SHARP REGULARITY FOR EVOLUTIONARY OBSTACLE PROBLEMS, INTERPOLATIVE GEOMETRIES AND REMOVABLE SETS

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ABSTRACT. In this paper we prove, by showing that solutions have exactly the same degree of regularity as the obstacle, optimal regularity results for obstacle problems involving evolutionary p-Laplace type operators. A main ingredient, of independent interest, is a new intrinsic interpolative geometry allowing for optimal linearization principles via blow-up analysis at contact points. This also opens the way to the proof of a removability theorem for solutions to evolutionary p-Laplace type equations. A basic feature of the paper is that no differentiability in time is assumed on the obstacle; this is in line with the corresponding linear results.

RÉSUMÉ - Dans ce papier, nous montrons, en utilisant le fait que les solutions ont le même degré de régularité que l'obstacle, des résultats de régularité optimale pour des problèmes d'obstacles dans lesquels interviennent des opérateurs d'évolution de type p-Laplace. Un des ingrédients principaux, interressant en tant que tel, est une nouvelle géométrie interpolative intrinsèque avec laquelle des principes de linéarisation optimale par l'analyse d'explosions aux points de contacts peuvent être utilisés. Cela ouvre la voie à la démonstration d'un théorème d'élimination pour les solutions d'équations d'évolutions de type p-Laplace. On notera que dans ce papier, l'obstacle n'est pas supposé différentiable en temps, ce qui est cohérent avec les résultats correspondants dans le cas linéaire.

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1. Introduction and results

1.1. **Results.** This paper is devoted to the study of regularity of solutions to obstacle problems involving quasilinear parabolic operators of the type

$$- Hu := u_t - \operatorname{div} a(Du) \tag{1.1}$$

and considered in cylindrical domains of the form $\mathcal{O} = \Omega \times (0, T) \subset \mathbb{R}^n \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $T > 0$ and $n \geq 2$. The vector field $a \colon \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular and satisfying the following *growth and ellipticity assumptions*:

$$\begin{cases} |a(z)| + |\partial a(z)|(|z|^2 + s^2)^{1/2} \leq L(|z|^2 + s^2)^{(p-1)/2} \\ \nu(|z|^2 + s^2)^{(p-2)/2}|\xi|^2 \leq \langle \partial a(z)\xi, \xi\rangle \end{cases} \tag{1.2}$$

whenever $z, \xi \in \mathbb{R}^n$. Here $0 < \nu \leq L$ and $s \geq 0$ are fixed parameters. In order to emphasize the main new ideas we in this paper concentrate on the case $p \geq 2$; the case $2n/(n+2) < p < 2$ can also be treated starting form the techniques introduced here, and will be presented elsewhere. Needless to say, a chief model example of the operators considered in this paper is given by *the evolutionary p-Laplace operator*

$$u \mapsto u_t - \operatorname{div}(|Du|^{p-2}Du)\,.$$

For a good introduction to the regularity theory of the p-Laplacean operator we for instance refer to the basic work of Manfredi [34, 35] and to introductory notes of Lindqvist [32].

In the following we let $\partial_{\mathrm{P}}\mathcal{O} := \big(\bar\Omega \times \{0\}\big) \cup (\partial\Omega \times [0,T])$ denote the parabolic boundary of $\mathcal{O}$. Given continuous boundary datum $b : \bar{\mathcal{O}} \to \mathbb{R}$ and a continuous obstacle $\psi : \bar{\mathcal{O}} \to \mathbb{R}$ such that $b \geq \psi$ on $\partial_{\mathrm{P}}\mathcal{O}$, we consider the problem

$$\begin{cases} \max\{Hu, \psi - u\} = 0 & \text{in } \mathcal{O} \\ u = b & \text{on } \partial_{\mathrm{P}}\mathcal{O}, \end{cases} \tag{1.3}$$

and we are particularly interested in the optimal regularity of the solution u conditioned on the regularity of b and ψ. For the definition of the concept of solution adopted in this paper we refer to Section 2.1. The goal of the paper is to prove that solutions to (1.3) *have the same degree of regularity as the data* b, ψ and we emphasize that a key point of this paper is that *we assume no differentiability of the obstacle* ψ *with respect to time*, something which is not usual when considering nonlinear regularity problems. See for instance [3], and related references, where the time differentiability of the obstacle must be assumed in order to obtain regularity results. As discussed below, the problems studied in this paper have up to now been open issues in the case of the degenerate parabolic equations we consider.

In order to state our results we need to briefly describe the by now classical approach to regularity of solutions to the degenerate evolutionary p-Laplace operator as first introduced by DiBenedetto (see for instance [11, 42]). Equations of the type

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0 \tag{1.4}$$

have an obvious lack of isotropy and, as a consequence, already the very issue of scaling properties become more involved compared to the linear ($p = 2$) case. For this reason the classical regularity analysis based on straightforward scaling and decay estimates on shrinking balls/cylinders does not apply in this case. To overcome this one is led to study the local regularity properties via decay analysis on shrinking cylinders *whose size depends on the solution itself.* This is the basic idea of DiBenedetto's *intrinsic geometry* and for this reason the cylinders considered are referred to as *intrinsic cylinders.* More specifically, one is led to consider cylinders of the type

$$B(x,r) \times (t - \lambda^{2-p}r^p, t + \lambda^{2-p}r^p) \quad \text{or} \quad B(x,r) \times (t - \lambda^{2-p}r^2, t + \lambda^{2-p}r^2)\,,$$

where $\lambda > 0$ is a parameter related to the size of the solution *on the same cylinder.* Here $B(x,r)$ is the standard Euclidean ball in $\mathbb{R}^n$, centered at x and with radius $r > 0$. Note that when $p = 2$ both of the above cylinders reduce to the standard parabolic cylinders used in the context of the heat equation. In Section 1.3 below we will in more detail describe the way intrinsic geometries are used to obtain regularity results. Here we just to draw the attention of the reader to the fact that the necessity of using such intrinsic cylinders implies that the relevant notions of regularity also have to be introduced in a suitable intrinsic way.

To introduce the function spaces used in the paper we let, given $A \subset \mathbb{R}^{n+1}$ and a function $f : A \to \mathbb{R}^m$, $m \geq 1$,

$$\operatorname*{osc}_A f := \sup_{(x_0,t_0),(x,t)\in A} |f(x_0,t_0) - f(x,t)|$$

denote the oscillation of f on A. Given $(x_0,t_0) \in \mathbb{R}^{n+1}$ and $r, \lambda > 0$ we introduce the cylinders

$$Q_r^\lambda(x_0,t_0) = \{(x,t) \in \mathbb{R}^{n+1} \;:\; |x_0 - x| < r,\; |t_0 - t| < \lambda^{2-p}r^p\}. \tag{1.5}$$

Furthermore, we let $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a concave modulus of continuity, i.e., a concave nondecreasing function such that

$$\omega(1) = 1 \qquad \text{and} \qquad \omega(0) := \lim_{r\downarrow 0} \omega(r) = 0\,.$$

Given a function f defined on $\mathcal{O} = \Omega \times (0,T) \subset \mathbb{R}^n \times \mathbb{R}$, we set

$$[f]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O})} := \inf \left\{ \lambda > 0 \ : \ \sup_{Q_r^{\lambda\omega(r)} \subset \mathbb{R}^n \times \mathbb{R}} \left(\frac{1}{\lambda\omega(r)} \operatorname*{osc}_{Q_r^{\lambda\omega(r)} \cap \mathcal{O}} f \right) \leq 1 \right\}. \tag{1.6}$$

For time independent functions we define the semi-norm related to $\omega(\cdot)$ as

$$[f]_{C^{\omega(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 \ : \ \sup_{B(x,r) \subset \mathbb{R}^n} \left(\frac{1}{\lambda\omega(r)} \operatorname*{osc}_{B(x,r)\cap\Omega} f \right) \leq 1 \right\}. \tag{1.7}$$

Needless to say, the localized version of the above spaces is defined in the usual way and we write, for instance, $f \in \widetilde{C}^{\omega(\cdot)}_{\text{loc}}(\mathcal{O})$ if and only if $f \in \widetilde{C}^{\omega(\cdot)}(\mathcal{O}')$ whenever $\mathcal{O}' \Subset \mathcal{O}$. Moreover, we let $C^0(\mathcal{O})$, $C^0(\Omega)$ denote the set of functions which are continuous on $\mathcal{O}$ and Ω, respectively. We note that in the special case $\omega(r) = r^\alpha$, $\alpha \in (0,1]$, then the definitions in (1.6) and (1.7) reduce to a notion of Hölder continuity:

$$\omega(r) = r^\alpha\,, \quad \alpha \in (0,1]\,, \quad [f]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O})} < \infty \quad \Longleftrightarrow \quad \sup_{z_1,z_2\in O} \frac{|f(z_1)-f(z_2)|}{\|z_1-z_2\|_\alpha^\alpha} < \infty\,,$$

where the parabolic metric is defined as

$$\|(x_1,t_1)-(x_2,t_2)\|_\alpha := \max\left\{ |x_1-x_2|, |t_1-t_2|^{1/[p-\alpha(p-2)]} \right\}. \tag{1.8}$$

In particular, the metric is depending on the degree of regularity considered. Note also that when $p = 2$, then these spaces coincide with the spaces of functions which are Hölder continuous of order α with respect to the standard parabolic metric.

We are now ready to state our first result which concerns optimal interior regularity in the obstacle problem assuming that the obstacle is in the space $\widetilde{C}^{\omega(\cdot)}$.

Theorem 1.1 (Interior regularity). *Let H be as in* (1.1), (1.2), *let $\psi \in \widetilde{C}^{\omega(\cdot)}(\mathcal{O})$ and let u solve* (1.3). *Let $\mathcal{O}' \subset \mathcal{O}$ be a bounded space-time cylinder such that $\bar{\mathcal{O}}' \cap \partial_{\mathrm{P}}\mathcal{O} = \emptyset$. Then $u \in \widetilde{C}^{\omega(\cdot)}(\mathcal{O}')$ and*

$$[u]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O}')} \leq c\left(n,p,\nu,L,\omega(\cdot),\mathcal{O},\mathcal{O}',\operatorname*{osc}_{\mathcal{O}} b, [\psi]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O})}\right).$$

While Theorem 1.1 concerns optimal interior regularity we also establish optimal regularity up to the initial state. In particular, in this case we prove $\widetilde{C}^{\omega(\cdot)}$-estimates on $\mathcal{O}' = \Omega' \times (0,T)$ for every $\Omega' \Subset \Omega$. We explicitly remark that in this case $\mathcal{O}'$ is not a compact subset of $\mathcal{O}$. In this context our main result is the following.

Theorem 1.2 (Initial time regularity). *Let H be as in* (1.1), (1.2), *let $\psi \in \widetilde{C}^{\omega(\cdot)}(\mathcal{O})$, $b(\cdot,0) \in C^{\omega(\cdot)}(\Omega)$ and let u solve* (1.3). *Let $\Omega' \Subset \Omega$ and $\mathcal{O}' = \Omega' \times (0,T)$. Then $u \in \widetilde{C}^{\omega(\cdot)}(\mathcal{O}')$ and*

$$[u]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O}')} \leq c\left(n,p,\nu,L,\omega(\cdot),\Omega,\Omega',\operatorname*{osc}_{\mathcal{O}} b, [b(\cdot,0)]_{C^{\omega(\cdot)}(\Omega)}, [\psi]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O})}\right).$$

As a by-product of the techniques developed to prove Theorem 1.1 and Theorem 1.2 we also obtain a regularity result for solutions to the Cauchy-Dirichlet problem

$$\begin{cases} Hu = 0 & \text{in } \mathcal{O} \\ u = b & \text{on } \partial_{\mathrm{P}}\mathcal{O}\,. \end{cases} \tag{1.9}$$

Corollary 1.1 (Initial time regularity without obstacles)**.** *Let H be as in* (1.1), (1.2)*, let $b(\cdot,0)\in C^{\omega(\cdot)}(\Omega)$ and let u solve* (1.9)*. Let $\Omega'\Subset\Omega$ and $\mathcal{O}'=\Omega'\times(0,T)$. Then $u\in\widetilde{C}^{\omega(\cdot)}(\mathcal{O}')$ and*

$$[u]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O}')}\leq c\left(n,p,\nu,L,\omega(\cdot),\Omega,\Omega',\operatorname*{osc}_{\mathcal{O}}b,[b(\cdot,0)]_{C^{\omega(\cdot)}(\Omega)}\right).$$

We again remark that Theorem 1.1 also implies the following endpoint result:

Corollary 1.2. *Let u be a solution to* (1.3) *with $D\psi, b\in L^\infty(\mathcal{O})$; then $Du\in L^\infty_{\rm loc}(\mathcal{O})$.*

To put out results slightly in perspective we note that when $p=2$ the operator appearing in (1.4) coincides with the heat operator $Hu=u_t-\triangle u$ and hence the obstacle problem in (1.3) becomes an obstacle problem for the heat equation. In the case of linear and uniformly parabolic equations we note that there is an extensive literature on the existence and regularity of generalized solutions to the obstacle problem in Sobolev spaces and we refer to [16] for details. Furthermore, optimal regularity of the solution to the obstacle problem for the Laplace equation was first proved in [4, 6], see also [5], and we note that the technique is based on the Harnack inequality for harmonic functions and the control of a harmonic function by its Taylor expansion. For obstacle problems involving the heat equation we instead refer to [7] while we refer to [41] for p-parabolic free boundary problems. We note that the parabolic obstacle problems for linear Kolmogorov type operators have been treated in [15, 37], papers where blow-up arguments of the type developed here are also used.

1.2. **A removability theorem for weak solutions.** Using Theorem 1.1 we are able to establish sharp removability conditions for sets in the context of weak solutions. Recall the notion of cylinders introduced in (1.5). Given a concave modulus of continuity $\omega(\cdot)$ as introduced in the previous section, we define a Hausdorff measure as follows. We let, for fixed δ, $0<\delta<r_0$ and $E\subset\mathbb{R}^{n+1}$, $L(\delta,\omega(\cdot);E)=\{Q^{\omega(r_i)}_{r_i}(x_i,t_i)\}$ be a family of cylinders such that $E\subseteq\bigcup Q^{\omega(r_i)}_{r_i}(x_i,t_i)$ and $0<r_i<\delta$ for $i=1,2,...$ Using this notation we let

$$\mathcal{H}^{\omega(\cdot)}(E):=\lim_{\delta\downarrow 0}\inf_{L(\delta,\omega(\cdot);E)}\left\{\sum r_i^n\omega(r_i)\ :\ E\subseteq\bigcup Q^{\omega(r_i)}_{r_i}(x_i,t_i)\right\}, \tag{1.10}$$

where the infimum is taken with respect to all possible coverings $L(\delta,\omega(\cdot);E)$ of E.

We prove the following result.

Theorem 1.3 (Removable singularities)**.** *Let $\mathcal{O}\subset\mathbb{R}^{n+1}$ be a cylindrical domain and let $E\subset\mathcal{O}$ be a closed set. Let $a\colon\mathbb{R}^n\to\mathbb{R}^n$ be a vector field satisfying* (1.2)*. Assume that u is a weak solution to*

$$u_t-\operatorname{div}a(Du)=0\qquad in\ \ \mathcal{O}\setminus E$$

and that $u\in\widetilde{C}^{\omega(\cdot)}_{\rm loc}(\mathcal{O})$. Assume also that $\mathcal{H}^{\omega(\cdot)}(E)=0$. Then the set E is removable, i.e., u can be extended to be a weak solution in $\mathcal{O}$.

Note that through the definition in (1.10) different $\omega(\cdot)$ correspond to different choices of Hausdorff measures (and dimensions) related to gauge functions and metrics, see, for instance, [38] for the basics on Hausdorff measures. The peculiarity, in our case, is that every time we define $\mathcal{H}^{\omega(\cdot)}$ we consider a metric - or, equivalently, the cylinders used for the coverings - and a gauge function that relate to each other. To get a closer comparison to the situation where standard parabolic Hausdorff measures are used we note that in the case $\omega(r)=r^\alpha$, $\alpha\in(0,1]$, one is led to the Hausdorff measures

$$\mathcal{H}^\sigma_\alpha(E):=\lim_{\delta\downarrow 0}\inf_{L(\delta,r^\alpha;E)}\left\{\sum r_i^\sigma\ :\ E\subseteq\bigcup Q^{r_i^\alpha}_{r_i}(x_i,t_i)\right\}.$$

In this case the Lebesgue measure of the cylinder $Q_{r_i}^{r_i^{\alpha}}(x_i, t_i)$ used in the covering of E is essentially $r_i^{n+\alpha(2-p)+p}$ and the assumption in Theorem 1.3 on the set E is that $\mathcal{H}_{\alpha}^{n+\alpha}(E) = 0$. The "Lipschitz" case of the previous result, that is when $\alpha = 1$, amounts to consider the standard parabolic Hausdorff measure

$$\mathcal{H}^{\sigma}(E) := \lim_{\delta \downarrow 0} \inf_{L(\delta,r;E)} \left\{ \sum r_i^{\sigma} \ : \ E \subseteq \bigcup B(x_i, r_i) \times (t_i - r_i^2, t_i + r_i^2) \right\}.$$

In this case we highlight the following corollary to Theorem 1.3.

Corollary 1.3. *Let $\mathcal{O}$, E, and a be as in Theorem 1.3. Assume that u is a weak solution to*

$$u_t - \operatorname{div} a(Du) = 0 \qquad \text{in } \ \mathcal{O} \setminus E$$

and that $u \in \widetilde{C}_{\text{loc}}^{\omega(\cdot)}(\mathcal{O})$ with $\omega(r) = r$, $r \geq 0$. Let $N = n + 2$ and assume that $\mathcal{H}^{N-1}(E) = 0$. Then the set E is removable, i.e., u can be extended to be a weak solution in $\mathcal{O}$.

Note that $N = n + 2$ is the standard parabolic dimension. Corollary 1.3 is the optimal parabolic analog of a series of results known in the elliptic case and we recall that Carleson [8] was the first to prove that a sufficient condition for a set $E \subset \mathbb{R}^n$ to be removable with respect to a Lipschitz harmonic function is that $\mathcal{H}^{n-1}(E) = 0$. Generalization of this result to the nonlinear setting of operators of p-Laplace type first came with the fundamental work of Serrin [40], while more recent work under assumptions of Lipschitz and Hölder continuity of the solution can be found in [9, 18, 24].

