time argument used above, from now on we shall restrict our attention to the indexes $j \ge i_e$.

Step 2: Upper and lower bounds, and a decay estimate. For $j \ge i_e$ we now consider the following condition:

$$\mathrm{Ind}(j): \qquad \max\left\{\left(\fint_{B_{j-1}}|Du|^s\,dx\right)^{1/s}, \left(\fint_{B_j}|Du|^s\,dx\right)^{1/s}\right\} \le \lambda \tag{90}$$

and determine a few consequences of it to be proven shortly. The first two are the following comparison estimates:

$$\mathrm{Ind}(j) \implies \left(\fint_{B_{j-1}}|Du - Dw_{j-1}|^s\,dx\right)^{1/s} \le \frac{\sigma^{n(k+5)}\lambda}{2^n 10^6} \tag{91}$$

and

$$\mathrm{Ind}(j) \implies \left(\fint_{\frac{1}{2}B_j}|Du - Dv_j|^s\,dx\right)^{1/s} \le \frac{\sigma^{n(k+5)}\lambda}{2^n 10^6}\,. \tag{92}$$

We also prove a series of upper and lower bounds, namely

$$\begin{aligned}&\mathrm{Ind}(j) \implies \\ &\quad \sup_{\frac{1}{2}B_{j-1}}|Dw_{j-1}| \le A\lambda\,, \quad \sup_{\frac{1}{2}B_j}|Dw_j| \le A\lambda \quad \text{and} \quad \sup_{\frac{1}{4}B_j}|Dv_j| \le A\lambda\end{aligned} \tag{93}$$

$$\mathrm{Ind}(j) \implies \frac{\lambda}{A} \le \inf_{B_j} |Dw_{j-1}| \quad \text{and} \quad \frac{\lambda}{A} \le \sup_{B_{j+1}} |Dv_j| \,. \tag{94}$$

Finally, we prove the following decay estimate for the excess functional $E(\cdot)$:

$$\mathrm{Ind}(j) \implies E_{j+1} \le \frac{1}{4}E_j + \frac{2c_8\lambda}{\sigma^n}\omega(r_j) + \frac{2c_8\lambda^{2-p}}{\sigma^n}\mu_{j-1} \,. \tag{95}$$

We now proceed with the proofs, starting by the one of (91) in the case $p \ge 2$, when by definition of s it is $s = p' \le p$:

$$\begin{aligned}\left(\fint_{B_{j-1}} |Du - Dw_{j-1}|^s \, dx\right)^{1/s} &\overset{(46)}{\le} c_5\mu_{j-1}^{1/(p-1)} \\ &\overset{(85)}{\le} c_5[S(x_0, r, \sigma)]^{1/(p-1)} \\ &\overset{(77)}{\le} \frac{c_5\lambda}{H_2} \\ &\overset{(82)}{\le} \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} \,.\end{aligned}$$

In the case $p \in (1, 2]$, when instead $s = p$, we have

$$\begin{aligned}\left(\fint_{B_{j-1}} |Du - Dw_{j-1}|^s \, dx\right)^{1/s} &\overset{(72)}{\le} c_5\lambda^{2-p}\mu_{j-1} \\ &\overset{(85)}{\le} c_5\lambda^{2-p}S(x_0, r, \sigma) \\ &\overset{(77)}{\le} \frac{c_5\lambda}{H_2^{p-1}} \\ &\overset{(82)}{\le} \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} \,.\end{aligned}$$

Hence (91) holds in the full range $p > 1$. For future convenience, we also record a similar estimate, that is

$$\left(\fint_{B_j} |Du - Dw_j|^s \, dx\right)^{1/s} \le \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} \,. \tag{96}$$

The proof of (92) will also involve the first two inequalities appearing in (93), so we start proving them. Using (96) gives

$$\begin{aligned}\left(\fint_{B_j} |Dw_j|^s \, dx\right)^{1/s} &\le \left(\fint_{B_j} |Du|^s \, dx\right)^{1/s} + \left(\fint_{B_j} |Du - Dw_j|^s \, dx\right)^{1/s} \\ &\le 2\lambda \,.\end{aligned}$$

Applying (23) yields

$$\sup_{\frac{1}{2}B_j} |Dw_j| \leq c_2 \left(\fint_{B_j} |Dw_j|^s \, dx \right)^{1/s} \leq 2c_2\lambda \overset{(78)}{\leq} A\lambda \tag{97}$$

so that the second inequality in (93) follows. Obviously, the first inequality in (93) follows applying the same argument to w_{j-1} in B_{j-1}. We are now ready for the proof of (92); we start by treating the case $p \geq 2$, when $s = p' \leq p$. We have

$$\begin{aligned}
&\left(\fint_{\frac{1}{2}B_j} |Du - Dv_j|^s \, dx \right)^{1/s} \\
&\quad \leq \left(\fint_{\frac{1}{2}B_j} |Du - Dw_j|^s \, dx \right)^{1/s} + \left(\fint_{\frac{1}{2}B_j} |Dv_j - Dw_j|^p \, dx \right)^{1/p} \\
&\quad \overset{(96),(50)}{\leq} \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} + c_7[\omega(r_j)]^{2/p} \left(\fint_{\frac{1}{2}B_j} |Dw_j|^p \, dx \right)^{1/p} \\
&\quad \leq \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} + c_7[\omega(r_j)]^{2/p} \sup_{\frac{1}{2}B_j} |Dw_j| \\
&\quad \overset{(97)}{\leq} \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} + 2c_2c_7[\omega(r_j)]^{2/p}\lambda \\
&\quad \overset{(87)}{\leq} \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} + \frac{\sigma^{n(k+5)}\lambda}{4^n 10^6} \\
&\quad \leq \frac{\sigma^{n(k+5)}\lambda}{2^n 10^6} \, .
\end{aligned}$$

