LOCAL BEHAVIOUR OF SOLUTIONS TO DOUBLY NONLINEAR PARABOLIC EQUATIONS

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Abstract. We give a relatively simple and transparent proof for Harnack's inequality for certain degenerate doubly nonlinear parabolic equations. In particular, we consider the case where the Lebesgue measure is replaced with a doubling Borel measure which supports a Poincaré inequality.

1. Introduction

Our purpose is to study the local behaviour of nonnegative weak solutions to the doubly nonlinear parabolic equation

$$\operatorname{div}(|Du|^{p-2}Du) = \frac{\partial(u^{p-1})}{\partial t}, \qquad 1 < p < \infty. \tag{1.1}$$

When $p = 2$ we have the standard heat equation. Observe that the solutions to (1.1) can be scaled by nonnegative factors, but due to the nonlinearity of the term $(u^{p-1})_t$ we cannot add a constant to a solution. As far as we know, equation (1.1) has first been studied by Trudinger in [Tru], where he proved a Harnack inequality for nonnegative weak solutions. The proof was based on Moser's celebrated work [Mo1] and used a parabolic version of the John-Nirenberg lemma. Twenty years later the proof of the parabolic John-Nirenberg lemma was simplified by Fabes and Garofalo, see [FaGa]. However, the parabolic BMO still remains technically demanding. Our main objective is to give a relatively simple and transparent proof for Harnack's inequality using the approach of Moser in [Mo2]. In particular, the parabolic John-Nirenberg lemma is replaced with a lemma due to Bombieri in [BoGi] and [Bomb]. Let us point out a slightly unexpected phenomenon, which is related to the parabolic BMO. In the case $p = 2$ it is known that if u is a nonnegative solution, then $\log u$ is a subsolution to the same equation. However, if $p \neq 2$, then $\log u$ is not a subsolution to equation (1.1). Instead it is a subsolution to an equation of the p-parabolic type studied in [DiBe].

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To show that our proof is based on a general principle we consider the case where the Lebesgue measure is replaced with a more general Borel measure. The measure is assumed to be doubling and to support a Poincaré inequality. The precise definitions will be given below. The corresponding result in the elliptic case for measures induced by Muckenhoupt's weights has been studied by Fabes, Kenig and Serapioni in [FKS]. See also [ChFr]. The weighted theory in the parabolic case has been studied by Chiarenza and Serapioni in [ChSe]. However, in their approach the role of the measure is somewhat different. For the heat equation Grigor'yan and Saloff-Coste observed that the doubling condition and the Poincaré inequality are not only sufficient but also necessary conditions for a scale invariant parabolic Harnack principle on Riemannian manifolds, see [SaCo1], [SaCo2] and [Gri]. Our contibution is to show the sufficiency for the general $p \neq 2$ in a Euclidean space. It is a very interesting question whether also the necessity holds in this case. Moreover, the doubling condition and the Poincaré inequality are rather standard assumptions in analysis on metric spaces, see for example [HaK] and references therein. It is well known that Moser's technique is essentially based on a combination of a Sobolev and a Caccioppoli type inequalities. We take a full advatange of a metric space result, which states that the doubling property and the Poincaré inequality imply a Sobolev type inequality, see [BCLS] [HaK], [SaCo1], [SaCo2].

Our argument applies to more general equations of the type

$$\mathrm{div} A(x,t,u,Du) = \frac{\partial (u^{p-1})}{\partial t},$$

where A is a Caratheodory function and satisfies the standard structural conditions (see for example [DiBe], [DBUV], [WZYL])

$$A(x,t,u,Du)\cdot Du \geq C_0 |Du|^p,$$
$$|A(x,t,u,Du)| \leq C_1 |Du|^{p-1},$$

where C_0 and C_1 are positive constants. However, for expository purposes, we only consider equation (1.1).

2. Preliminaries

In this section we describe our assumptions and results more precisely. Let μ be a Borel measure and suppose that Ω is an open set in $\mathbb{R}^n$. The Sobolev space $H^{1,p}(\Omega,\mu)$ is defined to be the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p,\Omega} = \Big(\int_\Omega |u|^p \, d\mu\Big)^{1/p} + \Big(\int_\Omega |Du|^p \, d\mu\Big)^{1/p}.$$

A function belongs to the local Sobolev space $H^{1,p}_{loc}(\Omega,\mu))$ if it belongs to $H^{1,p}(\Omega',\mu)$ for every open subset Ω' of Ω, whose closure is a compact subset of Ω. The Sobolev space with zero boundary values $H^{1,p}_0(\Omega,\mu)$ is the closure of $C_0^\infty(\Omega)$ with respect to the Sobolev norm. For the basic properties of weighted Sobolev spaces we refer to [HKM].

We denote by $L^p(t_1,t_2;H^{1,p}(\Omega))$, $t_1<t_2$, the space of functions such that for almost every t, $t_1\le t\le t_2$, the function $x\mapsto u(x,t)$ belongs to $H^{1,p}(\Omega,\mu)$ and

$$\int_{t_1}^{t_2}\int_\Omega \big(|u(x,t)|^p+|Du(x,t)|^p\big)\,d\mu(x)\,dt<\infty.$$

Notice that the time derivative u_t is deliberately avoided. The definition for the space $L^p_{loc}(t_1,t_2;H^{1,p}_{loc}(\Omega,\mu)$ is clear.

Let $t_1<t_2$ and $1<p<\infty$. A nonnegative function u which belongs to $L^p_{loc}(t_1,t_2;H^{1,p}_{loc}(\Omega,\mu))$ is a weak solution to (1.1) in $\Omega\times(t_1,t_2)$ if

$$\int_{t_1}^{t_2}\int_\Omega \Big(|Du|^{p-2}Du\cdot D\eta-u^{p-1}\frac{\partial\eta}{\partial t}\Big)\,d\mu\,dt=0 \tag{2.1}$$

for all $\eta\in C_0^\infty(\Omega\times(t_1,t_2))$. Further, we say that u is a supersolution to (1.1), if the integral (2.1) is nonnegative for all $\eta\in C_0^\infty(\Omega\times(t_1,t_2))$ with $\eta\ge 0$. If this integral is nonpositive, we say that u is a subsolution.

The measure μ is doubling if there exists a universal constant $D_0\ge 1$ such that

$$\mu(B(z,2R))\le D_0\mu(B(z,R)) \tag{2.2}$$

for every $z\in\mathbb{R}^n$ and $R>0$. Here $B(z,R)$ denotes the open ball with center z and radius R. The dimension of the measure is defined as $d_\mu=\log_2 D_0$. Note that in the case of the Lebesgue measure the dimension is n.

The measure is said to support a weak $(1,p)$-Poincaré inequality if there exist constants $P_0>0$ and $\tau\ge 1$ such that

$$\fint_{B(z,R)}|v-v_{B(z,R)}|\,d\mu\le P_0R\Big(\fint_{B(z,\tau R)}|Dv|^p\,d\mu\Big)^{1/p}, \tag{2.3}$$

for every $v\in H^{1,p}_{loc}(\mathbb{R}^n,\mu)$, $z\in\mathbb{R}^n$ and $R>0$. Here we use the notation

$$v_{B(z,R)}=\fint_{B(z,R)}v\,d\mu=\frac{1}{\mu(B(z,R))}\int_{B(z,R)}v\,d\mu.$$

The word weak refers to the possibility that $\tau>1$. If $\tau=1$, the space is said to support a $(1,p)$-Poincaré inequality. Indeed, in the Euclidean case the weak Poincaré inequality implies the Poincaré inequality, see Theorem 3.4 in [HaK]. Thus we may take $\tau=1$ in (2.3).