1.3. **Interpolative intrinsic geometries.** In this section we want to briefly describe some new technical inputs of the paper, and, in particular, something we are going to call *interpolative intrinsic geometries*. To better explain the situation let us here concentrate on the model case in (1.4). As mentioned in Section 1.1, a crucial ingredient in the regularity theory for the equation in (1.4) is the use of DiBenedetto's intrinsic geometry when deriving local estimates. This amounts to use cylinders whose size depends on the solution itself. As already mentioned earlier, this is necessary since equations as the one considered in (1.4) show a strong anisotropy when $p \neq 2$ as the multiplication of a solution by a constant does not yield a solution to a similar equation. One consequence of this is the lack of *homogeneous* a priori estimates and the hence the impossibility to use such estimates in iterative schemes in line with the standard regularity techniques. Instead, the lack of homogeneity must be locally corrected by using intrinsic geometries.

A first point we want to emphasize here is that the type of cylinders used must depend on the type of regularity one is currently using. Let us first explain this fact by analyzing the type of local geometry used in the case one is interested in proving gradient regularity starting from higher integrability of the gradient [12, 11, 21, 1, 3]. In this case the relevant cylinders are

$$\widetilde{Q}_{r,-}^{\lambda} := B(x_0, r) \times (t_0 - \lambda^{2-p} r^2, t_0) \quad \text{with} \quad \left(\fint_{\widetilde{Q}_{r,-}^{\lambda}} |Du|^p \, dx \, dt \right)^{1/p} \approx \lambda \,. \tag{1.11}$$

The last relation encodes the fact that on $\widetilde{Q}_{r,-}^{\lambda}$ we have, in some integral sense, $|Du| \approx \lambda$. The heuristic approach now proceeds as follows. On $\widetilde{Q}_{r,-}^{\lambda}$ we formally identify

$$u_t - \operatorname{div}(|Du|^{p-2} Du) \approx u_t - \lambda^{p-2} \triangle u \,.$$

Therefore, with this heuristics $v(x,t) := u(x_0 + rx, t_0 + \lambda^{2-p} r^2 t)$ solves the heat equation $v_t - \triangle v = 0$ in $B(0,1) \times (-1, 0)$ and *homogenous* estimates can be derived which are suitable for regularity procedures. While making this opaque and rough argument rigorous is far from being trivial, the point we want to emphasize here is that for this procedure to work, along the iteration, *the gradient must remain bounded.* In other words, *the type of intrinsic geometry considered depends on the kind of regularity one is proving.* Exactly for this reason, when proving regularity results for u (see for instance [10, 13, 14, 27]), rather than for Du, one is led to use the geometry dictated by

$$Q_{r,-}^{\lambda} := B(x_0,r)\times(t_0-\lambda^{2-p}r^p,t_0) \quad \text{with} \quad \operatorname*{osc}_{Q_{r,-}^{\lambda}} u\approx\lambda\,. \tag{1.12}$$

Our starting observation is that the two geometries considered in (1.11)-(1.12) are two particular, actually extremal, cases of a class of intermediate/interpolative intrinsic geometries, suited to the regularity we want to prove. In this paper we are interested in proving, as for instance in Theorem 1.1, the continuity of solutions with a certain degree of smoothness given by $\omega(\cdot)$. In analogy with (1.11), let's now use a heuristic "dimensional analysis" on an ansatz of the cylinders of the form

$$Q := B(x_0,r)\times(t_0-\lambda^{2-p}h(r),t_0)\,,$$

for some function $h(r)$ to be choose in order to have the equation $u_t-\operatorname{div}(|Du|^{p-2}Du)=0$ behaving as the heat equation in Q. The number λ *is this time controlled, along the iteration, by the type of regularity we are bound to prove.* More precisely, assuming that the following quantity is under control

$$\lambda\approx\frac{\operatorname{osc}_Q u}{\omega(r)}\approx\left[\frac{r}{\omega(r)}\right]|Du|\,,$$

we then formally identify

$$u_t-\operatorname{div}(|Du|^{p-2}Du)\approx u_t-\lambda^{p-2}\left[\frac{r}{\omega(r)}\right]^{2-p}\triangle u \quad \text{in } Q\,.$$

Let us now set $v(x,t) := u(x_0+rx,t_0+\lambda^{2-p}h(r)t)$ for $(x,t)\in B(0,1)\times(-1,0)$. We then have, formally, that the choice $h(r)=[\omega(r)]^{2-p}r^p$ gives $v_t-\triangle v=0$. This ansatz, that is

$$Q_{r,-}^{\lambda\omega(r)} := B(x_0,r)\times(t_0-\lambda^{2-p}[\omega(r)]^{2-p}r^p,t_0) \quad \text{with} \quad \lambda\approx\frac{\operatorname{osc}_{Q_{r,-}^{\lambda\omega(r)}} u}{\omega(r)}\,, \tag{1.13}$$

reveals to be the correct one in order to treat the $\widetilde{C}^{\omega(\cdot)}$ regularity of solutions to the obstacle problem via a particularly neat blow-up technique (see Section 4). The geometry in (1.13) formally gives back either (1.11) or (1.12) by taking $\omega(r)=r$ or $\omega(r)\equiv 1$, respectively (the last one conceived as a limit case of $\omega(r)=r^\alpha$ as $\alpha\to 0$). In exactly the same way, the limiting cases of the parabolic metric used in (1.8) are, in the case $\alpha=1$

$$\|(x_1,t_1)-(x_2,t_2)\|_1=\max\left\{|x_1-x_2|,|t_1-t_2|^{1/2}\right\}$$

and this the usual parabolic metric used to study the regularity of the gradient, and, when $\alpha\to 0$

$$\|(x_1,t_1)-(x_2,t_2)\|_0=\max\left\{|x_1-x_2|,|t_1-t_2|^{1/p}\right\}\,,$$

which is instead the metric that turns out to be relevant in the study of Hölder continuity of solutions (see again [11]).

Although the proofs used in this paper to establish the main results concerning the obstacle problem are indirect, we could also proceed in a more direct fashion. Indeed, one can use a sequence of shrinking cylinders of the type $Q_{r_i,-}^{\lambda\omega(r_i)}$, where

$r_i \to 0$ geometrically, and such that the specific form of the geometry allows one to ensure, at each stage, that a bound of the form

$$\frac{\operatorname{osc}_{Q_{r_i,-}^{\lambda\omega(r_i)}} u}{\omega(r_i)} \lesssim \lambda$$

holds. This argument can be made rigorous giving the desired regularity of u.

We conclude by remarking that the implementation of the regularization procedure put forward in this paper requires a delicate combinations of several ingredients from regularity theory of degenerate parabolic equations. In particular, once the interpolative intrinsic geometry is adopted as the right set-up for the blow-up procedure, the proofs heavily use bounds and oscillation estimates for the gradient of solutions to p-parabolic equations (initially developed by DiBenedetto [11] and in the formulation used essentially taken from [29]), certain Gaussian estimates for solutions to nonlinear parabolic equations with linear growth and of the type considered by Moser [36], and recent Harnack inequalities for degenerate parabolic equations developed in [13, 14] and [27]. Certain trace estimates established in [31] also turn out to be important when dealing with regularity at the initial state.

2. Existence theory and preliminaries

2.1. **Concept of solutions.** If $U \subset \mathbb{R}^n$ is open and $1 \le q \le \infty$, then by $W^{1,q}(U)$, we denote the space of equivalence classes of functions f with distributional gradient $Df = (f_{x_1}, \dots, f_{x_n})$, both of which are q-th power integrable on U. Let

$$\|f\|_{W^{1,q}(U)} = \|f\|_{L^q(U)} + \| \, |Df| \, \|_{L^q(U)}$$

be the norm in $W^{1,q}(U)$ where $\| \cdot \|_{L^q(U)}$ denotes the usual Lebesgue q-norm in U. Given $t_1 < t_2$ we denote by $L^q(t_1, t_2, W^{1,q}(U))$ the space of functions such that for almost every t, $t_1 \le t \le t_2$, the function $x \to u(x,t)$ belongs to $W^{1,q}(U)$ and

$$\|u\|_{L^q(t_1,t_2,W^{1,q}(U))} := \left(\int_{t_1}^{t_2} \int_U \left(|u(x,t)|^q + |Du(x,t)|^q \right) dx dt \right)^{1/q} < \infty.$$

In the following we here first describe the concept of weak solutions to

$$-Hw = w_t - \operatorname{div} a(Dw) = 0 \tag{2.1}$$

when the underlying domain considered is not necessarily a cylinder.

Definition 1. Let H be as in (2.1) and assume (1.2). We say that a function w is a weak supersolution (subsolution) to (2.1) in an open set $\Xi \Subset \mathbb{R}^{n+1}$ if, whenever $\Xi' = U \times (t_1, t_2) \Subset \Xi$ with $U \subset \mathbb{R}^n$ and $t_1 < t_2$, then $w \in L^p(t_1, t_2; W^{1,p}(U))$ and

$$\int_{\Xi'} \left(\langle a(Dw), D\phi \rangle - w\phi_t \right) \, dx \, dt \ \ge \ (\le) \ 0 \tag{2.2}$$

for all nonnegative $\phi \in C_0^\infty(\Xi')$. A weak solution is a distributional solution satisfying (2.2) with equality and without sign restrictions for the test functions.

Note, in particular, that in Definition 1 no assumption on the time derivative of w is made. We are now ready to give the definition of solutions to the obstacle problem. In the following we assume that the obstacle ψ and boundary value function b are continuous on $\bar{\mathcal{O}}$ and that $b \ge \psi$ on the parabolic boundary of $\mathcal{O} = \Omega \times (0,T)$.

Definition 2. A function u is a solution to (1.3) if it satisfies the following three properties:

(i) u is continuous on $\bar{\mathcal{O}}$, $u \ge \psi$ in $\mathcal{O}$ and $u = b$ on $\partial_{\mathrm{P}}\mathcal{O}$,
(ii) u is a weak supersolution in $\mathcal{O}$,

(iii) u is a weak solution in $\mathcal{O} \cap \{u > \psi\}$.

As for the property (iii), we recall that u is a weak solution in $\mathcal{O} \cap \{u > \psi\}$ means that u is a standard distributional solution in the sense of Definition 1 in every space-time cylinder contained in $\mathcal{O} \cap \{u > \psi\}$. We note that a solution to the obstacle problem as in Definition 2 exists by the results in [26]. To be precise, in [26] the boundary values were given by the obstacle itself but it is straightforward to modify the argument in [26] to obtain the existence result for general boundary values assuming $b \geq \psi$ on the parabolic boundary. Moreover, the solution is easily seen to be unique by an "elliptic" comparison principle for weak solutions, see Lemma 2.1 below. There are naturally other ways to obtain existence. An argument arising from potential theory is given in [33] and by uniqueness arguments this solution coincides with the solution obtained in [26]. In fact, from [33] one finds an argument for an existence result when the obstacle belongs merely to a parabolic Sobolev space. If the obstacle, on the other hand, belongs to parabolic Sobolev space and has time derivative in L^2, then the existence follows from [2] and by an approximation argument this approach can be used to obtain the unique solution also in the case when the obstacle is merely continuous; related existence results under regularity assumptions on the obstacle, such as in the existence of ψ_t in suitable Lebesgue spaces, can be found in [3]. Furthermore, an additional approach is given by viscosity solutions in which case the existence is rather easy to obtain. It turns out that a viscosity solution to the obstacle problem is also a so-called a-superparabolic function in $\mathcal{O}$, see [22, 25, 20, 19], and a continuous weak solution in $\mathcal{O} \cap \{u > \psi\}$. Every bounded superparabolic function is also a weak supersolution by [22, 25] and therefore any viscosity solution is a solution in the above sense and hence unique.

Concerning the notion of solution considered above we will several time use the following very useful result (see for example [22] and [25, Corollary 4.6]).

Lemma 2.1 ("Elliptic comparison"). *Let $S \subset \mathbb{R}^{n+1}$ be an open and bounded set and let $T \in \mathbb{R}$. Let $S_T := S \cap \{t < T\}$. Let u be a weak supersolution in S_T and let v be a weak subsolution in S_T. Assume further that u and v are continuous on the closure of S_T. If $v \leq u$ on $\partial S_T \setminus \{t = T\}$, then $v \leq u$ in S_T.*

The strength of the previous result is that it allows for a comparison principle also in non-cylindrical domains (recall here that ∂S denotes the usual topological boundary of S).

2.2. **Notation.** In this paper, following a standard notation, we let

$$(f)_A := \fint_A f(x,t)\,dx\,dt = \frac{1}{|A|}\int_A f(x,t)\,dx\,dt$$

whenever $A \subset \mathbb{R}^{n+1}$ has positive measure and f is a measurable function defined on A. Moreover, we shall denote by c a generic constant, always larger than one, whose value might change from line to line. Relevant dependence upon parameters will be displayed in parentheses. Given $(x,t) \in \mathbb{R}^{n+1}$ and $r, \lambda, \sigma > 0$, we introduce the space-time cylinders

$$\sigma Q_r^\lambda(x,t) := B(x,\sigma r) \times (t - \lambda^{2-p}(\sigma r)^p, t + \lambda^{2-p}(\sigma r)^p)\,,$$

$$\sigma Q_{r,-}^\lambda(x,t) := B(x,\sigma r) \times (t - \lambda^{2-p}(\sigma r)^p, t)\,, \tag{2.3}$$

$$\sigma Q_{r,+}^\lambda(x,t) := B(x,\sigma r) \times (t, t + \lambda^{2-p}(\sigma r)^p)\,. \tag{2.4}$$

In a context where the dependence on (x,t) is not important we will often write σQ_r^λ, $\sigma Q_{r,-}^\lambda$, $\sigma Q_{r,+}^\lambda$ for $\sigma Q_r^\lambda(x,t)$, $\sigma Q_{r,-}^\lambda(x,t)$, $\sigma Q_{r,+}^\lambda(x,t)$, respectively. Furthermore, if $\sigma = 1$ we will often simply write Q_r^λ, $Q_{r,\pm}^\lambda$ for the cylinders $1Q_r^\lambda(x,t)$, $1Q_{r,\pm}^\lambda(x,t)$, respectively. To avoid a too cumbersome notation we by $\bar{Q}_{R,+}^{\lambda\omega(R)}(x_0,t_0)$ ($\bar{Q}_{R,+}^{\lambda\omega(R)}$) denote the closure of $Q_{R,+}^{\lambda\omega(R)}(x_0,t_0)$ and so on in order to avoid the use of too long bars. Thoroughout the paper we let $\omega:\mathbb{R}_+\mapsto\mathbb{R}_+$ be a nondecreasing concave function such that $\omega(1)=1$ and $\omega(0):=\lim_{r\downarrow 0}\omega(r)=0$. Concavity of $\omega(\cdot)$ implies that

$$\frac{\omega(r)}{r}\leq\frac{\omega(\varrho)}{\varrho}\qquad\text{whenever } 0<\varrho<r. \tag{2.5}$$

Cylinders we are often considering relate to $\omega(\cdot)$ and they are of the form $Q_r^{\lambda\omega(r)}$. Usually λ will be chosen to reflect the intrinsic behavior of the problem and the goal will be to establish intrinsic relations of the type

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}} u\leq\lambda\omega(r)$$

when u is a given function in $Q_r^{\lambda\omega(r)}$. We can use (2.5) to obtain

$$\omega(\varrho)^{2-p}\varrho^p=\left(\frac{\omega(r)}{\omega(\varrho)}\frac{\varrho}{r}\right)^{p-2}\omega(r)^{2-p}r^{p-2}\varrho^2\leq\left(\frac{\varrho}{r}\right)^2\omega(r)^{2-p}r^p \tag{2.6}$$

whenever $0<\varrho<r$. It readily follows that

$$(r/\varrho)^{2/p}\,Q_\varrho^{\lambda\omega(\varrho)}\subset Q_r^{\lambda\omega(r)}\qquad\text{for every } \varrho\in(0,r)\,.$$

Moreover, applying again (2.5), we get

$$\omega(r)^{2-p}\varrho^p=\left(\frac{\omega(\varrho)}{\omega(r)}\right)^{p-2}\omega(\varrho)^{2-p}\varrho^p\geq\left(\frac{\varrho}{r}\right)^{p-2}\omega(\varrho)^{2-p}\varrho^p\,,$$

in turn implying the following inclusion:

$$(\varrho/r)^{(p-2)/p}\,Q_\varrho^{\lambda\omega(\varrho)}\subset Q_\varrho^{\lambda\omega(r)}\qquad\text{for every } \varrho\in(0,r)\,. \tag{2.7}$$

3. A priori estimates for weak solutions

In this section we first consider scaling properties of (weak) solutions to the equation (2.1) and then collect a number of estimates involving the gradient of for (weak) solutions to the equation (2.1). Finally we establish certain refined Gaussian estimates for (weak) solutions to equations of p-parabolic type, with variable coefficients, but with linear growth.

3.1. **Scaling of solutions.** We here describe scaling properties of (weak) solutions to the equation (2.1). Let $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}$, $R>0$, and suppose that w solves (2.1) in either of the cylinders $Q_{R,\pm}^{\lambda\omega(R)}(x_0,t_0)$ introduced in (2.3), (2.4). Consider $r\leq R,\lambda>0$, and define

$$\widetilde{w}(x,t):=\frac{w(x_0+rx,t_0+(\lambda\omega(r))^{2-p}r^pt)}{\lambda\omega(r)} \tag{3.1}$$

$$\widetilde{a}(z):=\frac{a((\lambda\omega(r)/r)z)}{(\lambda\omega(r)/r)^{p-1}}\,,\qquad z\in\mathbb{R}^n\,. \tag{3.2}$$

Then $\widetilde{w}$ solves the equation $\widetilde{H}\widetilde{w}:=\widetilde{w}_t-\operatorname{div}\widetilde{a}(D\widetilde{w})=0$ in

$$Q_{R/r,\pm}^{\widetilde{\omega}(R/r)}\qquad\text{where}\qquad\widetilde{\omega}(\gamma):=\frac{\omega(\gamma r)}{\omega(r)}\ \text{ for } \gamma>0. \tag{3.3}$$

In particular, in the case $r=R$, we have that $\widetilde{w}$ is a solution in $Q_{1,\pm}^1$. The new vector field $\widetilde{a}(\cdot)$ satisfies bounds

$$\begin{cases}|\widetilde{a}(z)|+|\partial\widetilde{a}(z)|(|z|^2+\widetilde{s}^2)^{1/2}\leq L(|z|^2+\widetilde{s}^2)^{(p-1)/2}\\ \nu(|z|^2+\widetilde{s}^2)^{(p-2)/2}|\xi|^2\leq\langle\partial\widetilde{a}(z)\xi,\xi\rangle\,,\end{cases}\qquad\widetilde{s}:=\frac{sr}{\lambda\omega(r)}\,, \tag{3.4}$$

for all $z,\xi \in \mathbb{R}^n$. In particular, we remark that the assumptions in (1.2) imply the existence of $\bar{\nu} \in (0,1)$, and $c, \bar{L} \geq 1$, depending on n,p,ν,L, such that the following growth and coercivity assumptions do hold for every choice $z \in \mathbb{R}^n$:

$$|\tilde{a}(z)| \leq \bar{L}(|z|^2+\tilde{s}^2)^{(p-1)/2}\,, \qquad \langle \tilde{a}(z), z\rangle \geq \bar{\nu}|z|^p - c\tilde{s}^p\,. \tag{3.5}$$

3.2. **Gradient estimates.** The first auxiliary theorem stated below gives an estimate of the local supremum of the gradient in the form of a reverse Hölder inequality. In the case of the equation in (1.4) the estimate can be found in [11, Chapter 8, Theorem 5.1] and in the form suitable for the more general equations considered in this paper it can be retrieved by a small modification from [28, Theorem 5.1]. The second estimate below is a consequence of the first estimate, a simple covering argument and (2.7). We emphasize that when we in the following say that a constant only depend on n,p,ν,L, then the constant is, in particular, independent of s, the solution w considered and of the vector field $a(\cdot)$.

