OBSTACLE PROBLEM FOR NONLINEAR PARABOLIC EQUATIONS

RIIKKA KORTE, TUOMO KUUSI, AND JUHANA SILJANDER

Abstract. We show the existence of a continuous solution to a nonlinear parabolic obstacle problem with a continuous time-dependent obstacle. The solution is constructed by an adaptation of the Schwarz alternating method. Moreover, if the obstacle is Hölder continuous, we prove that the solution inherits the same property.

1. Introduction

Consider the obstacle problem

$$\frac{\partial u}{\partial t} \geq \nabla \cdot \mathcal{A}(x,t,\nabla u), \quad u \geq \psi,$$

where $\mathcal{A}(x,t,\xi) \approx |\xi|^{p-2}\xi$, $\xi \in \mathbb{R}^n$, $p > 2n/(n+2)$, and ψ is a continuous obstacle depending on both space and time variables. We define the solution to the obstacle problem as the smallest weak supersolution above the given obstacle. Our definition is motivated by nonlinear potential theory where the obstacle problem is a basic tool. It is essential when proving convergence and comparison results as well as pointwise behaviour of weak supersolutions and superparabolic functions, see [3], [5], and [6].

Starting from the obstacle, we apply a modification of the Schwarz alternating method and construct an increasing sequence of functions using continuous solutions to Dirichlet boundary value problems. We show that the limit of the sequence is the unique continuous solution to the obstacle problem. Moreover, we show that the solution to the obstacle problem attains continuous boundary values continuously provided that the complement of the domain is thick enough. If, in addition, the obstacle is Hölder continuous, we prove that the solution to the obstacle problem is Hölder continuous as well.

The existence of solutions to the parabolic obstacle problems via variational inequalities has been studied by Lions [8]. The method is based on a time discretization and the semi-group property of the corresponding differential quotient. See also [1], [9], [?], and [?]. In these works, a

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crucial assumption on the obstacle seems to be a suitable monotonicity or regularity condition. In the case of smooth obstacles, our definition of the solution to the parabolic obstacle problem coincides with the standard definition via variational inequalities. Our method, however, provides a new constructive way to obtain the solution to the general parabolic obstacle problem. In particular, we also consider obstacles which are merely continuous functions in time.

2. Preliminaries

Our notation is standard. In what follows, Q will stand for a space-time box

$$Q = (a_1, b_1) \times \ldots \times (a_n, b_n) \times (t_1, t_2)$$

in $\mathbb{R}^n \times \mathbb{R}$. We also use the notation

$$K(x, r) = (x_1 - r, x_1 + r) \times \ldots \times (x_n - r, x_n + r)$$

for the cube centered at $x \in \mathbb{R}^n$.

Let Ω be an open set in $\mathbb{R}^n$. The parabolic boundary of a cylinder $\Omega \times (t_1, t_2) \subset \mathbb{R}^n \times \mathbb{R}$ is

$$\partial_p(\Omega \times (t_1, t_2)) = (\overline{\Omega} \times \{t_1\}) \cup (\partial\Omega \times (t_1, t_2]).$$

For the cylindrical domain, we often use the notation $\Omega_T := \Omega \times (0, T]$, where $0 < T < \infty$. If D' is a bounded open subset of D and the closure of D' belongs to D, we denote $D' \Subset D$.

We now state our main assumptions. Let Ξ be an open set in $\mathbb{R}^n \times \mathbb{R}$. We assume that $\mathcal{A} : \Xi \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a Carathéodory function, that is, $(x, t) \mapsto \mathcal{A}(x, t, \xi)$ is measurable for every ξ in $\mathbb{R}^n$ and $\xi \mapsto \mathcal{A}(x, t, \xi)$ is continuous for almost every $(x, t) \in \Xi$. In addition, $\mathcal{A}$ satisfies the growth bounds

$$\mathcal{A}(x, t, \xi) \cdot \xi \geq \alpha|\xi|^p \quad \text{and} \quad |\mathcal{A}(x, t, \xi)| \leq \beta|\xi|^{p-1}, \tag{2.1}$$

for almost every $(x, t) \in \Xi$ and every $\xi \in \mathbb{R}^n$. Here α and β are positive constants. Furthermore, we assume that $\mathcal{A}$ is monotonic in a sense that

$$\big(\mathcal{A}(x, t, \xi_1) - \mathcal{A}(x, t, \xi_2)\big) \cdot (\xi_1 - \xi_2) > 0 \tag{2.2}$$

whenever $(x, t, \xi_i) \in \Xi \times \mathbb{R}^n$, $i = 1, 2$, and $\xi_1 \neq \xi_2$.

