A NOTE ON THE WOLFF POTENTIAL ESTIMATE FOR SOLUTIONS TO ELLIPTIC EQUATIONS INVOLVING MEASURES

RIIKKA KORTE AND TUOMO KUUSI

Abstract. We present a new proof for a pointwise upper bound in terms of Wolff potential for $\mathcal{A}$-superharmonic functions, which are the pointwise defined solutions to elliptic equations involving nonnegative measure data.

1. Introduction

The $\mathcal{A}$-superharmonic functions are defined as lower semicontinuous functions that satisfy comparison principle with $\mathcal{A}$-harmonic functions. There is a nonnegative Radon measure μ associated to each $\mathcal{A}$-superharmonic function u via the equation

$$-\operatorname{div}(\mathcal{A}(x,\nabla u)) = \mu.$$

The solution u to this equation is generally unbounded. However, there is a pointwise upper bound of $\mathcal{A}$-superharmonic functions in terms of the Wolff potential

$$\mathbf{W}_p^{\mu}(x,R) = \int_0^R \big(s^{p-n}\mu(B(x,s))\big)^{1/(p-1)}\,\frac{ds}{s}$$

of the corresponding measure. The result is due to Kilpeläinen and Malý, see [3] and [4]. The upper bound is sharp since the same potential also gives the lower bound as was shown in [3]. The purpose of this work is to present an alternative proof for the upper bound. Another approach is available by Trudinger and Wang [7]. Their method uses Poisson modification, as does ours, and various Harnack inequalities. We have also added here, for the sake of completeness, a proof which closely follows elegant ideas of their proof. We have tried to emphasize the key points.

Mikkonen [6] studied the estimate in the weighted case. The proofs in [4] and [6] are based on the same method and on a delicate choice of a test function leading to an iterative scheme of truncated functions. One of our motivations to introduce a new proof has been to find an interpretation for the truncation levels appearing in their proof. Indeed, our method is natural in view of the fundamental solution, as explained later. Our proof is based on a choice of a test function, which appears to be new in this context. Our main tools include Poisson modification of $\mathcal{A}$-superharmonic functions, Caccioppoli estimates, reverse Hölder inequality for $\mathcal{A}$-subsolutions, Sobolev embedding theorem, and the weak Harnack's inequality for $\mathcal{A}$-supersolutions.

Date: December 2, 2009.

2000 *Mathematics Subject Classification.* 35J60, 31C15, 31C05.

The pointwise upper and lower estimates have several consequences, such as the Wiener test for the regularity of boundary points, demonstrated in [4]. The result was originally proved by Lindqvist and Martio in [5] in the case $p > n-1$.

2. Preliminaries

A continuous Sobolev function $u \in W^{1,p}_{loc}(\Omega)$ is an $\mathcal{A}$-harmonic function in an open set Ω, if it is a weak solution to the equation

$$-\operatorname{div}(\mathcal{A}(x,\nabla u)) = 0.$$

Here $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that $x \mapsto \mathcal{A}(x,\xi)$ is measurable for all $\xi \in \mathbb{R}^n$, $\xi \mapsto \mathcal{A}(x,\xi)$ is continuous for almost every $x \in \mathbb{R}^n$, and it satisfies the following structural assumptions:

$$\begin{aligned}
&\big\langle \mathcal{A}(x,\xi),\xi \big\rangle \geq \mathcal{A}_0|\xi|^p,\\
&|\mathcal{A}(x,\xi)| \leq \mathcal{A}_1|\xi|^{p-1},\\
&\big\langle \mathcal{A}(x,\xi_1)-\mathcal{A}(x,\xi_2),\xi_1-\xi_2 \big\rangle > 0,
\end{aligned}$$

whenever $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$ for almost every $x \in \mathbb{R}^n$. A function $u \in W^{1,p}_{loc}(\Omega)$ is an $\mathcal{A}$-supersolution (subsolution), if

$$-\operatorname{div}(\mathcal{A}(x,\nabla u)) \geq (\leq)\, 0$$

weakly in Ω, i.e.,

$$\int_\Omega \big\langle \mathcal{A}(x,\nabla u),\nabla\varphi \big\rangle\, dx \geq (\leq)\, 0$$

for every nonnegative $\varphi \in C_0^\infty(\Omega)$.

The class of $\mathcal{A}$-superharmonic functions comprises lower semicontinuous functions u that are finite in a dense subset of Ω and satisfy the following comparison principle: Let $D \Subset \Omega$. Then for each $\mathcal{A}$-harmonic function $h \in C(\overline{D})$ in D, $u \geq h$ on ∂D implies $u \geq h$ in D. By Corollary 7.20 [2], continuous $\mathcal{A}$-superharmonic functions are $\mathcal{A}$-supersolutions. For further properties of $\mathcal{A}$-superharmonic functions and $\mathcal{A}$-supersolutions, see Sections 3 and 7 in [2].

For each nonnegative Radon measure μ, there is an $\mathcal{A}$-superharmonic function u such that

$$\int_\Omega \big\langle \mathcal{A}(x,Du),\nabla\varphi \big\rangle\, dx = \int_\Omega \varphi\, d\mu,$$

where $Du = \lim_{k\to\infty} \nabla \min\{u,k\}$, see Theorem 2.4 in [3]. See also [1] for the existence of very weak solutions. Conversely, each $\mathcal{A}$-superharmonic function generates a nonnegative Radon measure μ_u by Riesz representation theorem.