Theorem 3.1. *Suppose that w is a weak solution to* (2.1) *in $Q^{\lambda r}_{r,-}$ for some $r,\lambda>0$ and let $\varepsilon>0$ be a degree of freedom. Then there exists a constant $c_\varepsilon \geq 1$, depending only on n,p,ν,L,ε, such that*

$$\sup_{Q^{\lambda r}_{r/2,-}} |Dw| \leq \varepsilon\lambda + c_\varepsilon \lambda^{2-p} \fint_{Q^{\lambda r}_{r,-}} (|Dw|+s)^{p-1}\,dx\,dt$$

holds. In particular, if $\lambda = \widetilde{\lambda}\omega(r)/r$, for some $r,\tilde{\lambda}>0$, then

$$\sup_{Q^{\widetilde{\lambda}\omega(r/2)}_{r/2,-}} |Dw| \leq \varepsilon\widetilde{\lambda}\omega(r)/r + c_\varepsilon(\widetilde{\lambda}\omega(r)/r)^{2-p} \fint_{Q^{\widetilde{\lambda}\omega(r)}_{r,-}} (|Dw|+s)^{p-1}\,dx\,dt\,. \tag{3.6}$$

The next and fundamental regularity result was obtained by DiBenedetto and Friedman for evolutionary parabolic equations [12]. We refer to [29, Theorem 3.2] and [30, Theorem 3.2] for the scalar case and for more details.

Theorem 3.2. *Suppose that w is a weak solution to* (2.1) *in a space-time cylinder $\mathcal{O}$. Then Dw has the Hölder continuous representative in $\mathcal{O}$. Moreover, let $Q^{\lambda r}_{r,-} \subset \mathcal{O}$, for some $r,\lambda>0$ such that*

$$s + \sup_{Q^{\lambda r}_{r,-}} |Dw| \leq A\lambda$$

holds for a constant $A \geq 1$. Then there exists $\widetilde{\alpha} \equiv \widetilde{\alpha}(n,p,\nu,L,A) \in (0,1]$ such that

$$\operatorname*{osc}_{Q^{\lambda\varrho}_{\varrho,-}} Dw \leq 4A\lambda\left(\frac{\varrho}{r}\right)^{\widetilde{\alpha}}. \tag{3.7}$$

holds for all $\varrho \in (0,r)$. Here $Q^{\lambda\varrho}_{\varrho,-} \subset Q^{\lambda r}_{r,-}$, for $0<\rho\leq r$, is an intrinsic cylinder sharing its center with $Q^{\lambda r}_{r,-}$.

In the intrinsic geometry suited for the general modulus of continuity the above Hölder estimates takes the following form.

Corollary 3.1. *Let w be as in Theorem 3.2 with $\lambda = \widetilde{\lambda}\omega(r)/r$, for some $r,\widetilde{\lambda}>0$. Then*

$$\operatorname*{osc}_{Q^{\widetilde{\lambda}\omega(\varrho)}_{\varrho,-}} Dw \leq 4A\widetilde{\lambda}\frac{\omega(r)}{r}\left(\frac{\varrho}{r}\right)^{\widetilde{\alpha}}$$

holds for all $\varrho \in (0,r)$ and with $\tilde{\alpha}$ as in Theorem 3.2.

Proof. Applying (3.7) we obtain

$$\operatorname*{osc}_{Q_{\varrho,-}^{(\widetilde{\lambda}\omega(r)/r)\varrho}} Dw \le 4A\widetilde{\lambda}\frac{\omega(r)}{r}\left(\frac{\varrho}{r}\right)^{\widetilde{\alpha}}$$

for all $\varrho \in (0,r)$. Using (2.5) we have

$$\left(\widetilde{\lambda}\frac{\omega(r)}{r}\varrho\right)^{2-p}\varrho^p = \widetilde{\lambda}^{2-p}\left(\frac{\omega(\varrho)}{\omega(r)}\frac{r}{\varrho}\right)^{p-2}\omega(\varrho)^{2-p}\varrho^p \ge \left[\widetilde{\lambda}\omega(\varrho)\right]^{2-p}\varrho^p\,.$$

Hence $Q_{\varrho,-}^{\widetilde{\lambda}\omega(\varrho)} \subset Q_{\varrho,-}^{(\widetilde{\lambda}\omega(r)/r)\varrho}$, and the proof is complete. □

3.3. **Energy and zero order estimates.** The following Harnack estimate can be retrieved from [13, 14] and [27].

Theorem 3.3. *Suppose that w is a nonnegative weak solution to* (2.1) *in a space-time cylinder $\mathcal{O}$. There are constants $c_i \equiv c_i(n,p,\nu,L)$, $i \in \{1,2\}$, such that if*

$$B(x_0,2r)\times(t_0 - c_1 w(x_0,t_0)^{2-p}r^p, t_0 + c_1 w(x_0,t_0)^{2-p}r^p) \Subset \mathcal{O}\,,$$

then

$$w(x_0,t_0) \le c_2\left(\inf_{x\in B(x_0,r)} w\left(x, t_0 + c_1 w(x_0,t_0)^{2-p}r^p\right) + sr\right)\,.$$

The next result is a standard energy estimate applied in $Q_{r,-}^{\lambda\omega(r)}(x_0,t_0)$ (see [11, Proposition 3.1, Chapter 2]), together with an L^∞ bound for the solution which can be inferred from [11, Theorem 4.1, Chapter 5], with some small variants.

Lemma 3.1. *Suppose that w is a nonnegative weak subsolution to* (2.1) *in $Q_r \equiv Q_{r,-}^{\lambda\omega(r)}(x_0,t_0)$. Then there exists a constant $c \equiv c(n,p,\nu,L)$ such that*

$$\begin{aligned}\int_{Q_{r/2}} |Dw|^p\,dx\,dt + \sup_{t_0-(\lambda\omega(r/2))^{2-p}(r/2)^p<t<t_0}\int_{B(x_0,r/2)} w^2\,dx \\ \le \frac{c}{r^p}\int_{Q_r}\left[w^p + (\omega(r)\lambda)^{p-2}w^2 + (rs)^p\right]dx\,dt\end{aligned} \tag{3.8}$$

holds. Furthermore, let $\varepsilon > 0$ be a degree of freedom. Then there exists a constant $c_\varepsilon \ge 1$, depending only on n,p,ν,L,ε, such that

$$\sup_{Q_{r/2}} w \le \varepsilon\omega(r)\lambda + c_\varepsilon(\omega(r)\lambda)^{2-p}\fint_{Q_r} w^{p-1}\,dx\,dt + c_\varepsilon rs\,. \tag{3.9}$$

The parameter $s \ge 0$ has been introduced in (1.2).

The following lemma is a consequence of of De Giorgi's iteration at the initial state, see [31] for related results.

Lemma 3.2. *Let $R,T > 0$ be given and let w be a continuous nonnegative weak subsolution to $Hw = 0$ in $Q := B(x_0,R)\times(t_0,t_0+T)$. Suppose further that w attains locally continuously zero initial values at t_0, i.e., on any set of the form $B(x_0,r)\times\{t_0\}$, for some $r < R$. Then there exists a constant $c \equiv c(n,p,\nu,L)$ such that*

$$\sup_{B(x_0,R/2)\times(t_0,t_0+T)} w \le c\frac{T}{R^p}\fint_Q w^{p-1}\,dx\,dt + csR\,. \tag{3.10}$$

The parameter $s \ge 0$ has been introduced in (1.2).

Proof. We can assume, without loss of generality, that $t_0 = 0$. To start the proof we set

$$\begin{aligned} & r_i := R(1+2^{-i})/2,\ B_i := B(0,r_i),\ Q_i := B_i \times (0,T), \\ & k_i := K(1-2^{-i}), \end{aligned} \tag{3.11}$$

for some $K \geq \varepsilon + sR$, $\varepsilon > 0$, and for all $i = 0, 1, \ldots$. Note that $k_{i+1} - k_i = 2^{-i-1}K$ and that ε is a degree of freedom to be chosen eventually. Let, for $i = 0, 1, \ldots$, $\phi_i \in C_0^\infty(B_i)$ be such that $0 \leq \phi_i \leq 1$, $\phi_i \equiv 1$ on B_{i+1} and $|D\phi_i| \leq c2^i R$. Let $0 < \tau < T$ and consider the function $(w-k_i)_+\phi_i^p \chi_{\{t<\tau\}}$ where, we recall the standard notation

$$(w-k)_+ = \max\{w-k, 0\} \qquad k \in \mathbb{R}$$

and $\chi_{\{t<\tau\}}$ is the indicator function for the set $\{t < \tau\}$. Recall the definition of weak subsolutions in Definition 1 and note that a regularized, in time, version of $(w-k_i)_+\phi_i^p\chi_{\{t<\tau\}}$ can be used as a test function in the definition of w being a subsolution to (2.1) in Q. In particular, since $(w-k_i)_+\phi_i^p\chi_{\{t<\tau\}}$ vanishes on the lateral boundary and at times 0 and τ it can be made an admissible test function after a standard regularization in time via convolutions or Steklov averages. Using these facts, appealing to the Caccioppoli inequality (the one used in [31, Remark 3.10] with $\varepsilon = 1$), and finally averaging, we see that

$$\begin{aligned} & \fint_{Q_i} |D((w-k_i)_+\phi_i)|^p \, dx\, dt + \frac{1}{T} \sup_{0<t<T} \fint_{B_i} (w-k_i)_+^2 \phi_i^p \, dx \\ & \quad \leq c \fint_{Q_i} \left((w-k_i)_+^p |D\phi_i|^p + s^p \chi_{\{w>k_i\}} \phi_i^p \right) dx\, dt \end{aligned}$$

whenever $i \geq 0$. Also notice that whenever w is a weak subsolution, $(w-k)_+$ is also a weak subsolution. Furthermore, using that

$$s^p \chi_{\{w>k_i\}} \leq s^p \frac{(w-k_{i-1})_+^p}{(k_i-k_{i-1})^p} = 2^{ip}\left(\frac{s}{K}\right)^p (w-k_{i-1})_+^p \leq \frac{2^{(i+1)p}}{r_i^p}(w-k_{i-1})_+^p\,,$$

where again $\chi_{\{w>k_i\}}$ denotes the indicator function of the set $\{w > k_i\}$, we can put the estimates of the last to displays together and conclude that

$$\begin{aligned} & \fint_{Q_i} |D((w-k_i)_+\phi_i)|^p \, dx\, dt + \frac{1}{T} \sup_{0<t<T} \fint_{B_i} (w-k_i)_+^2 \phi_i^p \, dx \\ & \quad \leq c \frac{2^{ip}}{r_i^p} \fint_{Q_{i-1}} (w-k_{i-1})_+^p \, dx\, dt, \end{aligned} \tag{3.12}$$

whenever $i \geq 1$. Let $\theta_i := (w-k_i)_+$ for $i = 0, 1, \ldots$ Then, using Hölder's inequality, Sobolev's embedding and finally Young's inequality, we deduce that

$$\begin{aligned} \fint_{Q_{i+1}} \theta_i^{p+2/n} \, dx\, dt & \leq \frac{1}{T}\int_0^T \left(\fint_{B_{i+1}} \theta_i^2 \phi_i^p \, dx \right)^{1/n} \left(\fint_{B_{i+1}} (\theta_i\phi_i)^{\frac{np}{n-1}} \, dx \right)^{\frac{n-1}{n}} dt \\ & \leq c \left(\frac{T}{R^p}\right)^{1/n} \left(\frac{r_i^p}{T} \sup_{0<t<T} \fint_{B_i} \theta_i^2 \phi_i^p \, dx \right)^{1/n} r_i^p \fint_{Q_i} |D(\theta_i\phi_i)|^p \, dx\, dt \\ & \leq c \left(\frac{T}{R^p}\right)^{1/n} \left(\frac{r_i^p}{T} \sup_{0<t<T} \fint_{B_i} \theta_i^2 \phi_i^p \, dx + r_i^p \fint_{Q_i} |D(\theta_i\phi_i)|^p \, dx\, dt \right)^{1+1/n} . \end{aligned} \tag{3.13}$$

We recall that the form of Sobolev embedding used in the lines above is the following one:

$$\left(\fint_{B_i} |v|^{\frac{np}{n-1}} \, dx \right)^{\frac{n-1}{np}} \leq c r_i \left(\fint_{B_i} |Dv|^p \, dx \right)^{\frac{1}{p}} \tag{3.14}$$

which works whenever $v \in W_0^{1,p}(B_i)$, and we applied slicewise with the choice $v = \theta_i\phi_i$. Notice in (3.14) the constant c is stable as long as p varies in a compact subset of $(1,\infty)$. The inequality in display (3.14) standardly follows by the usual Sobolev inequality for functions in $W^{1,1}$ together with Hölder's inequality.

Combining the content of displays (3.12)-(3.13), we can conclude that

$$\begin{aligned} &\fint_{Q_{i+1}} (w-k_i)_+^{p+2/n}\,dx\,dt \\ &\leq c2^{ip(1+1/n)}\left(\frac{T}{R^p}\right)^{1/n}\left[\fint_{Q_{i-1}} (w-k_{i-1})_+^p\,dx\,dt\right]^{1+1/n}, \end{aligned} \tag{3.15}$$

for $i = 1, \ldots$. Let

$$Y_i := \fint_{Q_{2i}} (w-k_{2i})_+^p\,dx\,dt.$$

Then, using (3.15) and noting that

$$\begin{aligned} (w-k_i)_+^{p+2/n} &\geq (w-k_{i+1})_+^p(k_{i+1}-k_i)_+^{2/n} \\ &\geq 2^{-2(i+1)/n}K^{2/n}(w-k_{i+1})_+^p \end{aligned}$$

we see that

$$Y_{i+1} \leq \tilde{c}2^{ip(1+3/n)}\left(\frac{T}{K^2R^p}\right)^{1/n} Y_i^{1+1/n}, \qquad i = 0,1,2,\ldots,$$

and for a constant $\tilde{c} \equiv \tilde{c}(n,p,\nu,L)$. Then, combing this with a standard iteration argument which dates back to De Giorgi (see for instance [17, Lemma 7.1]), it follows that if

$$Y_0 \leq \frac{1}{\tilde{c}^n 2^{p(1+3/n)n^2}}\left(\frac{K^2R^p}{T}\right), \tag{3.16}$$

then $Y_i \to 0$ as $i \to \infty$. We now let

$$K := \varepsilon + sR, \quad \text{where} \quad \varepsilon := c\left(\frac{T}{R^p}\fint_Q w^p\,dx\,dt\right)^{1/2}, \tag{3.17}$$

and for a constant $c \equiv c(n,p,\nu,L)$. We note that we can assume, without loss of generality, that $\varepsilon > 0$. Then, by adjusting the constant c and using the specification in (3.17) we can ensure (3.16) and the condition $K \geq sR$. Putting everything together we arrive at

$$\sup_{\frac{1}{2}Q} w \leq c\left(\frac{T}{R^p}\fint_Q w^p\,dx\,dt\right)^{1/2} + sR.$$

The same argument as above can also be used to interpolate between different cubes. Specifically, let $\sigma Q = B(x_0,\sigma R)\times(t_0,t_0+T)$, $\sigma > 0$, fix σ', $1/2 \leq \sigma' < \sigma \leq 1$, fix $r_i := [\sigma' + (\sigma-\sigma')/2^i]R$ and define, accordingly, the cut-off functions $\phi_i \in C_0^\infty(B_i)$. The final outcome of the argument above is that

$$\sup_{\sigma' Q} w \leq c\left(\frac{1}{(\sigma-\sigma')^p}\frac{T}{R^p}\fint_{\sigma Q} w^p\,dx\,dt\right)^{1/2} + sR$$

with $c \equiv c(n,p,\nu,L)$. Extracting the supremum of w from the integral above, and using Young's inequality, it follows that

$$\sup_{\sigma' Q} w \leq \frac{1}{2}\sup_{\sigma Q} w + \frac{c}{(\sigma-\sigma')^p}\frac{T}{R^p}\fint_Q w^{p-1}\,dx\,dt + sR.$$

The proof of (3.10) now follows from the inequality in the last display by appealing to another standard iteration argument as outlined in [28, Lemma 5.1]. □

We close this section by proving an oscillation reduction result for weak solutions.

Lemma 3.3. *Let $R > 0$ and $M_1 > 0$. There exists a constant $c_{3.3} \equiv c_{3.3}(n, p, \nu, L)$ such that if*

$$\lambda \geq c_{3.3} \max\{M_1/\omega(R), sR/\omega(R)\}$$

and $w \in C^0(\bar{Q}^{\lambda\omega(R)}_{R,-})$ is a weak solution to (2.1) in $Q^{\lambda\omega(R)}_{R,-}$ satisfying an intrinsic relation

$$\operatorname*{osc}_{Q^{\lambda\omega(R)}_{R,-}} w \leq M_1 \,,$$

then

$$\operatorname*{osc}_{Q^{\lambda\omega(r)}_{r,-}} w \leq \lambda\omega(r) \qquad \textit{holds for every } r \in (0, R)\,, \tag{3.18}$$

where $Q^{\lambda\omega(r)}_{r,-} \equiv Q^{\lambda\omega(r)}_{r,-}(x_0, t_0)$ for $r \leq R$.

