A BOUNDARY HARNACK INEQUALITY FOR SINGULAR EQUATIONS OF p-PARABOLIC TYPE

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Abstract. We prove a boundary Harnack type inequality for non-negative solutions to singular equations of p-parabolic type, $2n/(n+1) < p < 2$, in time-independent cylinder whose base is $C^{1,1}$-regular. Simple examples show, using the corresponding estimates valid for the heat equation as a point of reference, that this type of inequalities can not, in general, be expected to hold in the degenerate case ($2 < p < \infty$).

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1. Introduction and results

In recent years the study of boundary estimates, and boundary Harnack inequalities in particular, for p-harmonic functions, $p \neq 2$, $1 < p < \infty$, in Lipschitz domains, and in domains which are well approximated by Lipschitz domains in the Hausdorff distance sense, have been advanced, see [LN1]-[LN4]. These estimates have subsequently been used, see [LN5]-[LN7], to solve several problems concerning regularity and free boundary regularity for the p-Laplace operator. In this note we initiate the study of the corresponding parabolic theory. In particular, we consider boundary estimates involving quasilinear parabolic operators of the type

$$Hu := u_t - \operatorname{div} a(Du), \tag{1.1}$$

in time-independent domains of the form $\mathcal{O} = \Omega \times (0,T) \subset \mathbb{R}^n \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $n \geq 2$. The vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular and is assumed to satisfy

$$\begin{cases} |a(\eta)| + |\partial a(\eta)||\eta| \leq L|\eta|^{p-1} \\ \nu|\eta|^{p-2}|\xi|^2 \leq \langle \partial a(\eta)\xi, \xi\rangle \end{cases} \tag{1.2}$$

whenever $\eta, \xi \in \mathbb{R}^n$, for some fixed parameters $0 < \nu \leq L$, and where, in general, p is allowed to vary in the range $1 < p < \infty$. The prototype for this type of operators is the operator

$$u \to u_t - \operatorname{div}(|Du|^{p-2}Du). \tag{1.3}$$

The operator in (1.3) is often referred to as the p-parabolic operator or the evolutionary p-Laplace operator. It is well-known, see [DB], that solutions to the p-parabolic equation exhibit quite different behaviors in the parameter regimes $2 < p < \infty$ (degenerate case) and $1 < p < 2$ (singular case). In particular, in the degenerate case the phenomenon of finite speed propagation is present and in the singular case solutions will go extinct. Furthermore, the singular case is often divided into the regimes $2n/(n+1) < p < 2$ (super-critical case) and $1 < p \leq 2n/(n+1)$ (sub-critical case) and we will here, in the singular case and due to the lack of theory in the sub-critical case, exclusively consider the super-critical case. We also note that when $p = 2$ then the evolutionary p-Laplace operator coincides with the familiar heat operator.

1.1. **Background on the linear theory.** To outline the type of results we are aiming at, and to put this ambition into context, we here first briefly discuss the corresponding linear theory and the corresponding estimates for the heat equation ($p = 2$) in the setting of bounded Lipschitz domains. We say that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain if there exists a finite set of (standard Euclidean) balls $\{B(z_i, r_i)\}$, with $z_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(z_i, r_i)\}$ constitutes a covering of an open neighbourhood of $\partial\Omega$ and such that, for each i,

$$\begin{aligned} \Omega \cap B(z_i, 4r_i) &= \{x = (x', x_n) \in \mathbb{R}^n : x_n > \phi_i(x')\} \cap B(z_i, 4r_i), \\ \partial\Omega \cap B(z_i, 4r_i) &= \{x = (x', x_n) \in \mathbb{R}^n : x_n = \phi_i(x')\} \cap B(z_i, 4r_i), \end{aligned} \tag{1.4}$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i. The Lipschitz constants of Ω are defined to be $M = \max_i \| |\nabla\phi_i| \|_\infty$ and $r_0 = \min_i r_i$. Given a bounded Lipschitz domain Ω with constants M, r_0, we let, for any $(x_0, t_0) \in \partial\Omega \times (0, T)$ and $r < \min\{r_0/\max\{2, 2M\}, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}$,

$$A_r(x_0, t_0) = (x_0 + 2Mre_n, t_0)\,, \qquad A_r^{\pm}(x_0, t_0) = (x_0 + 2Mre_n, t_0 \pm 2r^2)\,,$$

where e_n is the unit vector pointing in the positive x_n-direction and defined through the local coordinate system. Then each of these three points are contained in $\Omega \times (0, T)$ and

$$c^{-1}r < d_p((x_0, t_0), P) < cr, \quad \text{and} \quad d_p(P, \partial\Omega \times (0, T)) \geq c^{-1}r,$$

for some $c = c(n, M)$, $1 \leq c < \infty$, whenever $P \in \{A_r(x_0, t_0), A_r^+(x_0, t_0), A_r^-(x_0, t_0)\}$. Here d_p denotes the standard parabolic distance function, i.e., $d_p((x, t), (y, s)) = \max\{|x - y|, |t - s|^{1/2}\}$ whenever (x, t), $(y, s) \in \mathbb{R}^n \times \mathbb{R}$, and $d_p((x, t), \partial\Omega \times (0, T))$ is the parabolic distance from (x, t) to $\partial\Omega \times (0, T)$. We let $C_r(x, t) = B(x, r) \times (t - r^2, t + r^2)$ whenever $(x, t) \in \mathbb{R}^{n+1}$ and $r > 0$. Consider now a bounded Lipschitz domain Ω as above with constants M and r_0. Furthermore, let u and v be two nonnegative solutions to the heat equation in $(\Omega \times (0, T)) \cap C_{2r_0}(x_0, t_0)$ and assume that both u and v vanish continuously on $(\partial\Omega \times (0, T)) \cap C_{2r_0}(x_0, t_0)$. The following result is essentially due to [FGS], see also [G], [FGS], [N], [FS], [FSY], [S] and [SY] for more: There exist constants $c_1 \equiv c_1(n, M, m_u^+/m_u^-, m_v^+/m_v^-)$, $c_2 \equiv c_2(n, M)$, $1 \leq c_1, c_2 < \infty$, and $\sigma \equiv \sigma(n, M, m_u^+/m_u^-, m_v^+/m_v^-)$, $0 < \sigma < 1$, where $m_u^{\pm} = u(A^{\pm}_{r_0/c_2}(x_0, t_0))$, $m_v^{\pm} = v(A^{\pm}_{r_0/c_2}(x_0, t_0))$, such that

$$\left| \frac{u(x, t)}{v(x, t)} - \frac{u(y, s)}{v(y, s)} \right| \leq c_1 \frac{u(A_r(x_0, t_0))}{v(A_r(x_0, t_0))} \left(\frac{d_p((x, t), (y, s))}{r} \right)^{\sigma}$$

holds whenever $(x, t), (y, s) \in (\Omega \times (0, T)) \cap C_{r/4}(x_0, t_0)$ and $0 < r < r_0/c_2$. An important feature of this result is that the statement is both forward and backward in time – something which initially may seem unnatural for the heat equation considering the time-lag generally appearing in the parabolic Harnack inequality. However, the fact that u and v both vanish continuously on a large portion of $(\partial\Omega \times (0, T))$ enables one to establish an elliptic type Harnack inequality for them and subsequently the above result, see [FGS].