In the case $1 < p \leq 2$, we recall that $s = p$ and get

$$\begin{aligned}
\left(\fint_{\frac{1}{2}B_j} |Du - Dv_j|^s \, dx \right)^{1/s} &\overset{(71)}{\leq} c_8\omega(r_j)\lambda + c_8\lambda^{2-p}\mu_{j-1} \\
&\overset{(87),(85)}{\leq} \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} + c_8\lambda^{2-p}S(x_0, r, \sigma) \\
&\overset{(77)}{\leq} \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} + \frac{c_8\lambda}{H_2^{p-1}} \\
&\overset{(82)}{\leq} \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} + \frac{\sigma^{n(k+5)}\lambda}{6^n 10^6} \\
&\leq \frac{\sigma^{n(k+5)}\lambda}{2^n 10^6}
\end{aligned}$$

so that (92) is finally proved in the full range $p > 1$.

As for (93), it remains to prove the third estimate, since the first two have been obtained in due course of the proof of (92). We have

$$\left(\fint_{\frac{1}{2}B_j}|Dv_j|^s\,dx\right)^{1/s} \le 2^n\left(\fint_{B_j}|Du|^s\,dx\right)^{1/s} + \left(\fint_{\frac{1}{2}B_j}|Du-Dv_j|^s\,dx\right)^{1/s} \overset{(92)}{\le} 2^{n+1}\lambda\,.$$

Therefore, by using this time (33) we have

$$\sup_{\frac{1}{4}B_j}|Dv_j| \le c_3\left(\fint_{\frac{1}{2}B_j}|Dv_j|^s\,dx\right)^{1/s} \le 2^{n+1}c_3\lambda \overset{(78)}{\le} A\lambda\,,$$

i.e., the third inequality in (93), which is now completely established. We now turn to (94). By (93) we can apply Theorem 2 to w_{j-1}, and recalling that $B_j = \sigma B_{j-1} \subset \sigma_1 B_{j-1}$, we have

$$\operatorname*{osc}_{B_j} Dw_{j-1} \le \frac{\lambda}{10^5}\,. \tag{98}$$

For a similar bound on Dv_j we use again (93) that in turn allows to employ (34) thereby concluding with

$$\operatorname*{osc}_{B_{j+k}} Dv_j \le c_4\sigma^{\alpha k}\lambda \overset{(80)}{\le} \frac{\sigma^n\lambda}{10^6}\,. \tag{99}$$

Therefore, using also (8) we have

$$\begin{aligned}
2\sigma^{-n}E_{j+k} &\le 4\sigma^{-n}E(Dv_j, B_{j+k}) + 4\sigma^{-n}\left(\fint_{B_{j+k}}|Du-Dv_j|^s\,dx\right)^{1/s}\\
&\le 4\sigma^{-n}\operatorname*{osc}_{B_{j+k}} Dv_j + 4\sigma^{-n-k}\left(\fint_{\frac{1}{2}B_j}|Du-Dv_j|^s\,dx\right)^{1/s}\\
&\overset{(99),(92)}{\le} \frac{\lambda}{10^6} + \frac{4\lambda}{10^6} \le \frac{\lambda}{10^5}\,.
\end{aligned}$$

Recalling that $C_{j+k} > \lambda/1000$ for $j \ge i_e$, we then have

$$\sum_{m=-1}^{0}\left(\fint_{B_{j+m+k}}|Du|^s\,dx\right)^{1/s} \ge \frac{\lambda}{2000}\,.$$

Triangle inequality then yields

$$\frac{\lambda}{2000} \le \sum_{m=-1}^{0}\left(\fint_{B_{j+m+k}}|Du|^s\,dx\right)^{1/s}$$

$$\leq \frac{2}{\sigma^{n(k+1)}}\left(\fint_{B_{j-1}}|Du-Dw_{j-1}|^s\,dx\right)^{1/s}$$

$$+\sum_{m=-1}^{0}\left(\fint_{B_{j+m+k}}|Dw_{j-1}|^s\,dx\right)^{1/s}$$

$$\overset{(91)}{\leq} \frac{\lambda}{10^6}+\sum_{m=-1}^{0}\left(\fint_{B_{j+m+k}}|Dw_{j-1}|^s\,dx\right)^{1/s} \tag{100}$$

so that, as $k \geq 2$, we also get

$$2\sup_{B_j}|Dw_{j-1}| \geq \sum_{m=-1}^{0}\left(\fint_{B_{j+m+k}}|Dw_{j-1}|^s\,dx\right)^{1/s} \geq \frac{\lambda}{2000}-\frac{\lambda}{10^6} \geq \frac{3\lambda}{10^4}\,.$$