Let $B = B(x,r)$. For $0 < \alpha < \infty$, we use the notation

$$\alpha B = B(x,\alpha r),$$

and, for $0 < \sigma < 1$,

$$\sigma A_r = \frac{5+\sigma}{4}B(x_0,r) \setminus \frac{5-\sigma}{4}B(x_0,r).$$

Let $u \in W_0^{1,p}(B(x_0,R))$. Then Sobolev's embedding theorem states that there exists $\kappa = \kappa(n,p) > p$ such that

$$\left(\fint_{B(x_0,R)} |u|^\kappa \, dx \right)^{1/\kappa} \leq CR \left(\fint_{B(x_0,R)} |\nabla u|^p \, dx \right)^{1/p}. \tag{2.1}$$

Here we use the abbreviation

$$\fint_A f \, dx = \frac{1}{|A|} \int_A f \, dx.$$

The following Caccioppoli estimate follows by testing the equation of an $\mathcal{A}$-subsolution u with the test function $u_+\phi^p$.

Lemma 2.2. *Let u be an $\mathcal{A}$-subsolution in a domain Ω and let $\phi \in C_0^\infty(\Omega)$ be nonnegative. Then there is a constant $C = C(p, \mathcal{A}_0, \mathcal{A}_1)$ such that*

$$\int_\Omega |\nabla u_+|^p \phi^p \, dx \leq C \int_\Omega u_+^p |\nabla \phi|^p \, dx.$$

By using either De Giorgi's or Moser's method, an application of Sobolev's embedding theorem together with the Caccioppoli estimate leads to the reverse Hölder's inequality.

Lemma 2.3. *Let u be an $\mathcal{A}$-subsolution in $B(x_0,2R) \setminus \overline{B}(x_0,R)$. Then there is a constant $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ such that*

$$\left(\fint_{\sigma A} u_+^s \, dx \right)^{1/s} \leq \left(\frac{C}{(\tau-\sigma)^C} \fint_{\tau A} u_+^q \, dx \right)^{1/q}$$

for all $0 < q < s \leq +\infty$ and $0 < \sigma < \tau < 1$.

Applying logarithmic estimate together with John-Nirenberg Lemma, one can prove the weak Harnack inequality for $\mathcal{A}$-supersolutions.

Theorem 2.4. *Let $u \geq 0$ be $\mathcal{A}$-supersolution in $B(x_0,2R)$, or in A_R. Then there is a constant $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ such that*

$$\fint_{B(x_0,R)} u \, dx \leq C \operatorname*{ess\,inf}_{B(x_0,R)} u,$$

or

$$\fint_{\frac{1}{2}A_R} u \, dx \leq C \operatorname*{ess\,inf}_{\frac{1}{2}A_R} u,$$

respectively.

This, together with the reverse Hölder's inequality, leads to the Harnack inequality.

Theorem 2.5. *Let $u \geq 0$ be $\mathcal{A}$-harmonic in A_R. Then there is a constant $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ such that*

$$\operatorname*{ess\,sup}_{\frac{1}{2}A_R} u \leq C \operatorname*{ess\,inf}_{\frac{1}{2}A_R} u.$$

For the proofs of Lemmas 2.2 and 2.3, and Theorems 2.4 and 2.5, see [2].

3. Potential estimates

Theorem 3.1. *Let u be a nonnegative $\mathcal{A}$-superharmonic function in the ball $B(x_0, 2R)$. Then there is a constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ such that*

$$\frac{1}{C} W_p^{\mu_u}(x_0, R) \le u(x_0) \le C \inf_{B(x_0,R)} u + C W_p^{\mu_u}(x_0, R).$$

We may reduce the proof of the upper bound to more restricted case. Namely, we only consider class of continuous $\mathcal{A}$-superharmonic functions. Indeed, since u is by the definition lower semicontinuous, there is an increasing sequence of continuous functions converging to u pointwise. Solving obstacle problems with these continuous functions gives an increasing sequence (u_j) of continuous $\mathcal{A}$-superharmonic functions converging to u pointwise. This implies by Theorem 1.17 in [3] that $\nabla u_j \to Du$ as $j \to \infty$, possibly passing to a subsequence. Hence we also have the weak convergence of corresponding measures μ_{u_j} to μ_u. It follows that

$$\limsup_{j\to\infty} \mu_{u_j}(\overline{B(x_0, s)}) \le \mu_u(\overline{B(x_0, s)})$$

and

$$\liminf_{j\to\infty} \mu_{u_j}(B(x_0, s)) \ge \mu_u(B(x_0, s)).$$

Using these together with the fact

$$\int_0^{R-\varepsilon} \left(s^{p-n} \mu_u(\overline{B(x_0, s)})\right)^{1/(p-1)} \frac{ds}{s} \le C W_p^{\mu_u}(x_0, R),$$

for all $\varepsilon > 0$ and the pointwise convergence of u_j to u, allows us to proceed in the proof with u_j instead of u and therefore reduces the analysis to the continuous case.