1.2. **Degenerate versus singular.** As mentioned above it is well-known that solutions to the p-parabolic equation exhibit quite different behaviors in the degenerate case compared to the singular case. In particular, in the degenerate case the phenomenon of finite speed propagation is present and due to this simple examples show that one can not expect the result in Section 1.1 to generalize to the p-parabolic equation in the degenerate case without imposing additional conditions on u and v. Indeed, simply consider the setting of the half space

$\mathbb{R}^{n+1}_+ := \mathbb{R}^n_+ \times \mathbb{R} = \{(x_1, ..., x_n, t) \in \mathbb{R}^n \times \mathbb{R} : \ x_n > 0\}$. Since $u = x_n$ is solution to the p-parabolic equation in $\mathbb{R}^{n+1}_+$, and since $u = x_n$ vanishes continuously on the boundary of $\mathbb{R}^{n+1}_+$, it is obvious that in the degenerate case two non-negative solutions to the p-parabolic equation in $\mathbb{R}^{n+1}_+$ need not have the same decay at the boundary since we could, in general, have a solution which is zero in a neighborhood of the boundary. In this case one could say that the solution decays exponentially at the boundary and there is no chance to control its boundary behavior, from above and below, using the linear function $u = x_n$. A slightly more advanced counterexample to a conjecture on linear growth at the boundary of the half space $\mathbb{R}^{n+1}_+$ is supplied by the function $v(x,t) = c_p(T-t)^{-1/(p-2)}x_n^{p/(p-2)}$, $p > 2$, $T > 0$ fixed, for an appropriate constant c_p. This discussion gives at hand, in particular, that in the degenerate case one can not in general expect results in the spirit of Section 1.1 unless imposing additional restrictions on the set of functions considered to enforce some initial estimates to proceed from. Currently it is not clear to us what these estimates should be.

The singular case differs considerably from the degenerate case and we emphasize that we here only consider p in the range

$$2n/(n+1) < p < 2 \tag{1.5}$$

to ensure the validity of suitable Harnack inequalities, see [DB]. In the singular case there is, though solutions may go extinct, a phenomena of infinite (in space) propagation and in the singular range the equation exhibits elliptic features as seen from the forward-backward Harnack inequality valid for positive solutions to the singular p-parabolic equation ($2n/(n+1) < p < 2$). To recall this important property we let $\mathcal{O} = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ a bounded domain and $T > 0$. Let p as in (1.5) be given and suppose that u is a nonnegative and continuous weak solution to (1.1) in $\mathcal{O}$, $(\tilde{x}_0, \tilde{t}_0) \in \mathcal{O}$, and assume that $u(\tilde{x}_0, \tilde{t}_0) > 0$. The following result has been proved in [DBGV1]. There are positive constants $c_i \equiv c_i(n, p, \nu, L)$, $i \in \{1, 2, 3\}$, such that if

$$B(\tilde{x}_0, 8r) \times (\tilde{t}_0 - c_1 u(\tilde{x}_0, \tilde{t}_0)^{2-p}(8r)^p, \tilde{t}_0 + c_1 u(\tilde{x}_0, \tilde{t}_0)^{2-p}(8r)^p) \Subset \mathcal{O}\,, \tag{1.6}$$

then

$$c_2^{-1}\big(\sup_{x\in B(\tilde{x}_0,r)} u\,(x,\tau_1)\big) \le u(\tilde{x}_0, \tilde{t}_0) \le c_2\big(\inf_{x\in B(\tilde{x}_0,r)} u\,(x,\tau_2)\big)\,, \tag{1.7}$$

for all τ_i, $i \in \{1, 2\}$, such that

$$\tau_1\,, \tau_2 \in (\tilde{t}_0 - c_3 u(\tilde{x}_0, \tilde{t}_0)^{2-p} r^p, \tilde{t}_0 + c_3 u(\tilde{x}_0, \tilde{t}_0)^{2-p} r^p)\,. \tag{1.8}$$

Hence, for p as in (1.5), a forward, backward and elliptic Harnack inequality is valid for nonnegative solutions. While this Harnack inequality still is intrinsic it distinguishes the range in (1.5) from the range $2 < p < \infty$ as in the latter case only the standard, but still intrinsic, forward in time Harnack inequality holds, see [DBGV]. Based on this discussion, and the simple examples above, we here limit ourselves to singular equations of p-parabolic type as defined in (1.1), (1.2), and for p as in (1.5), and our main result is a version of the result in Section 1.1 valid for p in the range $2n/(n+1) < p < 2$ and in the setting of time-independent $C^{1,1}$-regular cylinders. The argument outlined below does not generalize to time-independent Lipschitz or C^1-regular cylinders and we hope to develop different arguments to cope with these more challenging situations in future papers.

1.3. **Results.** We say that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$-regular domain, if there exists a finite set of balls $\{B(z_i, r_i)\}$, with $z_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(z_i, r_i)\}$ constitutes a covering of an open neighborhood of $\partial\Omega$ and such that, for each i, (1.4) holds in an appropriate coordinate system and for a $C^{1,1}$-regular function ϕ_i. Since we will prove a result which is local in space we will in the following, using that the family of operators in (1.1), (1.2) is closed under translations (in space and time) and rotations (in space), simply consider the situation when $0 \in \partial\Omega$ (translation) and $\nabla\phi(0) = 0$ (rotation), and

$$\begin{aligned} \Omega \cap B(0, 4r_0) &= \{x = (x', x_n) \in \mathbb{R}^n : x_n > \phi(x')\} \cap B(0, 4r_0)\,, \\ \partial\Omega \cap B(0, 4r_0) &= \{x = (x', x_n) \in \mathbb{R}^n : x_n = \phi(x')\} \cap B(0, 4r_0)\,, \end{aligned} \tag{1.9}$$

for some r_0 fixed and for some $C^{1,1}$-regular function ϕ. In particular, $\phi(0) = 0$ and we consider a fixed coordinate system and the existence of a $C^{1,1}$-regular function ϕ defining $\Omega \cap B(0, 4r_0)$ and $\partial\Omega \cap B(0, 4r_0)$. Using this fixed coordinate system we note that we can always, by the assumption that ϕ is $C^{1,1}$-regular and $\nabla\phi(0) = 0$, find small enough r_0 such that

$$\sup_{x' \in B'(0,4r_0)} |\nabla\phi(x')| \leq 1\,. \tag{1.10}$$

In (1.10), and in what follows, we let $x' = (x_1, \dots, x_{n-1})$ whenever $x \in \mathbb{R}^n$ and we let B' denote the orthogonal projection of B onto $\{x = (x', x_n) \in \mathbb{R}^n : x_n = 0\}$. Chosen this way, r_0 depends only on the $C^{1,1}$-norm of ϕ. Moreover, recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy a uniform inner ball condition if there exists r_0, $0 < r_0 < \infty$, such that the following is true. There exists, whenever $x_0 \in \partial\Omega$, a point $\tilde{x}_0 \in \Omega$ such that $B(\tilde{x}_0, r_0) \subset \Omega$ and $\overline{B(\tilde{x}_0, r_0)} \cap \partial\Omega = \{x_0\}$. Similarly, $\Omega \subset \mathbb{R}^n$ is said to satisfy a uniform outer ball condition if the above holds with Ω replaced by the complement of Ω. It is a well known fact, see for example [AKSZ] for a proof, that

(1.11) a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies a uniform inner (outer) ball condition for some r_0, $0 < r_0 < \infty$, if and only if Ω is $C^{1,1}$-regular.