From now on we assume that the measure μ is doubling and supports the weak $(1,p)$-Poincaré inequality. Moreover, we assume that the measure is nontrivial in the sense that the measure of every nonempty open set is strictly positive and measure of every bounded set is finite. These assumptions imply a weak (κ,p)-Sobolev-Poincaré inequality for some $\kappa > p$ possibly with a different τ, see [BCLS] and [HaK]. More precisely, there are $\kappa = \kappa(p, D_0, P_0) > p$, $C = C(p, D_0, P_0) > 0$ and $\tau' \geq 1$ such that

$$\Big(\fint_{B(z,R)} |v - v_{B(z,R)}|^\kappa \, d\mu\Big)^{1/\kappa} \leq CR\Big(\fint_{B(z,\tau' R)} |Dv|^p \, d\mu\Big)^{1/p}, \tag{2.4}$$

for every $z \in \mathbb{R}^n$ and $R > 0$. Again, by Theorem 3.4 in [HaK] we may take $\tau' = 1$ in (2.4).

For Sobolev functions with the zero boundary values we have the following version of Sobolev's inequality. Suppose that $u \in H_0^{1,p}(B(z,R),\mu)$. Then there exists constants $C = C(p, D_0, P_0)$ and $\kappa = \kappa(p, D_0, P_0) > p$ such that

$$\Big(\fint_{B(z,R)} |v|^\kappa \, d\mu\Big)^{1/\kappa} \leq CR\Big(\fint_{B(z,R)} |Dv|^p \, d\mu\Big)^{1/p}. \tag{2.5}$$

For the proof we refer, for example, to [KS].

$$\kappa = \begin{cases} \frac{d_\mu p}{d_\mu - p}, & p < d_\mu \\ \in (p,\infty), & p \geq d_\mu \end{cases}, \tag{2.6}$$

where d_μ is the dimension of the measure.

For any fixed $0 < \sigma \leq 1$, $\tau \in \mathbb{R}$ and for a ball $B(z,r) \subset \mathbb{R}^n$ with $r > 0$, we denote

$$\sigma U^+ = B(z,\sigma r) \times \Big(\tau + \frac{1}{2}r^p - \frac{1}{2}(\sigma r)^p, \tau + \frac{1}{2}r^p + \frac{1}{2}(\sigma r)^p\Big),$$

$$\sigma U^- = B(z,\sigma r) \times \Big(\tau - \frac{1}{2}r^p - \frac{1}{2}(\sigma r)^p, \tau - \frac{1}{2}r^p + \frac{1}{2}(\sigma r)^p\Big).$$

and

$$Q = B(z,r) \times (\tau - r^p, \tau + r^p).$$

We show that the following parabolic Harnack inequality holds for a weak solution to (1.1).

Theorem 2.1. *Let $u \geq \rho > 0$ be a weak solution to equation (1.1) in Q and let $0 < \sigma < 1$. Then we have*

$$\operatorname*{ess\,sup}_{\sigma U^-} u \leq C \operatorname*{ess\,inf}_{\sigma U^+} u, \tag{2.7}$$

where the constant C depends only on p, D_0, P_0 and σ.

Note carefully that the constant in (2.7) is independent of ρ. A modification of the proof shows that the technical assumption $u \geq \rho$ can be removed and the result holds for all nonnegative solutions.

It has come to our attention that a similar question has been studied in a recent work by Gianazza and Vespri [GiVe] using a different method.

It is well-known that the local Hölder continuity of a weak solution is a consequence of the Harnack inequality also in the parabolic case when $p = 2$, see [Mo1]. However, due to the nonlinearity of the term $(u^{p-1})_t$, it is not clear how to modify the same proof for the doubly nonlinear equation (1.1). The local Hölder continuity of the solution has been proved in [Ve] using different method.

2.1. **Preliminary results.** Suppose that $t_1 \leq \tau_1 < \tau_2 \leq t_2$. If the test function η vanishes only on the lateral boundary $\partial\Omega \times (\tau_1, \tau_2)$, then the boundary terms

$$\int_\Omega u(x,\tau_1)^{p-1}\eta(x,\tau_1)\,d\mu = \lim_{\sigma\to 0}\frac{1}{\sigma}\int_{\tau_1}^{\tau_1+\sigma}\int_\Omega u(x,t)^{p-1}\eta(x,t)\,d\mu\,dt$$

and

$$\int_\Omega u(x,\tau_2)^{p-1}\eta(x,\tau_2)\,d\mu = \lim_{\sigma\to 0}\frac{1}{\sigma}\int_{\tau_2-\sigma}^{\tau_2}\int_\Omega u(x,t)^{p-1}\eta(x,t)\,d\mu\,dt$$

have to be included. In the case of a supersolution to the doubly nonlinear equation (1.1) the condition becomes

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_\Omega &|Du|^{p-2}Du\cdot D\eta\,d\mu\,dt\\ &+\left[\int_\Omega u^{p-1}\eta\,d\mu\right]_{t=\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1}\frac{\partial\eta}{\partial t}\,d\mu\,dt \geq 0\end{aligned} \tag{2.8}$$

for almost every τ_1, τ_2 with $t_1 < \tau_1 < \tau_2 < t_2$.

There is a well-recognized difficulty with the test functions. Namely, in proving estimates we usually need a test function which depends on the solution itself. Then we cannot avoid that the "forbidden quantity" u_t shows up in the calculation of η_t. In most cases one can easily overcome this difficulty by using an equivalent definition in terms of Steklov averages, as on pages 18 and 25 in [DiBe] and in Chapter 2 of [WZYL]. Alternatively, one can proceed using convolutions with smooth mollifiers as on pages 199–121 in [AS].

We start with an elementary lemma.

Lemma 2.1. *Suppose that $u \geq \rho > 0$ is a supersolution. Then $v = u^{-1}$ is a subsolution.*

Proof. Let $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$. Formally we choose the test function $\eta = u^{2(1-p)}\varphi$. Then

$$D\eta = -2(p-1)u^{1-2p}\varphi Du + u^{2(1-p)} D\varphi$$

and

$$\frac{\partial \eta}{\partial t} = -2(p-1)u^{1-2p}\varphi \frac{\partial u}{\partial t} + u^{2(1-p)} \frac{\partial \varphi}{\partial t}.$$

A substitutton of these in (2.8) leads to

$$\begin{aligned} 0 \leq & -2(p-1)\int_{t_1}^{t_2}\int_\Omega |Du|^p u^{1-2p}\varphi \, d\mu \, dt \\ & + \int_{t_1}^{t_2}\int_\Omega u^{2(1-p)}|Du|^{p-2}Du \cdot D\varphi \; d\mu \, dt \\ & + 2(p-1)\int_{t_1}^{t_2}\int_\Omega u^{-p}\varphi \frac{\partial u}{\partial t}\, d\mu \, dt - \int_{t_1}^{t_2}\int_\Omega u^{1-p}\frac{\partial \varphi}{\partial t}\, d\mu \, dt. \end{aligned}$$

