NONLINEAR PARABOLIC CAPACITY AND POLAR SETS OF SUPERPARABOLIC FUNCTIONS

JUHA KINNUNEN, RIIKKA KORTE, TUOMO KUUSI AND MIKKO PARVIAINEN

Abstract. We extend the theory of the thermal capacity for the heat equation to nonlinear parabolic equations of the p-Laplacian type. We study definitions and properties of the nonlinear parabolic capacity and show that the capacity of a compact set can be represented via a capacitary potential. As an application, we show that polar sets of superparabolic functions are of zero capacity. The main technical tools used include estimates for equations with measure data and obstacle problems.

1. Introduction

The concept of capacity is of fundamental importance in the classical potential theory. For example, a Wiener type criterion for boundary regularity, a characterization of polar sets and removability results are expressed in terms of capacities. In the stationary case, capacity is related to the underlying Sobolev space, but the situation is more delicate for parabolic partial differential equations. Indeed, the definition of the true thermal capacity seems to be related more closely to the partial differential equation than to the underlying function space.

As far as we are aware, this work is the first attempt to extend the theory of the thermal capacity to nonlinear partial differential equations of the form

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x, t, \nabla u) = 0.$$

The principal prototype is the p-parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

with $1 < p < \infty$. When $p \neq 2$, linear tools such as Green's functions and representation formulas are not at our disposal. Hence the nonlinear parabolic capacity of a set $E \subset \mathbb{R}^{n+1}$ is defined as

$$\operatorname{cap}(E) = \sup\{\mu(\mathbb{R}^{n+1}) : 0 \leq u_\mu \leq 1, \operatorname{supp} \mu \subset E\},$$

2010 *Mathematics Subject Classification.* 35K55, 31C45.

Key words and phrases. Thermal capacity, polar set, supercaloric function, parabolic p-Laplace equation, parabolic obstacle problem, parabolic potential theory.

where μ is a nonnegative Radon measure, and u_μ is a weak solution to the measure data problem

$$\frac{\partial u_\mu}{\partial t} - \operatorname{div} \mathcal{A}(x, t, \nabla u_\mu) = \mu,$$

with zero boundary values on the parabolic boundary of the reference domain. The case where the reference set is the whole $\mathbb{R}^{n+1}$ can be reached via a limiting procedure.

Several parabolic capacities have been introduced in the quadratic case when $p = 2$. The thermal capacity related to the heat equation, and its generalizations have been studied, for example, by Lanconelli [27] and Watson [41]. For applications to boundary regularity and removability problems, we refer to Evans and Gariepy [12], Gariepy and Ziemer [13], [14] and Lanconelli [27], [28]. Boundary regularity has been also studied in [29] and polar sets in [37] and [41]. The monumental work [9] contains plenty of material about potential theory related to the heat equation. Capacities introduced in [2], [10], [11], [15], [34], [35], [36] and [42] are defined in terms of function spaces. Droniou, Porretta and Prignet [10], as well as Saraiva [35], [36], also consider the nonquadratic case. As examples in [14] show, some of these capacities may have different zero sets and, consequently, they are more restrictive than the classical thermal capacity. The main motivation for using the thermal capacity is that it gives optimal results for boundary regularity and removable sets.

One of our main results, Theorem 5.7, gives a representation of the capacity of a compact set through capacitary potentials. This extends Theorem 1.1 in [27]. As an application, we show that polar sets of superparabolic functions are of zero parabolic capacity. For the heat equation, we have supercaloric functions or supertemperatures, see [39]. In the nonlinear case, superparabolic functions are defined through the parabolic comparison principle, as proposed in [18], but there are also several alternative characterizations. For example, they can be defined as limits of increasing sequences of continuous supersolutions and every superparabolic function is a solution of a measure data problem, see [20], [21] and [23]. In contrast with the elliptic case in [31] and [16] (see also [17]), the class of superparabolic functions is not closed under scaling. Our argument is based on rather delicate estimates for scaled obstacle problems and convergence results.

2. Nonlinear parabolic PDEs

2.1. **Parabolic Sobolev Spaces.** Let Ω be a bounded smooth open set in $\mathbb{R}^n$ with $n \geq 2$. We denote

$$\Omega_\infty = \Omega \times (0, \infty)$$

and

$$\Omega_{t_1,t_2} = \Omega \times (t_1, t_2)$$

for $-\infty < t_1 < t_2 < \infty$. The parabolic boundary of Ω_{t_1,t_2} is

$$\partial_p \Omega_{t_1,t_2} = \big(\partial\Omega \times [t_1, t_2]\big) \cup (\overline{\Omega} \times \{t_1\}).$$

We emphasize that Ω_∞ is merely a reference set for us and the assumed smoothness properties are rather irrelevant. The smoothness assumption quarantees that the solution to an initial-boundary value problem obtains zero boundary values continuously on the parabolic boundary of the reference set Ω_∞.

As usual, $W^{1,p}(\Omega)$ denotes the Sobolev space of functions in $L^p(\Omega)$ whose first distributional partial derivatives belong to $L^p(\Omega)$ with the norm

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)} .$$

The Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$.

The parabolic space $L^p(0,\infty; W^{1,p}(\Omega))$ is the collection of measurable functions $u(x,t)$ such that for almost every $t \in (0,\infty)$, the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$, and

$$\int_0^\infty ||u||^p_{W^{1,p}(\Omega)} \, \mathrm{d}t < \infty$$

is finite. Analogously, the space $L^p(0,\infty; W_0^{1,p}(\Omega))$ is a collection of measurable functions $u \in L^p(0,\infty; W^{1,p}(\Omega))$ such that for almost every $t \in (0,\infty)$, the function $x \mapsto u(x,t)$ belongs to $W_0^{1,p}(\Omega)$. The local space $L^p_{\mathrm{loc}}(0,\infty; W^{1,p}_{\mathrm{loc}}(\Omega))$ consist of functions that belong to the parabolic Sobolev space in every space time cylinder $\Omega' \times (t_1, t_2) \Subset \Omega_\infty$.

2.2. **Stucture properties.** We consider capacities related to nonlinear parabolic partial differential equations of type

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x,t,\nabla u) = 0,$$

where $\mathcal{A} : \Omega_\infty \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following stuctural conditions:

(1) $(x,t) \mapsto \mathcal{A}(x,t,\xi)$ is measurable for every $\xi \in \mathbb{R}^n$,
(2) $\xi \mapsto \mathcal{A}(x,t,\xi)$ is continuous for almost every $(x,t) \in \Omega_\infty$,
(3) there exist constants $0 < \alpha \le \beta < \infty$ such that for every $\xi \in \mathbb{R}^n$ and for almost every $(x,t) \in \Omega_\infty$, we have

$$\mathcal{A}(x,t,\xi) \cdot \xi \ge \alpha\, |\xi|^p \quad \text{and} \quad |\mathcal{A}(x,t,\xi)| \le \beta\, |\xi|^{p-1} ,$$

and
(4) $\mathcal{A}$ satisfies the monotonicity condition

$$\big(\mathcal{A}(x,t,\xi_1) - \mathcal{A}(x,t,\xi_2)\big) \cdot (\xi_1 - \xi_2) > 0 \tag{2.1}$$

whenever $(x,t,\xi_i) \in \Omega_\infty \times \mathbb{R}^n$, $i = 1, 2$, and $\xi_1 \neq \xi_2$.