Let Ω be a bounded open set in $\mathbb{R}^n$. The Sobolev space $W^{1,p}(\Omega)$ is the space of real-valued functions f such that $f \in L^p(\Omega)$ and the distributional first partial derivatives $\partial f/\partial x_i$, $i = 1, 2, \ldots, n$, exist in Ω and belong to $L^p(\Omega)$. We use the norm

$$\|f\|_{1,p,\Omega} = \Big(\int_\Omega \big(|f|^p + |\nabla f|^p\big)\, dx \Big)^{1/p}.$$

The Sobolev space with zero boundary values, $W_0^{1,p}(\Omega)$, is the closure of $C_0^\infty(\Omega)$ with respect to the Sobolev norm. By the parabolic Sobolev space

$$L^p(t_1, t_2; W^{1,p}(\Omega)),$$

$t_1 < t_2$, we mean the space of functions u such that the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$ for almost every $t_1 < t < t_2$ and the norm

$$\left(\int_{t_1}^{t_2}\int_{\Omega}\big(|u(x,t)|^p + |\nabla u(x,t)|^p\big)\,dx\,dt\right)^{1/p}$$

is finite. The definition of the space $L^p(t_1,t_2;W_0^{1,p}(\Omega))$ is analogous.

Definition 2.3. Let Ξ be an open set in $\mathbb{R}^n \times \mathbb{R}$. A function u is a weak solution in Ξ provided that whenever $\Omega \times (\tau_1,\tau_2) \Subset \Xi$, then $u \in L^p(t_1,t_2;W^{1,p}(\Omega))$ and it satisfies the integral equality

$$\int_{\tau_1}^{\tau_2}\int_{\Omega}\big(\mathcal{A}(x,t,\nabla u)\cdot\nabla\phi - u\frac{\partial \phi}{\partial t}\big)\,dx\,dt \;= 0 \tag{2.4}$$

for all $\phi \in C_0^\infty(\Omega \times (\tau_1,\tau_2))$. We call a weak solution as $\mathcal{A}$-parabolic.

A function u is a weak supersolution (subsolution) in Ξ provided that whenever $\Omega \times (\tau_1,\tau_2) \Subset \Xi$, then $u \in L^p(t_1,t_2;W^{1,p}(\Omega))$ and the integral above is non-negative (non-positive) for all non-negative $\phi \in C_0^\infty(\Omega \times (\tau_1,\tau_2))$.

Now we can proceed to the exact definition of a solution to the obstacle problem.

Definition 2.5. A function $u \in C(\Omega_T)$ solves the obstacle problem with the obstacle $\psi \in C(\Omega_T)\cap L^\infty(\Omega_T)$, if it has the following properties

1. $u \geq \psi$ in Ω_T,
2. u is a weak supersolution in Ω_T,
3. u is a weak solution, i.e. $\mathcal{A}$-parabolic, in the set $\{u > \psi\}$,
4. u is the smallest weak supersolution above ψ, i.e. if v is a weak supersolution in Ω_T and $v \geq \psi$, then $v \geq u$.

Finally, we define so-called $\mathcal{A}$-superparabolic functions via comparison principle, see [3] and [5]. This is an essential class of functions in our proof.

Definition 2.6. Let Ξ be an open set in $\mathbb{R}^n \times \mathbb{R}$. A function $u : \Xi \to (-\infty,\infty]$ is called $\mathcal{A}$-superparabolic if

(i) u is lower semicontinuous,
(ii) u is finite in a dense subset of Ξ,
(iii) u satisfies the comparison principle on each space-time box $Q \Subset \Xi$: If h is $\mathcal{A}$-parabolic in Q and continuous on $\overline{Q}$, and, if $h \leq u$ on $\partial_p Q$, then $h \leq u$ in the whole Q.

A function u is $\mathcal{A}$-subparabolic if $-u$ is $\widetilde{\mathcal{A}}$-superparabolic, where

$$\widetilde{\mathcal{A}}(x,t,\xi) = -\mathcal{A}(x,t,-\xi), \quad (x,t,\xi) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n.$$

The monotonicity of the operator, see assumption (2.2), guarantees the comparison principle between lower semicontinuous weak supersolutions and upper semicontinuous weak subsolutions, see e.g. [3]. By [7], every weak supersolution has a lower semicontinuous representative. In particular, by the comparison principle, every weak supersolution has an $\mathcal{A}$-superparabolic representative.

In the proof, we construct an increasing sequence of uniformly bounded continuous weak supersolutions. The following theorem in [6] shows that also the limit is a weak supersolution. See also [5].

Theorem 2.7. *Let Ξ be an open set in $\mathbb{R}^n \times \mathbb{R}$. Suppose that u_i, $i = 1, 2, \ldots$, is an increasing sequence of uniformly locally bounded weak supersolutions to* (2.4) *in Ξ. Then the limit*

$$u = \lim_{i\to\infty} u_i$$

is a weak supersolution.

For the local Hölder continuity of $\mathcal{A}$-parabolic functions, we refer to DiBenedetto [2]. Define a weighted distance between points (x, s) and (y, t) as

$$\begin{aligned} &d_M((x, s), (y, t)) \\ &= M^{(p-2)/(2p)} \big(M^{-|p-2|/(2p)} |x - y| + M^{|p-2|/(2p)} |s - t|^{1/p} \big), \end{aligned}$$

where $M > 0$. The corresponding distance between the space-time cylinder $\Omega_T = \Omega \times (0, T)$ and an open set $\Xi \subset \Omega_T$ is defined as

$$(M,p) - \operatorname{dist}(\Xi, \Omega_T) = \inf_{(x,s)\in\Xi,\ (y,t)\in\partial_p\Omega_T} d_M((x, s), (y, t)).$$

Theorems 1.1 on pages 41 and 77 in [2] gives us the following theorem.