The second reduction we make in the proof is that we may modify u to be a weak solution in a countable union of disjoint annuli shrinking to the reference point x_0. The corresponding measure in each annulus concentrates on the boundary of the particular annulus, but in a controllable way, since the measure corresponding to the new solution stays also in the dual of $W^{1,p}(B(x_0, R))$. The advantage of the modification is that since being a solution is a local property, we now know that in each annulus local a priori estimates for weak solutions hold.

To proceed formally, let $R_k = 2^{1-k} R$ and $B_k = B(x_0, R_k)$, $k = 0, 1, \ldots$ Let

$$\omega = \bigcup_{k=1}^{\infty} \frac{3}{2} B_k \setminus \overline{B_k}$$

be the union of annuli. Define v to be

$$\begin{cases} \operatorname{div}\big(\mathcal{A}(x, \nabla v)\big) = 0 & \text{in } \omega, \\ v = u & \text{otherwise.} \end{cases} \tag{3.2}$$

The function v is called Poisson modification of u and it is an $\mathcal{A}$-superharmonic function, see Lemma 7.14 in [2]. Note that it is continuous by the assumed continuity of u and hence also an $\mathcal{A}$-supersolution. In ω, v is $\mathcal{A}$-harmonic. Moreover, v satisfies the equation

$$\operatorname{div}\big(\mathcal{A}(x, \nabla v)\big) = \mu_v,$$

where the nonnegative measure μ_v has the property

$$\mu_v(B_k) = \mu_u(B_k), \tag{3.3}$$

$k = 0, 1, \ldots$ This is seen by the inner regularity of μ_v and μ_u and by testing equations of u and v with $\phi \in C_0^\infty(B_k)$ such that $0 \le \phi \le 1$ and $\phi = 1$ on a compact set K in B_k containing $\overline{\frac{3}{2}B_{k+1}}$. Indeed, then

$$\begin{aligned}\int_{B_k} \phi \, d\mu_u &- \int_{B_k} \phi \, d\mu_v \\ &= \int_{B_k} \big(\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u)\big) \cdot \nabla\phi \, dx = 0,\end{aligned}$$

because $u = v$ in the support of $\nabla\phi$. By exhausting B_k with such K, (3.3) follows.

Proof of the upper bound based on a choice of a test function. Define

$$a_k = \inf\{a : |\{v > a\} \cap B_{k-1}| \le \delta |B_{k-1}|\}, \quad k = 1, 2, \ldots$$

We then have

$$|\{v \ge a_k\} \cap B_{k-1}| \ge \delta |B_{k-1}| \quad \text{and} \quad |\{v > a_k\} \cap B_{k-1}| \le \delta |B_{k-1}|.$$

Observe that if

$$v = \min\{\ell, |x - x_0|^{(p-n)/(p-1)}\}, \quad p < n,$$

i.e., v is the truncated fundamental solution for the p-Laplace equation centered at x_0, then the levels a_k are nothing else but

$$a_k = \min\{\ell, C(\delta) R_k^{(p-n)/(p-1)}\}.$$

Our goal is to show that

$$a_{k+1} - a_k \le C\left(R_{k-1}^{p-n}\mu_v(B_{k-1})\right)^{1/(p-1)} + \frac{1}{2}\big(\inf_{B_{k-1}} v - \inf_{B_{k-2}} v\big)$$

for small enough δ depending on n, p, $\mathcal{A}_0$, and $\mathcal{A}_1$.

If $a_{k+1} \le a_k$, the inequality holds trivially by the minimum principle. Hence we may suppose that $a_{k+1} > a_k$. We define the auxiliary function

$$v_k = \min\Big\{\frac{(v - a_k)_+}{a_{k+1} - a_k}, 1\Big\}.$$

Let $\phi_k \in C_0^\infty(B_{k-1})$, $0 \le \phi_k \le 1$, be a cut-off function such that $\phi_k = 1$ in $\frac{7}{6}B_k$, $\operatorname{supp} \nabla\phi_k \subset \frac{1}{3}A_{R_k}$, and $|\nabla\phi_k| \le C/R_k$. Recall the definition

$$\sigma A_r = \frac{5+\sigma}{4} B(x_0, r) \setminus \frac{5-\sigma}{4} B(x_0, r),$$

$0 < \sigma < 1$.

By (3.3) and testing the equation of v with $v_k\phi_k^p$, it follows that

$$\begin{aligned}\mu_u(B_{k-1}) = \mu_v(B_{k-1}) &\ge \int_{B_{k-1}} v_k \phi_k^p \, d\mu_v \\ &= \int_{B_{k-1}} \big\langle \mathcal{A}(x, \nabla v), \nabla(v_k\phi_k^p)\big\rangle \, dx.\end{aligned} \tag{3.4}$$

The test function $v_k\phi_k^p$ is admissible since v is continuous and belongs to $W^{1,p}(B_{k-1})$. Growth conditions imply that

$$\begin{aligned}
&\int_{B_{k-1}} \big\langle \mathcal{A}(x,\nabla v), \nabla(v_k\phi_k^p)\big\rangle\,dx \\
&\quad\geq \frac{\mathcal{A}_0}{a_{k+1}-a_k}\int_{\{a_k<v<a_{k+1}\}\cap B_{k-1}} |\nabla v|^p\phi_k^p\,dx \\
&\qquad+\int_{\{a_k<v<a_{k+1}\}\cap B_{k-1}} v_k\big\langle \mathcal{A}(x,\nabla v),\nabla\phi_k^p\big\rangle\,dx \\
&\qquad+\int_{\{v\geq a_{k+1}\}\cap B_{k-1}} \big\langle \mathcal{A}(x,\nabla v),\nabla\phi_k^p\big\rangle\,dx \\
&\quad= I_1+I_2+I_3.
\end{aligned} \tag{3.5}$$