Although this class of equations is relevant for all $p > 1$, we shall only consider the case

$$p > \frac{2n}{n+2}. \tag{2.2}$$

The same lower bound for p appears also in the regularity theory of parabolic equations of the p-Laplacian type, see [6] and [8].

Next we recall the definition of a weak solution. We shall use a shorthand notation $\mathcal{A}(\xi) = \mathcal{A}(x,t,\xi)$.

Definition 2.3. A function $u \in L^p_{\mathrm{loc}}(0,\infty; W^{1,p}_{\mathrm{loc}}(\Omega))$, $1 < p < \infty$, is a weak solution of

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(\nabla u) = 0 \tag{2.4}$$

in Ω_∞, if it holds that

$$\int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u) \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \Big) \, \mathrm{d}z = 0 \tag{2.5}$$

for every test function $\varphi \in C_0^\infty(\Omega_\infty)$. For short, we denote $z = (x,t)$ and $\mathrm{d}z = \mathrm{d}x \, \mathrm{d}t$. A function u is a supersolution if the integral in (2.5) is nonnegative for all nonnegative test functions. In a general open subset U of $\mathbb{R}^{n+1}$, the above notions are to be understood in a local sense, that is, u is a solution if it is a solution in every set $\Omega \times (t_2, t_2) \Subset U$.

It follows immediately from the definition that, if u is a weak (super)solution, then $u + \alpha$, $\alpha \in \mathbb{R}$, is a weak (super)solution. Observe that αu $\alpha \in \mathbb{R}$ is not a weak (super)solution in general. The sum of weak (super)solutions is not a weak (super)solution in general, but, however, the pointwise minimum of weak (super)solutions is a weak supersolution.

2.3. **Regularity.** Under the assumption (2.2), weak solutions are locally Hölder continuous after a possible redefinition on a set of measure zero, see DiBenedetto [6] and DiBenedetto, Gianazza and Vespri [7], [8]. See also Wu, Zhao, Yin, and Li [43]. Hence every weak solution has a continuous representative and a continuous weak solution is called an $\mathcal{A}$-parabolic function.

In this work we are mainly interested in weak supersolutions. According to the next result, every weak supersolution has a lower semicontinuous representative. Recall that the lower semicontinuous regularization of a function u is defined as

$$\widehat{u}(x,t) = \operatorname*{ess\,lim\,inf}_{(y,s)\to(x,t)} u(y,s) = \lim_{r\to 0} \operatorname*{ess\,inf}_{B_r(x)\times(t-r^p,t+r^p)} u. \tag{2.6}$$

For the proof of the following result we refer to [25].

Theorem 2.7. *Let u be a weak supersolution in Ω_∞. Then the lower semicontinuous regularization $\widehat{u}$ of u is a weak supersolution and $u = \widehat{u}$ almost everywhere in Ω_∞.*

2.4. **Superparabolic functions.** So called superparabolic functions, also called supertemperatures, play an important role in the parabolic potential theory.

Definition 2.8. A function $u : \Omega_\infty \to (-\infty, \infty]$ is $\mathcal{A}$-superparabolic in Ω_∞, if

(1) u is lower semicontinuous,
(2) u is finite in a dense subset, and
(3) If h is a solution of (2.4) in $\Omega'_{t_1,t_2} = \Omega' \times (t_1, t_2) \Subset \Omega_\infty$, continuous in $\overline{\Omega'}_{t_1,t_2}$, and $h \leq u$ on the parabolic boundary $\partial_p \Omega'_{t_1,t_2}$, then $h \leq u$ in Ω'_{t_1,t_2}.

The reader should carefully distinguish between weak supersolutions and superparabolic functions. Notice that a superparabolic function is defined at every point in its domain, but a weak supersolution is defined only up to a set of measure zero. However, the lower semicontinuous representative $\widehat{u}$ of a weak supersolution u is superparabolic, since the comparison principle holds for supersolutions, see [18].

It has been shown in [21] and [20], see also [23], that every locally bounded superparabolic function is a weak supersolution. Hence there are no other locally bounded superparabolic functions except weak supersolutions. A prime example of an unbounded superparabolic function with respect to to the p-parabolic equation is the Barenblatt solution $\mathcal{B}_p : \mathbb{R}^{n+1} \to [0, \infty)$,

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-n/\lambda}\left(c - \dfrac{p-2}{p}\lambda^{1/(1-p)}\left(\dfrac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\lambda = n(p-2) + p$, $p > 2$, and the constant c is usually chosen so that

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x,t)\, dx = 1$$

for every $t > 0$. There is also a corresponding formula for the case $2n/(n+1) < p \leq 2$. The Barenblatt solution is a weak solution of the p-parabolic equation in the upper half space. However, it is not a weak supersolution in $\mathbb{R}^{n+1}$ because it does not belong to the correct parabolic Sobolev space, see [20] and [21]. The truncations $\min\{\mathcal{B}_p, \lambda\}$, $\lambda > 0$, belong to the correct parabolic Sobolev space and, consequently, are weak supersolutions in $\mathbb{R}^{n+1}$. This shows that the the class of weak supersolutions is not closed with respect to an increasing convergence. In contrast, superparabolic functions have this property.

3. Measure data problems

Next we consider a measure data problem related to weak supersolutions and superparabolic functions.

Definition 3.1. Let μ be a nonnegative Radon measure on $\mathbb{R}^{n+1}$. A function $u \in L^p_{\mathrm{loc}}(0,\infty; W^{1,p}_{\mathrm{loc}}(\Omega))$ is a weak solution of

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(\nabla u) = \mu, \tag{3.2}$$

if

$$\int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u) \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \Big) \, \mathrm{d}z = \int_{\Omega_\infty} \varphi \, \mathrm{d}\mu, \tag{3.3}$$

for every test function $\varphi \in C_0^\infty(\Omega_\infty)$.

Observe that every weak solution to a measure data problem is a weak supersolution. Conversely, every weak supersolution is also a solution to a measure data problem. Indeed, if u is a weak supersolution in Ω_∞, we have

$$\int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u) \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \Big) \, \mathrm{d}z \geq 0$$

for every nonnegative $\varphi \in C_0^\infty(\Omega_\infty)$. The Riesz representation theorem implies that there exists a Radon measure μ_u such that

$$\int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u) \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \Big) \, \mathrm{d}z = \int_{\Omega_\infty} \varphi \, \mathrm{d}\mu_u,$$

for every test function $\varphi \in C_0^\infty(\Omega_\infty)$. The measure μ_u is called the Riesz measure of u. This shows that weak supersolutions and weak solutions to a measure data problem are the same class of functions. Moreover, by Theorem 2.7 we may assume that they are lower semicontinuous.