At this point the oscillation control in (98) gives $\lambda/10^5 \leq |Dw_{j-1}|$ in B_j, that in turn, recalling (78), implies the first inequality in (94). Arguing as for (100), but using (92), this time we gain

$$\frac{\lambda}{2000} \leq \sum_{m=-1}^{0}\left(\fint_{B_{j+m+k}}|Du|^s\,dx\right)^{1/s}$$

$$\leq \frac{\lambda}{10^6}+\sum_{m=-1}^{0}\left(\fint_{B_{j+m+k}}|Dv_j|^s\,dx\right)^{1/s}$$

and therefore

$$2\sup_{B_{j+1}}|Dv_j| \geq \sum_{m=-1}^{0}\left(\fint_{B_{j+m+k}}|Dv_j|^s\,dx\right)^{1/s} \geq \frac{\lambda}{10^3}$$

so that the second inequality in (94) follows recalling the definition in (78). It remains to check (95). We will do this by using all the inequalities appearing in (93) and (94). Keeping in mind the definition of σ in (79) we can apply Lemma 3 (with the choice $t = p'$ when $p \geq 2$ and $t = p$ when $1 < p < 2$) that gives

$$E(Dv_j, B_{j+1}) \leq \frac{1}{4^{n+4}}E(Dv_j,(1/2)B_j)\,.$$

Now, note that by (89) and since we are assuming $\operatorname{Ind}(j)$ we are able to apply Lemmas 6 and 7; therefore, by also using (8) we obtain

$$E(Du,B_{j+1}) \leq 2E(Dv_j,B_{j+1})+\sigma^{-n}\left(\fint_{\frac{1}{2}B_j}|Du-Dv_j|^s\,dx\right)^{1/s}$$

$$\overset{(58),(71)}{\leq} 2E(Dv_j,B_{j+1})+\frac{c_8\lambda}{\sigma^n}\omega(r_j)+\frac{c_8\lambda^{2-p}}{\sigma^n}\mu_{j-1}$$

and, similarly,

$$\begin{aligned} E(Dv_j,(1/2)B_j) &\leq 2^{n+1}E(Du,B_j)+\left(\fint_{\frac{1}{2}B_j}|Du-Dv_j|^s\,dx\right)^{1/s} \\ &\overset{(58),(71)}{\leq} 2^{n+1}E(Du,B_j)+c_8\omega(r_j)\lambda+c_8\lambda^{2-p}\mu_{j-1}\,. \end{aligned}$$

Combining the estimates in the above three displays yields (95).

Step 3: Final induction. Here we prove, by induction, that the inequality

$$a_j+E_j\leq\lambda \tag{101}$$

holds whenever $j\geq i_e$, thereby proving the theorem since x_0 is a Lebesgue point of Du and so

$$\lim_{j\to\infty}a_j=|Du(x_0)|\,.$$

Notice that (86) implies that (101) holds for $j=i_e$. Therefore, let us assume that (101) holds whenever $j\in\{i_e,\dots,i\}$ for some $i\geq i_e$ and prove that it holds for $j=i+1$ too. We notice that this implies the validity of $\mathrm{Ind}(j)$ for every $j\in\{i_e,\dots,i\}$. Indeed, notice first that if $j=i_e$ then $\mathrm{Ind}(i_e)$ is a direct consequence of (86), while when $j>i_e$ then $\mathrm{Ind}(j)$ follows in a straightforward manner from (101) and the definition of E_j:

$$\left(\fint_{B_j}|Du|^s\,dx\right)^{1/s}\leq a_j+E_j\leq\lambda$$

and

$$\left(\fint_{B_{j-1}}|Du|^s\,dx\right)^{1/s}\leq a_{j-1}+E_{j-1}\leq\lambda\,.$$

With $\mathrm{Ind}(j)$ being in force for $j\in\{i_e,\dots,i\}$ the inequality appearing in (95) holds for the corresponding indexes j. Summing up then yields

$$\sum_{j=i_e+1}^{i+1}E_j\leq\frac{1}{2}\sum_{i=i_e}^{i}E_j+\frac{2c_8}{\sigma^n}\sum_{i=i_e}^{i}\omega(r_j)\,\lambda+\frac{2c_8\lambda^{2-p}}{\sigma^n}\sum_{i=i_e-1}^{i-1}\mu_j$$

and therefore, recalling the definition of $S(x_0,r,\sigma)$ we have

$$\begin{aligned} \sum_{j=i_e}^{i+1}E_j &\leq 2E_{i_e}+\frac{4c_8}{\sigma^n}\sum_{i=0}^{\infty}\omega(r_j)\,\lambda+\frac{4c_8\lambda^{2-p}}{\sigma^n}S(x_0,r,\sigma) \\ &\leq \sigma^nC_{i_e}+\frac{\sigma^n\lambda}{10^5}+\frac{4c_8\lambda}{\sigma^nH_2^{p-1}}\leq\frac{\sigma^n\lambda}{500}\,. \end{aligned} \tag{102}$$

Notice that in order to perform the last estimation we have also used (86), (87) and (82). On the other hand, notice that

$$a_{i+1}-a_{i_e}=\sum_{j=i_e}^{i}(a_{j+1}-a_j)$$

$$\leq \sum_{j=i_e}^{i} \fint_{B_{j+1}} |Du - (Du)_{B_j}| \, dx$$

$$= \sigma^{-n} \sum_{j=i_e}^{i} E_j \overset{(102)}{\leq} \frac{\lambda}{500}$$

and therefore (86) gives

$$a_{i+1} \leq a_{i_e} + \frac{\lambda}{500} \leq C_{i_e} + \frac{\lambda}{500} \leq \frac{\lambda}{2} \, .$$

Connecting the last inequality with (102) yields $a_{i+1} + E_{i+1} \leq \lambda$ so that the induction step is verified and (101) holds for every $j \geq i_e$. This completes the proof of Theorem 4.