Theorem 2.8. *Let u be an $\mathcal{A}$-parabolic function in Ω_T and suppose that*

$$M = \operatorname*{osc}_{\Omega_T} u < \infty.$$

Let $\Xi \Subset \Omega_T$. Then there are constants $C > 1$ and $0 < \sigma < 1$ depending only on data such that

$$|u(x, s) - u(y, t)| \le CM \Big(\frac{d_M((x, s), (y, t))}{(M, p) - \operatorname{dist}(\Xi, \Omega_T)} \Big)^{\sigma}$$

for all $(x, s), (y, t) \in \Xi$.

The existence of solutions to the Dirichlet boundary value problem in space-time cylinders with the continuous boundary data follows by the monotonicity of the operator, see e.g. Lions [8] or Showalter [9]. For the continuity of the solution up to the boundary, we need to assume some geometric properties of the complement of the set. The complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ has positive geometric density at a point $x_0 \in \partial\Omega$ if there exist constants $0 < \alpha < 1$ and $\rho > 0$ such that for all $\delta < \rho$,

$$\big| B(x_0, \delta) \cap \Omega \big| \le \alpha \big| B(x_0, \delta) \big|.$$

The condition is enough to show that the weak solution to the Dirichlet boundary value problem attains continuously the continuous boundary values at the point (x_0, t_0), $0 < t_0 < T$. For the proof, see [2]. More generally, if the complement of Ω is p-thick at $x_0 \in \partial\Omega$, i.e.

$$\int_0^1 \Big(\frac{\operatorname{cap}(\Omega^c \cap B(x_0, r), B(x_0, 2r))}{\operatorname{cap}(B(x_0, r), B(x_0, 2r))}\Big)^{1/(p-1)} \frac{dr}{r} = \infty,$$

then the weak solution attains continuously the continuous boundary values at (x_0, t_0), $0 \le t_0 \le T$. In the case of evolutionary p-Laplace equation, the result is due to Kilpeläinen and Lindqvist [3]. For the general case, see Skrypnik [10] and the references therein. Recall that if the complement has positive geometric density at x_0, then it is also p-thick at x_0. We state the result as an existence theorem.

Theorem 2.9. *Let Ω be an open set and assume that Ω^c is p-thick at $x_0 \in \partial\Omega$. Let $\vartheta \in C(\overline{\Omega_T})$. Then there is a unique $\mathcal{A}$-parabolic function $u \in C(\Omega_T)$ such that u is continuous at (x_0, t_0) and $u(x_0, t_0) = \vartheta(x_0, t_0)$, $0 \le t_0 \le T$.*

3. The existence theorem

The following theorem is our main result.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let ψ be a continuous bounded obstacle in Ω_T. Then there exists a unique solution to the obstacle problem. If, in addition, Ω^c is p-thick at $x_0 \in \partial\Omega$ and ψ is continuous up to the boundary in a neighborhood of (x_0, t_0), then u is continuous at (x_0, t_0) and $u(x_0, t_0) = \psi(x_0, t_0)$.*

We split the proof of Theorem 3.1 into two steps. We first construct a candidate for a solution to the obstacle problem using solutions to the Dirichlet boundary value problem. We show that the obtained function is continuous and $\mathcal{A}$-superparabolic, and it satisfies properties (1) and (3) of Definition 2.5. To finish the proof, we need to show that the obtained function is also a weak supersolution. This we establish by showing that every continuous $\mathcal{A}$-superparabolic function is a weak supersolution. That the candidate is the smallest supersolution above the obstacle, follows by the construction.

Note that in [6] it is shown that every bounded $\mathcal{A}$-superparabolic function is a weak supersolution, see also [5]. However, the existence of a solution to the obstacle problem is used in the proof. Hence we present an alternative proof in the case of continuous $\mathcal{A}$-superparabolic functions.

We construct a candidate for a solution to the obstacle problem as follows.

Construction 3.2. Let $\mathcal{F} = \{Q^k\}_k$ be a dense family of space-time boxes in Ω_T ending at the instant $t = T$. For example, one can take all the space-time boxes

$$Q = (a_1, b_1) \times \ldots \times (a_n, b_n) \times (t, T),$$

where $a_i, b_i, t \in \mathbb{Q}$, $a_i < b_i$, $i = 1, \dots, n$, $0 \le t < T$ and $Q \subset \Omega_T$.

Construct sequences $(\varphi_k)_k$ as follows:

$$\varphi_0 = \psi, \quad \varphi_{k+1} = \max\{\varphi_k, v_k\}, \quad k = 0, 1, 2, \dots,$$

where v_k is $\mathcal{A}$-parabolic function in Q^k with the boundary values φ_k on $\partial_p Q^k$ and $v_k = \varphi_k$ in $\Omega_T \setminus Q^k$. The limit of the construction is

$$u = \lim_{k\to\infty} \varphi_k. \tag{3.3}$$

The construction has the following basic properties.

(i) The sequence $(\varphi_k)_k$ is pointwise increasing, and, thus, $\varphi_k \ge \psi$ for all $k = 1, 2, \dots$.