From now on r_0 is taken to be small enough so that both (1.10) and (1.11) hold. Without loss of generality we may assume that $r_0 \in (0, 1]$.

We need to introduce cubes adapted to ϕ. Indeed, given a point $(x_0, t_0) = (x_0', \phi(x_0'), t_0) \in \mathbb{R}^{n+1}$, $r \leq r_0/(10\sqrt{n})$, and $\lambda > 0$, we let

$$\begin{aligned} Q^{\lambda,+}_{r,\phi}(x_0, t_0) &= \big\{(x, t) \in \mathbb{R}^{n+1} : \ |x_i - (x_0)_i| < r \text{ for } i \in \{1, ..., n-1\}\,, \\ &\qquad \phi(x') < x_n < 10r + \phi(x')\,,\ |t - t_0| < \lambda^{2-p} r^p\big\}\,. \end{aligned} \tag{1.12}$$

We also let

$$\begin{aligned} \Delta^{\lambda}_{r,\phi}(x_0, t_0) &= \big\{(x, t) \in \mathbb{R}^{n+1} : \ |x_i - (x_0)_i| < r \text{ for } i \in \{1, ..., n-1\}\,, \\ &\qquad x_n = \phi(x')\,,\ |t - t_0| < \lambda^{2-p} r^p\big\}\,. \end{aligned} \tag{1.13}$$

Consider $\mathcal{O} = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is as above and $T > 0$. Let $(\tilde{x}_0, \tilde{t}_0) \in \Omega_T$, assume (1.9), and assume that $\tilde{x}_0 \in \Omega \cap B(0, r_0/(100\sqrt{n}))$. Let x_0 denote the projection of $\tilde{x}_0$ along e_n and onto $\partial\Omega$ and let $r = |x_0 - \tilde{x}_0|$, $t_0 = \tilde{t}_0$. Then $x_0 \in \partial\Omega \cap B(0, r_0/(50\sqrt{n}))$. Let u and v be two functions which are nonnegative and continuous in a neighborhood of $(\tilde{x}_0, \tilde{t}_0)$ and assume that

$$\theta_u := u(\tilde{x}_0, \tilde{t}_0) \quad \text{and} \quad \theta_v := v(\tilde{x}_0, \tilde{t}_0) \quad \text{are positive}\,. \tag{1.14}$$

We also assume that

$$\begin{aligned} \theta_u^{2-p}(8r)^p &< t_0 < T - \theta_u^{2-p}(8r)^p\,, \\ \theta_v^{2-p}(8r)^p &< t_0 < T - \theta_v^{2-p}(8r)^p\,. \end{aligned} \tag{1.15}$$

To formulate our result we will assume the existence of certain intrinsic parameters associated to u and v. Indeed, given u, $(\tilde{x}_0, \tilde{t}_0)$, and r as above we let Γ_u denote the set of all values of Λ_u, $0 < \Lambda_u < \infty$, for which the following three restrictions hold. Firstly,

$$(\Lambda_u)^{2-p}(8r)^p < t_0 < T - (\Lambda_u)^{2-p}(8r)^p \,. \tag{1.16}$$

Secondly, u is assumed to be a nonnegative solution to (1.1) in $Q^{\Lambda_u,+}_{8r,\phi}(x_0,t_0)$, continuous on the closure of this set and vanishing continuously on $\Delta^{\Lambda_u}_{8r,\phi}(x_0,t_0)$. Thirdly, we assume that

$$\sup_{Q^{\Lambda_u,+}_{4r,\phi}(x_0,t_0)} u \le \Lambda_u \,. \tag{1.17}$$

We define Γ_v analogously. Note that assuming (1.15) and using the fact that $(\tilde x_0,\tilde t_0) \in Q^{\Lambda_u,+}_{4r,\phi}(x_0,t_0)$, $(\tilde x_0,\tilde t_0) \in Q^{\Lambda_v,+}_{4r,\phi}(x_0,t_0)$, we see that any such Λ_u and Λ_v must satisfy $\theta_u \le \Lambda_u$ and $\theta_v \le \Lambda_v$. In the following we will assume that u, v, $(\tilde x_0,\tilde t_0)$, r, T are such that

$$\text{(1.15) holds and } \Gamma_u \neq \emptyset \text{ and } \Gamma_v \neq \emptyset \,. \tag{1.18}$$

Based on (1.18) we in the following let Λ_u and Λ_v denote the smallest values of Λ_u and Λ_v for which (1.17), and the corresponding statement for Λ_v, hold. Note that for these values of Λ_u, Λ_v, we can assume, without loss of generality, that

$$\sup_{Q^{\Lambda_u,+}_{4r,\phi}(x_0,t_0)} u = \Lambda_u \,, \quad \sup_{Q^{\Lambda_v,+}_{4r,\phi}(x_0,t_0)} v = \Lambda_v \,. \tag{1.19}$$

The relevance of this complexity, and of the parameters Λ_u, Λ_v, is outlined below. In this paper we prove the following theorem.

Theorem 1.1. *Let $\mathcal{O} = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$-regular domain and $T > 0$. Let p as in* (1.5) *be fixed, and let H be as in* (1.1), (1.2). *Let $(\tilde x_0,\tilde t_0)$, (x_0,t_0), and r be as above. Let u and v be solutions to* (1.1) *as stated above, and assume that they satisfy* (1.14) *and* (1.15). *Assume further* (1.18) *and let Λ_u and Λ_v be such that* (1.19) *holds. Assume in addition that there exist λ_u, λ_v, $1 \le \lambda_u < \infty$, $1 \le \lambda_v < \infty$, such that*

$$\theta_u \le \Lambda_u \le \lambda_u\theta_u \,, \quad \theta_v \le \Lambda_v \le \lambda_v\theta_v \,. \tag{1.20}$$

Then there exist constants $c_1 \equiv c_1(n,p,\nu,L,r_0,\lambda_u,\lambda_v)$, $c_2 \equiv c_2(n,p,\nu,L,r_0)$, $1 \le c_1, c_2 < \infty$, and $\sigma \equiv \sigma(n,p,\nu,L)$, $0 < \sigma < 1$, such that

$$\left|\frac{u(x,t)}{v(x,t)} - \frac{u(y,s)}{v(y,s)}\right| \le c_1\frac{\theta_u}{\theta_v}\left(\frac{|x-y|}{r} + \left(\frac{1}{\theta_{uv}}\right)^{(2/p-1)}\left(\frac{|s-t|}{r^p}\right)^{1/p}\right)^{\sigma}$$

holds whenever $(x,t),(y,s) \in Q^{\theta_{uv},+}_{r/c_2,\phi}(x_0,t_0)$, where $\theta_{uv} = \min\{\theta_u,\theta_v\}$.