An integration by parts gives

$$\begin{aligned} \int_{t_1}^{t_2}\int_\Omega u^{-p}\varphi \frac{\partial u}{\partial t}\, d\mu \, dt &= -\frac{1}{p-1}\int_{t_1}^{t_2}\int_\Omega \frac{\partial (u^{1-p})}{\partial t}\varphi \, d\mu \, dt \\ &= \frac{1}{p-1}\int_{t_1}^{t_2}\int_\Omega u^{1-p}\frac{\partial \varphi}{\partial t}\, d\mu \, dt. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} 0 &\leq \int_{t_1}^{t_2}\int_\Omega |Du|^{p-2}Du \cdot D\varphi \, u^{2(1-p)}\, d\mu \, dt + \int_{t_1}^{t_2}\int_\Omega u^{1-p}\frac{\partial \varphi}{\partial t}\, d\mu \, dt \\ &= -\int_{t_1}^{t_2}\int_\Omega \Big(|Dv|^{p-2}Dv \cdot D\varphi - v^{p-1}\frac{\partial \varphi}{\partial t}\Big)\, d\mu \, dt \end{aligned}$$

since $Du = -v^{-2}Dv$. □

The following weighted Poincaré inequality is a consequence of the doubling property (2.2) and the $(1,p)$-Poincaré inequality (2.3). For the proof we refer to [SaCo1].

Theorem 2.2. *Suppose that $u \in H^{1,p}(B(z,R),\mu)$. Let*

$$\phi(x) = \Big(1 - \frac{|x-z|}{R}\Big)_+^\theta,$$

where $\theta > 0$. Then there exists a constant $C = C(p, D_0, P_0, \theta)$ such that for all $0 < r < R$

$$\fint_{B(z,r)} |u - u_\phi|^p \phi \, d\mu \leq C r^p \fint_{B(z,r)} |Du|^p \phi \, d\mu,$$

where

$$u_\phi = \frac{\int_{B(z,r)} u\phi \, d\mu}{\int_{B(z,r)} \phi \, d\mu}.$$

The following abstract lemma is originally due to Bombieri, see [BoGi] and [Bomb]. The proof is an easy modification of the proof by Moser in [Mo2]. See also [SaCo1].

Lemma 2.2. *Let ν be a Borel measure and θ, A and γ be positive constants, $0 < \delta < 1$ and $0 < q \leq \infty$. Let U_σ be bounded measurable sets with $U_{\sigma'} \subset U_\sigma$ for $0 < \delta \leq \sigma' < \sigma \leq 1$. Moreover, if $q < \infty$, we assume that the doubling condition $\nu(U_1) \leq A\nu(U_\delta)$ holds. Let f be a positive measurable function on U_1 which satisfies the reverse Hölder inequality*

$$\Big(\fint_{U_{\sigma'}} f^q \, d\nu\Big)^{1/q} \leq \Big(\frac{A}{(\sigma - \sigma')^\theta} \fint_{U_\sigma} f^s \, d\nu\Big)^{1/s}$$

with $0 < s < q$. Assume further that f satisfies

$$\nu(\{x \in U_1 | \log f > \lambda\}) \leq \frac{A\nu(U_\delta)}{\lambda^\gamma}$$

for all $\lambda > 0$. Then

$$\Big(\fint_{U_\delta} f^q \, d\nu\Big)^{1/q} \leq C,$$

where C depends only on θ, γ, q and A.

Proof. We denote

$$\psi = \psi(\sigma) = \log \Big(\fint_{U_\sigma} f^q \, d\nu\Big)^{1/q}.$$

First, Hölder's inequality gives

$$\begin{aligned}
\fint_{U_\sigma} f^s \, d\nu =& \frac{1}{\nu(U_\sigma)} \int_{\log f \leq \psi/2} f^s \, d\nu + \frac{1}{\nu(U_\sigma)} \int_{\log f > \psi/2} f^s \, d\nu \\
\leq& \exp(\psi s/2) + \Big(\fint_{U_\sigma} f^q \, d\nu\Big)^{s/q} \Big(\frac{\nu(\{\log f > \psi/2\})}{\nu(U_\sigma)}\Big)^{(q-s)/q} \\
\leq& \exp(\psi s/2) + \exp(\psi s) \left(\frac{A}{(\psi/2)^\gamma}\right)^{(q-s)/q}.
\end{aligned}$$

Then, if $s = 2\psi^{-1} \log\big(\psi^\gamma / A2^\gamma\big)$, we have

$$\fint_{U_\sigma} f^s \, d\nu \leq 2 \exp(\psi s/2).$$

This is true if $\psi \geq 4A^{1/\gamma}$ and ψ is also so large that $s < q$. Consequently, the lower bound on ψ depends on A, γ and q. We call it A_1.

Next, we take a logarithm from the reverse Hölder inequality and use the estimate above:

$$\begin{aligned}\psi(\sigma') \leq& \frac{1}{s}\Big(\log\Big(\frac{2A}{(\sigma-\sigma')^\theta}\Big) + \psi(\sigma)s/2\Big)\\ =&\frac{\psi(\sigma)}{2}\Big(\log\Big(\frac{2A}{(\sigma-\sigma')^\theta}\Big)/\log\big(\psi^\gamma/A2^\gamma\big)+1\Big).\end{aligned}$$

Suppose further that

$$\psi^\gamma/A2^\gamma \geq \Big(\frac{2A}{(\sigma-\sigma')^\theta}\Big)^2$$

or

$$\psi \geq \frac{A_2}{(\sigma-\sigma')^{2\theta/\gamma}},$$

where A_2 depends only on A and γ. Then, for $\psi(\sigma)$ large we have an estimate

$$\psi(\sigma') \leq \frac{3}{4}\psi(\sigma).$$

On the other hand, if

$$\psi(\sigma) \leq \min\Big(A_1, \frac{A_2}{(\sigma-\sigma')^{2\theta/\gamma}}\Big)$$

we have from the doubling condition that

$$\psi(\sigma') \leq \log\Big(\frac{\nu(U_\sigma)}{\nu(U_{\sigma'})}\Big) + \psi(\sigma) \leq \log A + \min\Big(A_1, \frac{A_2}{(\sigma-\sigma')^{2\theta/\gamma}}\Big).$$

We collect the results obtained so far: There exists a constant C depending only on A, γ and q such that.

$$\psi(\sigma') \leq \frac{3}{4}\psi(\sigma) + C\Big(1+\frac{1}{(\sigma-\sigma')^{2\theta/\gamma}}\Big).$$

The assertion follows now by a standard iteration argument (see e.g. [Giaq]). □

3. Estimates for super- and subsolutions

3.1. **Caccioppoli estimates.** The following three lemmata are essentially consequences of choosing a correct test function in (2.1).

Lemma 3.1. *Suppose that $u \geq \rho > 0$ is a supersolution in $\Omega \times (t_1, t_2)$ and let $\varepsilon > 0$ with $\varepsilon \neq p-1$. Then there exists a constant $C = C(p, \varepsilon)$ such that*

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega |Du|^p u^{-\varepsilon-1}\varphi^p\,d\mu\,dt + \operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu\\ &\quad\leq C\int_{t_1}^{t_2}\int_\Omega u^{p-1-\varepsilon}|D\varphi|^p\,d\mu\,dt + C\int_{t_1}^{t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^{p-1}\Big|\frac{\partial\varphi}{\partial t}\Big|\,d\mu\,dt\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega\times(t_1,t_2))$ with $\varphi \geq 0$.