The first target is to obtain a lower bound for I_1. Then, later, we will show that I_2 is small compared to I_1. The term I_3 we estimate with the aid of Lemmas 2.2 and 2.3 for $\mathcal{A}$-subsolutions. Indeed, the reason why we study v instead of u is the term I_3. In particular, v is a $\mathcal{A}$-harmonic in the support of $\nabla\phi_k$.

To begin with, note that

$$\begin{aligned}
I_1 &= \mathcal{A}_0(a_{k+1}-a_k)^{p-1}\int_{B_{k-1}} |\nabla v_k|^p\phi_k^p\,dx \\
&\geq \frac{\mathcal{A}_0(a_{k+1}-a_k)^{p-1}}{2^p}\int_{B_{k-1}} |\nabla(\phi_k v_k)|^p\,dx \\
&\qquad - \mathcal{A}_0(a_{k+1}-a_k)^{p-1}\int_{B_{k-1}} |\nabla\phi_k|^p v_k^p\,dx.
\end{aligned} \tag{3.6}$$

We apply Sobolev's embedding theorem (2.1) to the first term above and obtain

$$\begin{aligned}
&\frac{\mathcal{A}_0(a_{k+1}-a_k)^{p-1}}{2^p}\int_{B_{k-1}} |\nabla(\phi_k v_k)|^p\,dx \\
&\quad\geq \frac{(a_{k+1}-a_k)^{p-1}|B_{k-1}|}{CR_k^p}\left(\fint_{B_{k-1}} (v_k\phi_k)^\kappa\,dx\right)^{p/\kappa} \\
&\quad\geq \frac{1}{C}(a_{k+1}-a_k)^{p-1}R_k^{n-p}\left(\frac{\big|\{v\geq a_{k+1}\}\cap B_k\big|}{\big|B_k\big|}\right)^{p/\kappa} \\
&\quad\geq \frac{1}{C}(a_{k+1}-a_k)^{p-1}R_k^{n-p}\delta^{p/\kappa}.
\end{aligned} \tag{3.7}$$

Here we used the fact that, by the choice of a_{k+1}, we have

$$\big|\{v\geq a_{k+1}\}\cap B_k\big| \geq \delta\big|B_k\big|.$$

Moreover, the choice of a_k leads to

$$\begin{aligned}\mathcal{A}_0(a_{k+1}-a_k)^{p-1}&\int_{B_{k-1}}|\nabla\phi_k|^p v_k^p\,dx\\&\le\frac{C(a_{k+1}-a_k)^{p-1}|\{v_k>0\}\cap B_{k-1}|}{R_k^p}\\&\le\frac{C(a_{k+1}-a_k)^{p-1}|B_{k-1}|\delta}{R_k^p}.\end{aligned}\tag{3.8}$$

Since $p/\kappa<1$, a combination of (3.7) and (3.8) yields that the second term in (3.6) is small compared to the first term provided that δ is small enough. Consequently,

$$I_1\ge\frac{1}{C}(a_{k+1}-a_k)^{p-1}R_k^{n-p}\delta^{p/\kappa}.\tag{3.9}$$

Furthermore, we estimate the second term in (3.5) as

$$\begin{aligned}|I_2|&\le p\mathcal{A}_1\int_{\{a_k<v<a_{k+1}\}\cap B_{k-1}}|\nabla v|^{p-1}\phi_k^{p-1}|\nabla\phi_k|\,dx\\&\le\frac{1}{4}\frac{\mathcal{A}_0}{a_{k+1}-a_k}\int_{\{a_k<v<a_{k+1}\}\cap B_{k-1}}|\nabla v|^p\phi_k^p\,dx\\&\qquad+C(a_{k+1}-a_k)^{p-1}\int_{\{a_k<v<a_{k+1}\}\cap B_{k-1}}|\nabla\phi_k|^p\,dx\\&\le\frac{1}{4}I_1+C(a_{k+1}-a_k)^{p-1}R_k^{n-p}\delta.\end{aligned}\tag{3.10}$$

Here we have applied growth conditions and Young's inequality. By (3.9), it follows that when δ is small enough,

$$|I_2|\le\frac{1}{2}I_1.$$

Thus, by (3.4) and (3.5), we have

$$\mu_u(B_{k-1})\ge\frac{1}{2}I_1+I_3.\tag{3.11}$$

We then estimate I_3. Observe carefully that

$$w_k=(v-a_{k+1})_+$$

is an $\mathcal{A}$-subsolution in $A_{R_k}=\frac{3}{2}B_k\setminus\overline{B}_k$. Let $\theta_k\in C_0^\infty(\frac{1}{2}A_{R_k})$, $0\le\theta_k\le1$, be such that $\theta_k=1$ in $\frac{1}{3}A_{R_k}$ (i.e. on the support of $\nabla\phi_k$) and $|\nabla\theta_k|\le C/R_k$. We set $\widetilde{\phi}_k=\theta_k\phi_k$. Note that $\phi_k\le\widetilde{\phi}_k$ and $|\nabla\phi_k|\le|\nabla\widetilde{\phi}_k|\le C/R_k$. Thus, by growth conditions and Hölder's inequality, we obtain