Note that when $p = 2$, then, formally and essentially, Theorem 1.1 coincides with the linear result described in Section 1.1. Indeed, in the case $p = 2$ we see that if both u and v vanish on a sufficiently large portion of the lateral boundary, so that we can, following [S], [FGS], ensure the validity of the forward-backward in time Harnack inequalities

$$\sup_{Q^{1,+}_{4r,\phi}(x_0,t_0)} u \le c\theta_u \,, \quad \sup_{Q^{1,+}_{4r,\phi}(x_0,t_0)} v \le c\theta_v \,, \tag{1.21}$$

for some c, $1 \le c < \infty$, independent of r and $(\tilde x_0,\tilde t_0)$, then

$$\Lambda_u := \sup_{Q^{1,+}_{4r,\phi}(x_0,t_0)} u \le c\theta_u \,, \ \Lambda_v := \sup_{Q^{1,+}_{4r,\phi}(x_0,t_0)} v \le c\theta_v \,. \tag{1.22}$$

Hence, formally the statement of Theorem 1.1 reduces to

$$\left|\frac{u(x,t)}{v(x,t)} - \frac{u(y,s)}{v(y,s)}\right| \le c_1\frac{u(\tilde x_0,\tilde t_0)}{v(\tilde x_0,\tilde t_0)}\left(\frac{d_p((x,t),(y,s))}{r}\right)^{\sigma}$$

whenever $(x,t),(y,s)\in Q^{1,+}_{r/c_2,\phi}(x_0,t_0)$. We emphasize that in Section 1.1 the involved dependence of the constants described enters through the constant c in (1.21). We also note that while, formally and when $p=2$, the statement of Theorem 1.1 reduces to a form similar to that in Section 1.1, we are not saying that the constants of Theorem 1.1 by necessity are stable as $p\to 2$.

1.4. **Intrinsic scaling parameters.** A crucial ingredient in the regularity theory for the operator in (1.3) is the use of DiBenedetto's intrinsic geometry when deriving local estimates. This amounts to the use of cylinders whose size depends on the solution itself and this is necessary since operators as the one considered in (1.3) show a strong anisotropy when $p\neq 2$ as the multiplication of a solution to the associated equation by a constant does not yield a solution to a similar equation. One consequence of this is the lack of homogeneous a priori estimates and hence the impossibility to use such estimates in iterative schemes in line with the standard regularity techniques. Instead, the lack of homogeneity must be locally corrected by using scaling parameters and intrinsic geometries and a key insight from regularity theory, see [DBF, DB, KiL, AM, KM1, KM2, KMN], is that in general the type of cylinders used must depend on the type of problem/regularity one is currently considering/using. In the context of Theorem 1.1 we see that Λ_u, Λ_v, λ_u, and λ_v serve as intrinsic (scaling) parameters. Indeed, concerning the conditions in (1.19) we note, focusing on u, assuming $(x_0,t_0)=(0,0)$, $r=1$, $(\tilde x_0,\tilde t_0)=(e_n,0)$, that if we define $\tilde u(x,t)=u(x,t\Lambda_u^{2-p})/\Lambda_u$ for $(x,t)\in Q^{1,+}_{4,\phi}(0,0)$ then, by construction,

$$\sup_{Q^{1,+}_{4,\phi}(0,0)} \tilde u = 1\,. \tag{1.23}$$

In particular, in this way we can simultaneously normalize the scale to unit scale and the supremum of the function on the unit scale to 1. This enables us to ensure the validity of estimates for the gradient of $\tilde u$ with constants depending only on n,p,ν,L. Furthermore, the parameter λ_u ensures a relation between the largest value of u on the large box $Q^{\Lambda_u,+}_{4r,\phi}(x_0,t_0)$ and the value of u at $(\tilde x_0,\tilde t_0)$, θ_u. This relation cannot in general be expected to hold, due to the phenomena of extinction present in the singular case, for some uniform λ not depending on u. Indeed, u could very well go extinct at $t=\tilde t_0+\varepsilon$ while $\theta_u\neq 0$. Restarting the Cauchy-Dirichlet problem at $t=\tilde t_0+2\varepsilon$ enforcing large positive data on parts of the lateral side of the cylinder from $t=\tilde t_0+2\varepsilon$ and putting zero data on the base of the cylinder at $t=\tilde t_0+2\varepsilon$, we can construct a solution u such that the supremum of u is large while θ_u is very small resulting in a very large value of λ_u. This question is related to the possibility of establishing Carleson type estimates, see [S], for nonnegative solutions to the p-parabolic equation and for an account of this type of estimates, for the p-parabolic equation and for the porous medium equation, we refer to [AGS]. The condition $\theta_u\le\Lambda_u\le\lambda_u\theta_u$ also allows us to construct elliptic type Harnack chains, depending on λ_u, to compare values of u close to the boundary and as outlined in the bulk of the paper. In particular, after appropriate normalizations, using (1.20), the construction of elliptic Harnack chains becomes analogous to the construction of Harnack chains for the Laplace operator, see [JK].

2. Preliminaries

2.1. **Weak solution and the Dirichlet problem.** If $U\subset\mathbb{R}^n$ is open and $1\le q\le\infty$, then by $W^{1,q}(U)$, we denote the space of equivalence classes of functions f with distributional gradient $Df=(f_{x_1},\dots,f_{x_n})$, both of which are q-th power integrable on U. Let

$$\|f\|_{W^{1,q}(U)}=\|f\|_{L^q(U)}+\|\,|Df|\,\|_{L^q(U)}$$

be the norm in $W^{1,q}(U)$, where $\|\cdot\|_{L^q(U)}$ denotes the usual Lebesgue q-norm in U. Given $t_1 < t_2$ we denote by $L^q(t_1, t_2, W^{1,q}(U))$ the space of functions such that for almost every t, $t_1 < t < t_2$, the function $x \to u(x,t)$ belongs to $W^{1,q}(U)$ and

$$\|u\|_{L^q(t_1,t_2,W^{1,q}(U))} := \left(\int_{t_1}^{t_2}\int_U \Big(|u(x,t)|^q + |Du(x,t)|^q\Big)dxdt\right)^{1/q} < \infty\,.$$

In the following we here first describe the concept of weak solutions to

$$Hu = u_t - \operatorname{div} a(Du) = 0 \tag{2.1}$$

when the underlying domain considered is not necessarily a cylinder.

Definition 1. Let H be as in (2.1) and assume (1.2). We say that a function u is a weak supersolution (subsolution) to (2.1) in an open set $\Xi \Subset \mathbb{R}^{n+1}$ if, whenever $\Xi' = U \times (t_1, t_2) \Subset \Xi$ with $U \subset \mathbb{R}^n$ and $t_1 < t_2$, then $u \in L^p(t_1, t_2; W^{1,p}(U))$ and

$$\int_{\Xi'} \left(\langle a(Du), D\phi\rangle - u\phi_t\right)\, dx\, dt \;\geq\; (\leq)\; 0 \tag{2.2}$$

for all nonnegative $\phi \in C_0^\infty(\Xi')$. A weak solution is a distributional solution satisfying (2.2) with equality and without sign restrictions for test functions.