Proof. Formally we choose the test function $\eta = u^{-\varepsilon}\varphi^p$ so that

$$D\eta = -\varepsilon u^{-\varepsilon-1}\varphi^p Du + u^{-\varepsilon} D\varphi^p$$

and

$$\frac{\partial \eta}{\partial t} = -\varepsilon u^{-\varepsilon-1}\varphi^p \frac{\partial u}{\partial t} + u^{-\varepsilon}\frac{\partial \varphi^p}{\partial t},$$

where $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$. Let $t_1 < \tau_1 < \tau_2 < t_2$. We first integrate by parts to get

$$\begin{aligned}
&-\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1}\frac{\partial \eta}{\partial t}\,d\mu\,dt + \left[\int_\Omega u^{p-1}\eta\,d\mu\right]_{t=\tau_1}^{\tau_2} \\
&=\frac{\varepsilon}{p-1-\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega \frac{\partial u^{p-1-\varepsilon}}{\partial t}\varphi^p\,d\mu\,dt - \int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1-\varepsilon}\frac{\partial \varphi^p}{\partial t}\,d\mu\,dt \\
&\quad + \left[\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} \\
&\leq \frac{p(p-1)}{|p-1-\varepsilon|}\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1-\varepsilon}\left|\frac{\partial \varphi}{\partial t}\right|\varphi^{p-1}\,d\mu\,dt \\
&\quad + \frac{p-1}{p-1-\varepsilon}\left[\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu\,dt\right]_{t=\tau_1}^{\tau_2}.
\end{aligned}$$

Hence a substitution of η in (2.8) gives

$$\begin{aligned}
0 \leq &-\varepsilon\int_{\tau_1}^{\tau_2}\int_\Omega |Du|^p u^{-\varepsilon-1}\varphi^p\,d\mu\,dt \\
&+ p\int_{\tau_1}^{\tau_2}\int_\Omega |Du|^{p-1}\varphi^{p-1}|D\varphi|u^{-\varepsilon}\,d\mu\,dt \\
&+ \frac{p(p-1)}{|p-1-\varepsilon|}\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1-\varepsilon}\left|\frac{\partial \varphi}{\partial t}\right|\varphi^{p-1}\,d\mu\,dt \\
&+ \frac{p-1}{p-1-\varepsilon}\left[\int_\Omega u^{p-1-\varepsilon}\varphi^p\,d\mu\,dt\right]_{t=\tau_1}^{\tau_2} \\
= &-\varepsilon I_1 + pI_2 + \frac{p(p-1)}{|p-1-\varepsilon|}I_3 + \frac{p-1}{p-1-\varepsilon}I_4.
\end{aligned}$$

Young's inequality implies

$$\begin{aligned}
I_2 &= \int_{\tau_1}^{\tau_2}\int_\Omega \left(|Du|\varphi u^{-(\varepsilon+1)/p}\right)^{p-1}\left(|D\varphi|u^{-\varepsilon+(\varepsilon+1)(p-1)/p}\right)d\mu\,dt \\
&\leq \gamma I_1 + c(\gamma)\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p u^{-\varepsilon p+(\varepsilon+1)(p-1)}\,d\mu\,dt \\
&= \gamma I_1 + c(\gamma)\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p u^{p-1-\varepsilon}\,d\mu\,dt,
\end{aligned}$$

where $\gamma > 0$. Thus we have

$$\begin{aligned} I_1 &- \frac{2(p-1)}{\varepsilon(p-1-\varepsilon)} I_4 \\ &\le \frac{2p\,c(\varepsilon/2)}{\varepsilon} \int_{\tau_1}^{\tau_2} \int_\Omega |D\varphi|^p u^{p-1-\varepsilon}\, d\mu\, dt + \frac{2p(p-1)}{\varepsilon|p-1-\varepsilon|} I_3, \end{aligned}$$

where we have chosen $\gamma = \varepsilon/2$. Furthermore, if $\varepsilon < p-1$ by choosing $\tau_2 = t_2$ and $\tau_1 = \tau > t_1$ such that

$$\int_\Omega u^{p-1-\varepsilon}(x,\tau)\varphi^p(x,\tau)\, d\mu \ge \frac{1}{2} \operatorname*{ess\,sup}_{t_1<t<t_2} \int_\Omega u^{p-1-\varepsilon}\varphi^p d\mu$$

we obtain

$$\begin{aligned} &\operatorname*{ess\,sup}_{t_1<t<t_2} \int_\Omega u^{p-1-\varepsilon}\varphi^p\, d\mu \\ &\le C \int_{\tau_1}^{t_2} \int_\Omega |D\varphi|^p u^{p-1-\varepsilon}\, d\mu\, dt + C \int_{\tau_1}^{t_2} \int_\Omega u^{p-1-\varepsilon} \Big|\frac{\partial\varphi}{\partial t}\Big| \varphi^{p-1}\, d\mu\, dt \\ &\le C \int_{t_1}^{t_2} \int_\Omega |D\varphi|^p u^{p-1-\varepsilon}\, d\mu\, dt + C \int_{t_1}^{t_2} \int_\Omega u^{p-1-\varepsilon} \Big|\frac{\partial\varphi}{\partial t}\Big| \varphi^{p-1}\, d\mu\, dt. \end{aligned}$$

We conclude the same estimate for $\varepsilon > p-1$, if we choose $\tau_1 = t_1$ and $\tau_2 = \tau$. Now the result follows with the constant C depending on ε and p. Remark that the constant blows up as ε tends to 0 or $p-1$. □

Next, we show a corresponding result for a subsolution. Observe that in the following lemma we may have quantities which are not finite a priori. Nevertheless, we can make our calculations with a truncated test function. After we have a control on the quantities, we obtain the result by letting the level of truncation go to infinity. In fact, this also justifies the formal calculations made in the proof of Lemma 3.5.

Lemma 3.2. *Suppose that $u \ge \rho > 0$ is a subsolution and let $\varepsilon > 0$. Then there exists a constant $C = C(\varepsilon, p)$ such that*

$$\begin{aligned} &\int_{t_1}^{t_2} \int_\Omega |Du|^p u^{\varepsilon-1}\varphi^p\, d\mu\, dt + \operatorname*{ess\,sup}_{t_1<t<t_2} \int_\Omega u^{p-1+\varepsilon}\varphi^p\, d\mu \\ &\le C \int_{t_1}^{t_2} \int_\Omega u^{p-1+\varepsilon}|D\varphi|^p\, d\mu\, dt + C \int_{t_1}^{t_2} \int_\Omega u^{p-1+\varepsilon}\varphi^{p-1} \Big|\frac{\partial\varphi}{\partial t}\Big|\, d\mu\, dt \end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega \times (t_1,t_2))$ with $\varphi \ge 0$.

Proof. This time we formally choose the test function $\eta = u^\varepsilon\varphi^p$. Otherwise the assertion follows as in the proof of Lemma 3.1. The constant C blows up as ε tends to 0. □

Finally, we show a Caccioppoli type estimate for the logarithm of a supersolution.

Lemma 3.3. *Suppose that $u \geq \rho > 0$ is a supersolution. Then there exists a constant $C = C(p)$ such that*

$$\int_{t_1}^{t_2}\int_\Omega |D(\log u)|^p \varphi^p \, d\mu \, dt + \operatorname*{ess\,sup}_{t_1<t<t_2} \left| \int_\Omega \log u \, \varphi^p \, d\mu \right|$$
$$\leq C \int_{t_1}^{t_2}\int_\Omega |D\varphi|^p \, d\mu \, dt + C \int_{t_1}^{t_2}\int_\Omega |\log u| \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| d\mu \, dt$$

for every $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$.