4. Proof of Theorem 1

By Theorem 4 we know that Du is locally bounded; since we are proving a local result, up to passing to open subsets compactly contained in Ω, we can assume w.l.o.g. that the gradient is globally bounded, thereby letting

$$\lambda := \|Du\|_{L^\infty(\Omega)} + 1 \, . \tag{103}$$

The strategy of the proof consists of gaining the continuity of Du by showing that Du is the locally uniform limit of a net of continuous maps - actually defined via averages

$$x_0 \mapsto (Du)_{B_\varrho(x_0)} \, . \tag{104}$$

To do this, we consider an open subset $\Omega_0 \Subset \Omega$ and prove that for every $\varepsilon > 0$ there exists a radius

$$r_\varepsilon \leq \operatorname{dist}(\Omega_0, \partial\Omega)/100 =: R_* \, , \tag{105}$$

depending only on $n, N, p, \nu, L, \omega(\cdot), \|F\|_{L(n,1)}, \varepsilon$, such that

$$|(Du)_{B_\varrho(x_0)} - (Du)_{B_\rho(x_0)}| \leq \lambda\varepsilon \qquad \text{holds for every } \varrho, \rho \in (0, r_\varepsilon] \tag{106}$$

whenever $x_0 \in \Omega_0$. This proves that Du is the uniform limit of continuous maps defined in display (104) and hence it is continuous. The rest of the proof sees now $\varepsilon > 0$ and Ω_0 defined as above, and it is now devoted to establish (106). The numbers s and q stay fixed as in (74) and (75), respectively.

Step 1: A decay estimate. We start as in Theorem 4, where the choice of the constants in (78) is now replaced by

$$A := \frac{1000^{5n} \max\{c_2, c_3\}}{\varepsilon} \, , \quad \delta := \frac{\varepsilon}{10^5} \, , \quad \bar{\varepsilon} := \frac{\varepsilon}{4^{n+4}} \, , \tag{107}$$

where, exactly as for Theorem 4, the constants $c_2 \equiv c_2(n, N, p, \nu, L)$ and $c_3 \equiv c_3(n, N, p)$ come from Theorem 2 and Theorem 3, respectively; in this way we have $A \equiv A(n,N,p,\nu,L,\varepsilon)$. The choice in (107) allows to determine $\sigma_1 \equiv \sigma_1(n,N,p,\nu,L,\omega(\cdot),\varepsilon)$ in Theorem 2, the constants $c_4 \equiv c_4(n,N,p,\nu,L,\varepsilon)$ and $\alpha \equiv \alpha(n,N,p) > 0$ in Theorem 3 and $\sigma_2 \equiv \sigma_2(n,N,p,\nu,L,\varepsilon)$ in Lemma 3. Similarly to as done in (79), we fix σ as

$$\sigma := \min\{\sigma_1,\sigma_2\} \in (0,1/4)\,. \tag{108}$$

In turn, this determines the constants c_5, c_6, c_7, c_8 in Lemmas 4-7. All in all we have determined constants depending only on $n,N,p,\nu,L,\omega(\cdot),\varepsilon$. Next, we fix some limitations on the radii considered; $R_1 \equiv R_1(n,N,p,\nu,L,\omega(\cdot))$ still denotes the radius considered in Theorem 2. Thanks to (6) and Lemma 1 we can select a new radius $R_3 \equiv R_3(n,N,p,\nu,L,\omega(\cdot),\|F\|_{L(n,1)},\varepsilon)$ in such a way that the following *smallness conditions* hold:

$$\sigma^{1-n/q}\int_0^{2R_3}[\varrho^q(|F|^q)^{**}(\omega_n\varrho^n)]^{1/q}\frac{d\varrho}{\varrho} \le \left(\frac{\sigma^{4n}\varepsilon}{6^n 10^6 c_f}\right)^{\max\{1,p-1\}}\lambda^{p-1} \tag{109}$$

and

$$[\omega(R_3)]^{\min\{1,2/p\}} + \int_0^{4R_3}\omega(\varrho)\,\frac{d\varrho}{\varrho} \le \frac{\sigma^{5n}\varepsilon}{6^n 10^6 c_f}$$

and we this time set

$$\bar{R}_0 := \min\{R_*, R_1, R_3\}/4\,,$$

with R_* that has been defined in (105). The constant c_f here is defined as in (81) with the new values of the various factors $c_2,\ldots,c_8$ determined here; therefore an additional dependence on ε appears for c_f. Notice that the possibility of satisfying the last inequality follows by (6) while (109) is allowed by (18). Notice also that, again as a consequence of the content of Section 2.3, we have $\bar{R}_0 \equiv \bar{R}_0(n,N,p,\nu,L,\omega(\cdot),\|F\|_{L(n,1)},\varepsilon,R_*)$.

Remark 2. The above lines contain a certain abuse of notation: when we say that the radius R_3 (and therefore $\bar{R}_0$) depends on the quantity $\|F\|_{L(n,1)}$ we are not really precise. Indeed, the dependence should be credited as simply on $F(\cdot)$ rather than on $\|F\|_{L(n,1)}$ as the choice of R_3 in (109) is made through the smallness of the left hand side in (109). On the other hand, since the integral appearing in (109) relates to the norm $\|F\|_{L(n,1)}$ via the discussion in Section 2.3, and in particular through Lemma 1, we then prefer to keep this ambiguity here as also in the following.