Proof. We assume that

$$\lambda \geq \delta^{-1} \max\{M_1/\omega(R), sR/\omega(R)\} > 0 \tag{3.19}$$

with $\delta \in (0, 1)$ to be chosen. Note that in the following all cylinders will share the same vertex and $Q^{\lambda\omega(r)}_{r,-} \equiv Q^{\lambda\omega(r)}_{r,-}(x_0, t_0)$ for $r \leq R$. We divide the proof into two steps.

Step 1: Bound for the sharp maximal function. Let $\varepsilon \in (0, 1)$ be a degree of freedom. With ε given, we in this step prove, for $\delta \in (0, \varepsilon]$ depending only on $n, p, \nu, L, \varepsilon$, that

$$E(r) := \frac{1}{\omega(r)} \left(\fint_{Q^{\lambda\omega(r)}_{r,-}} |w - (w)_{Q^{\lambda\omega(r)}_{r,-}}|^{p-1} \, dx \, dt \right)^{1/(p-1)} \leq c\varepsilon\lambda \tag{3.20}$$

whenever $r \in (0, R)$ and for a constant $c \equiv c(n, p, \nu, L)$. Note that from (2.5) and that $\delta \leq \varepsilon$ we immediately deduce that

$$\sup_{R/8 \leq r < R} E(r) \leq \frac{1}{\omega(R/8)} \operatorname*{osc}_{Q^{\lambda\omega(R)}_{R,-}} w \leq \frac{8M_1}{\omega(R)} \leq 8\varepsilon\lambda\,. \tag{3.21}$$

Hence we in the following only have to consider the case $r \in (0, R/8)$. To proceed we first note using the energy estimate, Lemma 3.1, and the fact that $\max\{\pm(w - w(x_0, t_0)), 0\}$ are nonnegative weak subsolutions, that

$$\begin{aligned}
\fint_{Q^{\lambda\omega(R/2)}_{R/2,-}} (|Dw| + s)^p \, dx \, dt &\leq c \left(\frac{M_1^p}{R^p} + \frac{M_1^2}{[\lambda\omega(R)]^{2-p} R^p} + s^p \right) \\
&\leq c \left[\left(\frac{M_1}{\lambda\omega(R)} \right)^p + \left(\frac{M_1}{\lambda\omega(R)} \right)^2 + \left(\frac{sR}{\lambda\omega(R)} \right)^p \right] \left(\lambda \frac{\omega(R)}{R} \right)^p \\
&\leq c_* \delta^2 \left(\lambda \frac{\omega(R)}{R} \right)^p
\end{aligned}$$

holds with $c_* \equiv c_*(n, p, \nu, L)$. Given $\varepsilon \in (0, 1)$, we let $c_{\varepsilon/2}$ be the constant in Theorem 3.1 corresponding to $\varepsilon/2$. We now want to ensure that $\delta \leq \varepsilon$ and that $c_{\varepsilon/2}(c_*\delta^2)^{(p-1)/p} \leq \varepsilon/2$. Hence we fix δ through the relation

$$\delta = \min\left\{ \left(\frac{1}{c_*} \left(\frac{\varepsilon}{2c_{\varepsilon/2}} \right)^{p/(p-1)} \right)^{1/2}, \frac{\varepsilon}{2} \right\}. \tag{3.22}$$

Then, using (3.6) of Theorem 3.1, Hölder's inequality and (3.22) we conclude that

$$\sup_{Q_{R/4,-}^{\lambda\omega(R/4)}} |Dw| \le \varepsilon\lambda \frac{\omega(R)}{R}\,. \tag{3.23}$$

Note that δ now only depends on $n, p, \nu, L, \varepsilon$, and in the second step of the proof we will fix ε. Using (3.23) and (2.5) we have

$$\sup_{Q_{r,-}^{\lambda\omega(r)}} |Dw| \le \varepsilon\lambda \frac{\omega(r)}{r}\left(\frac{r}{\omega(r)}\frac{\omega(R)}{R}\right) \le \varepsilon\lambda\frac{\omega(r)}{r} \qquad \text{for every } r \in (0, R/4)\,. \tag{3.24}$$

Now, as by standard regularity theory Dw is continuous in $Q_{R,-}^{\lambda\omega(R)}$, applying divergence theorem yields

$$\fint_{B(x_0,r)} w(x,t)\,dx\Big|_{t=t_1}^{t_2} = \frac{n}{r}\int_{t_1}^{t_2}\fint_{\partial B(x_0,r)} a(Dw(x,t))\cdot\frac{x-x_0}{|x-x_0|}\,d\mathcal{H}^{n-1}(x)\,dt$$

for all $t_0 - (\lambda\omega(r))^{2-p}r^p < t_1 < t_2 < t_0$, where $\mathcal{H}^{n-1}(x)$ denotes the $n-1$-dimensional Hausdorff measure. Applying then (3.24) together with the bound on $a(\cdot)$ we get

$$\begin{aligned} \operatorname*{osc}_{t_0-(\lambda\omega(r))^{2-p}r^p<t<t_0} \fint_{B(x_0,r)} w(x,t)\,dx &\le \frac{c(n,L)}{r}(\lambda\omega(r))^{2-p}r^p\left(\varepsilon\lambda\frac{\omega(r)}{r}+s\right)^{p-1} \\ &\le c\varepsilon^{p-1}\lambda\omega(r)\,. \end{aligned} \tag{3.25}$$

Here we also used the fact that $s \le \delta\lambda\omega(R)/R \le \varepsilon\lambda\omega(r)/r$. Set then

$$I(t) = \fint_{B(x_0,r)} w(x,t)\,dx \qquad\text{and}\qquad I = \fint_{Q_{r,-}^{\lambda\omega(r)}} w(x,t)\,dx\,dt\,.$$

On the one hand, from (3.25) we obtain that

$$\sup_{t_0-(\lambda\omega(r))^{2-p}r^p<t<t_0} |I - I(t)| \le c\varepsilon^{p-1}\lambda\omega(r)\,.$$

On the other hand, Poincaré's inequality and (3.24) yield

$$\fint_{B(x_0,r)} |w(x,t)-I(t)|^{p-1}\,dx \le cr^{p-1}\fint_{B(x_0,r)} |Dw(x,t)|^{p-1}\,dx \le c\,[\varepsilon\lambda\omega(r)]^{p-1}\,.$$

Combining above two displays, together with (3.21), proves (3.20).

Step 2: Continuity estimates. By (3.21), we need to prove (3.18) only for $r \in (0, R/8)$. Fix such $r \in (0, R/8)$ and recall that the functions $\max\{\pm(w-k), 0\}$ are both weak subsolutions whenever $k \in \mathbb{R}$. Hence, for any $\tilde\varepsilon \in (0,1)$, (3.9) gives

$$\begin{aligned} \sup_{Q_{r,-}^{\lambda\omega(r)}} |w-(w)_{Q_{r,-}^{\lambda\omega(r)}}| &\le \tilde\varepsilon\lambda\omega(2r) \\ &+ c_{\tilde\varepsilon}(\lambda\omega(2r))^{2-p}\fint_{Q_{2r,-}^{\lambda\omega(2r)}} |w-(w)_{Q_{r,-}^{\lambda\omega(r)}}|^{p-1}\,dx\,dt + crs \end{aligned} \tag{3.26}$$

and $c_{\tilde\varepsilon}$ depends only on $\tilde\varepsilon$ and n, p, ν, L. By the triangle inequality, (2.5) and (3.20) we see that

$$\begin{aligned} \fint_{Q_{2r,-}^{\lambda\omega(2r)}} |w-(w)_{Q_{r,-}^{\lambda\omega(r)}}|^{p-1}\,dx\,dt &\le c\fint_{Q_{2r,-}^{\lambda\omega(2r)}} |w-(w)_{Q_{2r,-}^{\lambda\omega(2r)}}|^{p-1}\,dx\,dt \\ &= c\,[E(2r)\omega(2r)]^{p-1} \le c\,[\varepsilon\lambda\omega(2r)]^{p-1}\,. \end{aligned} \tag{3.27}$$

Combining (3.26), (2.5) and (3.27) we conclude that

$$\sup_{Q_{r,-}^{\lambda\omega(r)}} |w-(w)_{Q_{r,-}^{\lambda\omega(r)}}| \le 2\tilde\varepsilon\lambda\omega(r) + c_{\tilde\varepsilon}\varepsilon\lambda\omega(r) + crs.$$

Furthermore, using (3.19) and (2.5) we have

$$rs = \frac{Rs}{\omega(R)} \frac{r\omega(R)}{R\omega(r)} \omega(r) \leq \varepsilon\lambda\omega(r)$$

by (3.22). By putting together the estimates in the last two displays we gain

$$\sup_{Q_{r,-}^{\lambda\omega(r)}} |w - (w)_{Q_{r,-}^{\lambda\omega(r)}}| \leq 2\tilde{\varepsilon}\lambda\omega(r) + c_{\tilde{\varepsilon}}\varepsilon\lambda\omega(r) + c\varepsilon\lambda\omega(r)\,. \tag{3.28}$$

We now let $\tilde{\varepsilon} = 1/8$ and then choose ε according to

$$\varepsilon = \min\{1/(8c_{\tilde{\varepsilon}}), 1/(8c)\}\,,$$

where c is the constant in the term to the right in (3.28). Hence, using (3.28) we see that

$$\begin{aligned} \operatorname*{osc}_{Q_{r,-}^{\lambda\omega(r)}} w &= \sup_{Q_r^{\lambda\omega(r)}(z)} \left(w - (w)_{Q_{r,-}^{\lambda\omega(r)}}\right) - \inf_{Q_{r,-}^{\lambda\omega(r)}} \left(w - (w)_{Q_{r,-}^{\lambda\omega(r)}(z)}\right) \\ &\leq \ 2 \sup_{Q_{r,-}^{\lambda\omega(r)}} |w - (w)_{Q_{r,-}^{\lambda\omega(r)}}| \leq \lambda\omega(r)\,. \end{aligned}$$

All in all, ε depends only on n, p, ν, L, and through (3.22) so does δ. This completes the proof of (3.18). □

3.4. **Gaussian estimates.** Here we consider general nonlinear equations, with linear growth and with measurable coefficients, of the type

$$v_t - \operatorname{div}\, \bar{a}(x,t,Dv) = 0 \tag{3.29}$$

under the assumptions that

$$|\bar{a}(x,t,z)| \leq \bar{L}(|z| + \bar{s})\,, \qquad \langle \bar{a}(x,t,z), z\rangle \geq \bar{\nu}|z|^2 - \bar{L}^2\bar{s}^2 \tag{3.30}$$

whenever $z \in \mathbb{R}^n$ and for almost every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Here $0 < \bar{\nu} \leq \bar{L}$ and $\bar{s} \geq 0$ are fixed parameters. Needless to say the vector field $\bar{a}\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be a Carathéodory function, i.e., measurable with respect to the first two variables and continuous with respect to the third parameter. We here prove the following lemma, which in some sense encodes the Gaussian behavior which is typical of weak solutions to parabolic equations with linear growth.

Lemma 3.4. *Suppose that v is a continuous weak solution to* (3.29) *in the cylinder $Q \equiv B(0,4) \times (-4,0)$ and assume that $0 \leq v \leq M$ in $Q \equiv B(0,4) \times (-4,0)$ for some $M \geq 4$. Then there exists $\delta \in (0,1)$, depending only on $n, \bar{\nu}, \bar{L}$ and M, such that*

$$v(0,0) + \bar{s} \leq \delta \qquad \Longrightarrow \qquad \sup_{B(0,1)\times(-1,0)} v \leq \frac{1}{4}\,.$$

The proof uses two basic ingredients: a specific Harnack estimate and the standard $L^\infty - L^1$ a priori estimate for solutions. We will use the following form of the Harnack estimate.

Theorem 3.4. *Let v be a nonnegative weak solution to* (3.29) *in the cylinder $B(0,2R)\times(-4(\varepsilon R)^2,0)$ for some $\varepsilon \in (0,1]$ and $R > 0$. Then there exists a constant $\bar{c} \equiv \bar{c}(n,\bar{\nu},\bar{L},\varepsilon)$, which is a decreasing function of ε, such that*

$$\sup_{B(0,R)\times(-3(\varepsilon R)^2,-2(\varepsilon R)^2)} v \leq \bar{c}\left(\inf_{B(0,R)\times(-(\varepsilon R)^2,0)} v + R\bar{s}\right)\,. \tag{3.31}$$

Remark 3.1. By a careful analysis of the proof of the parabolic Harnack's inequality for nonlinear equations, it can be proved that the constant $\bar{c}$ appearing in (3.31) is of the form $\bar{c} = c^{1/\varepsilon^2}$, with $c \equiv c(n,\nu,L)$. Let us add a few comments. Parabolic

Harnack inequalities are consequences of reverse Hölder inequalities for the supersolution u and for the subsolution u^{-1} applied together with the cross-over lemma of Moser [36]. The customary way to present the result is that the spatial variable scales as R and the time variable as R^2. A concise presentation can be found in the book of Saloff-Coste [39, Section 5]. The peculiar fact of Theorem 3.4 is the use of the time scale $(\varepsilon R)^2$ and that this causes an additional dependence on ε of the constant involved. In the linear case, the analysis of the fundamental solution reveals that the constant must grow like c^{1/ε^2} as $\varepsilon \to 0$ for some c depending only on $n, \bar{\nu}, \bar{L}$. This fact continues to hold for nonlinear equations as those in (3.29), (3.30). This can be verified by a careful tracking of the dependence of the constants in the proof of Moser [36] and by a modification of the argument based on coverings with space-times cylinders of the type $B(x,r) \times (t - \varepsilon^2 r^2, t)$ instead of the standard parabolic space-times cylinders.

Proof of Lemma 3.4. We first use the Harnack estimate in (3.31), with parameters $R = 2$ and $\varepsilon = \sqrt{-t/2}/2$, $t \in [-2, 0)$, to obtain

$$\sup_{x \in B(0,2)} v(x,t) \le \bar{c}(t)\,(v(0,0) + \bar{s}) \le \bar{c}(t)\delta \qquad \text{for} \quad -2 < t < 0\,.$$

Combining this last estimate with the standard $L^\infty - L^1$ a priori estimate

$$\sup_{B(0,1)\times(-1,0)} v \le c(n, \bar{\nu}, \bar{L}) \int_{-2}^{0} \int_{B(0,2)} (v + \bar{s})\, dx\, dt,$$

valid for nonnegative weak (sub)solutions, see (3.9), we arrive at

$$\begin{aligned} \sup_{B(0,1)\times(-1,0)} v \quad &\le \quad c\left(|B(0,2)|(\sigma M + 2\bar{s}) + \int_{-2}^{-\sigma} \int_{B(0,2)} v\, dx\, dt \right) \\ &\le \quad \tilde{c}(M\sigma + \delta + \bar{c}(\sigma)\delta) \end{aligned}$$

for any $\sigma \in (0,1)$. Above the constant $\tilde{c}$ depends only on $n, \bar{\nu}, \bar{L}$ and is larger than one. We have also employed the assumptions $v \le M$ and $\bar{s} \le \delta$. To conclude the proof, we choose $\sigma = 1/(16\tilde{c}M)$ and take $\delta \le \min\{1/\tilde{c}, 1/(\tilde{c}\bar{c}(\sigma))\}/16$. □

Remark 3.2. When referring to Lemma 3.4 and to the given solution v therein, assumptions (3.30) can be replaced by the following assumption to hold almost everywhere

$$\begin{aligned} &|\bar{a}(Dv(x,t))| \le \bar{L}(|Dv(x,t)| + s), \\ &\langle \bar{a}(Dv(x,t)), Dv(x,t) \rangle \ge \bar{\nu}|Dv(x,t)|^2 - \bar{L}^2 s^2\,. \end{aligned} \tag{3.32}$$

Indeed, we apply Lemma 3.4 under certain nondegeneracy condition for degenerate operators. Then the resulting vector field $\bar{a}$ does not satisfy (3.30) for all vectors $z \in \mathbb{R}^n$, but instead (3.32) for a particular solution v which is on the other hand the one of interest here.

4. Technical lemmas

In this section we give a stream of technical lemmas that, suitably combined, will eventually lead to the proof of the desired regularity results in the subsequent section. For this, we need to introduce a few classes of solutions. We recall that in the following H is any general operator of the type in display (1.1) with the vector field $a(\cdot)$ satisfying (1.2). In other words we are considering the whole class of operators determined by the structure parameters (n, p, ν, L) (usually referred to as "data"). We emphasize that we in this section, as throughout the paper, assume that $p \geq 2$. Moreover, in what follows we shall heavily use the intrinsic cylinders defined in Section 2.2. In particular, we shall use

$$Q_{r,-}^{\lambda\omega(r)} \equiv Q_{r,-}^{\lambda\omega(r)}(x_0,t_0) := B(x_0,r) \times (t_0 - \lambda^{2-p}[\omega(r)]^{2-p}r^p, t_0)\,,$$

$$Q_{r,+}^{\lambda\omega(r)} \equiv Q_{r,+}^{\lambda\omega(r)}(x_0,t_0) := B(x_0,r) \times (t_0, t_0 + \lambda^{2-p}[\omega(r)]^{2-p}r^p)$$

and

$$Q_{r}^{\lambda\omega(r)} \equiv Q_{r}^{\lambda\omega(r)}(x_0,t_0) := B(x_0,r) \times (t_0 - \lambda^{2-p}[\omega(r)]^{2-p}r^p, t_0 + \lambda^{2-p}[\omega(r)]^{2-p}r^p)\,.$$

The first definition considers solutions to Cauchy-Dirichlet problems under suitable oscillation bounds for the boundary datum and for the solution itself.