In a similar fashion, as shown in [22], every superparabolic function satisfies the equation with a finite Radon measure on the right-hand side, and conversely, for every finite Radon measure there exists a superparabolic function which is solution to the corresponding equation with measure data. The integrability of superparabolic functions [20], see also [3] and [4], and the convergence theorem in [23] play an essential role in this context.

The following convergence result will be an essential tool in this work.

Theorem 3.4. *Suppose that u_i, $i = 1, 2, \dots$, is a sequence of uniformly bounded weak supersolutions in Ω_∞ such that $u_i \to u$ almost everywhere in Ω_∞. Then u is a weak supersolution in Ω_∞ and*

$$\lim_{i\to\infty} \int_{\Omega_\infty} \varphi \, \mathrm{d}\mu_{u_i} = \int_{\Omega_\infty} \varphi \, \mathrm{d}\mu_u$$

for every $\varphi \in C_0^\infty(\Omega_\infty)$, i.e. $\mu_{u_i} \to \mu_u$ weakly as $i \to \infty$.

The proof of the previous result is a slight modification of the proof of Theorem 5.3 in [23], see also [30].

Remark 3.5. In general, the time derivative u_t does not exist in the Sobolev sense. This is a principal, well-recognized difficulty with the definition. Indeed, in proving estimates, usually a test function that depends on the solution itself is needed. Then the appearance of the forbidden u_t cannot be avoided. One way to overcome this difficulty is to use convolution in the time direction. Let

$$\varphi_h(x,t) = \int_{\mathbb{R}} \varphi(x, t-s)\zeta_h(s)\,\mathrm{d}s, \tag{3.6}$$

where $\varphi \in C_0^\infty(\Omega_\infty)$ and $\zeta_h(s)$ is a standard mollifier, whose support is contained in $(-h, h)$ with $h < \operatorname{dist}(\operatorname{supp}(\varphi), \Omega \times \{0\})$. We insert φ_h into (3.3), change variables and apply Fubini's theorem to obtain

$$\int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u)_h \cdot \nabla\varphi - u_h \frac{\partial \varphi}{\partial t}\Big)\,\mathrm{d}z = \int_{\Omega_\infty} \varphi_h\,\mathrm{d}\mu \tag{3.7}$$

and

$$\int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u)_h \cdot \nabla\varphi + \varphi \frac{\partial u_h}{\partial t}\Big)\,\mathrm{d}z = \int_{\Omega_\infty} \varphi_h\,\mathrm{d}\mu. \tag{3.8}$$

3.1. **Boundary data.** In this work we use weak solutions of (3.2) in Ω_∞ with zero boundary data, that is, zero boundary values on the lateral boundary $\partial\Omega \times (0, \infty)$ and zero initial values at $\Omega \times \{t = 0\}$. By this we mean that $u \in L^p(0, \infty; W_0^{1,p}(\Omega))$ and

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \int_\Omega |u|^2\,\mathrm{d}z = 0.$$

For existence results, in the case when μ belongs to the dual of the parabolic Sobolev space, we refer to [33]. See also [1] and [5]. General results for a finite Radon measure can be found in [3], [4] and [22].

3.2. **Asymptotical behaviour.** If the Riesz measure is compactly supported, then the corresponding solution u with zero boundary data tends uniformly to zero as $t \to \infty$. For the p-parabolic equation, this follows by a comparison with respect to the Barenblatt solution. Here we sketch an argument that applies for equations with more general structure.

Choose $T > 0$ so large that $\operatorname{supp}\mu \subset \Omega_T$ and let $T < t_1 < t_2 < \infty$. Then u_μ is a weak solution in $\Omega \times (T, \infty)$ and it has zero boundary values on the lateral boundary. We define a cutoff function η, which is independent of the space variable, by

$$\eta(t) = \begin{cases} 0, & t \le t_1 - h, \\ 1 - \frac{t_1 - t}{h}, & t_1 - h < t < t_1, \\ 1, & t_1 \le t \le t_2, \\ 1 - \frac{t - t_2}{h}, & t_2 \le t < t_2 + h, \\ 0, & t \ge t_2 + h. \end{cases}$$

Formally, we use ηu as a test function in (2.5) and obtain

$$\int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u) \cdot \nabla(\eta u) - u \frac{\partial}{\partial t}(\eta u)\Big)\,\mathrm{d}z = 0.$$

For the elliptic term, we have

$$\begin{aligned}\int_{\Omega_\infty} \mathcal{A}(\nabla u)\cdot\nabla(\eta u)\,\mathrm{d}z &= \int_{\Omega_\infty} \eta\mathcal{A}(\nabla u)\cdot\nabla u\,\mathrm{d}z\\ &\to \int_{t_1}^{t_2}\int_\Omega \mathcal{A}(\nabla u)\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

as $h\to 0$. For the remaining term, an integration by parts gives

$$\begin{aligned}-\int_{\Omega_\infty} u\frac{\partial}{\partial t}(\eta u)\,\mathrm{d}z = \int_{\Omega_\infty} \eta u\frac{\partial u}{\partial t}\,\mathrm{d}z &\to \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \frac{\partial(u^2)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{2}\int_\Omega u(x,t_2)^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega u(x,t_1)^2\,\mathrm{d}x\end{aligned}$$

as $h\to 0$. Hence we arrive at

$$\frac{1}{2}\int_\Omega u(x,t_1)^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega u(x,t_2)^2\,\mathrm{d}x = \int_{t_1}^{t_2}\int_\Omega \mathcal{A}(\nabla u)\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t$$

Let $\tau > T$. By denoting

$$I(\tau) = \frac{1}{2}\int_\Omega u(x,\tau)^2\,\mathrm{d}x,$$

we have

$$\frac{I(\tau+\delta)-I(\tau)}{\delta} = -\frac{1}{\delta}\int_\tau^{\tau+\delta}\int_\Omega \mathcal{A}(\nabla u)\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t$$

and by passing to the limit as $\delta\to 0$, we obtain

$$I'(\tau) = -\int_\Omega \mathcal{A}(\nabla u(x,\tau))\cdot\nabla u(x,\tau)\,\mathrm{d}x$$

for almost every $\tau > T$. The structure conditions and the Sobolev and Hölder inequalities imply that

$$\begin{aligned}\int_\Omega \mathcal{A}(\nabla u(x,\tau))\cdot\nabla u(x,\tau)\,\mathrm{d}x &\geq \alpha\int_\Omega |\nabla u(x,\tau)|^p\,\mathrm{d}x\\ &\geq C\Big(\int_\Omega u(x,\tau)^2\,\mathrm{d}x\Big)^{p/2} = CI(\tau)^{p/2},\end{aligned}$$

where the constant C depends only on Ω, the structure constants, p and n. From this we conclude that

$$I'(\tau) \leq -CI(\tau)^{p/2},$$

which together with the fact that $I(T)<\infty$ implies that $I(\tau)\to 0$ as $\tau\to\infty$. If $2n/(p+2) < p \leq 2$, then the differential inequality above implies extinction in finite time and the claim is clear. In the case $p > 2$, the claim follows from Lemma 3.24 in [26].

3.3. **Two comparison results.** Next we present two rather elementary, but extremely useful, technical results related to measure data problems.