$$\begin{aligned}|I_3| \le & p\mathcal{A}_1 \int_{\{v\ge a_{k+1}\}\cap B_{k-1}} |\nabla v|^{p-1}\phi_k^{p-1}|\nabla\phi_k|\,dx \\ \le & p\mathcal{A}_1 \int_{B_{k-1}} |\nabla w_k|^{p-1}\widetilde{\phi}_k^{p-1}|\nabla\widetilde{\phi}_k|\,dx \\ \le & C\big|B_{k-1}\big| \left(\fint_{B_{k-1}} |\nabla w_k|^p \widetilde{\phi}_k^p\,dx\right)^{(p-1)/p} \\ & \times \left(\|\nabla\widetilde{\phi}_k\|_\infty^p \frac{\big|\{w_k>0\}\cap B_{k-1}\big|}{\big|B_{k-1}\big|}\right)^{1/p} \\ \le & CR_k^{n-1}\delta^{1/p} \left(\fint_{B_{k-1}} |\nabla w_k|^p \widetilde{\phi}_k^p\,dx\right)^{(p-1)/p}.\end{aligned} \tag{3.12}$$

We use the Caccioppoli estimate, Lemma 2.2, in $\frac{3}{2}B_k\setminus\overline{B}_k$ and obtain

$$\fint_{\frac{3}{2}B_k\setminus\overline{B}_k} |\nabla w_k|^p\widetilde{\phi}_k^p\,dx \le C\fint_{\frac{3}{2}B_k\setminus\overline{B}_k} w_k^p|\nabla\widetilde{\phi}_k|^p\,dx. \tag{3.13}$$

Let $q=p/(p+1)$. Since $\operatorname{supp}(\nabla\widetilde{\phi}_k)\subset\frac{1}{2}A_{R_k}$, it follows by reverse Hölder's inequality, Lemma 2.3, that

$$\begin{aligned}\fint_{\frac{3}{2}B_k\setminus\overline{B}_k} w_k^p|\nabla\widetilde{\phi}_k|^p\,dx \le & C\|\nabla\widetilde{\phi}_k\|_\infty^p\fint_{\frac{1}{2}A_{R_k}} w_k^p\,dx \\ \le & \frac{C}{R_k^p}\left(\fint_{\frac{3}{2}B_k\setminus\overline{B}_k} w_k^q\,dx\right)^{p/q}.\end{aligned} \tag{3.14}$$

Hölder's inequality yields

$$\begin{aligned}&\Big(\fint_{\frac{3}{2}B_k\setminus\overline{B}_k} w_k^q\,dx\Big)^{p/q} \\ &\le \left(\frac{\big|\{w_k>0\}\cap B_{k-1}\big|}{\big|B_{k-1}\big|}\right)^{p(1-q)/q}\left(\fint_{\frac{3}{2}B_k\setminus\overline{B}_k} w_k\,dx\right)^p \\ &\le \delta\left(\fint_{\frac{3}{2}B_k\setminus\overline{B}_k} w_k\,dx\right)^p.\end{aligned} \tag{3.15}$$

We substitute (3.13), (3.14), and (3.15) into (3.12) and end up with

$$\begin{aligned}|I_3| \le & CR_k^{n-p}\delta^{(p-1)/p+1/p}\left(\fint_{\frac{3}{2}B_k\setminus\overline{B}_k} w_k\,dx\right)^{p-1} \\ \le & CR_k^{n-p}\delta\left(\fint_{B_{k-1}}(v-a_{k+1})_+\,dx\right)^{p-1}.\end{aligned} \tag{3.16}$$

Furthermore, the weak Harnack inequality, Theorem 2.4, implies that

$$\fint_{B_{k-1}}(v-a_{k+1})_+\,dx \le \fint_{B_{k-1}}(v-\inf_{B_{k-2}} v)\,dx \le C(\inf_{B_{k-1}} v - \inf_{B_{k-2}} v).$$

Using the estimates above, we conclude

$$|I_3| \leq C R_k^{n-p} \delta \big(\inf_{B_{k-1}} v - \inf_{B_{k-2}} v \big)^{p-1}. \tag{3.17}$$

A combination of (3.9), (3.11), and (3.17) gives

$$\frac{1}{C}(a_{k+1} - a_k)^{p-1} R_k^{n-p} \delta^{p/\kappa} \leq \mu_u(B_{k-1}) + C R_k^{n-p} \delta \big(\inf_{B_{k-1}} v - \inf_{B_{k-2}} v \big)^{p-1}.$$

This further implies that, when δ is small enough,

$$a_{k+1} - a_k \leq C \left(R_k^{p-n} \mu_u(B_{k-1}) \right)^{1/(p-1)} + \frac{1}{2} (\inf_{B_{k-1}} v - \inf_{B_{k-2}} v). \tag{3.18}$$