Definition 3. Let $Q_{R,+}^{\lambda\omega(R)} \equiv Q_{R,+}^{\lambda\omega(R)}(x_0,t_0)$ be a cylinder with

$$\lambda \geq \max\{M_1/\omega(R), M_2\}\,,$$

where M_1, M_2 are positive constants. Let $g \in C^0(\bar{Q}_{R,+}^{\lambda\omega(R)})$ and let $u \in C^0(\bar{Q}_{R,+}^{\lambda\omega(R)})$ be the unique solution to

$$\begin{cases} Hu = 0 & \text{in } Q_{R,+}^{\lambda\omega(R)} \\ u = g & \text{on } \partial_{\mathrm{P}} Q_{R,+}^{\lambda\omega(R)}\,. \end{cases} \tag{4.1}$$

We say that (u,g) belongs to the class $\mathscr{D}_0(H; Q_{R,+}^{\lambda\omega(R)}, \omega(\cdot), M_1, M_2)$ if

$$\operatorname*{osc}_{Q_{R,+}^{\lambda\omega(R)}} u \leq M_1 \qquad \text{and} \qquad \operatorname*{osc}_{B(x_0,r)} g(\cdot,t_0) \leq M_2\omega(r) \quad \text{for every } r \leq R\,. \tag{4.2}$$

In the same we say that $(u,g) \in \mathscr{D}_0(H; Q_{R}^{\lambda\omega(R)}, \omega(\cdot), M_1, \infty)$ if (4.1) holds in $Q_{R}^{\lambda\omega(R)}$ with $\lambda > M_1/\omega(R)$ and (only) the first inequality in (4.2) holds in $Q_{R}^{\lambda\omega(R)}$.

The next definition gives an analogous class, but this time for local solutions to obstacle problems.

Definition 4. Let $Q_{R}^{\lambda\omega(R)} \equiv Q_{R}^{\lambda\omega(R)}(x_0,t_0)$ be a cylinder with

$$\lambda \geq \max\{M_1/\omega(R), M_3\}\,,$$

where M_1, M_3 are positive constants. Let $\psi, g \in C^0(\bar{Q}_{R}^{\lambda\omega(R)})$, $g \geq \psi$ on $\partial_{\mathrm{P}} Q_{R}^{\lambda\omega(R)}$, and let $u \in C^0(\bar{Q}_{R}^{\lambda\omega(R)})$ be the unique solution to

$$\begin{cases} \max\{Hu, \psi - u\} = 0 & \text{in } Q_{R}^{\lambda\omega(R)} \\ u = g & \text{on } \partial_{\mathrm{P}} Q_{R}^{\lambda\omega(R)}\,. \end{cases}$$

We say that (u,g,ψ) belongs to the class $\mathscr{P}_0(H; Q_{R}^{\lambda\omega(R)}, \omega(\cdot), M_1, M_3)$ if

$$\operatorname*{osc}_{Q_{R}^{\lambda\omega(R)}} u \leq M_1 \qquad \text{and} \qquad \operatorname*{osc}_{Q_{r}^{\lambda\omega(r)}(x_0,t_0)} \psi \leq M_3\omega(r) \quad \text{for every } r \leq R\,.$$

The classes $\mathscr{P}_0(H; Q_{R,\pm}^{\lambda\omega(R)}(x_0,t_0), \omega(\cdot), M_1, M_3)$ are defined as above but with the cylinder $Q_{R}^{\lambda\omega(R)}$ now replaced by $Q_{R,\pm}^{\lambda\omega(R)}(x_0,t_0)$.

In the last definition we describe the class of Cauchy-Dirichlet problems in the presence of an obstacle.

Definition 5. Let $Q_{R,+}^{\lambda\omega(R)} \equiv Q_{R,+}^{\lambda\omega(R)}(x_0,t_0)$ be a cylinder with

$$\lambda \geq \max\{M_1/\omega(R), M_2, M_3\}\,,$$

where M_1, M_2, M_3 are positive constants. Let $\psi, g \in C^0(\bar{Q}^{\lambda\omega(R)}_{R,+})$, $g \geq \psi$ on $\partial_{\mathrm{P}} Q^{\lambda\omega(R)}_{R,+}$, and let $u \in C^0(Q^{\lambda\omega(R)}_{R,+})$ be the unique solution to

$$\begin{cases} \max\{Hu, \psi - u\} = 0 & \text{in } Q^{\lambda\omega(R)}_{R,+} \\ u = g & \text{on } \partial_{\mathrm{P}} Q^{\lambda\omega(R)}_{R,+} \,. \end{cases}$$

We say that (u, g, ψ) belongs to the class $\widetilde{\mathscr{P}}_0(H; Q^{\lambda\omega(R)}_{R,+}, \omega(\cdot), M_1, M_2, M_3)$ if

$$\operatorname*{osc}_{Q^{\lambda\omega(R)}_{R,+}} u \leq M_1$$

and

$$\operatorname*{osc}_{B(x_0,r)} g(\cdot, t_0) \leq M_2 \omega(r)\,, \qquad \operatorname*{osc}_{Q^{\lambda\omega(r)}_{r,+}(x_0,t_0)} \psi \leq M_3 \omega(r) \quad \text{for every } r \leq R\,.$$

The classes introduced are invariant under translation in the following sense. Consider, for instance, the class $\mathscr{D}_0$ and let $\mathcal{O} \subset \mathbb{R}^{n+1}$ be a space-time cylinder. Then $(u + k, g + k) \in \mathscr{D}_0(H; \mathcal{O}, \omega(\cdot), M_1, M_2)$ for all $k \in \mathbb{R}$ if $(u, g) \in \mathscr{D}_0(H; \mathcal{O}, \omega(\cdot), M_1, M_2)$. The similar fact also holds for the classes introduced in Definitions 4-5.

Lemma 4.1. *Let M_1, M_2, R be positive constants. There exists a constant $c_{4.1} \equiv c_{4.1}(n, p, \nu, L)$ such that if*

$$\lambda \geq c_{4.1} \max\{M_1/\omega(R), M_2, sR/\omega(R)\}$$

then the following holds: If

$$(u, g) \in \mathscr{D}_0(H; Q^{\lambda\omega(R)}_{R,+}(x_0, t_0), \omega(\cdot), M_1, M_2)\,,$$

then

$$\operatorname*{osc}_{Q^{\lambda\omega(r)}_{r,+}(x_0,t_0)} u \leq \lambda\omega(r) \qquad \text{for every } r \in (0, R)\,.$$

Lemma 4.2. *Let M_1, M_2, M_3, R be positive constants. There exists a constant $c_{4.2} \equiv c_{4.2}(n, p, \nu, L)$ such that if*

$$\lambda \geq c_{4.2} \max\{M_1/\omega(R), M_2, M_3, sR/\omega(R)\}$$

then the following holds: If

$$(u, g, \psi) \in \widetilde{\mathscr{P}}_0(H; Q^{\lambda\omega(R)}_{R,+}(x_0, t_0), \omega(\cdot), M_1, M_2, M_3)\,,$$

then

$$\operatorname*{osc}_{Q^{\lambda\omega(r)}_{r,+}(x_0,t_0)} u \leq \lambda\omega(r) \qquad \text{for every } r \in (0, R)\,.$$

Lemma 4.3. *Let M_1, M_3, R be positive constants. There exists a constant $c_{4.3} \equiv c_{4.3}(n, p, \nu, L)$ such that if*

$$\lambda \geq c_{4.3} \max\{M_1/\omega(R), M_3, sR/\omega(R)\}$$

then the following holds: If

$$(u, g, \psi) \in \mathscr{P}_0(H; Q^{\lambda\omega(R)}_{R,-}(x_0, t_0), \omega(\cdot), M_1, M_3) \quad \text{and} \quad u(x_0, t_0) = \psi(x_0, t_0)\,,$$

then

$$\operatorname*{osc}_{Q^{\lambda\omega(r)}_{r,-}(x_0,t_0)} u \leq \lambda\omega(r) \qquad \text{for every } r \in (0, R)\,.$$

Lemma 4.4. *Let M_1, M_3, R be positive constants. There exists a constant $c_{4.4} \equiv c_{4.4}(n,p,\nu,L)$ such that if*

$$\lambda \geq c_{4.4} \max\{M_1/\omega(R), M_3, sR/\omega(R)\}$$

then the following holds: If

$$(u,g,\psi) \in \mathscr{P}_0(H; Q_R^{\lambda\omega(R)}(x_0,t_0), \omega(\cdot), M_1, M_3) \quad \text{and} \quad u(x_0,t_0) = \psi(x_0,t_0)\,,$$

then

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}(x_0,t_0)} u \leq \lambda\omega(r) \qquad \text{for every } r \in (0,R)\,.$$

Lemma 4.5. *Let M_1, R be positive constants. There exists a constant $c_{4.5} \equiv c_{4.5}(n,p,\nu,L)$ such that if*

$$\lambda \geq c_{4.5} \max\{M_1/\omega(R), sR/\omega(R)\}$$

then the following holds: If

$$(u,g) \in \mathscr{D}_0(H; Q_R^{\lambda\omega(R)}(x_0,t_0), \omega(\cdot), M_1, \infty)\,,$$

then

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}(x_0,t_0)} u \leq \lambda\omega(r) \qquad \text{for every } r \in (0,R)\,.$$

4.1. **Proof of Lemmas 4.1–4.5.** The proofs of the above lemmas, though differing at important points, have several steps in common that we put together. Our proofs proceed in eight steps.

Step 1: Preliminary set-up. We will in the following use a positive constant Λ, whose definition changes according to the lemma in question. Specifically, we fix Λ as follows:

$$\begin{cases} \text{Lemma 4.1}: & \Lambda := 4\max\{M_1/\omega(R), M_2, sR/\omega(R)\} \\ \text{Lemma 4.2}: & \Lambda := 4\max\{M_1/\omega(R), M_2, M_3, sR/\omega(R)\} \\ \text{Lemma 4.3}: & \Lambda := 4\max\{M_1/\omega(R), M_3, sR/\omega(R)\} \\ \text{Lemma 4.4}: & \Lambda := 4\max\{M_1/\omega(R), M_3, sR/\omega(R)\} \\ \text{Lemma 4.5}: & \Lambda := 4\max\{M_1/\omega(R), sR/\omega(R)\}. \end{cases} \tag{4.3}$$

After translation, we may throughout the proof assume, without loss of generality, that

$$(x_0,t_0) = (0,0) \qquad \text{and} \qquad u(x_0,t_0) = 0\,.$$

With u being defined in $Q_{R,\pm}^{\lambda\omega(R)}$ (Lemma 4.1-Lemma 4.3) or $Q_R^{\lambda\omega(R)}$ (Lemma 4.4, Lemma 4.5), $u(0,0)=0$, we introduce the quantities

$$\begin{cases} S_k^+(u,\lambda) := \sup_{Q_{2^{-k}R,+}^{\lambda\omega(2^{-k}R)}} |u| \\ S_k^-(u,\lambda) := \sup_{Q_{2^{-k}R,-}^{\lambda\omega(2^{-k}R)}} |u| \\ S_k(u,\lambda) := \sup_{Q_{2^{-k}R}^{\lambda\omega(2^{-k}R)}} |u|\,. \end{cases} \tag{4.4}$$

Due to the normalization $u(0,0)=0$, the quantities above will play the role of an oscillation in the rest of the proof.

We shall show that there exists a constant $c \geq 4$, depending only on n,p,ν,L such that for

$$\lambda := c\Lambda, \qquad \text{where } \Lambda \text{ is as defined in (4.3)}\,,$$

we have

$$
(4.5)\qquad \begin{cases} \dfrac{S_k^+(u,\lambda)}{\omega(2^{-k}R)} & \leq \max\left\{\lambda, \dfrac{S_{k-1}^+(u,\lambda)}{\omega(2^{1-k}R)}, \dots, \dfrac{S_0^+(u,\lambda)}{\omega(R)}\right\} \\ \dfrac{S_k^-(u,\lambda)}{\omega(2^{-k}R)} & \leq \max\left\{\lambda, \dfrac{S_{k-1}^-(u,\lambda)}{\omega(2^{1-k}R)}, \dots, \dfrac{S_0^-(u,\lambda)}{\omega(R)}\right\} \\ \dfrac{S_k(u,\lambda)}{\omega(2^{-k}R)} & \leq \max\left\{\lambda, \dfrac{S_{k-1}(u,\lambda)}{\omega(2^{1-k}R)}, \dots, \dfrac{S_0(u,\lambda)}{\omega(R)}\right\}, \end{cases}
$$

for all $k \in \mathbb{N}$, depending on the Lemma considered. Specifically, $(4.5)_1$ comes into the play when considering Lemma 4.1 and Lemma 4.2; inequality $(4.5)_2$ intervenes in Lemma 4.3, while $(4.5)_3$ is used for Lemma 4.4 and Lemma 4.5. Indeed, if (4.5) holds for every positive integer k, then by induction it also follows that

$$
\begin{cases} S_m^+(u,\lambda) \leq \lambda\omega(2^{-m}R) \\ S_m^-(u,\lambda) \leq \lambda\omega(2^{-m}R) \\ S_m(u,\lambda) \leq \lambda\omega(2^{-m}R) \end{cases}
$$

for all $m \in \mathbb{N}$, and Lemmas 4.1–4.5 follow by an enlargement of the constant c by a fixed factor (for instance 10) and using (2.5).

Step 2: Set-up of the contradiction argument (Blow-up). We argue by contradiction assuming that (4.5) does not hold. Hence, with

$$
(4.6)\qquad \lambda_j = 2^j \Lambda_j\,, \qquad j \in \mathbb{N}\,,
$$

where Λ_j is described in a few lines below, there exist, for every $j \in \mathbb{N}$, a concave modulus of continuity $\omega_j(\cdot)$, functions u_j, g_j, ψ_j, intrinsic cylinders $Q_{R_j,\pm}^{\lambda_j\omega_j(R_j)}$ (and $Q_{R_j}^{\lambda_j\omega_j(R_j)}$, depending on the lemma we are considering), operators

$$
H_j w := w_t - \operatorname{div} a_j(Dw)\,,
$$

with vector fields $a_j(\cdot)$ uniformly satisfying (1.2) for fixed ν, L, and $s \equiv s_j \geq 0$, and finally numbers $M_{1,j}, M_{2,j}, M_{3,j}$, such that the following hold:

Lemma 4.1 : $(u_j, g_j) \in \mathscr{D}_0(H_j; Q_{R_j,+}^{\lambda_j\omega_j(R_j)}, \omega_j(\cdot), M_{1,j}, M_{2,j})$,
$u_j(0,0) = 0$,
$\Lambda_j := 4\max\{M_{1,j}/\omega_j(R_j), M_{2,j}, s_jR_j/\omega_j(R_j)\}$,

Lemma 4.2 : $(u_j, g_j, \psi_j) \in \widetilde{\mathscr{P}}_0(H_j; Q_{R_j,+}^{\lambda_j\omega_j(R_j)}, \omega_j(\cdot), M_{1,j}, M_{2,j}, M_{3,j})$,
$u_j(0,0) = 0 \geq \psi_j(0,0)$,
$\Lambda_j := 4\max\{M_{1,j}/\omega_j(R_j), M_{2,j}, M_{3,j}, s_jR_j/\omega_j(R_j)\}$,

Lemma 4.3 : $(u_j, g_j, \psi_j) \in \mathscr{P}_0(H_j; Q_{R_j,-}^{\lambda_j\omega_j(R)}, \omega_j(\cdot), M_{1,j}, M_{3,j})$,
$u_j(0,0) = \psi_j(0,0) = 0$,
$\Lambda_j := 4\max\{M_{1,j}/\omega_j(R_j), M_{3,j}, s_jR_j/\omega_j(R_j)\}$,

Lemma 4.4 : $(u_j, g_j, \psi_j) \in \mathscr{P}_0(H_j; Q_{R_j}^{\lambda_j\omega_j(R_j)}, \omega_j(\cdot), M_{1,j}, M_{3,j})$,
$u_j(0,0) = \psi_j(0,0) = 0$,
$\Lambda_j := 4\max\{M_{1,j}/\omega_j(R_j), M_{3,j}, s_jR_j/\omega_j(R_j)\}$,

Lemma 4.5 : $(u_j, g_j) \in \mathscr{D}_0(H_j; Q_{R_j}^{\lambda_j\omega_j(R_j)}, \omega_j(\cdot), M_{1,j}, +\infty)$,
$u_j(0,0) = 0$,

$$\Lambda_j := 4\max\{M_{1,j}/\omega_j(R_j), s_jR_j/\omega_j(R_j)\}\,,$$

and there exists, for every $j \in \mathbb{N}$, a positive integer k_j such that

$$\begin{cases} \dfrac{S^+_{k_j}(u_j,\lambda_j)}{\omega_j(2^{-k_j}R_j)} > \max\left\{\lambda_j, \dfrac{S^+_{k_j-1}(u_j,\lambda_j)}{\omega_j(2^{1-k_j}R_j)}, \dots, \dfrac{S^+_0(u_j,\lambda_j)}{\omega_j(R_j)}\right\} \\[2ex] \dfrac{S^-_{k_j}(u_j,\lambda_j)}{\omega_j(2^{-k_j}R_j)} > \max\left\{\lambda_j, \dfrac{S^-_{k_j-1}(u_j,\lambda_j)}{\omega_j(2^{1-k_j}R_j)}, \dots, \dfrac{S^-_0(u_j,\lambda_j)}{\omega_j(R_j)}\right\} \\[2ex] \dfrac{S_{k_j}(u_j,\lambda_j)}{\omega_j(2^{-k_j}R_j)} > \max\left\{\lambda_j, \dfrac{S_{k_j-1}(u_j,\lambda_j)}{\omega_j(2^{1-k_j}R_j)}, \dots, \dfrac{S_0(u_j,\lambda_j)}{\omega_j(R_j)}\right\}, \end{cases} \tag{4.7}$$

depending on the one we are considering amongst Lemmas 4.1-4.5, in the way described after (4.5). For each $j \in \mathbb{N}$ we let k_j be the smallest integer such that (4.7) holds. Note, in particular, that this choice of k_j implies that

$$\begin{cases} \lambda_j\omega_j(2^{-k_j}R_j) < S^+_{k_j}(u_j,\lambda_j) \le S^+_q(u_j,\lambda_j) \le \lambda_j\omega_j(2^{-q}R_j), \text{ or} \\[1ex] \lambda_j\omega_j(2^{-k_j}R_j) < S^-_{k_j}(u_j,\lambda_j) \le S^-_q(u_j,\lambda_j) \le \lambda_j\omega_j(2^{-q}R_j), \text{ or} \\[1ex] \lambda_j\omega_j(2^{-k_j}R_j) < S_{k_j}(u_j,\lambda_j) \le S_q(u_j,\lambda_j) \le \lambda_j\omega_j(2^{-q}R_j), \end{cases} \tag{4.8}$$

for all $q \in \{0, ..., k_j-1\}$ and depending on the lemma we are proving. Moreover, $k_j > j$ must hold. Indeed, assuming the contrary, then we would have

$$\|u_j\|_{L^\infty} \ge \lambda_j\omega_j(2^{-k_j}R_j) \ge \lambda_j\omega_j(2^{-j}R_j) = 2^j\Lambda_j\omega_j(2^{-j}R_j)\,.$$

Now using (2.5) we would have $\|u_j\|_{L^\infty} \ge \Lambda_j\omega_j(R_j)$ and this would contradict the definition of Λ_j.