Lemma 3.9. *If u and v are weak solutions of* (3.2) *in Ω_∞ with zero boundary data, and $\mu_v \le \mu_u$, then $v \le u$ in Ω_∞.*

Proof. We define a cutoff function η, which is independent of the space variable, by

$$\eta(t) = \begin{cases} 1, & t \le T, \\ 1 + \frac{T-t}{h}, & T < t < T+h, \\ 0, & t \ge T. \end{cases}$$

Formally, we use $\eta(v-u)_+$ as a test function in (3.3) for u and v. By subtracting the equations and using the assumption that $\mu_v \le \mu_u$, we obtain

$$\begin{aligned} 0 &\le \int_{\Omega_\infty} \eta(v-u)_+ \,\mathrm{d}\mu_u - \int_{\Omega_\infty} \eta(v-u)_+ \,\mathrm{d}\mu_v \\ &= \int_{\Omega_\infty} \big(\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)\big) \cdot \nabla\big(\eta(v-u)_+\big) \,\mathrm{d}z \\ &\quad - \int_{\Omega_\infty} (u-v) \frac{\partial}{\partial t}(\eta(v-u)_+) \,\mathrm{d}z. \end{aligned}$$

By monotonicity, the first term on the right hand side can be estimated as

$$\begin{aligned} &\int_{\Omega_\infty} \eta\big(\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)\big) \cdot \nabla(v-u)_+ \,\mathrm{d}z \\ &\qquad = - \int_{\Omega_\infty \cap \{v>u\}} \eta\big(\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)\big) \cdot (\nabla u - \nabla v) \,\mathrm{d}z \le 0. \end{aligned}$$

Since u has zero initial values at $t = 0$, an integration by parts implies that

$$\begin{aligned} 0 &\le - \int_{\Omega_\infty} (u-v) \frac{\partial}{\partial t}(\eta(v-u)_+) \,\mathrm{d}z \\ &= \frac{1}{2} \int_{\Omega_\infty} \frac{\partial \eta}{\partial t} \left[(v-u)_+^2\right] \,\mathrm{d}z = -\frac{1}{2h} \int_T^{T+h} \int_\Omega (v-u)_+^2 \,\mathrm{d}x \,\mathrm{d}t. \end{aligned}$$

By passing to the limit as $h \to 0$, by the Lebesgue differentiation theorem, we arrive at

$$\int_\Omega (v-u)_+^2(x,T) \,\mathrm{d}x \le 0$$

for almost every $T > 0$ and, consequently, $(v-u)_+ = 0$ almost everywhere in Ω_∞. This proves the claim. $\square$

The proof of the following lemma is very similar to the proof of Lemma 3.9. However, for the sake of completeness, we reproduce some details here.

Lemma 3.10. *Let u and v be weak solutions of* (3.2) *in Ω_∞ with zero boundary data. If $v \le u$ in Ω_∞, then*

$$\int_{\Omega_\infty} (u-v)\, d\mu_v \le \int_{\Omega_\infty} (u-v)\, d\mu_u.$$

Proof. Let η be the same cutoff function as in the proof of Lemma 3.9. We use $\eta(u-v)$ as a test function in (3.3), and obtain

$$\begin{aligned}\int_{\Omega_\infty} \eta(u-v)\,\mathrm{d}\mu_u - \int_{\Omega_\infty} \eta(u-v)\,\mathrm{d}\mu_v \\ = \int_{\Omega_\infty} \eta\big(\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)\big)\cdot\nabla(u-v)\,\mathrm{d}z \\ - \int_{\Omega_\infty} (u-v)\frac{\partial}{\partial t}(\eta(u-v))\,\mathrm{d}z.\end{aligned}$$

By monotonicity, the first term on the right hand side is nonnegative and, as in the proof of Lemma 3.9, we have

$$\int_{\Omega_\infty} (u-v)\frac{\partial}{\partial t}(\eta(u-v))\,\mathrm{d}z = -\frac{1}{2h}\int_T^{T+h}\int_\Omega (u-v)^2\,\mathrm{d}z.$$

By passing to the limit as $h \to 0$, by the Lebesgue differentiation theorem, we have

$$\int_0^T\int_\Omega (u-v)\,\mathrm{d}\mu_u - \int_0^T\int_\Omega (u-v)\,\mathrm{d}\mu_v \ge \frac{1}{2}\int_\Omega (u-v)^2(x,T)\,\mathrm{d}x \ge 0$$

for almost every $T > 0$. This proves the claim. □

4. Obstacle problems

Since we do not have representation formulas in the nonlinear parabolic potential theory, the obstacle problem is the main device to construct superparabolic functions with prescribed properties.

Definition 4.1. Let ψ be a bounded measurable function in Ω_∞, and consider the class

$$\Phi_\psi = \{v : v \text{ is superparabolic in } \Omega_\infty \text{ and } v \ge \psi \text{ in } \Omega_\infty\}.$$

Define

$$R_\psi = \inf\{v : v \in \Phi_\psi\}.$$

We say that R_ψ is the solution to the obstacle problem in Ω_∞ with the obstacle ψ. We also consider the lower semicontinuous regularization $\widehat{R}_\psi$.

For a bounded obstacle, the solution always exists and is unique. Moreover, the lower semicontinuous representative of a solution is superparabolic and, since it is bounded, it is also a weak supersolution in Ω_∞. Theorem 2.7 and [32] imply that

$$\widehat{R}_\psi = R_\psi$$

almost everywhere in Ω_∞. If, in addition, $\psi \in C(\overline{\Omega}_\infty)$, then the solution of the obstacle problem has the following properties:

- $R_\psi \in C(\overline{\Omega}_\infty)$,
- R_ψ is a weak solution in the set $\{R_\psi > \psi\}$, and
- R_ψ is the smallest superparabolic function above ψ , i.e. if v is a superparabolic function in Ω_∞ and $v \geq \psi$, then $v \geq R_\psi$.

For these results, see [24] and [32].

We shall see that the capacitary functions for compact sets are given by parabolic potentials. The potential of a compact subset K of Ω_∞ is defined to be the solution of the obstacle problem with the obstacle χ_K and we denote

$$R_K = R_{\chi_K}.$$

Again, we also consider the lower semicontinuous regularization $\widehat{R}_K$.

Both R_K and $\widehat{R}_K$ are weak supersolutions with zero boundary data in Ω_∞. Moreover, they both are weak supersolutions in Ω_∞ and weak solutions of in $\Omega_\infty \setminus K$. For the corresponding Riesz measures, we have

$$\mu_{\widehat{R}_K} = \mu_{R_K},$$

since $\widehat{R}_K = R_K$ almost everywhere in Ω_∞. Since Ω is a smooth and bounded open subset of $\mathbb{R}^n$ and K is a compact subset of Ω_∞, we conclude that

- R_K belongs to $L^p(0,\infty; W^{1,p}(\Omega))$,
- R_K is continuous outside K in $\overline{\Omega}_\infty$, and
- R_K takes zero boundary values continuously on the parabolic boundary $\partial_p\Omega_\infty$.