The lower semicontinuity of u leads to

$$u(x_0) \leq \liminf_{x \to x_0} u(x) \leq \liminf_{k \to \infty} \inf_{\partial B_k} u = \liminf_{k \to \infty} \inf_{\partial B_k} v \leq \limsup_{k \to \infty} a_k.$$

Moreover, the comparison principle implies $v \leq u$ and hence

$$\lim_{k \to \infty} \inf_{B_k} v \leq \lim_{k \to \infty} \inf_{B_k} u \leq u(x_0).$$

Thus, by summing up (3.18), we end up with

$$u(x_0) \leq 2a_3 + C \sum_{k=1}^{\infty} \left(R_k^{p-n} \mu_u(B_k) \right)^{1/(p-1)}.$$

By the weak Harnack principle and the comparison principle, we have

$$a_3 |\{v \geq a_3\} \cap B_2| \leq \int_{B_2} v \, dx \leq \int_{B_1} u \, dx \leq C |B_1| \inf_{B_1} u,$$

from which it follows by $\big|\{v \geq a_3\} \cap B_2\big| \geq \delta \big|B_2\big|$ that $a_3 \leq C \inf_{B_1} u$. The estimate

$$\begin{aligned} \sum_{k=1}^{\infty} \left(R_k^{p-n} \mu_u(B_k) \right)^{1/(p-1)} &= \sum_{k=1}^{\infty} \left(R_k^{p-n} \mu_u(B_k) \right)^{1/(p-1)} \frac{R_{k-1} - R_k}{R_k} \\ &\leq \int_0^R \left(s^{p-n} \mu_u(B(x_0, s)) \right)^{1/(p-1)} \frac{ds}{s} \end{aligned}$$

then concludes the proof. □

Proof of the upper bound following Trudinger and Wang. Let v be the $\mathcal{A}$-supersolution defined in (3.2). The main idea in the proof is to introduce comparison solutions with zero boundary values and measures given by μ_v. Let $w_k \in W_0^{1,p}(\frac{4}{3}B_{k+1})$ solve the equation

$$\operatorname{div} \big(\mathcal{A}(x, \nabla w_k) \big) = \mu_v \quad \text{in} \quad \frac{4}{3} B_{k+1},$$

$k = 0, 1, \ldots$ The existence of such a solution follows by the fact that μ_v belongs to $W^{-1,p'}(\frac{4}{3}B_{k+1})$. Subtraction of equations of v and w_k using the smooth approximation of the test function

$$\big(v - \max_{\partial \frac{4}{3} B_{k+1}} v - w_k \big)_+ \in W_0^{1,p}(\frac{4}{3} B_{k+1})$$

in weak formulations, yields

$$\int_{\{v-\max_{\partial\frac{4}{3}B_{k+1}} v\geq w_k\}} \big\langle \mathcal{A}(x,\nabla v)-\mathcal{A}(x,\nabla w_k),\nabla(v-w_k)\big\rangle\,dx=0.$$

It follows that $\nabla(v-\max_{\partial\frac{4}{3}B_{k+1}} v-w_k)_+=0$, and thus

$$w_k\geq v-\max_{\partial\frac{4}{3}B_{k+1}} v. \tag{3.19}$$

Next, since w_k is nonnegative $\mathcal{A}$-harmonic function in $\frac{3}{2}B_{k+2}\setminus\overline{B_{k+2}}$, the Harnack inequality, Theorem 2.5, gives

$$\max_{\partial\frac{4}{3}B_{k+2}} w_k\leq C\min_{\partial\frac{4}{3}B_{k+2}} w_k.$$

Assume first that $\min_{\partial\frac{4}{3}B_{k+2}} w_k=0$. The weak Harnack inequality then implies that $w_k=0$ in $\frac{3}{2}B_{k+2}$ and hence

$$\max_{\partial\frac{4}{3}B_{k+2}} v-\max_{\partial\frac{4}{3}B_{k+1}} v\leq 0. \tag{3.20}$$

Assume next the positivity of $\min_{\partial\frac{4}{3}B_{k+2}} w_k$. Since w_k takes continuously zero boundary values on $\partial\frac{4}{3}B_{k+1}$ and $w_k\geq 0$, it follows that

$$\mu_{\min\{w_k,\min_{\partial\frac{4}{3}B_{k+2}} w_k\}}(\frac{4}{3}B_{k+1})=\mu_{w_k}(\frac{4}{3}B_{k+1})=\mu_v(\frac{4}{3}B_{k+1}).$$

Thus, after approximating w_k, we have

$$\begin{aligned}
&(\min_{\partial\frac{4}{3}B_{k+2}} w_k)\mu_v(\frac{4}{3}B_{k+1})\\
&\qquad\geq\int_{\frac{4}{3}B_{k+1}}\min\{w_k,\min_{\partial\frac{4}{3}B_{k+2}} w_k\}\,d\mu_{\min\{w_k,\min_{\partial\frac{4}{3}B_{k+2}} w_k\}}\\
&\qquad=\int_{\frac{4}{3}B_{k+1}}\big\langle \mathcal{A}(x,\nabla\min\{w_k,\min_{\partial\frac{4}{3}B_{k+2}} w_k\}),\nabla\min\{w_k,\min_{\partial\frac{4}{3}B_{k+2}} w_k\}\big\rangle\,dx\\
&\qquad\geq\mathcal{A}_0\int_{\frac{4}{3}B_{k+1}}|\nabla\min\{w_k,\min_{\partial\frac{4}{3}B_{k+2}} w_k\})|^p\,dx\\
&\qquad\geq\mathcal{A}_0(\min_{\partial\frac{4}{3}B_{k+2}} w_k)^p\operatorname{cap}_p(\frac{4}{3}B_{k+2},\frac{4}{3}B_{k+1})\\
&\qquad\geq CR_k^{n-p}(\min_{\partial\frac{4}{3}B_{k+2}} w_k)^p.
\end{aligned}$$