(ii) By the maximum principle,

$$|\varphi_k(x,t)| \le \sup_{\Omega_T} |\psi|$$

for every $(x,t) \in \Omega_T$ and $k = 0, 1, 2, \dots$.

(iii) As a maximum of continuous functions, φ_k is continuous for all $k = 0, 1, 2, \dots$.

(iv) If $v \ge \psi$ is a weak supersolution, then $v \ge u$. Indeed, by the comparison principle, $\varphi_k \le v$ for every $k = 0, 1, 2, \dots$.

(v) The function φ_k is a weak subsolution in the set $\{\varphi_k > \psi\}$ for any $k = 1, 2, \dots$, because, in this set, it is obtained as a maximum of finitely many weak subsolutions.

(vi) As a limit of an increasing sequence of continuous functions, u is lower semicontinuous. Thus the set $\{u > \psi\}$ is open.

We begin the proof by showing that u is $\mathcal{A}$-superparabolic.

Lemma 3.4. *Suppose that ψ is a continuous obstacle. Then the limit u of Construction 3.2 satisfies the comparison principle in all space-time boxes $Q \subset \Omega_T$.*

Proof. We fix a space-time box $Q = (a_1, b_1) \times \dots (a_n, b_n) \times (t_1, t_2)$. Let h be an $\mathcal{A}$-parabolic function in Q such that it is continuous up to the parabolic boundary $\partial_p Q$ and $h \le u$ on $\partial_p Q$. To prove the lemma, we need to show that $h \le u$ in Q. Fix $\varepsilon > 0$. By the continuity of functions h and φ_k, the sets

$$E_k = \overline{Q} \cap \{\varphi_k > h - \varepsilon\}, \quad k = 1, 2, \dots,$$

are open with respect to the relative topology. Moreover, the collection of the sets E_k covers $\partial_p Q$. The compactness of $\partial_p Q$ and the monotonicity of the sequence $(\varphi_k)_k$ then implies that there is k_0 such that $\varphi_{k_0} > h - \varepsilon$ on $\partial_p Q$.

Since the sets E_k are open, there exists $Q^{k_1} \in \mathcal{F}$, $k_1 \ge k_0$, such that

$$\partial_p Q^{k_1} \cap \{t < t_2\} \subset E_{k_0}$$

and

$$(Q \setminus E_{k_0}) \subset Q^{k_1}.$$

Now

$$h \le \varphi_{k_0} + \varepsilon \le \varphi_{k_1} + \varepsilon \quad \text{on} \quad \partial_p Q^{k_1} \cap \{t < t_2\},$$

and, since v_{k_1} is $\mathcal{A}$-parabolic in $Q^{k_1} \in \mathcal{F}$ with the boundary values φ_{k_1} on $\partial_p Q^{k_1}$, we have

$$h \le v_{k_1} + \varepsilon \le \varphi_{k_1+1} + \varepsilon \quad \text{in} \quad Q^{k_1} \cap \{t < t_2\}$$

by the comparison principle. Thus it follows that $h \le u + \varepsilon$ in Q. The claim follows by letting $\varepsilon \to 0$. □

We next show that whenever the limit u of Construction 3.2 does not hinder the obstacle, it is $\mathcal{A}$-parabolic. The result shows Property (3) in Definition 2.5.

Lemma 3.5. *Suppose that ψ is a continuous obstacle. Let u be as in Construction 3.2. Then u is $\mathcal{A}$-parabolic in the set $\{u > \psi\}$.*

Proof. Let $z_0 = (x_0, t_0) \in \Omega_T$ be such that $u(z_0) > \psi(z_0)$. The set $\{u > \psi\}$ is open, and, hence, there is

$$Q_r = K(x_0, r) \times (t_0 - r^p, t_0 + r^p)$$

such that $Q_r \subset \{u > \psi\}$. Let φ_k, $k = 0, 1, \ldots$, be as in Construction 3.2. We cover $\overline{Q_{r/2}}$ with the open sets $Q_r \cap \{\varphi_k > \psi\}$, and by the compactness and monotonicity of $\{\varphi_k\}$, we find k_0 such that $\overline{Q_{r/2}} \subset \{\varphi_{k_0} > \psi\}$.

We collect from the construction all space-time boxes $Q^{k_i - 1}$, $k_{i-1} < k_i$, $i = 1, 2, \ldots$, such that

$$\partial_p\big(Q^{k_i-1} \cap \{t < t_0 + (r/2)^p\}\big) \subset Q_{r/2} \setminus Q_{r/4}.$$

There are infinitely many such space-time boxes.

Next, note that $\varphi_{k_i - 1}$ is a subsolution in $Q_{r/2}$. Then the comparison principle implies that

$$v_{k_i} \ge \varphi_{k_i-1} \quad \text{in} \quad Q^{k_i-1} \cap \{t < t_0 + (r/2)^p\}.$$

Hence φ_{k_i} is $\mathcal{A}$-parabolic in $Q^{k_i-1} \cap \{t < t_0 + (r/2)^p\}$. It follows that φ_{k_i} is $\mathcal{A}$-parabolic in $Q_{r/4}$ for every $i = 1, 2, \ldots$. By Theorem 2.8, the $\mathcal{A}$-parabolic functions φ_{k_i}, $i = 1, 2, \ldots$, and hence also u, have a uniform modulus of continuity in $Q_{r/8}$.