We now fix a ball $B \equiv B(x_0,r) \subset \Omega$ and accordingly to the set-up of Section 2.2 define

$$B_j \equiv B(x_0,r_j)\,, \qquad r_j := \sigma^j r\,, \qquad \text{with } r \in (\sigma\bar{R}_0,\bar{R}_0] \tag{110}$$

for $j \geq 0$, while the maps w_j and v_j are defined as in (10) and (11), respectively. We also keep the notation introduced in (84), in particular we have

$$E_i := \left(\fint_{B_i}|Du-(Du)_{B_i}|^s\,dx\right)^{1/s}\,, \quad \mu_i := r_i\left(\fint_{B_i}|F|^q\,dx\right)^{1/q}\,. \tag{111}$$

We now proceed with the proof, and notice that all the forthcoming arguments and constants are independent of the choice of the starting radius r in (110), i.e., they stay uniformly bounded as long as r varies in the range $[\sigma\bar{R}_0, \bar{R}_0]$; this fact will play a crucial role later. Applying (109) together with (18) we have

$$\sup_{0<\varrho\le R_3}\ \sup_{x\in\Omega_0} S(x,\varrho,\sigma) \le \left(\frac{\sigma^{4n}\varepsilon}{6^n 10^6 c_f}\right)^{\max\{1,p-1\}} \lambda^{p-1} \le \lambda^{p-1}\,. \tag{112}$$

Moreover, a computation similar to that in (88) then gives

$$[\omega(R_3)]^{\min\{1,2/p\}} + \sum_{i=0}^{\infty}\omega(r_i) \le \frac{\sigma^{4n}\varepsilon}{6^n 10^6 c_f}\,. \tag{113}$$

With the above definitions, and using (112) and (113), it turns out that the inequalities in (91) and (92) can be now replaced by

$$\left(\fint_{B_{j-1}} |Du - Dw_{j-1}|^s\,dx\right)^{1/s} \le \frac{\sigma^{4n}\lambda\varepsilon}{6^n 10^6} \tag{114}$$

and

$$\left(\fint_{\frac{1}{2}B_j} |Du - Dv_j|^s\,dx\right)^{1/s} \le \frac{\sigma^{4n}\lambda\varepsilon}{2^n 10^6}\,, \tag{115}$$

respectively, with exactly the same arguments as the ones for (91)-(92). We now consider the following condition:

$$\text{Ind*}(j): \qquad \left(\fint_{B_{j+1}} |Du|^s\,dx\right)^{1/s} \ge \frac{\lambda\varepsilon}{50} \tag{116}$$

for every $j \ge 1$ and start proving that

$$\text{Ind*}(j) \implies E_{j+1} \le \frac{\varepsilon}{4}E_j + \frac{2c_8\lambda}{\sigma^n}\omega(r_j) + \frac{2c_8\lambda^{2-p}}{\sigma^n}\mu_{j-1}\,. \tag{117}$$

In order to prove (117) we observe that the setting adopted here is completely similar to the one of Theorem 4 the only differences lying in two main points: the different choice of the constants in (107)-(108) with respect to (78)-(79) and the fact that the exit time information in (86) is not available. On the other hand, to rebalance such a lack of information we notice two further things: first, we notice that that (107)-(108) reduce to (78)-(79) for $\varepsilon = 1$ and therefore many of the inequalities implied by (78)-(79) can be proved in this setting since condition Ind(j) in (90) is satisfied for every j by (103). Second, the absence of (86) will be compensated by the new condition Ind*(j), which tells about a lower bound for $|Du|$ similar to the one eventually implied by (86) in the proof of Theorem 1. We now go for the details, starting by (93); by using (114)-(115) we see that all the inequalities appearing in (93), that is

$$\sup_{\frac{1}{2}B_{j-1}} |Dw_{j-1}| \leq A\lambda\,, \quad \sup_{\frac{1}{2}B_j} |Dw_j| \leq A\lambda\,, \quad \sup_{\frac{1}{4}B_j} |Dv_j| \leq A\lambda \tag{118}$$

can be obtained here with exactly the same proof, and for the value of A fixed in (107). We now pass to the analogs of (94), thereby proceeding as for (98)-(99). We take the new choices in (107)-(108) and into account and use (118); therefore applying Theorem 2 to w_{j-1} gives

$$\operatorname*{osc}_{B_j} Dw_{j-1} \leq \frac{\lambda\varepsilon}{10^5}\,. \tag{119}$$

Observe now that

$$\begin{aligned}
\frac{\lambda\varepsilon}{50} &\stackrel{\text{Ind}^*(j)}{\leq} \left(\fint_{B_{j+1}} |Du|^s\,dx\right)^{1/s} \\
&\leq \sigma^{-2n}\left(\fint_{B_{j-1}} |Du - Dw_{j-1}|^s\,dx\right)^{1/s} + \left(\fint_{B_{j+1}} |Dw_{j-1}|^s\,dx\right)^{1/s} \\
&\stackrel{(114)}{\leq} \frac{\lambda\varepsilon}{10^6} + \sup_{B_j} |Dw_{j-1}|
\end{aligned}$$

so that we infer the existence of a point $\tilde{x} \in B_j$ such that $|Dw_{j-1}(\tilde{x})| \geq \lambda\varepsilon/200$. This, together with the first inequality in (119) gives that

$$\frac{\lambda}{A} \leq \frac{\lambda\varepsilon}{400} \leq \inf_{B_j} |Dw_{j-1}|\,.$$