Note, in particular, that in Definition 1 no assumption on the time derivative of u is made. Note also that by parabolic regularity theory, see [DB] solutions are locally Hölder continuous after a redefinition on a set of measure zero. In particular, we can in the following assume that any solution u is continuous. Furthermore, we note that it is well known that if, for example, $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$-regular domain then the cylinder $\mathcal{O}$ is regular for the Dirichlet problem for the operators in (1.1), (1.2), see [KL]. In particular, given continuous boundary data $b : \bar{\mathcal{O}} \to \mathbb{R}$ the problem

$$\begin{cases} Hu = 0 & \text{in } \mathcal{O} \\ u = b & \text{on } \partial_{\mathrm{P}}\mathcal{O}\,, \end{cases} \tag{2.3}$$

has a unique weak solution which is continuous on the closure on $\mathcal{O}$. Here $\partial_{\mathrm{P}}\mathcal{O} := \big(\bar{\Omega} \times \{0\}\big) \cup (\partial\Omega \times [0,T])$ denotes the parabolic boundary of $\mathcal{O}$.

2.2. **Scaling of solutions.** We here describe scaling properties of (weak) solutions to the equation (2.1). Given $r, \lambda > 0$, we consider the cube

$$\begin{aligned} Q_r^\lambda \quad = \quad & \{(x,t) \in \mathbb{R}^{n+1} :\ |x_i| < r \text{ for } i \in \{1, ..., n-1\}\,, \\ & |x_n| < 10r\,,\ |t| < \lambda^{2-p} r^p\}\,, \end{aligned} \tag{2.4}$$

and given $(\tilde{x}_0, \tilde{t}_0) \in \mathbb{R}^{n+1}$ we let $Q_r^\lambda(\tilde{x}_0, \tilde{t}_0)$ be the cube Q_r^λ translated to the point $(\tilde{x}_0, \tilde{t}_0)$. Given $(\tilde{x}_0, \tilde{t}_0) \in \mathbb{R}^{n+1}$, $r \leq R, \lambda > 0$, suppose that u solves (2.1) in $Q_R^\lambda(\tilde{x}_0, \tilde{t}_0)$. Define

$$\tilde{u}(x,t) := \frac{u(\tilde{x}_0 + rx, \tilde{t}_0 + \lambda^{2-p} r^p t)}{\lambda}\,, \tag{2.5}$$

$$\tilde{a}(\eta) := \frac{a((\lambda/r)\eta)}{(\lambda/r)^{p-1}}\,, \qquad \eta \in \mathbb{R}^n\,. \tag{2.6}$$

Then $\tilde{u}$ solves the equation $\tilde{H}\tilde{u} := \tilde{u}_t - \operatorname{div} \tilde{a}(D\tilde{u}) = 0$ in $Q_{R/r}^1$. In particular, in the case $r = R$, we have that $\tilde{w}$ is a solution in Q_1^1. The new vector field $\tilde{a}(\cdot)$ satisfies bounds

$$\begin{cases} |\tilde{a}(\eta)| + |\partial\tilde{a}(\eta)||\eta| \leq L|\eta|^{(p-1)} \\ \nu|\eta|^{p-2}|\xi|^2 \leq \langle \partial\tilde{a}(\eta)\xi, \xi\rangle\,, \end{cases} \tag{2.7}$$

for all $\eta, \xi \in \mathbb{R}^n$. In particular, we remark that the assumptions in (1.2) imply the existence of $\bar{\nu} \in (0,1)$, and $c, \bar{L} \geq 1$, depending on n, p, ν, L, such that the following growth and coercivity assumptions hold for every choice $\eta \in \mathbb{R}^n$:

$$|\tilde{a}(\eta)| \leq \bar{L}|\eta|^{(p-1)}, \qquad \langle \tilde{a}(\eta), \eta \rangle \geq \bar{\nu}|\eta|^p . \tag{2.8}$$

2.3. **Gradient estimates.** We note that in the proof of Theorem 1.1 we can without loss of generality assume that

$$(x_0, t_0) = (0,0), \quad r = 1, \quad (\tilde{x}_0, \tilde{t}_0) = (e_n, 0) . \tag{2.9}$$

We here formulate, assuming (2.9), a boundary gradient estimate to be used in the proof of Theorem 1.1.

Lemma 2.1. *Let u be as in Theorem 1.1 and assume also that* (2.9) *holds. Let $\tilde{u}(x,t) = u(x, t\Lambda_u^{2-p})/\Lambda_u$ for $(x,t) \in Q^{1,+}_{4,\phi}(0,0)$ so that, by construction,*

$$\sup_{Q^{1,+}_{4,\phi}(0,0)} \tilde{u} = 1 . \tag{2.10}$$

Then $D\tilde{u}$ exists and is continuous up to $\Delta^1_{2,\phi}(0,0)$ and there exist constants $c \equiv c(n,p,\nu,L)$, $1 \leq c < \infty$, and $\sigma \equiv \sigma(n,p,\nu,L)$, $0 < \sigma \leq 1$, such that

$$|Du(x,t)| \leq c, \qquad |D\tilde{u}(x,t) - D\tilde{u}(y,s)| \leq c(|x-y| + |t-s|^{1/p})^{\sigma} \tag{2.11}$$

hold whenever $(x,t), (y,s) \in Q^{1,+}_{1,\phi}(0,0)$.

Proof. This is a special case of Theorem 0.1 in [L]. □

3. Proof of Theorem 1.1

To prove Theorem 1.1 we first use a barrier type argument to establish linear growth estimates at the boundary and here the assumption that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$-regular domain is important. Our proof of Theorem 1.1 is based on the following lemma. In the formulation r_0 is as described in Section 1.3.

Lemma 3.1. *Let u be as in Theorem 1.1 and assume also that* (2.9) *holds. Then there exist constants $c \equiv c(n,p,\nu,L,r_0)$, $1 \leq c < \infty$, and $\varrho_0 \equiv \varrho_0(n,p,\nu,L)$, $0 < \varrho_0 < 1$, such that*

$$c^{-1}\theta_u \cdot [x_n - \phi(x')] \leq u(x,t) \leq c\lambda_u\theta_u \cdot [x_n - \phi(x')] \tag{3.1}$$

holds whenever $(x,t) \in (\Omega \cap (B'(0,r_0) \times [-1,1]) \times (-\theta_u^{2-p}\varrho_0^p, \theta_u^{2-p}\varrho_0^p)$.