Since $\min_{\partial\frac{4}{3}B_{k+2}} w_k>0$, the Harnack inequality leads to

$$\max_{\partial\frac{4}{3}B_{k+2}} w_k\leq C\left(R_k^{p-n}\mu_v(\frac{4}{3}B_{k+1})\right)^{1/(p-1)}.$$

This, in view of (3.19), implies

$$\max_{\partial\frac{4}{3}B_{k+2}} v-\max_{\partial\frac{4}{3}B_{k+1}} v\leq C\left(R_k^{p-n}\mu_u(B_k)\right)^{1/(p-1)}.$$

Consequently, together with (3.20), we obtain

$$\limsup_{k\to\infty} \max_{\partial\frac{4}{3}B_{k+2}} v \le \max_{\partial\frac{4}{3}B_3} v + C\sum_{k=2}^{\infty}\Big(R_k^{p-n}\mu_u(B_k)\Big)^{1/(p-1)}.$$

Since v is $\mathcal{A}$-harmonic and nonnegative in $\frac{3}{2}B_3 \setminus \overline{B_3}$, the Harnack inequality and the comparison principle give

$$\max_{\partial\frac{4}{3}B_3} v \le C \min_{\partial\frac{4}{3}B_3} v \le C \min_{\partial\frac{4}{3}B_3} u = C \inf_{\frac{4}{3}B_3} u,$$

where the last equality follows by the minimum principle. The weak Harnack inequality for nonnegative $\mathcal{A}$-supersolutions further implies

$$\inf_{\frac{4}{3}B_3} u \le C \fint_{B(x_0,R)} u\,dx \le C \inf_{B(x_0,R)} u.$$

The lower semicontinuity of u, on the other hand, yields

$$\begin{aligned} u(x_0) &\le \lim_{k\to\infty} \inf_{B_k\setminus\overline{\frac{3}{2}B_{k+1}}} u \\ &= \lim_{k\to\infty} \inf_{B_k\setminus\overline{\frac{3}{2}B_{k+1}}} v \\ &\le \limsup_{k\to\infty} \max_{\partial\frac{4}{3}B_{k+2}} v \\ &\le C\sum_{k=2}^{\infty}\Big(R_k^{p-n}\mu_u(B_k)\Big)^{1/(p-1)} + C\inf_{B(x_0,R)} u. \end{aligned}$$

As before, this leads to the result. □

We next prove the lower bound. Observe that here we do not need to use the Poisson modification of u.

Proof of the lower bound following Trudinger and Wang. Take a smooth cut-off function $\theta_k \in C_0^\infty(\frac{5}{4}B_{k+1})$, $0 \le \theta_k \le 1$, such that $\theta_k = 1$ in B_{k+1}. Let $w_k \in W_0^{1,p}(B_k)$ solve the equation

$$\operatorname{div}\big(\mathcal{A}(x,\nabla w_k)\big) = \theta_k\mu_u \quad \text{in} \quad B_k,$$

$k = 0, 1, \ldots$. The existence of such solutions follows by the fact that μ_u belongs to $W^{-1,p'}(B_k)$ and θ_k is smooth. Note that by the minimum principle, w_k is nonnegative and, moreover, it is $\mathcal{A}$-harmonic in $B_k \setminus \overline{\frac{5}{4}B_{k+1}}$ taking continuously zero boundary values on ∂B_k.

By subtracting equations of u and w_k, while using the smooth approximation of the test function

$$\big(w_k - u + \min_{\partial B_k} u\big)_+ \in W_0^{1,p}(B_k),$$

in weak formulations, we obtain

$$\begin{aligned} 0 &\le \int_{B_k}\big(w_k - u + \min_{\partial B_k} u\big)_+\,d\mu_u - \int_{B_k}\big(w_k - u + \min_{\partial B_k} u\big)_+\,d\mu_{w_k} \\ &= \int_{\{u-\min_{\partial B_k} u\le w_k\}\cap B_k}\big\langle \mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w_k), \nabla(w_k - u)\big\rangle\,dx \le 0. \end{aligned}$$

Thus we have $\nabla(w_k - u + \min_{\partial B_k} u)_+ = 0$ and consequently

$$w_k \le u - \min_{\partial B_k} u. \tag{3.21}$$

Assume first the positivity of $\min_{\partial B_{k+1}} w_k$. This readily implies with the aid of the weak Harnack inequality that

$$\min_{\partial \frac{2}{3} B_k} w_k \ge \frac{1}{C} \fint_{\frac{2}{3}B_k} w_k \, dx \ge \frac{1}{C} \fint_{B_{k+1}} w_k \, dx \ge \frac{1}{C} \min_{\partial B_{k+1}} w_k > 0.$$