Proof. Let $\eta = u^{1-p}\varphi^p$, where $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ and $\varphi \geq 0$. We again integrate by parts and obtain

$$\begin{aligned}
-\int_{\tau_1}^{\tau_2}\int_\Omega & u^{p-1}\frac{\partial \eta}{\partial t} \, d\mu \, dt + \left[\int_\Omega u^{p-1} \eta \, d\mu \right]_{t=\tau_1}^{\tau_2} \\
=&(p-1)\int_{\tau_1}^{\tau_2}\int_\Omega \frac{\partial \log u}{\partial t} \varphi^p \, d\mu \, dt - \int_{\tau_1}^{\tau_2}\int_\Omega \frac{\partial \varphi^p}{\partial t} \, d\mu \, dt \\
&+ \left[\int_\Omega \varphi^p \, d\mu \right]_{t=\tau_1}^{\tau_2} \\
=& -p(p-1)\int_{\tau_1}^{\tau_2}\int_\Omega \log u \, \varphi^{p-1}\frac{\partial \varphi}{\partial t} \, d\mu \, dt \\
&+ (p-1)\left[\int_\Omega \log u \, \varphi^p \, d\mu \right]_{t=\tau_1}^{\tau_2},
\end{aligned}$$

where $t_1 < \tau_1 < \tau_2 < t_2$. We denote $v = \log u$ and substitute η in (2.8) to get

$$\begin{aligned}
0 \leq& -\int_{\tau_1}^{\tau_2}\int_\Omega |Dv|^p \varphi^p \, d\mu \, dt + \frac{p}{p-1}\int_{\tau_1}^{\tau_2}\int_\Omega |Dv|^{p-1}|D\varphi|\varphi^{p-1} \, d\mu \, dt \\
&+ \left[\int_\Omega v\varphi^p \, d\mu \right]_{t=\tau_1}^{\tau_2} + p\int_{\tau_1}^{\tau_2}\int_\Omega |v|\varphi^{p-1}\left| \frac{\partial \varphi}{\partial t} \right| d\mu \, dt.
\end{aligned}$$

We apply Young's inequality for the second term and obtain

$$\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_\Omega & (|Dv|\varphi)^{p-1}|D\varphi| \, d\mu \, dt \\
&\leq \frac{p-1}{2p}\int_{\tau_1}^{\tau_2}\int_\Omega |Dv|^p \varphi^p \, d\mu \, dt + C\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p \, d\mu \, dt.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_\Omega & |Dv|^p \varphi^p \, d\mu \, dt - \left[\int_\Omega v\varphi^p \, d\mu \right]_{t=\tau_1}^{\tau_2} \\
&\leq C\int_{\tau_1}^{\tau_2}\int_\Omega |D\varphi|^p \, d\mu \, dt + C\int_{\tau_1}^{\tau_2}\int_\Omega |v|\varphi^{p-1}\left| \frac{\partial \varphi}{\partial t} \right| d\mu \, dt.
\end{aligned} \tag{3.1}$$

Now the claim follows in the standard way as in the proof of Lemma 3.1. □

Remark. In (3.1) the test function φ does not need to have a compact support in time. We will use this fact in the future.

3.2. **Reverse Hölder inequality for a supersolution.** For $\tau_0 \in \mathbb{R}$ and a ball $B(z,r) \subset \mathbb{R}^n$ we set

$$Q_0 = Q_0(z,\tau_0,r) = B(z,r) \times (\tau_0 - r^p, \tau_0 + r^p).$$

We also define

$$\sigma Q = \sigma Q(z,\tau,r,T) = B(z,\sigma r) \times \big(\tau - T(\sigma r)^p, \tau + T(\sigma r)^p\big)$$

for any $\tau \in \mathbb{R}$, $0 < \sigma \leq 1$ and $T > 0$. The parameter T is going to be chosen so that the time intervals between different lemmata match.

In the following lemma our goal is to obtain a constant which is independent of the parameter s. In the standard approach of Moser [Mo1] only a finite iteration is needed. In that case there is no need to control the asymptotic behaviour of the constant. In our approach the number of iterations is not bounded and we have to make a geometrically convergent partition of the cylinder Q in order to obtain a uniform bound for the constant.

Lemma 3.4. *Suppose that $u \geq \rho > 0$ is a supersolution in Q_0 and let $Q \subset Q_0$ and $0 < \delta < 1$. Then there exist positive constants $C = C(p,q,D_0,P_0,T,\delta)$ and $\theta = \theta(p,D_0)$ such that*

$$\Big(\fint_{\sigma' Q} u^q \, d\mu \, dt\Big)^{1/q} \leq \Big(\frac{C}{(\sigma - \sigma')^\theta}\Big)^{1/s} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt\Big)^{1/s}$$

for all $0 < \delta \leq \sigma' < \sigma \leq 1$ and for all $0 < s < q < q_0$, where $q_0 = (p-1)(2 - p/\kappa)$ and $\kappa > p$ is as in (2.6).

Proof. The proof is based on the successive use of Sobolev's inequality and Caccioppoli's estimate. Let $\gamma = 2 - p/\kappa$. We fix σ and divide the interval (σ', σ) into k parts by setting

$$\sigma_0 = \sigma, \quad \sigma_k = \sigma', \quad \sigma_j = \sigma - (\sigma - \sigma')\frac{1 - \gamma^{-j}}{1 - \gamma^{-k}}.$$

We shall fix k later. We denote $Q_j = \sigma_j Q = B_j \times T_j$. We also choose test functions with the following properties:

$$\operatorname{supp}(\varphi_j) \subset Q_{j-1},$$

$$0 \leq \varphi_j \leq 1 \text{ in } Q_{j-1}, \quad \varphi_j = 1 \text{ in } Q_j,$$

$$|D\varphi_j| \leq C\frac{\gamma^j}{r(\sigma - \sigma')}, \quad \Big|\frac{\partial \varphi_j}{\partial t}\Big| \leq \frac{C}{T}\Big(\frac{\gamma^j}{r(\sigma - \sigma')}\Big)^p \text{ in } \sigma_j Q.$$

Furthermore, let $\alpha = p-1-\varepsilon$, $0<\varepsilon<p-1$. An application of Hölder's inequality yields

$$
\begin{aligned}
&\fint_{Q_{j+1}} u^{\gamma\alpha}\,d\mu\,dt \\
&\quad\le \fint_{T_{j+1}} \Big(\fint_{B_{j+1}} u^{\alpha}\varphi_j^p\,d\mu\Big)^{(\kappa-p)/\kappa}\Big(\fint_{B_{j+1}} \big(u^{\alpha/p}\varphi_j\big)^{\kappa}\,d\mu\Big)^{p/\kappa}\,dt \\
&\quad\le \frac{|T_j|\mu(B_j)}{|T_{j+1}|\mu(B_{j+1})}\Big(\operatorname*{ess\,sup}_{T_j}\fint_{B_j} u^{\alpha}\varphi_j^p\,d\mu\Big)^{(\kappa-p)/\kappa} \\
&\qquad\times \fint_{T_j}\Big(\fint_{B_j}\big(u^{\alpha/p}\varphi_j\big)^{\kappa}\,d\mu\Big)^{p/\kappa}\,dt.
\end{aligned}
$$