The obtained $\mathcal{A}$-parabolicity of the subsequence remains to the limit in $Q_{r/8}$. Indeed, it is easy to see that the limit is $\mathcal{A}$-sub- and $\mathcal{A}$-superparabolic in $Q_{r/8}$. By the comparison principle and the continuity of u, we obtain that u is also $\mathcal{A}$-parabolic in $Q_{r/8}$. Since being $\mathcal{A}$-parabolic is a local property, it follows that u is $\mathcal{A}$-parabolic in $\{u > \psi\}$. This finishes the proof. □

The next lemma tells that Construction 3.2 is stable. By this we mean that the limit does not change if we change the space-time boxes in the construction. We have the following uniqueness result.

Lemma 3.6. *Suppose that ψ is a continuous obstacle. Then the limit u of Construction 3.2 is unique, and, in particular, independent of the choices of space-time boxes.*

Proof. We take two limits of the construction, u_1 and u_2. Let φ_k^i and v_k^i, $k = 0, 1, \ldots$, generate u_i, $i = 1, 2$. Clearly, $u_1 \geq \varphi_0^2 = \psi$.

Suppose then that $u_1 \geq \varphi_k^2$ in Ω_T. Let Q be the space-time box, where v_k^2 is obtained from φ_k^2. The function v_k^2 solves the Dirichlet boundary value problem in Q with boundary values φ_k^2 on $\partial_p Q$. Since $u_1 \geq \varphi_k^2$ on $\partial_p Q$, we conclude that $u_1 \geq v_k^2$ in Q by Lemma 3.4. But

$$\varphi_{k+1}^2 = \max\{\varphi_k^2, v_k^2\} \qquad \text{in } Q,$$

and it follows that $u_1 \geq \varphi_{k+1}^2$ in Q. The induction argument then shows that $u_1 \geq u_2$. Interchanging the roles of u_1 and u_2 finishes the proof. □

The uniqueness leads to the comparison of limits.

Lemma 3.7. *Suppose that ψ_1 and ψ_2 are continuous obstacles. If $\psi_1 \leq \psi_2$, then the corresponding limits u_1 and u_2 of Construction 3.2 satisfy $u_1 \leq u_2$.*

Proof. Since the limits u_1 and u_2 do not depend on the choice of space-time boxes Q^k, $k = 1, 2, \ldots$, we can use the same family $\mathcal{F}$ to construct u_1 and u_2. Let φ_k^i and v_k^i, $i = 1, 2$, $k = 1, 2, \ldots$, generate u_1 and u_2. We have $\varphi_0^1 \leq \varphi_0^2$, because $\varphi_0^i = \psi_i$, $i = 1, 2$. Assume then that $\varphi_k^1 \leq \varphi_k^2$ for some $k \geq 0$. In particular, this means that $\varphi_k^1 \leq \varphi_k^2$ on $\partial_p Q_k$. It follows by the comparison principle for $\mathcal{A}$-superparabolic functions that $v_k^1 \leq v_k^2$ in Q_k and hence $\varphi_{k+1}^1 \leq \varphi_{k+1}^2$. The induction argument concludes the proof. □

Remark 3.8. Lemma 3.7 implies that if $\|\psi - \widetilde{\psi}\|_\infty \leq \varepsilon$, then also $\|u - \widetilde{u}\|_\infty \leq \varepsilon$. This can be seen by considering obstacle problems with obstacles $\psi - \varepsilon$, $\widetilde{\psi}$ and $\psi + \varepsilon$. Indeed, it follows from the construction that adding a constant to the obstacle changes the solution by the same constant.

We next show that the limit u is continuous in Ω_T whenever the obstacle is continuous. Moreover, if Ω^c is p-thick at some point, then u is continuous at that point. This shows the continuity in Theorem 3.1.

Lemma 3.9. *Suppose that ψ is a continuous obstacle. Then the limit u of Construction 3.2 is continuous in Ω_T. If, in addition, Ω^c is p-thick at $x_0 \in \partial\Omega$ and ψ is continuous up to the boundary in a neighborhood of (x_0, t_0), then u is continuous at (x_0, t_0) and $u(x_0, t_0) = \psi(x_0, t_0)$.*

Proof. Let $\varepsilon > 0$. Suppose first that $z_1 = (x_1, t_1)$ is an interior point, or in $\Omega \times \{T\}$. First, we denote

$$Q_r = K(x_1, r) \times (t_1 - r^p, \min\{t_1 + r^p, T\}).$$

Let r be so small that Q_r does not intersect $\partial_p\Omega_T$ and

$$\operatorname*{osc}_{Q_r}\psi := \max_{Q_r}\psi - \min_{Q_r}\psi \le \frac{\varepsilon}{4}.$$