Step 3: Scaling of solutions. Following (3.3) we define

$$\widetilde{\omega}_j(\gamma) := \frac{\omega_j(\gamma 2^{-k_j}R_j)}{\omega_j(2^{-k_j}R_j)} \qquad \text{for every } \gamma \ge 0\,.$$

Note that $\widetilde{\omega}_j(\cdot)$ remains concave with $\widetilde{\omega}_j(0) = 0$ and $\widetilde{\omega}_j(1) = 1$ and that

$$1 \le \widetilde{\omega}_j(r) \le r \tag{4.9}$$

holds for all $r \ge 1$ by (2.5). We also define

$$\widetilde{Q}^{\pm}_{m,j} := Q^{\widetilde{\omega}_j(2^m)}_{2^m,\pm}(0,0)\,, \qquad \widetilde{Q}_{m,j} := Q^{\widetilde{\omega}_j(2^m)}_{2^m}(0,0)\,, \qquad m \in \{0,1,...,k_j\}$$

and the scaled functions

$$\begin{aligned} \widetilde{u}_j(x,t) &:= \frac{u_j(2^{-k_j}R_jx, (\lambda_j\omega_j(2^{-k_j}R_j))^{2-p}(2^{-k_j}R_j)^pt)}{\lambda_j\omega_j(2^{-k_j}R_j)}\,, \\ \widetilde{g}_j(x,t) &:= \frac{g_j(2^{-k_j}R_jx, (\lambda_j\omega_j(2^{-k_j}R_j))^{2-p}(2^{-k_j}R_j)^pt)}{\lambda_j\omega_j(2^{-k_j}R_j)}\,, \\ \widetilde{\psi}_j(x,t) &:= \frac{\psi_j(2^{-k_j}R_jx, (\lambda_j\omega_j(2^{-k_j}R_j))^{2-p}(2^{-k_j}R_j)^pt)}{\lambda_j\omega_j(2^{-k_j}R_j)}\,, \end{aligned}$$

whenever $(x,t) \in \widetilde{Q}^+_{k_j,j}, \widetilde{Q}^-_{k_j,j}, \widetilde{Q}_{k_j,j}$ and depending on the lemma we are considering. Accordingly to the scaling already discussed in (3.1)-(3.2) (take $r = 2^{-k_j}R_j$, $R \equiv R_j$ there) $\widetilde{u}_j$ relates to the operator $\widetilde{H}_j$ defined by

$$\widetilde{H}_jw := w_t - \operatorname{div}\widetilde{a}_j(Dw)\,,$$

where

$$\widetilde{a}_j(z) := \frac{a_j(2^{k_j}R_j^{-1}\lambda_j\omega_j(2^{-k_j}R_j)z)}{(2^{k_j}R_j^{-1}\lambda_j\omega_j(2^{-k_j}R_j))^{p-1}}\,, \qquad z \in \mathbb{R}^n\,,$$

satisfies conditions (1.2) with s replaced by

$$\widetilde{s}_j := \frac{s_j 2^{-k_j} R_j}{\lambda_j \omega_j (2^{-k_j} R_j)} \leq \frac{s_j R_j}{\lambda_j \omega_j (R_j)} \leq 2^{-j}. \tag{4.10}$$

To conclude this we have here used (2.5) and that $\Lambda_j \geq 4 s_j R_j / \omega_j(R_j)$. Moreover, by (4.8) we have that

$$1 < \sup_{\widetilde{\mathcal{O}}_{0,j}} |\widetilde{u}_j| \leq \sup_{\widetilde{\mathcal{O}}_{m,j}} |\widetilde{u}_j| \leq \widetilde{\omega}_j(2^m) \leq 2^m\,, \qquad m \in \{1, ..., k_j\}, \tag{4.11}$$

where $\widetilde{\mathcal{O}}_{0,j}$ and $\widetilde{\mathcal{O}}_{m,j}$ equal $\widetilde{Q}^+_{0,j}, \widetilde{Q}^-_{0,j}$ or $\widetilde{Q}_{0,j}$, and, $\widetilde{Q}^+_{m,j}, \widetilde{Q}^-_{m,j}$ or $\widetilde{Q}_{m,j}$, respectively, depending on the lemma we are considering. With the above definitions, $\widetilde{u}_j$ solves, in the case of Lemma 4.1, the Cauchy problem

$$\begin{cases} \widetilde{H}_j \widetilde{u}_j = 0 & \text{in } \widetilde{Q}^+_{k_j,j} \\ \widetilde{u}_j = \widetilde{g}_j & \text{on } \partial_{\mathrm{P}} \widetilde{Q}^+_{k_j,j}\,, \end{cases}$$

and, in the case of Lemma 4.2, the Cauchy obstacle problem

$$\begin{cases} \max\left\{ \widetilde{H}_j \widetilde{u}_j, \widetilde{\psi}_j - \widetilde{u}_j \right\} = 0 & \text{in } \widetilde{Q}^+_{k_j,j} \\ \widetilde{u}_j = \widetilde{g}_j & \text{on } \partial_{\mathrm{P}} \widetilde{Q}^+_{k_j,j}\,. \end{cases}$$

In the case of Lemmas 4.3–4.4 the function $\widetilde{u}_j$ instead solves

$$\begin{cases} \max\left\{ \widetilde{H}_j \widetilde{u}_j, \widetilde{\psi}_j - \widetilde{u}_j \right\} = 0 & \text{in } \widetilde{Q}^-_{k_j,j} \\ \widetilde{u}_j = \widetilde{g}_j & \text{on } \partial_{\mathrm{P}} \widetilde{Q}^-_{k_j,j}\,, \end{cases}$$

and

$$\begin{cases} \max\left\{ \widetilde{H}_j \widetilde{u}_j, \widetilde{\psi}_j - \widetilde{u}_j \right\} = 0 & \text{in } \widetilde{Q}_{k_j,j} \\ \widetilde{u}_j = \widetilde{g}_j & \text{on } \partial_{\mathrm{P}} \widetilde{Q}_{k_j,j}\,, \end{cases}$$

respectively. Finally, in the case of Lemma 4.5, $\widetilde{u}_j$ solves

$$\begin{cases} \widetilde{H}_j \widetilde{u}_j = 0 & \text{in } \widetilde{Q}_{k_j,j} \\ \widetilde{u}_j = \widetilde{g}_j & \text{on } \partial_{\mathrm{P}} \widetilde{Q}_{k_j,j}\,. \end{cases}$$

We note that in the case of Lemma 4.1 and Lemma 4.2 we also have that

$$\operatorname*{osc}_{B(0,r)} g_j(\cdot, 0) \leq M_{2,j} \omega_j(r) \quad \text{for every } r \leq R_j\,,\ g_j(0,0) = 0\,,\ \lambda_j \geq 2^j M_{2,j},$$

and by (4.9), we see that

$$\sup_{x \in B(0,2^m)} |\widetilde{g}_j(x,0)| \leq 2^{m-j} \tag{4.12}$$

whenever $m \in \{0, 1, \ldots, k_j\}$. Similarly, in the case of Lemma 4.2, we have

$$\operatorname*{osc}_{Q^{\lambda_j \omega_j(r)}_{r,+}} \psi_j \leq M_{3,j} \omega_j(r) \text{ for every } r \leq R_j,\ \psi_j(0,0) \leq 0\,,\ \lambda_j \geq 2^j M_{3,j}.$$

In the case of Lemma 4.3 and Lemma 4.4 we have

$$\operatorname*{osc}_{Q^{\lambda_j \omega_j(r)}_{r,-}} \psi_j \leq M_{3,j} \omega_j(r) \text{ for every } r \leq R_j,\ \psi_j(0,0) = 0\,,\ \lambda_j \geq 2^j M_{3,j}$$

and

$$\operatorname*{osc}_{Q^{\lambda_j \omega_j(r)}_{r}} \psi_j \leq M_{3,j} \omega_j(r) \text{ for every } r \leq R_j,\ \psi_j(0,0) = 0\,,\ \lambda_j \geq 2^j M_{3,j},$$

respectively. In particular, in the case of Lemma 4.2 we only have that $\psi_j(0,0) \leq 0$ in contrast to $\psi_j(0,0) = 0$. Using all of this, and (4.9), we see that

$$
\begin{aligned}
&\text{Lemma 4.2 :} \qquad \sup_{\widetilde{Q}^+_{m,j}} \widetilde{\psi}_j \leq 2^{m-j}\,,\\
&\text{Lemma 4.3 :} \qquad \inf_{\widetilde{Q}^-_{m,j}} \widetilde{\psi}_j \geq -2^{m-j}\,,\\
&\text{Lemma 4.4 :} \qquad \inf_{\widetilde{Q}_{m,j}} \widetilde{\psi}_j \geq -2^{m-j}\,.
\end{aligned}
\tag{4.13}
$$

Step 4: Proof of Lemma 4.1. The proof is based on a comparison argument using suitable weak solutions and Lemma 3.2. Let l be a positive integer to be determined in a few lines and observe that if $j > l$, then $k_j > l$. Using (4.11) and (4.12) we immediately see that

$$
\sup_{\widetilde{Q}^+_{l,j}} |\widetilde{u}_j| \leq 2^l\,, \qquad\qquad \sup_{x\in B(0,2^l)} |\widetilde{g}_j(x,0)| \leq 2^{l-j}. \tag{4.14}
$$

Next, we set

$$
\partial^+_{\mathrm{P}} \widetilde{Q}^+_{l,j} = \partial_{\mathrm{P}} \widetilde{Q}^+_{l,j} \cap \{t > 0\} \qquad \text{and} \qquad \partial^-_{\mathrm{P}} \widetilde{Q}^+_{l,j} = \partial_{\mathrm{P}} \widetilde{Q}^+_{l,j} \cap \{t = 0\}\,,
$$

and we let the functions $\widetilde{v}^\pm_j$ (that is $\widetilde{v}^+_j$ and $\widetilde{v}^-_j$) solve

$$
\begin{cases}
\widetilde{H}_j \widetilde{v}^\pm_j = 0 & \text{in } Q^+_{l,j}\\
\widetilde{v}^\pm_j = \pm 2^l & \text{on } \partial^+_{\mathrm{P}} \widetilde{Q}^+_{l,j}\\
\widetilde{v}^\pm_j = \pm 2^{l-j} & \text{on } \partial^-_{\mathrm{P}} \widetilde{Q}^+_{l,j}\,.
\end{cases}
\tag{4.15}
$$

The existence of such a solutions, attaining the initial datum locally continuously (that is in any set of the type $B(0,\gamma) \times \{0\}$ for $\gamma < 2^l$) follows, for example, by a Perron method type argument (see [22]). By the standard comparison principle we see that

$$
-2^l \leq \widetilde{v}^-_j \leq \widetilde{u}_j \leq \widetilde{v}^+_j \leq 2^l \quad \text{in } Q^+_{l,j}. \tag{4.16}
$$

Observe that $\widetilde{w}^-_j = \max\{-\widetilde{v}^-_j - 2^{l-j}, 0\}$ and $\widetilde{w}^+_j = \max\{\widetilde{v}^+_j - 2^{l-j}, 0\}$ are both nonnegative weak subsolutions taking locally continuously zero initial values on $\partial^-_{\mathrm{P}} \widetilde{Q}^+_{l,j}$. Applying Lemma 3.2 with the choice $(x_0,t_0) \equiv (0,0)$, $T \equiv 1$, $R \equiv 2^l$, $s \equiv \widetilde{s}_j \leq 2^{-j}$, we arrive at

$$
\sup_{\widetilde{Q}^+_{0,l}} \widetilde{w}^\pm_j \leq \frac{c}{2^{pl}} \fint_{\widetilde{Q}^+_{l,j} \cap \{0<t<1\}} \left(\widetilde{w}^\pm_j\right)^{p-1} dx\,dt + c2^l \widetilde{s}_j \leq c\left(2^{-l} + 2^{l-j}\right)
$$

whenever $j > l$ and for a constant $c \equiv c(n,p,\nu,L)$. This leads to

$$
\sup_{\widetilde{Q}^+_{0,j}} |\widetilde{v}^\pm_j| \leq (1+c)2^{l-j} + c2^{-l}\,. \tag{4.17}
$$

Therefore, taking the smallest integer l such that $(1+c)2^{-l} \leq 1/8$, and then $j = 2l+3$, we can conclude that

$$
\sup_{\widetilde{Q}^+_{0,j}} |\widetilde{u}_j| \leq \frac{1}{4}\,, \tag{4.18}
$$

provided j is large enough. Obviously this contradicts (4.11) and hence the proof of Lemma 4.1 is complete.

Step 5: Proof of Lemma 4.2. We will prove the result with $c_{4.2} := 2c_{4.1}$, where $c_{4.1}$ is as in Lemma 4.1. Indeed, setting $\lambda := 2c_{4.1}\Lambda$, we will first prove that

$$\inf_{Q_{r,+}^{\lambda\omega(r)}} u \geq -\frac{\lambda}{2}\omega(r) \qquad \text{for every } r \in (0,R)\,. \tag{4.19}$$

In fact, let v be the solution to the Dirichlet problem (1.9) in the domain $\mathcal{O} = Q_{R,+}^{\lambda\omega(R)}(x_0,t_0)$ with $b = g$. Then, by the comparison principle, we see that $u \geq v$ and therefore (4.19) is a direct consequence of Lemma 4.1, because

$$(v,g) \in \mathscr{D}_0(H; Q_{R,+}^{\lambda\omega(R)}(x_j,t_j), \omega(\cdot), M_1, M_2).$$

Now, the proof of the lemma will be completed by proving the upper bound

$$\sup_{Q_{r,+}^{\lambda\omega(r)}} u \leq \frac{\lambda}{2}\omega(r) \qquad \text{for every } r \in (0,R)\,. \tag{4.20}$$

To prove (4.20) we repeat the steps above, starting with Step 2, but now considering the supremum of $\widetilde{u}_j$ rather than than the supremum of $|\widetilde{u}_j|$. In this case the heart of the matter is then to find a contradiction to

$$\sup_{\widetilde{Q}_{0,j}^{+}} \widetilde{u}_j > \frac{1}{2} \tag{4.21}$$

for j large enough. Note that since $\widetilde{u}_j(0,0) = 0$ the supremum is always positive. With this aim we fix, as in Step 4, a positive integer l to be chosen later and recall the validity of (4.14). Note that at this stage we do not have any control of the size of the scaled obstacle $\widetilde{\psi}_j$ since we are only assuming that $\widetilde{\psi}_j(0,0) \leq 0$, but not necessarily $\widetilde{\psi}_j(0,0) = 0$. Using (4.13) we obtain that $\tilde{\psi}_j \leq 2^{l-j}$ in the cylinder $\widetilde{Q}_{l,j}^{+}$. The plan now is to make use of the functions $\widetilde{v}_j^{+}$ of (4.15) as in Step 4, but accounting for the important difference that in the case we are now considering, $\widetilde{u}_j$ is not a solution but only a solution to an obstacle problem. To handle this we use the "elliptic" comparison principle given by Lemma 2.1. To proceed we first note, by the standard minimum principle, that $\widetilde{v}_j^{+} \geq 2^{l-j}$ in $\widetilde{Q}_{l,j}^{+}$. Since $\tilde{\psi}_j \leq 2^{l-j}$ in $\widetilde{Q}_{l,j}^{+}$ we see that $\widetilde{u}_j$ is a weak solution in $S := \widetilde{Q}_{l,j}^{+} \cap \{\widetilde{u}_j > 2^{l-j}\}$. We therefore conclude that $\widetilde{u}_j \leq \widetilde{v}_j^{+}$ in $\partial S \setminus \{t = [\widetilde{\omega}_j(2^l)]^{2-p}2^{lp}\}$ and Lemma 2.1 then implies that $\widetilde{u}_j \leq \widetilde{v}_j^{+}$ in S. All in all we get that $\widetilde{u}_j \leq \widetilde{v}_j^{+}$ in $\widetilde{Q}_{l,j}^{+}$. Using this and recalling (4.17) we therefore obtain

$$\sup_{\widetilde{Q}_{0,j}^{+}} \widetilde{u}_j \leq (1+c)2^{l-j} + c2^{-l}.$$

We can now argue as in Step 4, choosing l and j large enough, to get that $\widetilde{u}_j \leq 1/4$, in $\widetilde{Q}_{0,j}^{+}$. This now contradicts (4.21) and the proof of Lemma 4.2 is complete.