Since the measure μ is doubling and $\sigma_{j+1}\ge \min(\delta,(\gamma+1)^{-1})\sigma_j$, the first factor on the right hand side is bounded by a constant independent of j, r, σ and σ'. We now use Sobolev's inequality together with Caccioppoli's estimate and obtain

$$
\begin{aligned}
&\fint_{Q_{j+1}} u^{\gamma\alpha}\,d\mu\,dt \\
&\quad\le C\Big(\operatorname*{ess\,sup}_{T_j}\fint_{B_j} u^{\alpha}\varphi_j^p\,d\mu\Big)^{(\kappa-p)/\kappa} r^p \fint_{T_j}\fint_{B_j}\big|D(u^{\alpha/p}\varphi_j)\big|^p\,d\mu\,dt \\
&\quad\le C\Big(\operatorname*{ess\,sup}_{T_j}\fint_{B_j} u^{\alpha}\varphi_j^p\,d\mu \\
&\qquad + \frac{1}{\delta T}\int_{T_j}\fint_{B_j} \alpha^p|Du|^p u^{-\varepsilon-1}\varphi_j^p + u^{p-1-\varepsilon}|D\varphi_j|^p\,d\mu\,dt\Big)^{\gamma} \\
&\quad\le C\Big(\int_{T_j}\fint_{B_j} u^{p-1-\varepsilon}\Big(|D\varphi_j|^p + \Big|\frac{\partial\varphi_j}{\partial t}\Big|\Big)\,d\mu\,dt\Big)^{\gamma} \\
&\quad\le C\Big(\frac{\gamma^{jp}}{(\sigma-\sigma')^p}\fint_{Q_j} u^{\alpha}\,d\mu\,dt\Big)^{\gamma}.
\end{aligned}
\tag{3.2}
$$

Careful study of the proof of Lemma 3.1 shows that the constant C is indeed independent of α; the term α^p in the inequality above cancels the impact of the singularity of the constant in Lemma 3.1 when ε is close to $p-1$.

The next step in the proof is to iterate (3.2). Observe that the condition $0<\alpha<p-1$ must be satisfied. This gives an upper bound $q_0=\gamma(p-1)$ for q. For the iteration, we fix q and s with $q>s$, and k such that $s\gamma^{k-1}\le q\le s\gamma^k$. Let ρ_0 such that $\rho_0\le s$ and $q=\gamma^k\rho_0$. Denote

$\rho_j = \gamma^j \rho_0$ for $j = 0, \ldots, k$. Then we have

$$\Big(\fint_{Q_k} u^q \, d\mu \, dt\Big)^{1/q} \le \Big(\frac{C\gamma^k}{\sigma - \sigma'}\Big)^{p/\rho_{k-1}} \Big(\fint_{Q_{k-1}} u^{\rho_{k-1}} \, d\mu \, dt\Big)^{1/\rho_{k-1}}$$
$$\le \vdots$$
$$\le \Big(\frac{c_{prod}(k)}{(\sigma - \sigma')^{\gamma^*}} \fint_{\sigma Q} u^{\rho_0} \, d\mu \, dt\Big)^{1/\rho_0},$$

where

$$c_{prod}(k) = C^{\gamma^*} \prod_{j=0}^{k-1} \big(\gamma^{j+1}\big)^{p\gamma^{-j}}$$

and

$$\gamma^* = p \sum_{j=0}^{k-1} \gamma^{-j} = \frac{p\gamma}{\gamma - 1}(1 - \gamma^{-k}).$$

The constant C depends on q since the constant in Lemma 3.1 has a singularity at $\varepsilon = 0$. Obviously $c_{prod}(k)$ is uniformly bounded on k. From Hölder's inequality we obtain

$$\Big(\fint_{\sigma' Q} u^q \, d\mu \, dt\Big)^{1/q} \le \Big(\frac{C}{(\sigma - \sigma')^{\gamma^*}}\Big)^{1/\rho_0} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt\Big)^{1/s}.$$

Furthermore, since $s\gamma^{k-1} \le \rho_0 \gamma^k$, we have $\rho_0 \ge s/\gamma$ and consequently the required estimate follows with $\theta = p\gamma^2/(\gamma - 1)$. □

3.3. **Estimate for the essential supremum of a subsolution.**

Lemma 3.5. *Suppose that $u \ge \rho > 0$ is a subsolution in Q. Let $0 < \delta < 1$. Then there exist positive constants $C = C(p, D_0, P_0, T, \delta)$ and $\theta = \theta(p, D_0)$, such that*

$$\operatorname*{ess\,sup}_{\sigma' Q} u \le \Big(\frac{C}{(\sigma - \sigma')^{\theta}}\Big)^{1/s} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt\Big)^{1/s}$$

for all $0 < \delta \le \sigma' < \sigma \le 1$ and for all $s > 0$.

Proof. Without loss of generality we can choose $T = 1$. Let the choices of test functions and σ_j be the same as in the proof of Lemma 3.4 with an exception that

$$\sigma_j = \sigma - (\sigma - \sigma')(1 - \gamma^{-j}).$$

As in the proof of Lemma 3.4 we obtain from the Sobolev's inequality and from Lemma 3.2 that

$$\fint_{Q_{j+1}} u^{\gamma\alpha} \, d\mu \, dt \le C\Big(\frac{\alpha^p \gamma^{jp}}{(\sigma - \sigma')^p} \fint_{Q_j} u^\alpha \, d\mu \, dt\Big)^\gamma, \tag{3.3}$$

where
$$\gamma = 2 - \frac{p}{\kappa}, \quad \alpha = p - 1 + \varepsilon, \quad \varepsilon \geq 1.$$
In Lemma 3.2 the constant is singular as ε is close to 0. We deliberately avoid this singularity by choosing $\varepsilon \geq 1$. Moreover, we choose $\alpha_j = p\gamma^j$, $j = 0, 1, \dots$. We iterate the inequality above and obtain
$$\begin{aligned}&\Big(\fint_{Q_0} u^p \, d\mu \, dt\Big)^{1/p} \\ &\geq \Big(\frac{(\sigma - \sigma')}{C}\Big)^{\gamma^{-1}+\gamma^{-2}+\dots+\gamma^{-k}} \prod_{j=1}^{k} \gamma^{2j/\gamma^j} \Big(\fint_{Q_k} u^{\gamma^k p} \, d\mu \, dt\Big)^{1/\gamma^k p}\end{aligned}$$
We let k tend to infinity and get the result for $s \geq p$ from Hölder's inequality.

If $s < p$, then we have
$$\begin{aligned}\operatorname*{ess\,sup}_{\sigma' Q} u &\leq \Big(\frac{C}{(\sigma - \sigma')^\theta}\Big)^{1/p} \Big(\fint_{\sigma Q} u^p \, d\mu \, dt\Big)^{1/p} \\ &\leq \Big(\frac{p-s}{2p} \operatorname*{ess\,sup}_{\sigma Q} u\Big)^{(p-s)/p} \Big(\frac{2p}{p-s}\Big)^{(p-s)/p} \\ &\quad \times \Big(\frac{C}{(\sigma - \sigma')^\theta}\Big)^{1/p} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt\Big)^{1/p} \\ &\leq \frac{1}{2} \operatorname*{ess\,sup}_{\sigma Q} u + \Big((p-s)^{s-p} \frac{C}{(\sigma - \sigma')^\theta}\Big)^{1/s} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt\Big)^{1/s} \\ &\leq \frac{1}{2} \operatorname*{ess\,sup}_{\sigma Q} u + \Big(\frac{C}{(\sigma - \sigma')^\theta}\Big)^{1/s} \Big(\fint_{\sigma Q} u^s \, d\mu \, dt\Big)^{1/s},\end{aligned}$$
where we used Young's inequality. By a standard iteration argument (see e.g. [Giaq] Lemma 5.1) we obtain the result. □

3.4. **Logarithmic estimate for a supersolution.** We already have the reverse Hölder inequalities for both super- and subsolutions. Next we show that the condition for the logarithm in the assumptions of Lemma 2.2 holds.