Take a cut-off function $\phi \in C_0^\infty(B_k)$, $0 \le \phi \le 1$, such that $\phi = 1$ in $\frac{2}{3}B_k$, and $|\nabla \phi| \le C/R_k$. By the maximum principle for $\mathcal{A}$-harmonic functions, $w_k = \min\{w_k, \max_{\partial \frac{2}{3} B_k} w_k\}$ in the support of $\nabla\phi$. Moreover, the minimum principle implies

$$\min_{\partial \frac{2}{3} B_k} w_k \le \min\{w_k, \max_{\partial \frac{2}{3} B_k} w_k\}$$

in $\frac{2}{3}B_k$. Thus, by approximating w_k, we obtain

$$\begin{aligned}
&(\min_{\partial \frac{2}{3} B_k} w_k)\mu_u(B_{k+1}) \\
&\qquad \le \int_{B_k} \min\{w_k, \max_{\partial \frac{2}{3} B_k} w_k\} \phi^p \, d\mu_{w_k} \\
&\qquad = \int_{B_k} \big\langle \mathcal{A}(x, \nabla w_k), \nabla(\min\{w_k, \max_{\partial \frac{2}{3} B_k} w_k\}\phi^p)\big\rangle \, dx \\
&\qquad \le \mathcal{A}_1 \int_{B_k} |\nabla \min\{w_k, \max_{\partial \frac{2}{3} B_k} w_k\}|^p \phi^p \, dx \\
&\qquad\quad + p\mathcal{A}_1 \max_{\partial \frac{2}{3} B_k} w_k \int_{B_k} |\nabla \min\{w_k, \max_{\partial \frac{2}{3} B_k} w_k\}|^{p-1} \phi^{p-1} |\nabla\phi| \, dx.
\end{aligned}$$

The Caccioppoli estimate for the nonnegative weak subsolution

$$\max_{\partial \frac{2}{3} B_k} w_k - \min\{w_k, \max_{\partial \frac{2}{3} B_k} w_k\}$$

in B_k gives

$$\int_{B_k} |\nabla \min\{w_k, \max_{\partial \frac{2}{3} B_k} w_k\})|^p \phi^p \, dx \le C(\max_{\partial \frac{2}{3} B_k} w_k)^p \int_{B_k} |\nabla\phi|^p \, dx.$$

Thus, by applying Young's inequality and the Harnack inequality

$$\max_{\partial \frac{2}{3} B_k} w_k \le C \min_{\partial \frac{2}{3} B_k} w_k,$$

we end up with

$$(\min_{\partial \frac{2}{3} B_k} w_k)\mu_u(B_{k+1}) \le C R_k^{n-p} (\min_{\partial \frac{2}{3} B_k} w_k)^p.$$

Using the minimum principle and the positivity of $\min_{\partial \frac{2}{3} B_k} w_k$, we arrive at

$$\Big(R_k^{p-n} \mu_u(B_{k+1})\Big)^{1/(p-1)} \le \min_{\partial B_{k+1}} w_k.$$

Hence, by (3.21), we conclude that

$$\Big(R_k^{p-n} \mu_u(B_{k+1})\Big)^{1/(p-1)} \le C(\min_{\partial B_{k+1}} u - \min_{\partial B_k} u). \tag{3.22}$$

Assume next that $\min_{\partial B_{k+1}} w_k = 0$. Then the weak Harnack inequality implies that $w_k = 0$ in B_k. Especially, w_k, and hence also u, are $\mathcal{A}$-harmonic in B_{k+1}. Therefore, the Harnack inequality implies that

$$\min_{\partial B_{k+1}} u \le \inf_{B_{k+2}} u \le C \sup_{B_{k+2}} u \le Cu(x_0).$$

In this case, by summing up (3.22), we have by the nonnegativity of u that

$$u(x_0) \ge \frac{1}{C} \sum_{j=1}^{k} \Big(R_j^{p-n} \mu_u(B_j) \Big)^{1/(p-1)} = \frac{1}{C} \sum_{j=1}^{\infty} \Big(R_j^{p-n} \mu_u(B_j) \Big)^{1/(p-1)},$$

since $\mu_u(B_j) = 0$ for all $j > k$.

Consequently, in all cases, we obtain

$$u(x_0) \ge \lim_{k\to\infty} \min_{\partial B_k} u \ge \frac{1}{C} \sum_{k=1}^{\infty} \Big(R_k^{p-n} \mu_u(B_k) \Big)^{1/(p-1)}.$$

The estimate

$$\begin{aligned} \sum_{k=1}^{\infty} \big(R_k^{p-n} \mu_u(B_k) \big)^{1/(p-1)} &= \sum_{k=1}^{\infty} \big(R_k^{p-n} \mu_u(B_k) \big)^{1/(p-1)} \frac{R_{k-1} - R_k}{R_k} \\ &\ge \frac{1}{2} \int_0^R \big(s^{p-n} \mu_u(B(x_0, s)) \big)^{1/(p-1)} \frac{ds}{s} \end{aligned}$$

completes the proof. □

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TK, Helsinki University of Technology, Institute of Mathematics, P.O.Box 1100, FIN-02015 TKK, Finland

E-mail address: `rkorte@math.tkk.fi, tuomo.kuusi@tkk.fi`