Let h solve the Dirichlet boundary value problem with $h = \psi$ on $\partial_p Q_r$. The solution exists by Theorem 2.9, and $h \in C(\overline{Q_r})$. We define the following modified obstacle

$$\widetilde{\psi} = \begin{cases} \psi \quad \text{on} \quad \overline{\Omega_T \setminus Q_r}, \\ h \quad \text{in} \quad Q_{r/2}, \\ \frac{2(r-s)}{r}h + \frac{2s-r}{r}\psi \quad \text{on} \quad \partial Q_s, \quad r/2 \le s < r, \end{cases}$$

i.e. the interpolation between h and ψ. Clearly $\widetilde{\psi}$ is continuous. Moreover, by the maximum principle, we have

$$|\psi - \widetilde{\psi}| \le \frac{\varepsilon}{4} \quad \text{on} \quad \overline{\Omega_T}.$$

Let $\widetilde{u}$ be the limit of the construction with the obstacle $\widetilde{\psi}$, and let $\widetilde{\varphi}_k$, $k = 0, 1, \dots$, be the generating sequence. By the comparison of limits, see Remark 3.8, we have

$$|u - \widetilde{u}| \le \frac{\varepsilon}{4} \quad \text{on} \quad \overline{\Omega_T}.$$

Next, since $\widetilde{\psi}$ is $\mathcal{A}$-parabolic in $Q_{r/2}$, we obtain that $\widetilde{\varphi}_k$ is a weak subsolution in $Q_{r/2}$ for all $k = 0, 1, \dots$. This is based on the fact that if $\widetilde{\varphi}_{k-1}$ is a weak subsolution in $\Xi \subset \Omega_T$ and w is $\mathcal{A}$-parabolic in $Q \subset \Omega_T$ such that $w = \widetilde{\varphi}_{k-1}$ on $\partial Q \cap \Xi$, then

$$\widetilde{\varphi}_k = \begin{cases} \widetilde{\varphi}_{k-1}, & \text{in } \Xi \setminus Q, \\ w, & \text{in } \Xi \cap Q, \end{cases}$$

is a weak subsolution in Ξ, see the proof of Lemma 3.5 in [6]. Similarly as in the proof of Lemma 3.5, we conclude that $\widetilde{u}$ is continuous in $Q_{r/4}$. Therefore, there is $0 < \delta < r/4$ such that $\operatorname{osc}\widetilde{u} < \varepsilon/2$ in Q_δ. Consequently, we have

$$\operatorname{osc} u \le \operatorname{osc}\widetilde{u} + 2\sup|u - \widetilde{u}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{in} \quad Q_\delta.$$

This shows the continuity in the interior points.

Suppose then that Ω^c is p-thick at x_0 and ψ is continuous up to the boundary in a neighborhood of (x_0, t_0). Set $z_0 = (x_0, t_0)$. We denote

$$U_r = \big(K(x_0, r) \times (t_0 - r^p, t_0 + r^p)\big) \cap \Omega_T, \quad r > 0.$$

Let $r > 0$ be so small that ψ is continuous on $\overline{U_r}$ and

$$\operatorname*{osc}_{U_r}\psi \le \frac{\varepsilon}{4}.$$

Let $\widetilde{\psi}$ and $\widetilde{u}$ be defined as in the proof of interior points, but using U_r instead of Q_r. By the p-thickness of $(K(x_0, r) \cap \Omega)^c$, $\widetilde{\psi}$ is continuous at z_0. Since $\widetilde{\psi}$ is $\mathcal{A}$-parabolic in $U_{r/2}$, $\widetilde{\varphi}_k$ is a weak subsolution in $U_{r/2}$ for all $k = 1, 2, \ldots$.

To this end, let g solve the Dirichlet boundary value problem in $U_{r/2}$ with the boundary values g_b, where

$$g_b = \begin{cases} \sup_{\Omega_T} \widetilde{\psi} \quad \text{on} \quad \partial_p U_{r/2} \setminus \partial_p \Omega_T, \\ \widetilde{\psi} \quad \text{on} \quad \overline{U_{r/4}} \cap \partial_p \Omega_T, \\ \frac{2(r-2s)}{r} \widetilde{\psi} + \frac{4s-r}{r} \sup_{\Omega_T} \widetilde{\psi} \quad \text{on} \quad \overline{U_s} \cap \partial_p \Omega_T, \quad r/4 < s < r/2, \end{cases}$$

i.e. g_b equals $\sup_{\Omega_T} \widetilde{\psi}$ outside $U_{r/2}$, h in $U_{r/4}$ and, in $U_{r/2} \setminus U_{r/4}$, g_b is an interpolation between $\sup_{\Omega_T} \widetilde{\psi}$ and h. By the p-thickness of $(K(x_0, r/2) \cap \Omega)^c$ at x_0, the $\mathcal{A}$-parabolic function g is continuous at z_0.

Moreover, for the subsolutions $\widetilde{\varphi}_k$ we have that $\widetilde{\varphi}_k \le g$ on $\partial_p U_{r/2}$, and, consequently, we obtain $\widetilde{\varphi}_k \le g$ in $U_{r/2}$, $k = 1, 2, \ldots$, by the comparison principle. Hence also $\widetilde{u} \le g$ in $U_{r/2}$. But this means that $\widetilde{u}$ is between functions $\widetilde{\psi}$ and g in $C(U_r)$ which coincide and are continuous at z_0. Therefore, there is $0 < \delta < r/4$ such that

$$|\widetilde{u}(z_0) - \widetilde{u}(z)| < \frac{\varepsilon}{2}$$

for all $z \in U_\delta$. As before, this leads to the continuity of u at z_0, and concludes the proof. □

3.1. **The final step.** To prove Theorem 3.1, we still need to show that the limit is a weak supersolution. The following theorem gives the desired result.