Similarly, we have

$$\begin{aligned}
\frac{\lambda\varepsilon}{50} &\stackrel{\text{Ind}^*(j)}{\leq} \left(\fint_{B_{j+1}} |Du|^s\,dx\right)^{1/s} \\
&\leq \sigma^{-n}\left(\fint_{\frac{1}{2}B_j} |Du - Dv_j|^s\,dx\right)^{1/s} + \left(\fint_{B_{j+1}} |Dv_j|^s\,dx\right)^{1/s} \\
&\stackrel{(115)}{\leq} \frac{\lambda\varepsilon}{10^6} + \sup_{B_{j+1}} |Dv_j|
\end{aligned}$$

so that, recalling the choice in (107) we have

$$\frac{\lambda}{A} \leq \sup_{B_{j+1}} |Dv_j|\,.$$

Summarizing, we have proved that all the inequalities appearing in (93)-(94) hold for the new choice of A made in (107), and this in turn suffices to reproduce the proof for the estimate in (95), with the new choice of the constant $\bar{\varepsilon}$ made in (107) and keeping in mind the new definition of σ given in (108). But this is exactly the inequality appearing in (117), that is therefore proved.

Step 2: Smallness of the excess. We shall prove that for every $\varepsilon \in (0,1)$, there exists a radius $r_\varepsilon \equiv r_\varepsilon(n, N, p, \nu, L, \omega(\cdot), \|F\|_{L(n,1)}, \varepsilon, R_*) > 0$ such that

$$E(Du, B_\varrho) < \lambda\varepsilon \qquad \text{holds whenever } \varrho \in (0, r_\varepsilon] \tag{120}$$

and $B_\varrho \subset \Omega$. Let us first observe that

$$E_{j+1} < \lambda\varepsilon \qquad \text{holds for every } j \in \mathbb{N} \cap [1, \infty) \tag{121}$$

with the meaning fixed in (111). This is indeed a consequence of the estimates made in the previous step. If $\text{Ind}^*(j)$ does not hold then (121) follows trivially; if on the other hand $\text{Ind}^*(j)$ holds then we can apply (117) that, together with (103) and (112)-(113), yields

$$\begin{aligned} E_{j+1} &\leq \frac{\varepsilon}{4} E_j + \frac{2c_8\lambda}{\sigma^n}\omega(r_j) + \frac{2c_8\lambda^{2-p}}{\sigma^n}\mu_{j-1} \\ &\leq \frac{\lambda\varepsilon}{2} + \frac{\lambda\varepsilon}{100} + \frac{\lambda\varepsilon}{100} \leq \lambda\varepsilon \end{aligned}$$

that is (121). Now (120) follows with the choice $r_\varepsilon := \sigma^2 \bar{R}_0$. Indeed, consider $\varrho \leq \sigma^2 \bar{R}_0$; this means there exists an integer $m \geq 2$ such that $\sigma^{m+1}\bar{R}_0 < \varrho \leq \sigma^m \bar{R}_0$. Therefore we have $\varrho = \sigma^m r$ for some $r \in (\sigma\bar{R}_0, \bar{R}_0]$ and (120) follows from (121) with this particular choice of r in (110).

Step 3: Proof of (106). Thanks to (120), with $\varepsilon \in (0,1)$ always being the one fixed in (106), we can select a radius

$$R_4 \equiv R_4(n, N, p, \nu, L, \omega(\cdot), \|F\|_{L(n,1)}, \varepsilon) > 0$$

such that the following inequality is satisfied:

$$\sup_{0<\varrho\leq R_4} \sup_{x\in B_0} E(Du, B(x,\varrho)) \leq \frac{\sigma^{4n}\lambda\varepsilon}{10^5}\,. \tag{122}$$

Observe that it is possible to make the choice in (122) thanks to the result in (120), that also ensures that R_4 can be chosen in a way that makes it depending only on $n, N, p, \nu, L, \omega(\cdot), \|F\|_{L(n,1)}, \varepsilon$. This time we set

$$\tilde{R}_0 := \min\{R_*, R_1, R_3, R_4\}/4 \leq \bar{R}_0$$

and take everywhere $r \leq \tilde{R}_0$; notice that $\tilde{R}_0$ ultimately depends again on $n, N, p, \nu, L, \omega(\cdot), \|F\|_{L(n,1)}, \varepsilon, R_*$ only. The sequence of shrinking balls $\{B_j\}$ corresponding to the set-up defined in Section 2.2 is now defined as

$$B_j \equiv B(x_0, r_j)\,, \qquad r_j = \sigma^j \tilde{R}_0 \tag{123}$$

for $j \geq 0$. Again, we keep the notation in (111). Notice that since $\tilde{R}_0 \leq \bar{R}_0$ all the inequalities and the arguments developed through of Steps 1 and 2 apply here for the set-up in (123). Let us now assume for a moment that

$$|(Du)_{B_h} - (Du)_{B_k}| \leq \frac{\lambda\varepsilon}{12} \qquad \text{holds whenever } 2 \leq k \leq h\,. \tag{124}$$