Let $0 < \sigma \leq 1$, $\tau \in \mathbb{R}$, $T > 0$ and $B(z, r) \subset \mathbb{R}^n$. We define
$$\sigma Q^+ = \sigma Q^+(z, r, \tau) = B(z, \sigma r) \times (\tau, \tau + T(\sigma r)^p),$$
$$\sigma Q^- = \sigma Q^-(z, r, \tau) = B(z, \sigma r) \times (\tau - T(\sigma r)^p, \tau)$$
and
$$\sigma Q = B(z, \sigma r) \times (\tau - Tr^p, \tau + Tr^p).$$
Let $d\nu = d\mu \, dt$.

Lemma 3.6. *Suppose that $u \geq \rho > 0$ is a supersolution in Q. Furthermore, let*

$$\varphi(x,t) = \varphi(x) = \Big(1 - 2\frac{|x-z|}{(1+\sigma)r}\Big)_+,$$

where $0 < \sigma < 1$ and $(x,t) \in B(z,r) \times (\tau - (\sigma r)^p, \tau + (\sigma r)^p)$. Let

$$\beta = \int_{B(z,r)} \log u(x,\tau)\varphi^p(x)\, d\mu(x).$$

Then there exist constants $C = C(p, D_0, P_0, \sigma, T)$ and $C' = C'(p,D_0,\sigma,T)$ such that

$$\nu\big(\{(x,t) \in \sigma Q^- \,|\, \log u > \lambda + \beta + C'\}\big) \leq \frac{C}{\lambda^{p-1}}\nu(\sigma Q^-)$$

and

$$\nu\big(\{(x,t) \in \sigma Q^+ \,|\, \log u < -\lambda + \beta - C'\}\big) \leq \frac{C}{\lambda^{p-1}}\nu(\sigma Q^+).$$

for every $\lambda > 0$.

Proof. Let

$$N = \int_{B(z,r)} \varphi^p(x)\, d\mu(x),$$

so that we have

$$\Big(\frac{1-\sigma}{1+\sigma}\Big)^p \mu(B(z,\sigma r)) \leq N \leq \mu(B(z,r)).$$

We also denote

$$v(x,t) = \log u(x,t) - \beta, \quad V(t) = \frac{1}{N}\int_{B(z,r)} v(x,t)\varphi^p(x)\, d\mu(x).$$

Remark that $V(\tau) = 0$. Since φ is independent of t, we obtain from (3.1) that

$$\int_{t_1}^{t_2}\int_{B(z,r)} |Dv|^p\varphi^p\, d\mu\, dt - \Big[\int_{B(z,r)} v\varphi^p\, d\mu\Big]_{t=t_1}^{t_2} \leq C\int_{t_1}^{t_2}\int_{B(z,r)} |D\varphi|^p\, d\mu\, dt,$$

where $\tau - (\sigma r)^p \leq t_1 < t_2 \leq \tau + (\sigma r)^p$. Furthermore, the weighted Poincaré inequality 2.2 yields

$$\int_{B(z,r)} |Dv|^p\varphi^p\, d\mu \geq \frac{1}{Cr^p}\int_{B(z,r)} |v - V(t)|^p\varphi^p\, d\mu \geq \frac{(1-\sigma)^p}{Cr^p}\int_{B(z,\sigma r)} |v - V(t)|^p\, d\mu.$$

It follows that

$$\frac{1}{CNr^p}\int_{t_1}^{t_2}\int_{B(z,r)}|v-V(t)|^p\,d\mu\,dt+V(t_1)-V(t_2)$$
$$\le \frac{C(t_2-t_1)}{r^p}\frac{\mu(B(z,r))}{N}$$
$$\le C'\frac{t_2-t_1}{T(\sigma r)^p}.$$

By denoting

$$w(x,t)=v(x,t)+C'\frac{t-\tau}{T(\sigma r)^p},\quad W(t)=V(t)+C'\frac{t-\tau}{T(\sigma r)^p}$$

we obtain

$$\frac{1}{CNr^p}\int_{t_1}^{t_2}\int_{B(z,r)}|w-W(t)|^p\,d\mu\,dt+W(t_1)-W(t_2)\le 0.$$

From this we conclude that $W(t_1)\le W(t_2)$ for all $\tau-(\sigma r)^p\le t_1<t_2\le\tau+(\sigma r)^p$. Since W is a monotonic function it is differentiable almost everywhere. As a consequence we have

$$\frac{1}{CNr^p}\int_{B(z,r)}|w-W(t)|^p\,d\mu-W'(t)\le 0 \tag{3.4}$$

for almost every $t\in(t_1,t_2)$. Let

$$E_\lambda^-(t)=\{(x,t)\in\sigma Q^-|w(x,t)>\lambda\}.$$

We observe that

$$\int_{\sigma B}|w-W(t)|^p\,d\mu\ge(\lambda-W(t))^p\mu(E_\lambda^-(t))\ge\mu(E_\lambda^-(t))\lambda^p$$

because $W(t)\le W(\tau)=0$ as $\tau>t>t-(\sigma r)^p$. Thus we have

$$-\frac{W'(t)}{(\lambda-W(t))^p}+C\frac{\mu(E_\lambda^-(t))}{Nr^p}\le 0$$

for almost every $\tau>t>t-(\sigma r)^p$. We integrate this over the interval $(\tau-(\sigma r)^p,\tau)$ and obtain

$$\frac{\nu(E_\lambda^-)}{Nr^p}\le C\left[(\lambda-W(t))^{-(p-1)}\right]_{t=\tau-(\sigma r)^p}^{\tau}\le\frac{C}{\lambda^{p-1}}.$$

This yields

$$\nu\big(\{(x,t)\in\sigma Q^-\mid\log u>\lambda+\beta+C'\}\big)\le\frac{C\nu(\sigma Q^-)}{\lambda^{p-1}}.$$

Now, let

$$E_\lambda^+(t)=\{(x,t)\in\sigma Q^+|w(x,t)<-\lambda\}.$$

As in the case of Q^- we conclude that

$$\int_{\sigma B(z,r)}|w-W(t)|^p\,d\mu\ge\mu(E_\lambda^+(t))(\lambda+W(t))^p\ge\mu(E_\lambda^-(t))\lambda^p$$

because $W(t) \geq W(\tau) = 0$ as $\tau \leq t \leq \tau + (\sigma r^p)$. Thus, from (3.4), we have

$$-\frac{W'(t)}{(\lambda + W(t))^p} + C\frac{\mu(E_\lambda^+(t))}{\nu(Q^+)} \leq 0$$

for almost every $\tau < t \leq t + (\sigma r)^p$. An integration over the interval $(\tau, \tau + (\sigma r)^p)$ gives

$$\frac{\nu(E_\lambda^+)}{\nu(\sigma Q^+)} \leq -C\Big[(\lambda + W(t))^{-(p-1)}\Big]_{t=\tau}^{\tau+(\sigma r)^p} \leq \frac{C}{\lambda^{p-1}}.$$

Therefore

$$\nu\big(\{(x,t) \in \sigma Q^+ \mid \log u < -\lambda + \beta - C' \ \}\big) \leq \frac{C\nu(\sigma Q^+)}{\lambda^{p-1}}$$

and the claim follows. □

4. Harnack's inequality

First we prove a weak Harnack inequality. Here $U^\pm$ are as before Theorem 2.1.