Proof. First we notice that if $x \in \Omega \cap B(0,r_0)$ and $\hat{x} \in \partial\Omega$ is such that $|x - \hat{x}| = \operatorname{dist}(x, \partial\Omega)$, then

$$\operatorname{dist}(x,\partial\Omega) \leq |x_n - \phi(x')| \leq |x_n - \hat{x}_n| + |\phi(\hat{x}') - \phi(x')| \leq 2\operatorname{dist}(x,\partial\Omega)$$

holds by (1.10) since $\hat{x}_n = \phi(\hat{x}')$. Thus it is enough to prove (3.1) with $x_n - \phi(x')$ replaced with $\operatorname{dist}(x,\partial\Omega)$.

Focusing on the proof of the inequality on the right hand side in (3.1) we consider $\tilde{u}(x,t) = u(x, t\Lambda_u^{2-p})/\Lambda_u$ for all (x,t) such that $(x, t\Lambda_u^{2-p}) \in Q^{\Lambda_u,+}_{4,\phi}(0,0)$, i.e, for $(x,t) \in Q^{1,+}_{4,\phi}(0,0)$. Note that by construction

$$\sup_{Q^{1,+}_{4,\phi}(0,0)} \tilde{u} = 1 . \tag{3.2}$$

In accordance to (1.11) we in the following let $\delta \in (0, r_0/10]$ and we let

$$\Omega_\delta := \Omega \cap (-2,2)^n \cap \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \leq \delta\} . \tag{3.3}$$

where $(-2,2)^n$ denotes the n-dimensional cube $(-2,2) \times ... \times (-2,2)$. Consider then $(z,\tau) \in \Omega_\delta \times (-3,3) \subset Q^{1,+}_{4,\phi}(0,0)$. Given z, let $\hat{z} \in \partial\Omega$ be such that $|z - \hat{z}| = \operatorname{dist}(z,\partial\Omega) < \delta$, and for this $\hat{z} \in \partial\Omega$ we use (1.11) and let $\hat{z}_{r_0} \in \mathbb{R}^n \setminus \bar{\Omega}$ be such that

$B(\hat z_{r_0}, r_0) \subset (\mathbb{R}^n \setminus \bar\Omega)$ and $\overline{B(\hat z_{r_0}, r_0)} \cap \partial\Omega = \{\hat z\}$. Moreover, we set $I(\tau) = (\tau - 1, \tau]$. Focusing on the domain $\Omega_\delta \times I(\tau)$, considering $(z, \tau) \in \Omega_\delta \times I(\tau)$ fixed, we construct, following the construction in [SV, display (4.7)], a barrier as follows. We let

$$\begin{aligned} \psi(x,t) &:= C(1 - \eta(x,t))\,, \\ \eta(x,t) &:= \exp(-\delta^{-1}(|x - \hat z_{r_0}| - r_0)) \exp(t - \tau)\,, \end{aligned} \tag{3.4}$$

where C shall be fixed shortly. Consider the set

$$D_\delta := \{x \in \Omega : \ |x - \hat z_{r_0}| - r_0 < \delta\} \times I(\tau)\,. \tag{3.5}$$

If $(x,t) \in D_\delta$, then there is $y \in \partial\Omega$ such that $|x - \hat z_{r_0}| = |x - y| + |y - \hat z_{r_0}|$ and

$$\delta + r_0 > |x - \hat z_{r_0}| = |x - y| + |y - \hat z_{r_0}| \geq |x - y| + r_0\,,$$

and thus $D_\delta \subset \Omega_\delta \times I(\tau)$. Moreover, $(z, \tau) \in D_\delta$. We now want to prove that $\delta = \delta(p, \nu, L, r_0)$, $0 < \delta \ll 1$, can be chosen so that ψ satisfies

$$\tilde u(x,t) \leq \psi(x,t) \text{ whenever } (x,t) \in D_\delta\,. \tag{3.6}$$

Using the comparison principle we see that to achieve this it is enough to prove

$$\tilde u(x,t) \leq \psi(x,t) \qquad \text{on } \partial_P D_\delta\,, \tag{3.7}$$

$$\partial_t \psi - \operatorname{div} a(D\psi) \geq 0 \quad \text{in } D_\delta\,. \tag{3.8}$$

Appealing to (3.2) and the fact that $\tilde u = 0$ on $\partial\Omega \times (-4, 4)$, it is easy to see that (3.7) is verified if we let $C = (1 - \exp(-1))^{-1}$. Hence we focus on (3.8) and we first note that

$$\partial_t \psi - \operatorname{div} a(D\psi) = \partial_t \psi - \partial_{\eta_j} a_i(D\psi) \partial_{x_i x_j} \psi\,, \tag{3.9}$$

and

$$\partial_t \psi = -C\eta\,, \qquad D\psi = \delta^{-1} C\eta \frac{x - \hat z_{r_0}}{|x - \hat z_{r_0}|}\,. \tag{3.10}$$

Furthermore, we have that

$$\begin{aligned} (C\eta)^{-1} \partial_{x_i x_j} \psi \;&=\; -\delta^{-2} \frac{(x - \hat z_{r_0})_i}{|x - \hat z_{r_0}|} \frac{(x - \hat z_{r_0})_j}{|x - \hat z_{r_0}|} \\ &\quad + \delta^{-1} \frac{\delta_{ij}}{|x - \hat z_{r_0}|} - \delta^{-1} \frac{(x - \hat z_{r_0})_i (x - \hat z_{r_0})_j}{|x - \hat z_{r_0}|^3}\,. \end{aligned} \tag{3.11}$$

Let

$$\begin{aligned} A \;:=\; & -1 + \delta^{-2} \partial_{\eta_j} a_i(D\psi) \Big(\frac{(x - \hat z_{r_0})_i}{|x - \hat z_{r_0}|} \frac{(x - \hat z_{r_0})_j}{|x - \hat z_{r_0}|} \Big) \\ & + \delta^{-1} \partial_{\eta_j} a_i(D\psi) \Big(\frac{(x - \hat z_{r_0})_i (x - \hat z_{r_0})_j}{|x - \hat z_{r_0}|^3} - \frac{\delta_{ij}}{|x - \hat z_{r_0}|} \Big)\,. \end{aligned} \tag{3.12}$$

Then using (3.11) and the notation in (3.12) we see that

$$(C\eta)^{-1} (\partial_t \psi - \partial_{\eta_j} a_i(D\psi) \partial_{x_i x_j} \psi) = A\,, \tag{3.13}$$

and we are left with the task to prove that A can be constructed to be nonnegative. Using the assumption in (1.2) we see that

$$A \geq -1 + \delta^{-2} \nu |D\psi|^{p-2} + \delta^{-1} |D\psi|^{p-2} \frac{(\nu - L)}{|x - \hat z_{r_0}|}\,. \tag{3.14}$$

Next, since $|x - \hat z_{r_0}| \geq r_0$ in D_δ we can conclude, recalling $L \geq \nu > 0$, that

$$A \geq -1 + \delta^{-1} (\delta^{-1} \nu - L/r_0) |D\psi|^{p-2}\,. \tag{3.15}$$

Using that $|D\psi| = \delta^{-1} C\eta$ we see from the last display that

$$A \geq -1 + \delta^{-1} (\delta^{-1} \nu - L/r_0) (\delta^{-1} C\eta)^{p-2}. \tag{3.16}$$