Then (106) follows with the choice $r_\varepsilon := \sigma^2 \tilde{R}_0$. Indeed, whenever $0 < \rho < \varrho \leq r_\varepsilon$ there exist two integers, $2 \leq k \leq h$, such that

$$\sigma^{k+1}\tilde{R}_0 < \varrho \leq \sigma^k \tilde{R}_0 \quad \text{and} \quad \sigma^{h+1}\tilde{R}_0 < \rho \leq \sigma^h \tilde{R}_0\,.$$

Applying (122) we get, via Hölder's inequality

$$\begin{aligned}
|(Du)_{B_\varrho(x_0)} - (Du)_{B_{k+1}}| &\leq \fint_{B_{k+1}} |Du - (Du)_{B_\varrho(x_0)}|\,dx \\
&\leq \frac{|B_\varrho(x_0)|}{|B_{k+1}|} \fint_{B_\varrho(x_0)} |Du - (Du)_{B_\varrho(x_0)}|\,dx \\
&\leq \sigma^{-n} E(Du, B_\varrho(x_0)) \\
&\overset{(122)}{\leq} \frac{\lambda\varepsilon}{10}\,,
\end{aligned}$$

and, similarly,

$$|(Du)_{B_\rho(x_0)} - (Du)_{B_{h+1}}| \leq \frac{\lambda\varepsilon}{10}\,.$$

Using the last two estimates and (124) we conclude with (106). Therefore, in order to complete the proof we just need to establish (124). To this aim let us consider the set

$$\mathcal{L} := \left\{ j \in \mathbb{N} \;:\; \left(\fint_{B_j} |Du|^s\,dx \right)^{1/s} < \frac{\lambda\varepsilon}{50} \right\}\,,$$

and

$$\mathcal{C}_i^m = \{ j \in \mathbb{N} : i \leq j \leq i+m,\ i \in \mathcal{L},\ j \notin \mathcal{L} \text{ if } j > i \} \qquad \text{for } m \in \mathbb{N} \cup \{\infty\}$$

and, finally, the number $j_e := \min \mathcal{L}$. Note that it may happen that $j_e = \infty$; this means that the inequality in (116), considered for the balls B_j defined in (123), holds for every $j \geq 1$. We proceed with the proof of (124), obviously assuming $k < h$, and treating three different cases. The first case we analyze is when $k < h \leq j_e$; this means that Ind*(j) from (116) holds whenever $j \in \{k-1, \ldots, h-2\}$ and, as a consequence, by (117) we have that

$$E_{j+1} \leq \frac{\varepsilon}{4} E_j + \frac{2c_8}{\sigma^n}\omega(r_j)\lambda + \frac{2c_8\lambda^{2-p}}{\sigma^n}\mu_{j-1} \tag{125}$$

holds for every $j \in \{k-1, \ldots, h-2\}$. Summing up the previous inequalities and using (112)-(113) and (122) easily yields

$$\sum_{i=k}^{h-1} E_i \le E_{k-1} + \frac{4c_8}{\sigma^n} \sum_{j=0}^{\infty} \omega(r_j)\,\lambda + \frac{4c_8 \lambda^{2-p}}{\sigma^n} S(x_0, r, \sigma) \le \frac{\sigma^{2n}\lambda\varepsilon}{50}$$

therefore (124) follows since

$$\begin{aligned} |(Du)_{B_h} - (Du)_{B_k}| &\le \sum_{i=k}^{h-1} \fint_{B_{i+1}} |Du - (Du)_{B_i}|\,dx \\ &\le \sigma^{-n} \sum_{i=k}^{h-1} E_i \le \frac{\lambda\varepsilon}{50}\,. \end{aligned} \tag{126}$$

The second case we consider is when $j_e \le k < h$, where we prove (124) through the inequalities

$$|(Du)_{B_h}| \le \frac{\lambda\varepsilon}{25} \qquad \text{and} \qquad |(Du)_{B_k}| \le \frac{\lambda\varepsilon}{25}\,. \tag{127}$$

In (127), we prove the former, the argument for the latter being the same when $k > j_e$, otherwise $|(Du)_{B_k}| \le \lambda\varepsilon/25$ is trivial if $k = j_e \in \mathcal{L}$. If $h \in \mathcal{L}$, the first inequality in (127) follows immediately from the definition of $\mathcal{L}$. On the other hand, if $h \notin \mathcal{L}$, then, as $h > j_e$, it is possible to consider a set $\mathcal{C}_{i_h}^{m_h}$ with $m_h > 0$, such that $h \in \mathcal{C}_{i_h}^{m_h}$; notice that $h > i_h$ as $h \notin \mathcal{L} \ni i_h$. Then (117) gives that (125) holds whenever $j \in \{i_h, \ldots, i_h + m_h - 1\}$. Summing up the resulting versions of (125) and the usual elementary manipulations give

$$\sum_{i=i_h}^{i_h+m_h} E_i \le 2E_{i_h} + \frac{4c_8}{\sigma^n} \sum_{j=0}^{\infty} \omega(r_j)\,\lambda + \frac{4c_8 \lambda^{2-p}}{\sigma^n} S(x_0, r, \sigma) \le \frac{\sigma^{2n}\lambda\varepsilon}{50}$$

where again we have used (112)-(113) and (122). Therefore, as in (126), we have

$$|(Du)_{B_h} - (Du)_{B_{i_h}}| \le \sigma^{-n} \sum_{i=i_h}^{h-1} E_i \le \sigma^{-n} \sum_{i=i_h}^{i_h+m_h} E_i \le \frac{\lambda\varepsilon}{50}$$

and then, using that $|(Du)_{B_{i_h}}| \le \lambda\varepsilon/50$ as $i_h \in \mathcal{L}$, we have

$$|(Du)_{B_h}| \le |(Du)_{B_{i_h}}| + |(Du)_{B_h} - (Du)_{B_{i_h}}| \le \frac{\lambda\varepsilon}{25}$$

that is (127). The third and last case is when $k < j_e < h$, that can be actually treated by a combination of the first two. It suffices to prove that the inequalities in display (127) still hold. The former follows exactly as in the second case. As for the latter, let us remark that, since $j_e \in \mathcal{L}$ then