Next we show that by approximating the characteristic function by a decreasing sequence of continuous functions, we obtain a sequence of solutions to the obstacle problem that converges to the potential. This kind of approximation property also holds, more generally, for upper semicontinuous obstacles as shown in [32].

Lemma 4.2. *Let K be a compact subset of Ω_∞ and assume that $\psi_i \in C_0^\infty(\Omega_\infty)$, $i = 1, 2, \dots$, is a decreasing sequence such that $\psi_i \to \chi_K$ pointwise in Ω_∞ as $i \to \infty$. Then $R_{\psi_i} \to R_K$ pointwise in Ω_∞ and $\mu_{R_{\psi_i}} \to \mu_{R_K}$ weakly as $i \to \infty$.*

Proof. It follows immediately from the definition of the obstacle problem that R_{ψ_i}, $i = 1, 2, \dots$, is a decreasing sequence of continuous weak supersolutions. By Theorem 3.4, the pointwise limit function u is an upper semicontinuous weak supersolution in Ω_∞. The weak convergence of the corresponding Riesz measures follows from Theorem 3.4 as well.

We are left to show that $u = R_K$ in Ω_∞. Since $u_i \geq R_K$ and $u_i \to u$ pointwise as $i \to \infty$, we see that $u \geq R_K$ in Ω_∞. To establish the reverse inequality, we may use the comparison principle and show that every superparabolic function that lies above χ_K must also lie above u.

To this end, let v be a superparabolic function such that $v \geq \chi_K$. Since v is lower semicontinuous, u is upper semicontinous, and u takes zero boundary values continuously on $\partial_p\Omega_\infty$, it follows that the set

$$F = \{u \geq v + \varepsilon\}$$

is closed for every $\varepsilon > 0$. Since $u \leq 1$, the sets K and F are disjoint and hence F is a subset of $\Omega_\infty \setminus K$. There is a neighborhood $U \Subset \Omega_\infty \setminus K$ of F such that u is a weak solution in U and $u < v + \varepsilon$ on ∂U. The upper semicontinuity of u and lower semicontinuity of v in Ω_∞ imply that

$$\limsup_{U \ni y \to z} u(y) \leq u(z) < v(z) + \varepsilon \leq \liminf_{U \ni y \to z} v(y) + \varepsilon$$

for all $z \in \partial U$. The comparison principle, see [23], then gives that $u \leq v + \varepsilon$ in U and thus $u \leq v + \varepsilon$ in Ω_∞. This holds for every $\varepsilon > 0$ and hence $u \leq v$. We have thus shown that $u \leq R_K$. This completes the proof. □

5. Parabolic capacity

We shall mainly work with capacities of compact sets, but we begin with a general definition. Since we are interested in local properties, we restrict our attention to Ω_∞, where Ω is a bounded smooth open subset of $\mathbb{R}^n$. As already observed, this is convenient in arguments based on comparison principles and we also have regularity results up to the boundary.

Definition 5.1. The parabolic p-capacity of an arbitrary subset E of Ω_∞ is

$$\operatorname{cap}(E) = \sup\{\mu(\Omega_\infty) : 0 \leq u_\mu \leq 1, \operatorname{supp}\mu \subset E\},$$

where μ is a Radon measure, and u_μ is a weak solution to the measure data problem (3.2) in Ω_∞ with zero boundary data. If the set, over which the supremum is taken, is not bounded from above, then we set $\operatorname{cap}(E) = \infty$.

Remark 5.2. (1) Observe, that in the definition of the parabolic capacity, the solution u_μ of the measure data problem can be assumed to be superparabolic after a possible redefinition on a set of measure zero. Hence the parabolic capacity can be expressed in terms of superparabolic functions as

$$\operatorname{cap}(E) = \sup \mu_u(\Omega_\infty),$$

where the supremum is taken over all superparabolic functions u in Ω_∞ with $0 \le u \le 1$ and $\operatorname{supp} \mu_u \subset E$.

(2) Since $\operatorname{supp} \mu \subset E$ in the definition of the parabolic capacity, nothing changes if we consider the capacity relative to $\Omega \times (-\infty, \infty)$. Indeed, we can always take the zero extension of u_μ to the lower half space so that the the Riesz measure remains unchainged. This also explains, why we may assume that superparabolic functions, vanish on the initial boundary.

On the other hand, if E is a bounded subset of Ω_∞, then $E \Subset \Omega_T$ for some $T > 0$ and we can consider the capacity of E relative to Ω_T. In this case, we can always extend u_μ to $\Omega \times [T, \infty)$ by taking the solution to the boundary value problem with the initial values u_μ at $\Omega \times \{t = T\}$ and zero boundary values on the later boundary. In practice, this means that the different definitions give the same concept of capacity and, for simplicity, we have chosen to work with Ω_∞.

(3) The case when the reference set is the whole $\mathbb{R}^{n+1}$ can be obtained by a limiting procedure. Indeed, we can exhaust $\mathbb{R}^n$ with an expanding sequence of bounded and smooth open sets Ω_i, $i = 1, 2, \dots,$ and solve the measure data problem with zero boundary values in each $\Omega_i \times (-\infty, \infty)$. We obtain an increasing sequence of superparabolic functions and hence the limit function is superparabolic. The general theory can be based on this observation, but we do not need this feature here.

It also follows immediately from the definition that if $E_1 \subset E_2$, then

$$\operatorname{cap}(E_1) \le \operatorname{cap}(E_2).$$

Thus the parabolic capacity is a monotonic set function. The next result shows that the parabolic capacity is also countably subadditive.

Theorem 5.3. *Let E_i, $i = 1, 2, \dots,$ be arbitrary subsets of Ω_∞ and $E = \bigcup_{i=1}^\infty E_i$. Then*

$$\operatorname{cap}(E) \le \sum_{i=1}^\infty \operatorname{cap}(E_i).$$

Proof. Suppose first that $\operatorname{cap}(E) < \infty$. Then for every $\varepsilon > 0$ there is a Radon measure μ such that $0 \le u_\mu \le 1$, $\operatorname{supp} \mu \subset E$ and

$$\mu(\Omega_\infty) \ge \operatorname{cap}(E) - \varepsilon.$$

Let μ_i be the restriction of μ to the set E_i. Then Lemma 3.9 implies that $0 \le u_{\mu_i} \le u_\mu \le 1$ in Ω_∞ and, consequently, we have

$$\mu_i(\Omega_\infty) \le \operatorname{cap}(E_i)$$

for every $i = 1, 2, \dots$ It follows that

$$\operatorname{cap}(E) \le \mu(\Omega_\infty) + \varepsilon \le \sum_{i=1}^\infty \mu_i(\Omega_\infty) + \varepsilon \le \sum_{i=1}^\infty \operatorname{cap}(E_i) + \varepsilon,$$

and the claim follows by letting $\varepsilon \to 0$.