We now restrict δ so that δ^{-1} is larger than $2L/(\nu r_0)$. Consequently we obtain

$$A \geq -1 + \frac{1}{2}\delta^{-p}(C\eta)^{p-2}. \tag{3.17}$$

Using (3.17), together with the facts that $p < 2$ and $\eta \leq 1$ in D_δ, we can conclude that

$$A \geq -1 + \frac{1}{2}(1 - \exp(-1))^{2-p}\delta^{-p}\,. \tag{3.18}$$

Hence, if we let $\delta = \min\{r_0/10, \nu r_0/(2L), 4^{-1/p}\}$, then $A \geq 0$. In particular, we can conclude the validity of (3.6). We now apply (3.6) with $(x,t) \equiv (z,\tau)$ and, appealing to the elementary inequalities

$$(1 - \exp(-1))s \leq 1 - \exp(-s) \leq s \qquad \forall\, s \in [0,1]\,, \tag{3.19}$$

we obtain

$$\begin{aligned} \tilde{u}(z,\tau) &\leq \psi(z,\tau) \\ &= C\left(1 - \exp(-\delta^{-1}\operatorname{dist}(z,\partial\Omega)\right) \\ &\leq C\delta^{-1}\operatorname{dist}(z,\partial\Omega)\,. \end{aligned}$$

This allows us to conclude the proof of the right hand side inequality in (3.1) after scaling back to the original solution u and applying (1.20).

Focusing then on the proof of the inequality on the left hand side in (3.1), we consider $\tilde{u}(x,t) = u(x, t\theta_u^{2-p})/\theta_u$ for all (x,t) such that $(x, t\theta_u^{2-p}) \in Q_{4,\phi}^{\theta_u,+}(0,0)$, i.e, for $(x,t) \in Q_{4,\phi}^{1,+}(0,0)$. We then have that

$$\tilde{u}(e_n, 0) = 1\,. \tag{3.20}$$

Using the Harnack inequality outlined in (1.6)-(1.8) we see, using also (3.20), that there exists $\kappa \equiv \kappa(n,p,\nu,L)$, $0 < \kappa < 1$, and $c_2 \equiv c_2(n,p,\nu,L)$, $1 < c_2 < \infty$, such that

$$c_2^{-1} \leq \inf_{t \in (-4\kappa^2, 4\kappa^2)} \tilde{u}(e_n, t) \leq \sup_{t \in (-4\kappa^2, 4\kappa^2)} \tilde{u}(e_n, t) \leq c_2\,. \tag{3.21}$$

We let $\delta \in (0, r_0/10]$. Consider $(z,\tau) \in (\Omega_\delta \cap B(0,r_0)) \times (-\kappa^2, \kappa^2) \subset Q_{\kappa,\phi}^{2,+}(0,0)$, where Ω_δ is defined in (3.3). Given z, we again let $\hat{z} \in \partial\Omega$ be such that $|z - \hat{z}| = \operatorname{dist}(z,\partial\Omega)$ and we use (1.11) to find $\hat{z}^{r_0} \in \Omega$ such that $B(\hat{z}^{r_0}, r_0) \subset \Omega$ and $\overline{B(\hat{z}^{r_0}, r_0)} \cap \partial\Omega = \{\hat{z}\}$. Since the normal vector of the surface $\{x_n = \phi(x')\}$ is continuous, we necessarily have that $z - \hat{z}$ and $\hat{z}^{r_0} - \hat{z}$ are parallel and consequently, due to the orientation,

$$r_0 > |z - \hat{z}^{r_0}| = r_0 - \operatorname{dist}(z,\partial\Omega) > r_0 - \delta\,. \tag{3.22}$$

Given $\tau \in (-\kappa^2, \kappa^2)$, we also let $I_\kappa(\tau) = (\tau - \kappa^2, \tau]$. Furthermore, we set

$$m_\delta := \inf_{E^\delta \times (-2\kappa^2, 2\kappa^2)} \tilde{u}(x,t)\,, \tag{3.23}$$

where $E^\delta := \Omega \cap (-1,1)^n \cap \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \geq \delta\}$, $(-1,1)^n$ is the n-dimensional cube $(-1,1) \times ... \times (-1,1)$. With (z,τ) fixed, we construct, following the proof of [SV, Proposition 4.2], a barrier as follows. We let

$$\begin{aligned} &\tilde{\psi}(x,t) := \exp(-1)m_\delta(\tilde{\eta}(x,t) - 1)\,, \\ &\tilde{\eta}(x,t) := \exp(-\delta^{-1}(|x - \hat{z}^{r_0}| - r_0))\exp((t-\tau)/\kappa^2)\,. \end{aligned} \tag{3.24}$$

In this case we consider the set

$$\tilde{D}_\delta := \{x \in \Omega : \; r_0 - \delta < |x - \hat{z}^{r_0}| < r_0\} \times I_\kappa(\tau)\,.$$

In particular, by (3.22) we have that $z \in \tilde D_\delta$. We now want to prove that δ can be chosen so that $\tilde\psi$ satisfies

$$\tilde\psi(x,t) \le \tilde u(x,t) \text{ whenever } (x,t) \in \tilde D_\delta\,. \tag{3.25}$$

Again, using the comparison principle we see that to achieve this it is enough to prove that

$$\tilde\psi(x,t) \le \tilde u(x,t) \qquad \text{on } \partial_P \tilde D_\delta\,, \tag{3.26}$$

$$\partial_t\tilde\psi - \operatorname{div} a(D\tilde\psi) \le 0 \ \text{ in } \tilde D_\delta\,. \tag{3.27}$$

To obtain (3.26), note first that if $(x,t) \in \{x \in \Omega : |x - \hat z^{r_0}| = r_0\} \times I_\kappa(\tau)$, then $\tilde\psi(x,t) \le 0$, and for $(x,t) \in \{x \in \Omega : |x - \hat z^{r_0}| = r_0 - \delta\} \times I_\kappa(\tau)$ we have that $\tilde\psi(x,t) \le (1 - \exp(-1))m_\delta \le \tilde u(x,t)$. Furthermore, if $(x,t) \in \{x \in \Omega : r_0 - \delta < |x - \hat z^{r_0}| < r_0\} \times \{t : t = \tau - \kappa^2\}$, then we see that $\tilde\psi(x,t) \le 0$. Hence, it only remains to prove that we can choose $\delta \equiv \delta(p,\nu,L,r_0,\kappa)$, $0 < \delta \ll 1$, small enough so that we can prove (3.27). However, this can be verified by a direct calculations along the lines of the corresponding argument used to prove the right hand side inequality in (3.1) and we omit the details. Applying (3.25) to $(x,t) = (z,\tau)$ and using (3.19) and (3.22) we see that

$$\tilde u(z,\tau) \ge \tilde\psi(z,\tau) \ge \exp(-1)(1 - \exp(-1))m_\delta \operatorname{dist}(z,\partial\Omega)\,. \tag{3.28}$$