Theorem 4.1. *Let $u \geq \rho > 0$ be a supersolution in Q. Then there exist constants $C = C(p, D_0, P_0, q, \delta)$ and $q_0 = (p-1)(2 - p/\kappa)$, $\kappa > p$ as in (2.6), such that*

$$\Big(\fint_{\delta U^-} u^q \, d\mu \, dt\Big)^{1/q} \leq C \operatorname*{ess\,inf}_{\delta U^+} u,$$

for $0 < \delta < 1$ and $0 < q < q_0$.

Proof. We fix $0 < \delta < 1$. Let φ be as in the assumptions of Lemma 3.6 and let β and C' be the corresponding constants. We define $v^+ = u^{-1}e^{\beta - C'}$ and $v^- = u\, e^{-\beta - C'}$. We apply Lemma 3.6 for the function u and have

$$\nu\Big(\{(x,t) \in \frac{1+\delta}{2}U^+ | \log(v^+) > \lambda\}\Big) \leq \frac{C}{\lambda^{p-1}}\nu\Big(\frac{1+\delta}{2}U^+\Big)$$

and

$$\nu\Big(\{(x,t) \in \frac{1+\delta}{2}U^- | \log(v^-) > \lambda\}\Big) \leq \frac{C}{\lambda^{p-1}}\nu\Big(\frac{1+\delta}{2}U^-\Big).$$

We also used a fact that $\nu(B(z,\sigma R)\times(\tau, \tau \pm (\sigma R)^p)) \leq C\nu(\delta U^\pm)$. From Lemma 2.1 we obtain that v^+ is a subsolution in Q. Consequently, Lemma 3.5 yields

$$\operatorname*{ess\,sup}_{\sigma' U^+} v^+ \leq \Big(\frac{C}{(\sigma - \sigma')^\theta}\fint_{\sigma U^+} (v^+)^s \, d\mu \, dt\Big)^{1/s}$$

for all $\delta \le \sigma' < \sigma \le (1+\delta)/2$ and for all $s > 0$. Note that we have chosen a suitable parameter T to match the time scales between lemmata. We now use Lemma 2.2 and obtain

$$\operatorname*{ess\,sup}_{\delta U^+} v^+ \le C. \tag{4.1}$$

Furthermore, we have from the corollary of Lemma 3.4 for v^- that

$$\Big(\fint_{\sigma' U^-} (v^-)^q \, d\mu \, dt\Big)^{\frac{1}{q}} \le \Big(\frac{C}{(\sigma-\sigma')^\theta} \fint_{\sigma U^-} (v^-)^s \, d\mu \, dt\Big)^{\frac{1}{s}}$$

for every $\delta \le \sigma' < \sigma \le (1+\delta)/2$, $0 < s < q < q_0$. From Lemma 2.2 we obtain

$$\Big(\fint_{\delta U^-} (v^-)^q \, d\mu \, dt\Big)^{1/q} \le C.$$

Multiplying this with (4.1) gives

$$\Big(\fint_{\delta U^-} u^q \, d\mu \, dt\Big)^{1/q} \le C \operatorname*{ess\,inf}_{\delta U^+} u$$

and the result follows. □

Now we are ready to prove Harnack's inequality.

Proof of theorem 2.1. We apply Lemma 4.1 with $\delta = (1+\sigma)/2$. The result follows now from Lemma 3.5. □

References

[AS] D.G. Aronsson, J. Serrin, Local behaviour of solutions of quasilinear parabolic equations, Arch. Rat. Mech. Anal 25, 81–122 (1967)

[BCLS] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, Sobolev inequalities in disguise, Indiana Univ. Math. J. 44, 1033–1074 (1995)

[Bomb] E. Bombieri, Theory of minimal surfaces and a counterexample to the bernstein conjecture in high dimension, Mimeographed Notes of Lectures Held at Courant Institute, New York University (1970)

[BoGi] E. Bombieri, E. Giusti, Harnack's inequality for elliptic differential equations on minimal surfaces, Invent. Math. 15, 24–46 (1972)

[ChFr] F. Chiarenza, M. Frasca, A note on weighted Sobolev inequality, Proc. Amer. Math. Soc. 93, 703–704 (1985)

[ChSe] F. Chiarenza, R. Serapioni, A Harnack inequality for degenerate parabolic equations, Comm. Partial Differential Equations 9(8), 719–749 (1984)

[DiBe] E. DiBenedetto, Degenerate parabolic equations, Springer-Verlag (1993)

[DBUV] E. DiBenedetto, J.M. Urbano, V. Vespri, Current issues on singular and degenerate evolution equations, Handbook of differential equations, Elsevier, 169–286 (2004)

[FaGa] E. Fabes, N. Garofalo, Parabolic B.M.O. and Harnack's inequality, Proc. Amer. Math. Soc. 50, no. 1, 63–69 (1985)

[FKS] E. Fabes, C. Kenig, R. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. Partial Differential Equations 7, no. 1, 77–116 (1982)

[GiVe] U. Gianazza, V. Vespri, A Harnack inequality for solutions of doubly nonlinear parabolic equation, J. Appl. Funct. Anal (to appear)
[Giaq] M. Giaquinta, Introduction to regularity theory for nonlinear elliptic systems, Birkhäuser Verlag (1993)
[Gri] A. Grigor'yan, The heat equation on non-compact Riemannian manifolds, Matem. Sbornik 182, 55–87 (1991). Engl. Transl. Math. USSR Sb. 72, 47–77 (1992)
[HaK] P. Hajłasz, P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. 688 (2000)
[HKM] J. Heinonen, T. Kilpeläinen and O. Martio, Nonliear Potential Theory of Degenerate Elliptic Equations, Oxford University Press, Oxford (1993)
[KS] J. Kinnunen and N. Shanmugalingam, Regularity of quasi-minimizers on metric spaces, Manuscripta Math. 105, 401–423 (2001)
[Mo1] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17, 101–134 (1964), and correction in Comm. Pure Appl. Math. 20, 231–236 (1967)
[Mo2] J. Moser, On a pointwise estimate for parabolic equations, Comm. Pure Appl. Math. 24, 727–740 (1971)
[SaCo1] L. Saloff-Coste, Aspects of Sobolev-type inequalities, London Mathematical Society Lecture Note Series 289, Cambridge University Press (2002)
[SaCo2] L. Saloff-Coste, A note on Poincaré, Sobolev and Harnack inequalities, Duke Math. J. 65, IMRN 2, 27–38 (1992)
[Tru] N.S. Trudinger, Pointwise estimates and quasilinear parabolic equations, Comm. Pure Appl. Math. 21, 205–226 (1968)
[Ve] V. Vespri, On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations, Manuscripta Math. 75, 65–80 (1992)
[WZYL] Z. Wu, J. Zhao, J. Yin, H.Li, Nonlinear diffusion equations, World Scientific (2001)

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