$|(Du)_{B_{j_e}}| \leq \lambda\varepsilon/50$. On the other hand, we can use the first case $k < h \leq j_e$ with $h = j_e$ and this yields

$$|(Du)_{B_{j_e}} - (Du)_{B_k}| \leq \frac{\lambda\varepsilon}{50}\,.$$

At this stage the second estimate in (127) follows via triangle inequality. The proof of Theorem 1 is complete.

References

1. Bensoussan A. & Frehse J.: *Regularity results for nonlinear elliptic systems and applications.* Applied Mathematical Sciences 151, Springer-Verlag, Berlin, 2002.
2. Cianchi A.: Maximizing the L^∞ norm of the gradient of solutions to the Poisson equation. *J. Geom. Anal.* 2 (1992), 499–515.
3. Cianchi A. & Maz'ya V.: Global Lipschitz regularity for a class of quasilinear elliptic equations. *Comm. PDE* 36 (2011), 100–133.
4. DiBenedetto E.: *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York, 1993.
5. DiBenedetto E. & Manfredi J.J.: On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems. *Amer. J. Math.* 115 (1993), 1107–1134.
6. Duzaar F. & Mingione G.: Local Lipschitz regularity for degenerate elliptic systems. *Ann. Inst. H. Poincaré Anal. Non Linèaire* 27 (2010), 1361–1396.
7. Duzaar F. & Mingione G.: Gradient continuity estimates. *Calc. Var. & PDE* 39 (2010), 379–418.
8. Fusco N. & Hutchinson J.E.: Partial regularity for minimisers of certain functionals having nonquadratic growth. *Ann. Mat. Pura Appl. (IV)* 155 (1989), 1–24.
9. Hamburger C.: Regularity of differential forms minimizing degenerate elliptic functionals. *J. Reine Angew. Math. (Crelles J.)* 431 (1992), 7–64.
10. Hardy G. H. & Littlewood J. E. & Pólya G.: *Inequalities. Reprint of the 1952 edition.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.
11. Hunt L.I.: On $L(p,q)$ spaces. *L'Einsegnement Math.* 12 (1966), 249–276.
12. Iwaniec T. & Onninen J.: Continuity estimates for n-harmonic equations. *Indiana Univ. Math. J.* 56 (2007), 805–824.
13. Jin T. & Maz'ya V. & Van Schaftingen J.: Pathological solutions to elliptic problems in divergence form with continuous coefficients. *Comptes Rendus Mathematique* 347 (2009), 773-778.
14. Kilpeläinen T. & Malý J.: The Wiener test and potential estimates for quasilinear elliptic equations. *Acta Math.* 172 (1994), 137–161.
15. Kinnunen J. & Zhou S.: A local estimate for nonlinear equations with discontinuous coefficients. *Comm. PDE* 24 (1999), 2043–2068.
16. Kristensen J. & Mingione G.: The singular set of minima of integral functionals. *Arch. Ration. Mech. Anal.* 180 (2006), 331–398.
17. Kuusi T. & Mingione G.: New perturbation methods for nonlinear parabolic problems. *J. Math. Pures Appl. (IX)* 98 (2012), 390–427.
18. Kuusi T. & Mingione G.: Linear potentials in nonlinear potential theory. *Arch. Ration. Mech. Anal.* 207 (2013), 215–246.
19. Manfredi J.J.: Regularity for minima of functionals with p-growth. *J. Differential Equations* 76 (1988), 203–212.
20. Maz'ya V. & McOwen R.: Differentiability of solutions to second-order elliptic equations via dynamical systems. *J. Differential Equations* 250 (2011), 1137–1168.

21. Mingione G.: The Calderón-Zygmund theory for elliptic problems with measure data. *Ann Scu. Norm. Sup. Pisa Cl. Sci. (V)* 6 (2007), 195–261.
22. Stein E. M.: Editor's note: the differentiability of functions in $\mathbb{R}^n$. *Ann. of Math. (II)* 113 (1981), 383–385.
23. Stein E. M. & Weiss G.: *Introduction to Fourier analysis on Euclidean spaces.* Princeton Math. Ser., 32, Princeton Univ. Press, Princeton, N.J. 1971.
24. Trudinger N.S. & Wang X.J.: On the weak continuity of elliptic operators and applications to potential theory. *Amer. J. Math.* 124 (2002), 369–410.
25. Trudinger N.S. & Wang X.J.: Quasilinear elliptic equations with signed measure data. *Disc. Cont. Dyn. Systems* 124 (2002), 369–410.
26. Uhlenbeck K.: Regularity for a class of non-linear elliptic systems. *Acta Math.* 138 (1977), 219–240.
27. Ural'tseva N.N.: Degenerate quasilinear elliptic systems. *Zap. Na. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 7 (1968), 184–222.