Step 6: Proof of Lemma 4.3. The proof of Lemma 4.3 is more involved compared to the proof of Lemma 4.1 and Lemma 4.2. To prove the lemma we again introduce suitable positive solutions for comparison and we can do it in such a way that the value of the solutions for comparison are as small as we please at the origin. This smallness, in combination with Harnack estimates, shows that the solutions for comparison are small, at the origin, also for earlier times. However, since we know that the supremums of the solutions for comparison are much larger than the small values at origin, we can conclude that there must be a point, by the mean value principle, where the modulus of the gradient is large. Then, using Hölder estimates for the gradient we are able to prove that the moduli of gradients of the solutions for comparison are actually large in the whole cylinder and thus the equations for the solutions for comparison become non-degenerate. Using this we are then able to apply the Gaussian estimates from Section 3.4 to conclude the proof. This is the heuristics of the proof which we now intend to make rigorous. To do this we derive a contradiction to (4.11). Since $\widetilde{\psi}_j(0,0)=0$, we have $\|\widetilde{\psi}_j\|_{L^\infty(\widetilde{Q}^-_{0,j})} \le 2^{-j}$ by (4.13), and $\widetilde{u}_j \ge -1/2$ follows by taking $j=1$. Using this we see that it is enough to prove, for j large enough, that

$$\sup_{\widetilde{Q}^-_{0,j}} \widetilde{u}_j \le \frac{1}{2}. \tag{4.22}$$

Let $\widetilde{v}_j$ now solve

$$\begin{cases} \widetilde{H}_j \widetilde{v}_j = 0 & \text{in } \widetilde{Q}^-_{l,j} \\ \widetilde{v}_j = \max\left\{\widetilde{u}_j, 2^{l-j}\right\} & \text{on } \partial_{\mathrm{P}} \widetilde{Q}^-_{l,j}\,. \end{cases}$$

Clearly $\widetilde{v}_j \ge 2^{l-j}$ by the minimum principle. Furthermore,

$$\widetilde{u}_j + 2^{1+l-j} \ge \widetilde{v}_j \ge \widetilde{u}_j \qquad \text{on} \quad \partial_{\mathrm{P}} \widetilde{Q}^-_{l,j}\,,$$

and $\widetilde{v}_j \ge \widetilde{u}_j$ on the coincidence set $\widetilde{Q}^-_{l,j} \cap \{\widetilde{u}_j = \widetilde{\psi}_j\}$ since, by (4.13), we have $\|\widetilde{\psi}_j\|_{L^\infty(\widetilde{Q}^-_{l,j})} \le 2^{l-j}$. Since $\widetilde{u}_j$ is a weak solution in $S := \widetilde{Q}^-_{l,j} \setminus \{\widetilde{u}_j = \widetilde{\psi}_j\}$ and $\widetilde{v}_j \ge \widetilde{u}_j$ on the topological boundary of S, with the top $\{t=0\}$ excluded, the "elliptic" comparison principle of Lemma 2.1 implies that $\widetilde{v}_j \ge \widetilde{u}_j$. Moreover, the standard comparison principle on standard space-time cylinders yields $\widetilde{u}_j + 2^{1+l-j} \ge \widetilde{v}_j$ in $\widetilde{Q}^-_{l,j}$, because $\widetilde{u}_j$ (and therefore $\widetilde{u}_j + 2^{1+l-j}$) is a weak supersolution in $\widetilde{Q}^-_{l,j}$. Thus, all in all, we have

$$\widetilde{u}_j \le \widetilde{v}_j \le \widetilde{u}_j + 2^{1+l-j} \qquad \text{in } \widetilde{Q}^-_{l,j}\,. \tag{4.23}$$

Hence, since $\widetilde{u}_j(0,0)=0$, we have

$$\widetilde{v}_j(0,0) \le 2^{1+l-j} \tag{4.24}$$

and by (4.11) it follows that

$$\|\widetilde{v}_j\|_{L^\infty(\widetilde{Q}^-_{m,j})} \le \widetilde{\omega}_j(2^m) + 2^{1+l-j} \le 2\widetilde{\omega}_j(2^m) \le 2^{m+1} \tag{4.25}$$

for $m \in \{0,1,\dots,l\}$ and $j \ge l+1$. We next estimate $\widetilde{v}_j(0,\tau)$ for $\tau \in [-1,0]$. To do this we first note, using the Harnack inequality in Theorem 3.3, that

$$\widetilde{v}_j(0,\tau) \le \widetilde{c}_2 \widetilde{v}_j(0, \tau + \widetilde{c}_1 \widetilde{\varrho}^p (\widetilde{v}_j(0,\tau))^{2-p})) + \widetilde{c}_2 \widetilde{\varrho} \widetilde{s}_j \tag{4.26}$$

provided $\widetilde{\varrho} \le 2^{l-2}$ and

$$-\widetilde{\omega}_j(2^{l-2})^{2-p} 2^{(l-2)p} < \tau - \widetilde{c}_1 \widetilde{\varrho}^p (\widetilde{v}_j(0,\tau))^{2-p} < \tau + \widetilde{c}_1 \widetilde{\varrho}^p (\widetilde{v}_j(0,\tau))^{2-p} \le 0\,.$$

Here $\widetilde{c}_1 \ge 1$ and $\widetilde{c}_2 \ge 1$ are constants depending only on n,p,ν,L. Suppose now that $\widetilde{v}_j(0,\tau) > 1/4$ holds for some $\tau \in (-1,0)$. Using this together with (4.25), for $m=0$, we see that in this case $1/4 < \widetilde{v}_j(0,\tau) \le 2$ also must hold. Let now $\widetilde{\varrho}$ solve the equation

$$\widetilde{c}_1 \widetilde{\varrho}^p (\widetilde{v}_j(0,\tau))^{2-p} = -\tau$$

and note that the root $\widetilde{\varrho}$ for this equation is less than two, hence we need to assume $l \ge 4$ to apply (4.26). Consequently, using (4.24), (4.10) and (4.26), we see that

$$\begin{aligned} \widetilde{v}_j(0,0) &\ge \widetilde{c}_2^{-1} \widetilde{v}_j(0, -\widetilde{c}_2 \widetilde{\varrho}^p (\widetilde{v}_j(0,\tau))^{2-p})) - \widetilde{\varrho} \widetilde{s}_j \\ &= \widetilde{c}_2^{-1} \widetilde{v}_j(0,\tau) - \widetilde{\varrho}\widetilde{s}_j \ge \frac{1}{4\widetilde{c}_2} - \frac{\widetilde{\varrho}}{2^j} \ge \frac{1}{4\widetilde{c}_2} - \frac{1}{2^{j-1}}\,. \end{aligned} \tag{4.27}$$

However, this contradicts (4.24) if $j \geq j_0 + l$ for some j_0 (which may be chosen to depend only on n, p, ν, L). Hence we can conclude, using also continuity, that

$$0 \leq \widetilde{v}_j(0,\tau) \leq 1/4 \qquad \text{whenever } \tau \in [-1,0] \text{ and } j \geq j_0 + l\,. \tag{4.28}$$

If now

$$\sup_{\widetilde{Q}^-_{0,j}} |D\widetilde{v}_j| \leq \frac{1}{4}$$

then by the mean value principle, we would also have that

$$\sup_{x\in B_1} \widetilde{v}_j(x,\tau) - \widetilde{v}_j(0,\tau) \leq \sup_{x\in B_1} |\widetilde{v}_j(x,\tau) - \widetilde{v}_j(0,\tau)| \leq \sup_{\widetilde{Q}^-_{0,j}} |D\widetilde{v}_j| \leq \frac{1}{4}$$

for all $\tau \in [-1,0]$. As a consequence, by (4.23) and (4.28), we would then have that

$$\sup_{\widetilde{Q}^-_{0,j}} \widetilde{u}_j \leq \sup_{\widetilde{Q}^-_{0,j}} \widetilde{v}_j \leq \frac{1}{2}\,,$$

giving the desired contradiction in (4.18). Hence, in order to complete the proof, we can in the following assume that

$$\sup_{\widetilde{Q}^-_{0,j}} |D\widetilde{v}_j| \geq \frac{1}{4}\,. \tag{4.29}$$

To proceed we first calculate an upper bound for the gradient. The energy estimate of Lemma 3.1, (4.25), (4.10), and (2.5) give

$$\begin{aligned}
\fint_{\widetilde{Q}^-_{l-1,j}} (|D\widetilde{v}_j| + \widetilde{s}_j)^p \, dx\, dt \quad &\leq \quad c \fint_{\widetilde{Q}^-_{l,j}} \left(\frac{\widetilde{v}_j^p}{2^{lp}} + \frac{\widetilde{v}_j^2}{[\widetilde{\omega}_j(2^l)]^{2-p} 2^{lp}} \right) dx\, dt + c\widetilde{s}_j^p \\
&\leq \quad c2^{-lp} \left\{ [\widetilde{\omega}_j(2^l)]^p + 2^{lp-jp} \right\} \leq c \left[2^{-l} \widetilde{\omega}_j(2^l) \right]^p
\end{aligned}$$

for all $j > l$. The estimate in the last display, (3.6) with $\varepsilon \equiv 1$, Hölder's inequality, and (2.5), yield that

$$\begin{aligned}
&\widetilde{s}_j + \sup_{\widetilde{Q}^-_{l-2,j}} |D\widetilde{v}_j| \\
&\quad \leq 2^{-j} + 2^{1-l}\widetilde{\omega}_j(2^{l-1}) + c[2^{1-l}\widetilde{\omega}_j(2^{l-1})]^{2-p} \fint_{\widetilde{Q}^-_{l-1,j}} (|D\widetilde{v}_j| + \widetilde{s}_j)^{p-1} \, dx\, dt \\
&\quad \leq c2^{2-l}\widetilde{\omega}_j(2^{l-2})
\end{aligned}$$

for all $j > l$. The constant c depends only on n, p, ν, L. Having established the gradient bound, we are in position to apply Corollary 3.1, with parameters $r \equiv 2^{l-2}$, $\varrho = 2^4$, $\widetilde{\lambda} \equiv 1$, $\omega(\cdot) \equiv \widetilde{\omega}_j(\cdot)$ and $A \equiv c$. We thus obtain, using also (4.29) and (4.9), that

$$1/4 \leq \sup_{\widetilde{Q}^-_{4,j}} |D\widetilde{v}_j| \leq \frac{c\widetilde{\omega}_j(2^{l-2})}{2^{l-2}} \leq c$$

and

$$\operatorname*{osc}_{\widetilde{Q}^-_{4,j}} D\widetilde{v}_j \leq \frac{4c\widetilde{\omega}_j(2^{l-2})}{2^{l-2}} 2^{(6-l)\widetilde{\alpha}} \leq c2^{-l\widetilde{\alpha}}$$

where $c \equiv c(n,p,\nu,L)$ and $\widetilde{\alpha} \equiv \widetilde{\alpha}(n,p,\nu,L)$ are positive constants. Let l to be the smallest integer such that $c2^{-l\widetilde{\alpha}} \leq 1/8$, and note that this determines l as a function depending only on the parameters n, p, ν, L. Put together we can conclude that

$$1/8 \leq |D\widetilde{v}_j(x,t)| \leq c \qquad \text{for all } (x,t) \in \widetilde{Q}^-_{4,j}. \tag{4.30}$$

Now, (4.30) ensures that the rescaled equation satisfied by $\widetilde{v}_j$ is nondegenerate in $\widetilde{Q}^-_{4,j}$. Indeed, by (3.5), we have that

$$|\widetilde{a}_j(D\widetilde{v}_j(x,t))| \le L(|D\widetilde{v}_j(x,t)|^2 + \widetilde{s}_j^2)^{(p-1)/2} \le \bar{L}(|D\widetilde{v}_j(x,t)| + \widetilde{s}_j),$$
$$\langle \widetilde{a}_j(D\widetilde{v}_j(x,t)), D\widetilde{v}_j(x,t)\rangle \ge \nu |D\widetilde{v}_j(x,t)|^p - c\widetilde{s}_j^p \ge \bar{\nu}|D\widetilde{v}_j(x,t)|^2 - \bar{L}^2\widetilde{s}_j^2,$$

for some $\bar{\nu} \in (0,1)$ and $c, \bar{L} \ge 1$, all depending only on n,p,ν,L, and for all $(x,t) \in \widetilde{Q}^-_{4,j}$. Keeping the content of Remark 3.2 in mind we are able to apply the results of Section 3.4. Since $B(0,4)\times(-4,0) \subset \widetilde{Q}^-_{4,j}$ by (4.9) and $v_j \le 32$ in $\widetilde{Q}^-_{4,j}$ by (4.25), we then find δ as in Lemma 3.4 – corresponding parameters $\bar{\nu}, \bar{L}$ – depending only on n,p,ν,L as well. We arrive at

$$\widetilde{v}_j(0,0) + \widetilde{s}_j \le 2^{1+l-j} + 2^{-j} \le \delta$$

for $j \ge j_1$, where $j_1 \equiv j_1(n,p,\nu,L)$ is the smallest integer satisfying $2^{1+l-j_1} + 2^{-j_1} \le \delta$. Therefore Lemma 3.4 gives

$$\sup_{\widetilde{Q}^-_{0,j}} \widetilde{u}_j \le \sup_{\widetilde{Q}^-_{0,j}} \widetilde{v}_j \le \frac{1}{2}\,,$$

consequently proving the contradiction in (4.18). This completes the proof of Lemma 4.3.

Step 7: Proof of Lemma 4.4. The estimate in $Q^{\omega(R)}_{R,-}$ has already been obtained in Lemma 4.3 and we note that Lemma 4.3 also implies that

$$\operatorname*{osc}_{B(x_0,r)} u(\cdot,0) \le \lambda\omega(r) \qquad \text{for every } r \le R\,. \tag{4.31}$$

Therefore, with an appropriate setting of the constants, we may apply Lemma 4.2 in $Q^{\omega(R)}_{R,+}$ with initial values $g \equiv u$ to obtain the oscillation estimate also in the upper cylinder $Q^{\omega(R)}_{R,+}$ and thereby choosing a constant of the specific form $c_{4.4} := 10\max\{c_{4.2}, c_{4.3}\}^2$. This completes the proof of of Lemma 4.4.

Step 8: Proof of Lemma 4.5. To prove the lemma, we, as in the proof of Lemma 4.4, first obtain the estimate in the lower part of the cylinder and then in the upper part of the cylinder. The estimate in $Q^{\omega(R)}_{R,-}$ follows from Lemma 3.3 which also provides (4.31) in our case. To conclude with the estimate in the upper part of the cylinder $Q^{\omega(R)}_{R,+}$ we can, thanks to (4.31), apply Lemma 4.1 with initial value $g \equiv u$ to obtain the result. We can here take $c_{4.5} := 10\max\{c_{3.3}, c_{4.1}\}^2$. This completes the proof of of Lemma 4.5.

5. Proof of Theorems 1.1-1.2 and Corollary 1.1

Proof of Theorem 1.1. Recall that $\mathcal{O} = \Omega\times(0,T)$ is a space-time cylinder and that we consider the problem

$$\begin{cases} \max\{Hu, \psi - u\} = 0 & \text{in } \mathcal{O} \\ u = b & \text{on } \partial_{\mathrm{P}}\mathcal{O}, \end{cases}$$

where $b : \bar{\mathcal{O}} \to \mathbb{R}$ is continuous, $\psi \in \widetilde{C}^{\omega(\cdot)}(\mathcal{O})$, and $b \ge \psi$ on $\partial_{\mathrm{P}}\mathcal{O}$. Let $\mathcal{O}' \subset \mathcal{O}$ be a space-time cylinder such that $\bar{\mathcal{O}}' \cap \partial_{\mathrm{P}}\mathcal{O} = \emptyset$. We prove Theorem 1.1 by proceeding in three steps.

Step 1: Extension of solution. We define

$$\bar\psi(\cdot,t)=\begin{cases}\psi(\cdot,t) & t\in[0,T)\\ \psi(\cdot,T) & t\in[T,\infty)\,,\end{cases}\qquad \bar b(\cdot,t)=\begin{cases}b(\cdot,t) & t\in[0,T)\\ b(\cdot,T) & t\in[T,\infty)\,.\end{cases}$$

Then $\operatorname{osc}_{\Omega\times(0,\infty)}\bar b=\operatorname{osc}_{\bar\Omega\times[0,T]}b$ and

$$[\bar\psi]_{\widetilde C^{\omega(\cdot)}(\Omega\times(0,\infty))}=[\psi]_{\widetilde C^{\omega(\cdot)}(\Omega\times(0,T])}\,.$$

Let $\bar u$ be the unique solution to

$$\begin{cases}\max\left\{H\bar u,\bar\psi-\bar u\right\}=0 & \text{in }\Omega\times(0,\infty)\\ \bar u=\bar b & \text{on }\partial_{\mathrm P}(\Omega\times(0,\infty))\,.\end{cases}$$

By the uniqueness, $\bar u=u$ in $\Omega\times[0,T)$, and hence $\bar u$ is an extension of u. In the following we can therefore, without loss of generality, assume that $\mathcal O=\Omega\times(0,\infty)$. We also find out, by the very definition of the space $\widetilde C^{\omega(\cdot)}(\mathcal O)$ and (2.5), that

$$\operatorname*{osc}_{\Omega\times(0,\infty)}\bar\psi=\operatorname*{osc}_{\bar\Omega\times[0,T]}\psi\le\omega(\bar R)\,[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\le\bar R\,[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\equiv c\left(\Omega,T,[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\right)$$

with $\bar R=\max\{1,\operatorname{diam}\Omega,T^{1/2}\,[\psi]^{(p-2)/2}_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\}$ as clearly

$$T\le\left(\omega(R)\,[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\right)^{2-p}R^p\le[\psi]^{2-p}_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\,R^2$$

whenever $R\ge1$, again by (2.5). Note, in particular, that maximum and minimum principle then implies that

$$\operatorname*{osc}_{\mathcal O}u\le\max\left\{\operatorname*{osc}_{\mathcal O}b,\operatorname*{osc}_{\mathcal O}\psi\right\}\le\widetilde c\left(\Omega,T,[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)},\operatorname*{osc}_{\mathcal O}b\right)\,. \tag{5.1}$$

Step 2: Intrinsic geometry. After a possible enlargement of $\mathcal O'$, we may assume that $\mathcal O':=\Omega'\times(\tau,T)$ where $\Omega'\Subset\Omega$ and $\tau>0$. We let R be a number subject to the restrictions

$$R\le\operatorname{dist}(\Omega',\partial\Omega)\,,\qquad \tau\ge R^p\max\left\{\operatorname*{osc}_{\mathcal O}b,[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\,\omega(R),sR\right\}^{2-p}\,.$$

Using (2.5), and the fact that $\omega(1)=1$, we see that these conditions are satisfied if we take

$$\begin{aligned}R\;=\;\min\Bigg\{&\operatorname{dist}(\Omega',\partial\Omega)\,,\max\left\{\tau^{1/p}\widetilde c\left(\Omega,T,[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)},\operatorname*{osc}_{\mathcal O}b\right)^{(p-2)/p}\right.,\\ &\left.\tau^{1/p}\big([\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\big)^{(p-2)/p},\tau^{1/2}\big([\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)}\big)^{(p-2)/2},\tau^{1/2}s^{(p-2)/2}\right\}\Bigg\}\,,\end{aligned}\tag{5.2}$$

where $\widetilde c$ is as in (5.1). Letting $\bar c:=\max\{c_{4.1},\dots,c_{4.5}\}$, where $c_{4.1},\dots,c_{4.5}$ are the constants appearing in Lemmas 4.1–4.5, and taking

$$\lambda:=\bar c\max\left\{\widetilde c\left(\Omega,T,[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)},\operatorname*{osc}_{\mathcal O}b\right)/\omega(R),[\psi]_{\widetilde C^{\omega(\cdot)}(\mathcal O)},sR/\omega(R)\right\}\,, \tag{5.3}$$

it follows that $Q_R^{\lambda\omega(R)}(z)\subset\mathcal O$ whenever $z\in\bar{\mathcal O}'$.