If $\operatorname{cap}(E) = \infty$, then for any $M > 0$ there exists a Radon measure μ such that $0 \le u_\mu \le 1$, $\operatorname{supp}\mu \subset E$ and $\mu(\Omega_\infty) \ge M$. Then, as above, we have

$$M \le \mu(\Omega_\infty) \le \sum_{i=1}^{\infty} \mu_i(\Omega_\infty) \le \sum_{i=1}^{\infty} \operatorname{cap}(E_i),$$

and since M can be taken as large as we wish, we conclude that

$$\sum_{i=1}^{\infty} \operatorname{cap}(E_i) = \infty.$$

$\square$

Lemma 5.4. *Let E_i, $i = 1, 2, \ldots$, be subsets of Ω_∞ with the property $E_1 \subset E_2 \subset \ldots$, and denote $E = \bigcup_{i=1}^{\infty} E_i$. Then*

$$\lim_{i\to\infty} \operatorname{cap}(E_i) = \operatorname{cap}(E).$$

Proof. By monotonicity, we have

$$\lim_{i\to\infty} \operatorname{cap}(E_i) \le \operatorname{cap}(E).$$

To prove the opposite inequality, first we assume that $\operatorname{cap}(E) < \infty$. Then for every $\varepsilon > 0$, there is a Radon measure μ such that $0 \le u_\mu \le 1$, $\operatorname{supp}\mu \subset E$ and

$$\mu(E) \ge \operatorname{cap}(E) - \varepsilon.$$

Since μ is Borel regular, we have

$$\lim_{i\to\infty} \mu(E_i) = \mu(E).$$

This implies that

$$\operatorname{cap}(E) \le \mu(E) + \varepsilon = \lim_{i\to\infty} \mu(E_i) + \varepsilon.$$

Let μ_i be the restriction of μ to the set E_i. By Lemma 3.9, we conclude that $0 \le u_{\mu_i} \le u_\mu \le 1$ in Ω_∞ and consequently

$$\mu(E_i) = \mu_i(\Omega_\infty) \le \operatorname{cap}(E_i).$$

This implies that

$$\operatorname{cap}(E) \le \lim_{i\to\infty} \operatorname{cap}(E_i) + \varepsilon$$

and the claim follows by letting $\varepsilon \to 0$.

Finally, if $\operatorname{cap}(E) = \infty$, then for any $M > 0$ there exists μ such that $0 \le u_\mu \le 1$, $\operatorname{supp}\mu \subset E$ and $\mu(E) \ge M$. Then a similar reasoning as above shows that

$$M \le \mu(E) = \lim_{i\to\infty} \mu(E_i) \le \lim_{i\to\infty} \operatorname{cap}(E_i).$$

Since M can be chosen as large as we wish, we have

$$\lim_{i\to\infty} \operatorname{cap}(E_i) = \infty.$$

□

The next result shows that the capacity is inner regular.

Lemma 5.5. *Let $E \subset \Omega_\infty$ be a Borel set. Then*

$$\operatorname{cap}(E) = \sup\{\operatorname{cap}(K) : K \subset E,\ K \text{ compact}\}$$

Proof. By monotonicity, we have

$$\operatorname{cap}(E) \geq \sup\{\operatorname{cap}(K) : K \subset E,\ K \text{ compact}\}$$

To prove the reverse inequality, first assume that $\operatorname{cap}(E) < \infty$. For every $\varepsilon > 0$, there is a Radon measure μ such that $0 \leq u_\mu \leq 1$, $\operatorname{supp}\mu \subset E$ and

$$\mu(E) \geq \operatorname{cap}(E) - \frac{\varepsilon}{2}.$$

Since $\mu(E) < \infty$, there is a compact set $K \subset E$ such that

$$\mu(K) \geq \mu(E) - \frac{\varepsilon}{2}.$$

Let ν be the restriction of μ to the set K. Lemma 3.9 implies that $0 \leq u_\nu \leq u_\mu \leq 1$ in Ω_∞ and consequently

$$\mu(K) = \nu(\Omega_\infty) \leq \operatorname{cap}(K).$$

From this we obtain

$$\operatorname{cap}(K) \geq \mu(K) \geq \mu(E) - \frac{\varepsilon}{2} \geq \operatorname{cap}(E) - \varepsilon.$$

The claim follows in the case $\operatorname{cap}(E) < \infty$.

If $\operatorname{cap}(E) = \infty$, then for any $M > 0$ there exists μ such that $0 \leq u_\mu \leq 1$, $\operatorname{supp}\mu \subset E$ and $\mu(E) \geq M$. For every $\varepsilon > 0$, there exists $r > 0$ such that

$$\mu(E \cap B(0,r)) \geq M - \frac{\varepsilon}{2}.$$

Since $\mu(E \cap B(0,r)) < \infty$, there is a compact set $K \subset (E \cap B(0,r))$ such that

$$\mu(K) \geq \mu(E \cap B(0,r)) - \frac{\varepsilon}{2} \geq M - \varepsilon.$$

As above, this implies that $\operatorname{cap}(K) \geq M - \varepsilon$ and the claim follows. □

The following lemma is useful in proving the main result, Theorem 5.7, of this section. In the elliptic case, similar estimates have been obtained in [38].

Lemma 5.6. *Let K is a compact subset of Ω_∞. Assume that u and v are lower semicontinuous weak supersolutions in Ω_∞ and that u continuous in $\overline{\Omega}_\infty$, outside some compact subset of Ω_∞. Moreover, assume that $u > 1$ in K, $u = 0$ on $\partial_p\Omega_\infty$ and $0 \leq v \leq 1$ in Ω_∞. Then*

$$\mu_v(K) \leq \mu_u(\Omega_\infty).$$

Here μ_u and μ_v are the Riesz measures of u and v, respectively.

Proof. The lower semicontinuity of u and compactness of K imply that $\min_K u > 1$ and hence

$$\varepsilon = \frac{1}{4}(\min_K u - 1)$$

is a positive number. With this choice, we see that $U = \{u > 1 + \varepsilon\}$ is an open set with $K \subset U \Subset \Omega_\infty$.

Denote

$$v_\varepsilon = v + \varepsilon \quad \text{and} \quad w_\varepsilon = \min\{v_\varepsilon, u\}.$$

Observe, that both v_ε and w_ε are weak supersolutions in Ω_∞. Since $v_\varepsilon \le 1 + \varepsilon$, we have $w_\varepsilon = v_\varepsilon$ in U. On the other hand, since u vanishes on the parabolic boundary and $v_\varepsilon \ge \varepsilon$, we have $w_\varepsilon = u$ in $\Omega_\infty \setminus K'$, where $K' \Subset \Omega_\infty$ is a compact set, which can be chosen to be so large that $U \subset K'$. Hence w_ε is a weak supersolution which coincides with u near the parabolic boundary and v_ε inside the domain.