To complete the proof it now only remains to use (3.21) to bound m_δ from below with a constant depending only on n,p,ν,L,r_0. For this we assume the contrary, i.e., we assume that there is a point

$$(\tilde z_0, \tilde t_0) \in E^\delta \times (-2\kappa^2, 2\kappa^2) \quad \text{such that} \quad \tilde u(\tilde z_0, \tilde t) \le \varepsilon$$

for some $\varepsilon \in (0,1)$ to be chosen suitably small. We show that an elliptic Harnack chain can be constructed, similar to the case of the Laplace equation, see for instance [JK], to obtain contradiction with (3.21). To this end, take $\varrho = \delta/10$ and let k be the smallest integer such that $k \ge 4^{n+1}10/\delta$. We may choose $(z^j)_{j=0}^{k+1}$ so that $|z^{j+1} - z^j| < \varrho/2$, $\operatorname{dist}(z^j, \partial\Omega) \ge 10\varrho$ for all $j \in \{0,1,\dots,k\}$, and so that $z^{k+1} = e_n$. Using the elliptic Harnack inequality as outlined in (1.6)-(1.8), together with (1.15) and (1.11), we find that $u(e_n,\tilde t) \le \varepsilon c_2^{k+1} = c_2^{-2} < c_2^{-1} < 1$ provided that we choose $\varepsilon := c_2^{-k-3}$. Chosen this way ε depends only on n,p,ν,L,r_0. An important feature in the construction is that $u(z^j,\tilde t) \le 1$ for all $j \in \{0,1,\dots,k\}$ and hence also $u(z^j,\tilde t)^{2-p} \le 1$. Thus we obtain a contradiction with (3.21) and the proof is complete. □

Remark 3.1. Note that the inequality on the right hand side in (3.1) in Lemma 3.1 could also be proved by referring to the gradient estimate in Lemma 2.1. However, we have chosen to present a barrier type argument.

3.1. **The final argument.** Let u be as in Theorem 1.1 and assume also that (2.9) holds. Define

$$\tilde u(x,t) = u(x, t\Lambda_u^{2-p})/\Lambda_u,\ \ \tilde v(x,t) = v(x, t\Lambda_v^{2-p})/\Lambda_v$$

whenever $(x,t) \in Q^{1,+}_{4,\phi}(0,0)$. Then

$$\sup_{Q^{1,+}_{4,\phi}(0,0)} \tilde u = 1 = \sup_{Q^{1,+}_{4,\phi}(0,0)} \tilde v\,. \tag{3.29}$$

Using Lemma 2.1 we see that $D\tilde w$ exists and is continuous up to $\Delta^1_{2,\phi}(0,0)$ for $\tilde w \in \{\tilde u, \tilde v\}$ and that there exist $\rho = \rho(n,p,\nu,L)$, $0 < \rho \le 1$, $c = c(n,p,\nu,L)$, $1 \le c < \infty$, and $\sigma = \sigma(n,p,\nu,L)$, $0 < \sigma \le 1$ such that

$$|D\tilde w(x,t) - D\tilde w(y,s)| \le c(|x-y| + |t-s|^{1/p})^\sigma \tag{3.30}$$

holds whenever $(x,t),(y,s)\in Q^{1,+}_{\rho,\phi}(0,0)$ and for $\tilde w\in\{\tilde u,\tilde v\}$. In the following we let h be the function $h(x,t)=h(x',x_n,t)=x_n-\phi(x')$. Consider $(x,t),(y,s)\in Q^{1,+}_{\rho,\phi}(0,0)$ and $\tilde w\in\{\tilde u,\tilde v\}$. Then, using that $\tilde w(x',\phi(x'),t)=0$ for $(x',\phi(x'),t)\in\Delta^1_{2,\phi}(0,0)$, we see, using the fundamental theorem of calculus, that

$$\frac{\tilde w(x,t)}{h(x,t)}=\int_0^1 \tilde w_{x_n}(x',\tau x_n+(1-\tau)\phi(x'),t)\,d\tau\,. \tag{3.31}$$

Hence, using (3.30), the last display and (1.10), we can conclude that

$$\left|\frac{\tilde w(x,t)}{h(x,t)}-\frac{\tilde w(y,s)}{h(y,s)}\right|\le c(|x-y|+|t-s|^{1/p})^{\sigma}\,. \tag{3.32}$$

Using (3.32), for $\tilde u$ and $\tilde v$, and scaling and translating back, we see that

$$\left|\frac{u(x,t)}{h(x,t)}-\frac{u(y,s)}{h(y,s)}\right|\le \hat c\frac{\Lambda_u}{r}\left(\frac{|x-y|}{r}+\left(\frac{1}{\Lambda_u}\right)^{(2/p-1)}\left(\frac{|s-t|}{r^p}\right)^{1/p}\right)^{\sigma} \tag{3.33}$$

whenever $(x,t),(y,s)\in Q^{\Lambda_u,+}_{\rho r,\phi}(x_0,t_0)$ and that

$$\left|\frac{v(x,t)}{h(x,t)}-\frac{v(y,s)}{h(y,s)}\right|\le \hat c\frac{\Lambda_v}{r}\left(\frac{|x-y|}{r}+\left(\frac{1}{\Lambda_v}\right)^{(2/p-1)}\left(\frac{|s-t|}{r^p}\right)^{1/p}\right)^{\sigma} \tag{3.34}$$

whenever $(x,t),(y,s)\in Q^{\Lambda_v,+}_{\rho r,\phi}(x_0,t_0)$. Furthermore, using Lemma 3.1 we see that that there exist constants $c_1=c_1(n,p,\nu,L)$, $1\le c_1<\infty$, $\varrho_0\equiv\varrho_0(n,p,\nu,L,r_0)$, $0<\varrho_0<1$, such that

$$c_1^{-1}\theta_u(x_n-\phi(x'))\le u(x,t)\le c_1\lambda_u\theta_u(x_n-\phi(x')) \tag{3.35}$$

whenever$(x,t)\in(\Omega\cap(B'(0,r_0)\times[-1,1]))\times(-\theta_u^{2-p}\varrho_0^p,\theta_u^{2-p}\varrho_0^p)$, and

$$c_1^{-1}\theta_v(x_n-\phi(x'))\le v(x,t)\le c_1\lambda_v\theta_v(x_n-\phi(x')) \tag{3.36}$$

whenever $(x,t)\in(\Omega\cap(B'(0,r_0)\times[-1,1]))\times(-\theta_v^{2-p}\varrho_0^p,\theta_v^{2-p}\varrho_0^p)$. Theorem 1.1 is now a consequence of (3.33)–(3.36), and the identity

$$\begin{aligned}\left(\frac{u(x,t)}{v(x,t)}-\frac{u(y,s)}{v(y,s)}\right) &= \frac{h(x,t)}{v(x,t)}\left(\frac{u(x,t)}{h(x,t)}-\frac{u(y,s)}{h(y,s)}\right)\\ &\quad+\frac{u(y,s)}{h(y,s)}\frac{h(x,t)}{v(x,t)}\frac{h(y,s)}{v(y,s)}\left(\frac{v(y,s)}{h(y,s)}-\frac{v(x,t)}{h(x,t)}\right).\end{aligned}$$

□

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