Step 3: Continuity estimate. In this step we prove that the following holds whenever $z_0\in\mathcal O'$:

$$\operatorname*{osc}_{Q_r^{c\lambda\omega(r)}(z_0)}u\le c\lambda\omega(r)\qquad\text{for every }r\in(0,R/2)\,. \tag{5.4}$$

Here $c\ge2$ is a constant with the same dependence as the constant $\bar c$ introduced in Step 2. Note that if $r\ge R/2$, then

$$\operatorname*{osc}_{Q_r^{c\lambda\omega(r)}(z_0)\cap\mathcal O}u\le\operatorname*{osc}_{\mathcal O}u=\frac{\operatorname{osc}_{\mathcal O}u}{\omega(r)}\omega(r)\le\frac{\operatorname{osc}_{\mathcal O}u}{\omega(R/2)}\omega(r)\le2\lambda\omega(r) \tag{5.5}$$

by (2.5) and the definition of λ and (5.1). The proof of (5.4) will complete the proof of Theorem 1.1 since, by (5.4) and (5.5), it follows that

$$[u]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O}')} \leq c\lambda$$

with a constant c depending only on n, p, ν, L and λ as in (5.3) having the desired dependencies.

To prove (5.4) we fix $z_0 = (x_0, t_0) \in \mathcal{O}'$ and we let $\mathscr{F} := \bar{\mathcal{O}}' \cap \{u = \psi\}$. For every $\tilde{z} = (\tilde{x}, \tilde{t}) \in \mathscr{F}$ we obtain, using Lemma 4.4 with boundary datum $g = u$, that

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}(\tilde{z})} u \leq \lambda\omega(r) \qquad \text{for every } r \in (0, R)\,. \tag{5.6}$$

This proves (5.4) if $z_0 \in \mathscr{F}$. Next, suppose that $z_0 \in \mathcal{O}' \setminus \mathscr{F}$ and define $\tilde{r} > 0$ through

$$\tilde{r} := \sup\left\{r \,:\, Q_r^{\lambda\omega(r)}(z_0) \cap \mathscr{F} = \emptyset\right\}. \tag{5.7}$$

If $\tilde{r} \geq R/2$, then the desired estimate (5.4) follows immediately from Lemma 4.5. If $\tilde{r} < R/2$ then we take $\tilde{z}_0 \in \partial Q_{\tilde{r}}^{\lambda\omega(\tilde{r})}(z_0) \cap \mathscr{F}$ and first use (5.6) to conclude that

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}(\tilde{z}_0)} u \leq 2\lambda\omega(r) \qquad \text{for every } r \in (0, R)\,. \tag{5.8}$$

In particular, since $Q_r^{\lambda\omega(r)}(z_0) \subset Q_{2r}^{\lambda\omega(2r)}(\tilde{z}_0)$ by (2.6) and for all $r \in (\tilde{r}, R/2)$, we can use (5.8) and (2.5) to conclude that

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}(z_0)} u \leq \operatorname*{osc}_{Q_{2r}^{\lambda\omega(2r)}(\tilde{z}_0)} u \leq 2\lambda\omega(2r) \leq 4\lambda\omega(r) \qquad \text{for every } r \in [\tilde{r}, R/2)\,. \tag{5.9}$$

Furthermore, Lemma 4.5 applied to $Q_{\tilde{r}}^{\lambda\omega(\tilde{r})}(z_0)$ implies that

$$\operatorname*{osc}_{Q_r^{\tilde{\lambda}\omega(r)}(z_0)} u \leq \tilde{\lambda}\omega(r) \qquad \text{for every } r \in (0, \tilde{r})\,, \tag{5.10}$$

where

$$\tilde{\lambda} = c_{4.5} \max\{4\lambda\omega(\tilde{r})/\omega(\tilde{r}), s\tilde{r}/\omega(\tilde{r})\} \leq 4\bar{c}\max\{\lambda, sR/\omega(R)\} = c\lambda\,.$$

Combining (5.9)–(5.10) we see that the proof of (5.4) is complete when $z_0 \in \mathcal{O}' \setminus \mathscr{F}$. This completes the proof of Theorem 1.1. □

Proof of Theorems 1.2 and Corollary 1.1. We only prove Theorem 1.2 since Corollary 1.1 follows from Theorem 1.2 with $\psi = \inf b - 1$. To prove Theorem 1.2 we argue similarly as in the proof of Theorem 1.1 and we are here therefore quite brief and only sketch the main differences. After extending u as in Step 1 in the above proof we simply choose $R = \operatorname{dist}(\Omega', \partial\Omega)$ and define

$$\lambda := \bar{c}\max\left\{\widetilde{c}/\omega(R), [b]_{C^{\omega(\cdot)}(\Omega)}\,, [\psi]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O})}\,, sR/\omega(R)\right\}\,,$$

where $\widetilde{c} \equiv \widetilde{c}\left(\Omega, T, [\psi]_{\widetilde{C}^{\omega(\cdot)}(\mathcal{O})}\,, \operatorname{osc}_{\mathcal{O}} b\right)$ and $\bar{c} := \max\{c_{4.1}, \ldots, c_{4.5}\}$ are as in the proof of Theorem 1.1. We let $\mathscr{G} := \bar{\Omega}' \times \{0\}$ and using Lemma 4.2, with $g = b$, we see that

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}(z)\cap\mathcal{O}'} u \leq 2\lambda\omega(r) \qquad \text{for every } r \in (0, R)\,, \tag{5.11}$$

whenever $z \in \mathscr{G}$. Consider $z_1 \in \mathscr{F} := \bar{\mathcal{O}}' \cap \{u = \psi\} \cap \{t > 0\}$ and define

$$\bar{r} \equiv \bar{r}(z_1) := \sup\left\{r \leq R \,:\, Q_r^{\lambda\omega(r)}(z_1) \cap \mathscr{G} = \emptyset\right\}\,.$$

If $\bar{r} \geq R/2$, then Lemma 4.4 (applied in $Q_r^{\lambda\omega(r)}(z_1)$ with $g = u$) gives

$$\operatorname*{osc}_{Q_r^{2\lambda\omega(r)}(z_1)} u \leq 2\lambda\omega(r) \qquad \text{for every } r \in (0, R/2)\,.$$

If, on the other hand, $\bar r < R/2$, then we can find $\tilde z_1 \in \partial Q_{\bar r}^{\lambda\omega(\bar r)}(z_1)\cap \mathscr{G}$, and using (5.11) we see that

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}(\tilde z_1)\cap \mathcal{O}'} u \le 2\lambda\omega(r) \qquad \text{for every } r\in(0,R)\,.$$

Since $Q_r^{\lambda\omega(r)}(z_1)\subset Q_{2r}^{\lambda\omega(2r)}(\tilde z_1)$ by (2.6) and for all $r\in(\bar r, R/2)$, it furthermore follows that

$$\operatorname*{osc}_{Q_r^{\lambda\omega(r)}(z_1)\cap \mathcal{O}'} u \le \operatorname*{osc}_{Q_{2r}^{\lambda\omega(2r)}(\tilde z_1)\cap \mathcal{O}'} u \le 2\lambda\omega(2r)\le 4\lambda\omega(r) \quad \forall\, r\in[\bar r, R/2)\,. \tag{5.12}$$

In the final deduction we have here also used (2.5). Lemma 4.2 applied with

$$\bar\lambda = \bar c\max\{4\lambda\omega(\bar r)/\omega(\bar r), [\psi]_{\widetilde C^{\omega(\cdot)}}, s\bar r/\omega(\bar r)\}\le 4\bar c\max\{\lambda, sR/\omega(R)\} = c\lambda\,,$$

implies that

$$\operatorname*{osc}_{Q_r^{c\lambda\omega(r)}(z_1)} u \le c\lambda\omega(r) \qquad \text{for every } r\in[0,\bar r)\,. \tag{5.13}$$

Combining (5.12) and (5.13), keeping (5.11) in mind, we see that

$$\operatorname*{osc}_{Q_r^{c\lambda\omega(r)}(z_1)\cap \mathcal{O}'} u \le c\lambda\omega(r) \qquad \text{for every } r\in[0,R/2) \tag{5.14}$$

whenever $z_1\in \big(\bar{\mathcal{O}}'\cap\{u=\psi\}\big)\cup\big(\bar\Omega'\times\{0\}\big)$. Suppose now that

$$z_1\in \bar{\mathcal{O}}'\cap\{u>\psi\}\cap\{t>0\}\,.$$

In this case we combine the previous argument with the one from Theorem 1.1. More specifically, instead of (5.7) we consider

$$\bar r\equiv \bar r(z_1) := \sup\Big\{r\le R\,:\, Q_r^{\lambda\omega(r)}(z_1)\cap(\mathscr{F}\cup\mathscr{G})=\emptyset\Big\} \tag{5.15}$$

and we distinguish more cases. Firstly, if $\bar r\ge R/2$ then we use Lemma 4.5 as in the proof of Theorem 1.1. Secondly, if $\bar r<R/2$ then we distinguish two cases. Indeed, let $\bar z_1\in(\mathscr{F}\cup\mathscr{G})$ be such that $\bar z_1\in\partial_p Q_r^{\lambda\omega(r)}(z_1)$. In the first case we assume $\bar z_1\in\mathscr{G}$ and in this case we proceed as above using Lemma 4.2, see (5.11). In the second case we assume $\bar z_1\in\mathscr{F}$ and we proceed, using (5.14) instead of (5.8), as in Theorem 1.1. □

6. Proof of Theorem 1.3

We let u weakly solve

$$u_t - \operatorname{div} a(Du) = 0 \qquad \text{in } \mathcal{O}\setminus E\,,$$

and assume that $u\in \widetilde C^{\omega(\cdot)}_{\mathrm{loc}}(\mathcal{O})$ and $\mathcal{H}^{\omega(\cdot)}(E)=0$. Let $\mathcal{O}_2\Subset\mathcal{O}_1\Subset\mathcal{O}$ be two arbitrary, but fixed, smooth space-time cylinders. To prove Theorem 1.3 we only need to prove the conclusion in $\mathcal{O}_1$ since the one of being a weak solution is a local property. By the assumption $u\in\widetilde C^{\omega(\cdot)}_{\mathrm{loc}}(\mathcal{O})$ there exists $M>0$ such that

$$\operatorname*{osc}_{\mathcal{O}_1} u\le M \qquad \text{and} \qquad \operatorname*{osc}_{Q_r^{M\omega(r)}\cap\mathcal{O}_1} u\le M\omega(r)\,. \tag{6.1}$$

In what follows, we shall denote by c a constant which may vary from line to line, but which only depend on n,p,ν,L. Using the existence result in [26] we see that there exists a unique solution continuous v the obstacle problem

$$\begin{cases}\max\{Hv, u-v\}=0 & \text{in } \mathcal{O}_1\\ v=u & \text{on } \partial_{\mathrm{P}}\mathcal{O}_1\,.\end{cases}$$

Let μ be the nonnegative Riesz measure associated to v. Note that the existence of μ follows by standard arguments since v is a supersolution (see for instance [23, Theorem 2.1]). Let $F := \{(x,t) \in \mathcal{O}_1 : v(x,t) = u(x,t)\}$. We first prove that

$$\text{the support of } \mu \text{ is contained in } F \cap E. \tag{6.2}$$

To prove (6.2) it is sufficient to show that v is a weak solution to $Hv = 0$ in $\mathcal{O}_1 \setminus (F \cup E)$ in the sense of Definition 1. By Definition 2 we already know that $Hv = 0$ in $\mathcal{O}_1 \setminus F$ and it therefore remains to show that $Hv = 0$ in $Q := \mathcal{O}_1 \setminus E$. To this aim, we show that if $Q^* \Subset Q$ is a cylinder and $w \in C^0(\bar{Q}^*)$ is a weak solution to $Hw = 0$ in Q^* with $w = v$ on $\partial_{\mathrm{P}} Q^*$, then actually v must coincide with w in Q^*. Note that such a solution w exists and it is unique. We immediately see by the comparison principle that $v \geq w$ in Q^*, because v is a weak supersolution. To show that $v \leq w$ we instead argue as follows: since $u \leq v$, we also have $u \leq w$ on $\partial_{\mathrm{P}} Q^*$ and, as u solves $Hu = 0$ weakly in $Q \Supset Q^*$, the comparison principle yields $u \leq w$ in Q^*. We thus conclude that $v \leq w$ on $\partial_{\mathrm{P}} Q^* \cup F$. We are therefore in position to apply the "elliptic" comparison principle of Lemma 2.1 (with $S = Q^* \setminus F$) to deduce that $v \leq w$ in $Q^* \setminus F$ and hence also in the whole of Q^*. Therefore $v = w$ and consequently also $Hv = 0$ holds in Q^*. This completes the proof of (6.2) as Q^* can be chosen arbitrarily.

Next, using (6.1), Theorem 1.1, and a covering argument, we can conclude that there exists c, depending only on $n, p, \nu, L, M, \omega(\cdot), \mathcal{O}_1, \mathcal{O}_2$, such that

$$\operatorname*{osc}_{\mathcal{O}_1} v \leq c \qquad \text{and} \qquad \operatorname*{osc}_{Q_r^{\omega(r)}} v \leq c\omega(r)\,, \tag{6.3}$$

whenever $Q_r^{\omega(r)} \subset \mathcal{O}_2$. Consider concentric cylinders $Q_r^{\omega(r)} \subset Q_{2r}^{\omega(2r)} \subset \mathcal{O}_2$. In the following we will use the short notation $Q_r \equiv Q_r^{\omega(r)}$. Let $\phi \in C_0^\infty(Q_{2r})$ be such that

$$0 \leq \phi \leq 1\,, \quad |D\phi| \leq \frac{c}{r} \quad |\phi_t| \leq \frac{c[\omega(r)]^{p-2}}{r^p} \quad \text{and} \quad \phi \equiv 1 \quad \text{on} \quad Q_r\,.$$

Let k denote the supremum of v on Q_{2r}. Using the equation for v we have

$$\begin{aligned}
0 \quad &\leq \quad \mu(Q_r) \leq \int_{Q_{2r}} \phi^p \, d\mu \\
&= \quad \int_{Q_{2r}} \left(\langle a(Dv), D\phi^p\rangle + (k-v)(\phi^p)_t\right) \, dx\, dt \\
&\leq \quad c\int_{Q_{2r}} (|Dv| + s)^{p-1} \, |D\phi|\phi^{p-1} \, dx\, dt + \int_{Q_{2r}} |k-v||(\phi^p)_t| \, dx\, dt \\
&\leq \quad c\left(\int_{Q_{2r}} (|D(k-v)| + s)^p \, \phi^p \, dx\, dt\right)^{(p-1)/p} \left(\int_{Q_{2r}} |D\phi|^p \, dx\, dt\right)^{1/p} \\
&\quad + \int_{Q_{2r}} |k-v||(\phi^p)_t| dx\, dt\,.
\end{aligned} \tag{6.4}$$

Using (3.5) and energy estimates similar to Lemma 3.1, for the nonnegative weak subsolution $k - v$, we see that

$$\int_{Q_{2r}} |D(k-v)|^p \phi^p \, dx\, dt \leq \bar{c} \int_{Q_{2r}} \left(|k-v|^p |D\phi|^p + |k-v|^2 |(\phi^p)_t| + s^p \phi^p\right) \, dx\, dt$$

for some $\bar{c} = \bar{c}(n, p, \nu, L) \geq 1$. By (6.3) we have

$$\sup_{Q_{2r}} |k-v| \leq \operatorname*{osc}_{Q_{2r}} v \leq c\omega(r)$$

and putting the estimates together we conclude that

$$\mu(Q_r) \quad \leq \quad c\left\{[\omega(r)^2 r^n + s^p [\omega(r)]^{2-p} r^{n+p}]\right\}^{(p-1)/p} \left\{[\omega(r)^{2-p} r^n\right\}^{1/p}$$

$$+c\omega(r)r^n \leq c(1+s)^{p-1}\omega(r)r^n\,. \tag{6.5}$$

Here we have also used the estimate $[\omega(r)]^{2-p} \leq r^{2-p}$ for $r \leq 1$ and (2.5). Now, let $\mathcal{O}_3 \Subset \mathcal{O}_2$ be a space-time cylinder. We will prove that $\mu(\mathcal{O}_3) = 0$. To do this we first note, using that (6.5), that

$$\mu(Q_r^{\omega(r)}) \leq \tilde{c}r^n\omega(r) \tag{6.6}$$

whenever $Q_{2r}^{\omega(2r)} \subset \mathcal{O}_2$ and for a constant $\tilde{c} \geq 1$ which is independent of r and the center of $Q_{2r}^{\omega(2r)}$. Next, since $\mathcal{H}^{\omega(\cdot)}(E) = 0$ we obtain, for $\varepsilon > 0$ and $\delta > 0$ given (to be taken smaller that $\operatorname{dist}(\partial\mathcal{O}_3, \mathcal{O}_2)/4$), then there exists a countable family

$$\{Q_{r_i}^{\omega(r_i)}\} \equiv \{Q_{r_i}^{\omega(r_i)}(x_i, t_i)\}$$

of cylinders with $0 < r_i < \delta$, $i = 1, 2, ..$, such that $Q_{2r_i}^{\omega(2r_i)} \Subset \mathcal{O}_2$ and

$$E \cap \mathcal{O}_3 \subseteq \bigcup_i Q_{r_i}^{\omega(r_i)} \qquad \text{and} \qquad \sum_i r_i^n\omega(r_i) < \varepsilon\,. \tag{6.7}$$

Hence, using (6.6), we have

$$\mu(F \cap E \cap \mathcal{O}_3) \leq \sum_i \mu(Q_{r_i}^{\omega(r_i)}) \leq c\sum_i r_i^n\omega(r_i) < c\varepsilon\,, \tag{6.8}$$

proving that $\mu(F \cap E \cap \mathcal{O}_3) = 0$. Referring to (6.2), (6.8) and the fact that both $\mathcal{O}_2$ and $\mathcal{O}_3$ are arbitrary, we can conclude that $\mu(\mathcal{O}_1) = 0$. Thus v is a solution in $\mathcal{O}_1$. Finally, applying the above argument with u replaced by $-u$ we deduce that there exist two solutions v_1 and v_2, i.e., $Hv_1 = Hv_2 = 0$, such that $v_1 \leq u \leq v_2$ and $v_1 = v_2$ on $\partial_{\mathrm{P}}\mathcal{O}_1$. It follows that $v_1 = v_2 = u$ and the proof of Theorem 1.3 is now complete as also the open subset $\mathcal{O}_1$ is arbitrary.

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