Let $\varphi \in C_0^\infty(U)$ be such that $0 \le \varphi \le 1$ and $\varphi = 1$ on K. Since $v_\varepsilon = w_\varepsilon$ in U, we have

$$\begin{aligned}\mu_v(K) &\le \int_U \varphi \,\mathrm{d}\mu_v = \int_U \varphi \,\mathrm{d}\mu_{v_\varepsilon} \\ &= \int_U \varphi \,\mathrm{d}\mu_{w_\varepsilon} \le \mu_{w_\varepsilon}(U) \le \mu_{w_\varepsilon}(K').\end{aligned}$$

On the other hand, let $\varphi \in C_0^\infty(\Omega_\infty)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ on K'. The fact that that both the gradient and the time derivative of φ vanish in K' together with $w_\varepsilon = u$ in $\Omega_\infty \setminus K'$ gives

$$\begin{aligned}&\int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_{w_\varepsilon} - \int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_u \\ &\quad = \int_{\Omega_\infty \setminus K'} \Big(\big(\mathcal{A}(\nabla w_\varepsilon) - \mathcal{A}(\nabla u)\big) \cdot \nabla\varphi - (w_\varepsilon - u)\frac{\partial\varphi}{\partial t} \Big) \,\mathrm{d}z = 0\end{aligned}$$

and hence

$$\mu_{w_\varepsilon}(K') \le \int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_{w_\varepsilon} = \int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_u \le \mu_u(\Omega_\infty).$$

This completes the proof. □

The following theorem gives a characterization of the parabolic capacity of compact sets through capacitary potentials. We state the result for the superparabolic function $\widehat{R}_K$. For the case $p = 2$, see Lanconelli [27]. The proof is based on Lemma 5.6 and Theorem 3.4 above.

Theorem 5.7. *Let K be a compact subset of Ω_∞. Then*

$$\operatorname{cap}(K) = \mu_{\widehat{R}_K}(K),$$

where $\mu_{\widehat{R}_K}$ is the Riesz measure of $\widehat{R}_K$.

Proof. Since $\widehat{R}_K$ is a superparabolic function with the property $0 \le \widehat{R}_K \le 1$, it follows immediately from the definition of the capacity that

$$\mu_{\widehat{R}_K}(K) \le \operatorname{cap}(K).$$

In order to see the inequality in the other direction, we choose a decreasing sequence $\varepsilon_i \to 0$. Let $\psi_i \in C_0^\infty(\Omega_\infty)$, $i = 1, 2, \dots$, be a decreasing sequence functions such that $\psi_i \to \chi_K$ pointwise in Ω_∞ as $i \to \infty$,

$$\psi_i = 1 + \varepsilon_i \quad \text{on} \quad K,$$

and $\psi_i = 0$ outside K' for some compact K' with $K \subset K' \Subset \Omega_\infty$. We denote by u_i, $i = 1, 2, \dots$, the solutions of the corresponding obstacle problems with the obstacles ψ_i.

Let v be a weak supersolution in Ω_∞ with the property $0 \le v \le 1$. Lemma 5.6, and the fact that u_i is a weak solution of (2.4) in $\Omega_\infty \setminus K'$, imply that

$$\mu_v(K) \le \mu_{u_i}(\Omega_\infty) = \mu_{u_i}(K').$$

On the other hand, by Lemma 4.2, we conclude that $u_i \to \widehat{R}_K$ almost everywhere in Ω_∞ and $\mu_{u_i} \to \mu_{\widehat{R}_K}$ weakly as $i \to \infty$. Here we use the fact that Lemma 4.2 holds also for the lower semicontinuous representative $\widehat{R}_K$, if we replace the pointwise convergence with convergence almost everywhere, see Theorem 2.7. Since $\widehat{R}_K$ is a weak solution of (2.4) in $\Omega_\infty \setminus K$, we obtain

$$\limsup_{i \to \infty} \mu_{u_i}(K') \le \mu_{\widehat{R}_K}(K') = \mu_{\widehat{R}_K}(K).$$

Combining the previous inequalities, we arrive at

$$\mu_v(K) \le \mu_{\widehat{R}_K}(K),$$

and, by taking a supremum on the left hand side, we obtain

$$\operatorname{cap}(K) \le \mu_{\widehat{R}_K}(K).$$

This completes the proof. □

The following result is a version of the standard limiting theorem for capacities of a shrinking sequence of compact sets.

Lemma 5.8. *Let $K_i \subset \Omega_\infty$, $i = 1, 2, \dots$, be compact sets such that $K_1 \supset K_2 \supset \dots$ and $K = \bigcap_{i=1}^\infty K_i$. Then*

$$\lim_{i \to \infty} \operatorname{cap}(K_i) = \operatorname{cap}(K).$$

Proof. Observe that R_{K_i}, $i = 1, 2 \dots$, is a bounded and decreasing sequence of weak supersolutions in Ω_∞. As in the proof of Lemma 4.2, we conclude that $R_{K_i} \to R_K$ pointwise in Ω_∞ and that the measures $\mu_{R_{K_i}}$ converge to μ_{R_K} weakly as $i \to \infty$. The weak convergence of measures implies that with $\varphi \in C_0^\infty(\Omega_\infty)$ such that $\varphi = 1$ on K_1, we have

$$\begin{aligned}\lim_{i\to\infty} \operatorname{cap}(K_i) &= \lim_{i\to\infty} \mu_{R_{K_i}}(K_1) = \lim_{i\to\infty} \int_{\Omega_\infty} \varphi \, \mathrm{d}\mu_{R_{K_i}} \\ &= \int_{\Omega_\infty} \varphi \, \mathrm{d}\mu_{R_K} = \mu_{R_K}(K_1) = \operatorname{cap}(K).\end{aligned}$$

Here we also applied Theorem 5.7 twice. □

The next result is a version of Theorem 5.7 for open sets.

Lemma 5.9. *Let $U \Subset \Omega_\infty$ be an open set. Then*

$$\operatorname{cap}(U) = \mu_{R_U}(\Omega_\infty).$$

Proof. We first notice that, if K_i, $i = 1, 2, \ldots$, is an expanding sequence of compact sets such that $U = \bigcup_{i=1}^\infty K_i$, then by Lemma 5.4 and Theorem 5.7, we have

$$\operatorname{cap}(U) = \lim_{i\to\infty} \operatorname{cap}(K_i) = \lim_{i\to\infty} \mu_{u_i}(K_i),$$

where $u_i = \widehat{R}_{K_i}$. Note that the sequence u_i, $i = 1, 2, \ldots$, is increasing, and hence it converges pointwise to a function u, which is, by Theorem 3.4, a weak supersolution and lower semicontinuous as a supremum of lower semicontinuous functions. By the weak convergence, we obtain

$$\begin{aligned}\operatorname{cap}(U) &= \lim_{i\to\infty} \operatorname{cap}(K_i) = \lim_{i\to\infty} \int_{\Omega_\infty} \varphi \, \mathrm{d}\mu_{u_i} \\ &= \lim_{i\to\infty} \int_{\Omega_\infty} \varphi \, \mathrm{d}\mu_{u_i} = \int_{\overline{U}} \varphi \, d\mu_u\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega_\infty)$ such that $\varphi = 1$ on $\overline{U}$. The outer regularity of μ_u then implies that

$$\mu_u(\overline{U}) = \operatorname{cap}(U).$$

Finally, by construction $u \geq \chi_U$ so that $u \geq R_U$, and on the other hand $u_i = \widehat{R}_{K_i} \leq R_U$ implies that $u \leq R_U$. This shows that $u = R_U$ and proves the assertion. □

6. Polar sets of superparabolic functions

In this section we show that the infinity set of a superparabolic function is of zero capacity. In the time independent case, superharmonic functions can be scaled to obtain appropriate test functions for the capacity. However, the class of superparabolic functions is not closed under scaling and hence we derive estimates for scaled obstacles instead.