Theorem 3.10. *A continuous $\mathcal{A}$-superparabolic function is a weak supersolution.*

To prove Theorem 3.10, we construct an increasing sequence of supersolutions u_k that converge pointwise to u. Then, by Theorem 2.7, u is a supersolution as a limit of an increasing sequence of uniformly bounded supersolutions. First, we need the following lemma.

Lemma 3.11. *Let Q be a space-time box, $r \in \mathbb{R}$ and $1 \le k \le n$. Define $Q^1 = \{(x,t) \in Q : x_k < r\}$ and $Q^2 = \{(x,t) \in Q : x_k > r\}$. If v is a continuous $\mathcal{A}$-superparabolic function in Q, and v is a weak supersolution in Q^1 and in Q^2, then v is a weak supersolution in Q.*

Proof. Let $U^\varepsilon = Q \cap \{r - \varepsilon < x_k < r + \varepsilon\}$ and construct functions

$$v_\varepsilon = \begin{cases} h_\varepsilon, & \text{in } U_\varepsilon, \\ v, & \text{in } Q \setminus U_\varepsilon, \end{cases}$$

where h_ε is the solution of the Dirichlet problem in U^ε with boundary values v. Since v is $\mathcal{A}$-superparabolic in Q, we have by the comparison principle that v_ε is an increasing sequence and $v_\varepsilon \to v$ pointwise as

$\varepsilon \to 0$. Hence, by Theorem 2.7, it is enough to show that v_ε is a weak supersolution.

By a similar argument as in Lemma 3.5 of [6], it is straightforward to show that v_ε is a weak supersolution both in Q^1 and Q^2. Since v_ε is also a weak supersolution in U^ε and being a supersolution is a local property, the result of the claim follows. □

We now generate an increasing sequence of weak supersolutions approximating the continuous $\mathcal{A}$-superparabolic function. Let

$$K_0 = K(x_0, r_0) \Subset \Omega$$

be a dyadic cube. Let $\{K_k^j\}_{j=1}^{2^{nk}}$ be the set of dyadic subcubes of K_0 of kth generation. Set $Q_k^j = K_k^j \times (0,T)$ and $Q_0 = K_0 \times (0,T)$

Let u_k solve the Dirichlet boundary value problem in Q_k^j with $u_k = u$ on $\partial_p Q_k^j$, $k = 1, 2, \ldots$, for every $j = 1, \ldots, 2^{nk}$. The function u_k is continuous in Ω_T and $\mathcal{A}$-parabolic in each Q_k^j, for every $k = 1, 2, \ldots$ and $j = 1, \ldots, 2^{nk}$.

Lemma 3.12. *The function u_k is a continuous weak supersolution in Q_0.*

Proof. If we can show that u_k is $\mathcal{A}$-superparabolic in Q_0, the result follows by Lemma 3.11. We do this directly from the definition. First of all, due to the construction of u_k, it is clear that u_k is lower semicontinuous as well as finite in a dense subset of Q_0. Hence, we only need to show the comparison principle.

Fix a space-time box $Q \subset Q_0$ and an $\mathcal{A}$-parabolic function h for which $h \le u_k$ in $\partial_p Q$ and $h \in C(\overline{Q})$. Since u is $\mathcal{A}$-superparabolic and $u_k \le u$, we have $h \le u$ in Q. Moreover, since $u = u_k$ on $\partial_p Q_k^j$, we obtain $h \le u_k$ also in $\partial_p Q_k^j \cap Q$. Thus $h \le u_k$ on $\partial_p (Q \cap Q_k^j)$. As u_k is $\mathcal{A}$-parabolic in every Q_k^j, the comparison principle yields $h \le u_k$ in $Q \cap Q_k^j$ for every j separately, and hence also in the whole Q. □

Since being a weak supersolution is a local property, the following lemma together with Theorem 2.7 shows that u is a weak supersolution.

Lemma 3.13. *The sequence u_k, $k = 1, 2, \ldots$, is increasing, and $u_k \to u$ almost everywhere in $K(x_0, r_0/2) \times (0,T)$ as $k \to \infty$.*

Proof. Let $k_1 < k_2$. On $\partial_p Q_{k_2}^j$, $j = 1, 2, \ldots, 2^{k_2 n}$, we have by the comparison principle that $u_{k_1} \le u$. Hence it follows, again by the comparison principle, that $u_{k_1} \le u_{k_2}$ in $Q_{k_2}^j$, $j = 1, 2, \ldots, 2^{k_2 n}$. Therefore, the sequence is increasing.

Let then $\varepsilon, \delta > 0$. We set

$$U = K_0(x_0, r_0/2) \times (\delta, T - \delta) \Subset \Omega_T.$$

Let $\widetilde{u}$ be a smooth function such that $|u - \widetilde{u}| < \varepsilon/2$ on $\overline{U}$. This is possible by the continuity of u. We set

$$w_k = (\widetilde{u} - u_k - \varepsilon/2)_+.$$

Clearly $w_k = 0$ on $\partial K_k^j \times [\delta, T-\delta]$ for every $j = 1, 2, \dots, 2^{nk}$.

Next, we apply Sobolev-Poincaré inequality and obtain that

$$\int_\delta^{T-\delta} \int_{K_k^j} w_k^p \, dx \, dt \le C 2^{-kp} \int_\delta^{T-\delta} \int_{K_k^j} |\nabla w_k|^p \, dx \, dt.$$

Consequently, by summing these estimates up, we obtain

$$\int_U w_k^p \, dx \, dt \le C 2^{-kp} \int_U |\nabla w_k|^p \, dx \, dt. \tag{3.14}$$

Since u_k is a weak supersolution in Q_0, we conclude by the energy estimate, see Proposition 3.1, p. 24 in [2], for the nonnegative subsolution $\sup_{\Omega_T} \psi - u_k$ that

$$\int_U |\nabla u_k|^p \, dx \, dt \le C(K_0, T) \Big(\frac{\max\{1, \operatorname{osc}_{\Omega_T} \psi\}}{\min\{1, \delta, r_0\}} \Big)^p.$$

This implies that there is a constant C independent of k such that

$$\int_U |\nabla w_k|^p \, dx \, dt \le C.$$

Therefore, we obtain by (3.14) that

$$\big| U \cap \{ w_k > \gamma \} \big| \le 2^{-kp} \gamma^{-p} C$$

for all $\gamma > 0$. This means that w_k converges in measure to 0 as $k \to \infty$. Thus there is a subsequence such that $w_{k_i} \to 0$ almost everywhere in U as $i \to \infty$. This implies by the monotonicity of u_k that

$$\lim_{k \to \infty} u_k \ge \widetilde{u} - \varepsilon/2 > u - \varepsilon \quad \text{almost everywhere in } U.$$

The result follows, since $u_k \le u$ for all k and the inequality above holds for all positive ε and δ. □

4. Hölder continuity of the solution

The following theorem characterizes the Hölder continuity of the solution to the obstacle problem provided that the obstacle is Hölder continuous.

Theorem 4.1. *Suppose u is the solution for the obstacle problem and the obstacle ψ is Hölder continuous with the Hölder exponent α. Then also u is Hölder continuous with the Hölder exponent $\frac{\sigma\alpha}{\alpha+\sigma}$. Here σ is as in Theorem 2.8.*

Proof. Since u is $\mathcal{A}$-parabolic in the set $\{u > \psi\}$ and ψ is Hölder continuous, the only points in which the Hölder condition can fail are the boundary points of the set $\{u > \psi\}$. Suppose z_0 belongs to this boundary and consider a space-time cube Q_ρ centered at z_0. Since ψ is Hölder continuous in Ω_T, we have

$$\operatorname*{osc}_{Q_\rho} \psi \leq C_1 \rho^\alpha$$

for some $C_1 > 0$ independent of ρ. Moreover, by the maximum principle, we can choose a modified obstacle $\widetilde{\psi}$ below ψ in such a way that

$$\|\widetilde{\psi} - \psi\|_\infty < 2 \operatorname*{osc}_{Q_\rho} \psi$$

and $\widetilde{\psi}$ is $\mathcal{A}$-parabolic in $Q_{\rho/2}$. Fix $\delta > 0$ and $\rho = \delta^{\sigma/(\alpha+\sigma)}$. Let δ be small enough so that $\delta < \rho/2$ and $Q_\rho \subset \Omega_T$. Now, similarly as in Lemma 3.5, the solution $\widetilde{u}$ for the modified obstacle $\widetilde{\psi}$ is $\mathcal{A}$-parabolic in Q_δ and hence, by Theorem 2.8,

$$\operatorname*{osc}_{Q_\delta} \widetilde{u} \leq C \left(\frac{\delta}{\rho}\right)^\sigma .$$

Note that u is solution to the obstacle problem in Ω_T with the continuous obstacle ψ and therefore

$$M = \operatorname*{osc}_{\Omega_T} u \leq \operatorname*{osc}_{\Omega_T} \psi.$$

By Remark 3.8, this yields

$$\begin{aligned} \operatorname*{osc}_{Q_\delta} u &\leq \operatorname*{osc}_{Q_\delta} \widetilde{u} + 2 \sup_{Q_\delta} |u - \widetilde{u}| \\ &\leq C \left(\frac{\delta}{\rho}\right)^\sigma + 2\|\widetilde{\psi} - \psi\|_\infty \leq C \left(\frac{\delta}{\rho}\right)^\sigma + C\rho^\alpha \leq C\delta^{\alpha\sigma/(\alpha+\sigma)} \end{aligned}$$

as required. □

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R.K., T.K., Institute of Mathematics, P.O. Box 1100, FI-02015 Helsinki University of Technology, Finland.

E-mail address: riikka.korte@tkk.fi, tuomo.kuusi@tkk.fi

J.S., Department of Mathematical Sciences, University of Oulu, P.O. Box 3000, 90014 University of Oulu, Finland.

E-mail address: juhana.siljander@oulu.fi