Our strategy is first consider a compact set which is a finite union of closed space time boxes. Such a set is regular enough so that the solution of the obstacle problem has desired continuity properties. Finally, a general compact set can always be approximated by a shrinking sequence of such sets. According to the next result, the Riesz measure of the solution of such an obstacle problem does not charge the tops of the boxes.

Lemma 6.1. *Let $Q_j \subset \Omega$, $j = 1, 2, \dots, N$, be a finite collection of closed cubes and assume that*

$$K = \bigcup_{j=1}^{N} Q_j \times [t_{2j-1}, t_{2j}] \subset \Omega_\infty,$$

where $0 < t_{2j-1} < t_{2j} < \infty$ for every $j = 1, 2, \dots, N$. If u is a solution to the obstacle problem in Ω_∞ with the obstacle $\lambda\chi_K$, $\lambda > 0$, then

$$\lim_{h \to 0} \sum_{j=1}^{N} \mu_u(Q_j \times [t_{2j} - h, t_{2j}]) = 0.$$

Here μ_u is the Riesz measure of u.

Proof. Since the obstacle is bounded and compactly supported in Ω_∞, we have $u \in L^p(0, \infty; W_0^{1,p}(\Omega))$. Moreover, since Ω is smooth, the function u is continuous in $\overline{\Omega}_\infty \setminus K$ and $u = 0$ on $\partial_p \Omega_\infty$. Since K satisfies a uniform measure density condition, by Chapters 3 and 4 of DiBenedetto's monograph [6], we conclude that u is continuous in

$$\bigcup_{j=1}^{N} Q_j \times [t_{2j} - h, t_{2j} + h],$$

where $h < h_0$ with h_0 small enough.

For $j = 1, 2, \dots, N$, define a cutoff function

$$\chi^h = \begin{cases} 1 + (t - t_{2j} + 2h)/h, & t_{2j} - 3h < t \le t_{2j} - 2h, \\ 1 & t_{2j} - 2h < t < t_{2j} + 2h, \\ 1 + (t_{2j} + 2h - t)/h, & t_{2j} + 2h < t \le t_{2j} + 3h, \\ 0 & \text{otherwise}, \end{cases}$$

where $0 < h < h_0/3$.

Let then u_h stand for the standard mollification in the time variable as in (3.6). We test the equation for u with $\varphi = (u_h \chi^h)_h$, which is clearly admissible. Since $0 \le u \le \lambda$, $u = \lambda$ and $\chi^h = 1$ in $Q_j \times [t_{2j} - h, t_{2j}]$, we have

$$\frac{\lambda}{4} \le \varphi \le \lambda \quad \text{in} \quad Q_j \times [t_{2j} - h, t_{2j}].$$

Thus (3.7) gives

$$\begin{aligned} \mu_u(Q_j \times [t_{2j} - h, t_{2j}]) &\le \frac{4}{\lambda} \int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_u \\ &= \frac{4}{\lambda} \int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u)_h \cdot \nabla(u_h \chi^h) - u_h \frac{\partial(u_h \chi^h)}{\partial t} \Big) \,\mathrm{d}z. \end{aligned}$$

Integrating the second term on the right hand side by parts, we see that

$$-\int_{\Omega_\infty} u_h \frac{\partial(u_h\chi^h)}{\partial t}\,\mathrm{d}z = -\frac{1}{2}\int_{\Omega_\infty} u_h^2 \frac{\partial \chi^h}{\partial t}\,\mathrm{d}z.$$

By continuity of u and symmetry of χ_h, we have

$$\int_{\Omega_\infty} u_h^2 \frac{\partial \chi^h}{\partial t}\,\mathrm{d}z \to 0$$

as $h \to 0$. On the other hand, by the standard properties of the mollifiers, for the elliptic term we obtain

$$\int_{\Omega_\infty} \mathcal{A}(\nabla u)_h \cdot \nabla(u_h\chi^h)\,\mathrm{d}z = \int_{\Omega_\infty} \mathcal{A}(\nabla u)_h \cdot \nabla(u_h)\chi^h\,\mathrm{d}z \to 0$$

as $h \to 0$. Hence we conclude that

$$\lim_{h\to 0} \mu_u(Q_j \times [t_{2j} - h, t_{2j}]) = 0$$

for $j = 1, 2, \ldots, N$, from which the claim follows. □

In the proof of the next result we utilize a forward in time mollification

$$u^*(x,t) = \frac{1}{h}\int_t^\infty u(x,s)e^{(t-s)/h}\,\mathrm{d}s, \qquad h > 0.$$

Notation hides the dependence on h. It is rather straightforward to show that $u^* \to u$ and $\nabla u^* \to \nabla u$ in $L^p(\Omega_\infty)$ as $h \to 0$, if u and ∇u belong to $L^p(\Omega_\infty)$. Observe also that

$$\frac{\partial u^*}{\partial t} = \frac{u^* - u}{h}. \tag{6.2}$$

For further properties and more details, we refer, for example, to [20].

Lemma 6.3. *Assume that K is a finite union of boxes as in Lemma 6.1. Let u_1 be the solution of the obstacle problem in Ω_∞ with the obstacle χ_K and let u_λ be the solution of the corresponding problem with $\lambda\chi_K$, $\lambda > 0$. Then*

$$\int_{\Omega_\infty} |\nabla u_1|^p\,\mathrm{d}z \le C\big(\lambda^{-p} + \lambda^{-p/(p-1)}\big)\int_{\Omega_\infty} |\nabla u_\lambda|^p\,\mathrm{d}z$$

and

$$\int_{\Omega_\infty} |\nabla u_\lambda|^p\,\mathrm{d}z \le C\big(\lambda^{p} + \lambda^{p/(p-1)}\big)\int_{\Omega_\infty} |\nabla u_1|^p\,\mathrm{d}z.$$

The constant C depends only on the structure constants of the equation and p.

Proof. Denote

$$\varphi_1 = \lambda u_\lambda - \lambda^2 u_1 \quad \text{and} \quad \varphi_\lambda = \lambda u_1 - u_\lambda.$$

Observe that φ_1 and φ_λ vanish on $\partial_p\Omega_\infty$ and also on K. We use the test functions $(\varphi_1)^*$ in the equation for u_1 and $(\varphi_\lambda)^*$ in the equation for u_λ. By summing up the equations we obtain

$$\begin{aligned}\int_{\Omega_\infty} (\varphi_1)^* \,\mathrm{d}\mu_{u_1} &+ \int_{\Omega_\infty} (\varphi_\lambda)^* \,\mathrm{d}\mu_{u_\lambda} \\ &= \int_{\Omega_\infty} \big(\mathcal{A}(\nabla u_1)\cdot\nabla(\varphi_1)^* + \mathcal{A}(\nabla u_\lambda)\cdot\nabla(\varphi_\lambda)^*\big)\,\mathrm{d}z \\ &\qquad - \int_{\Omega_\infty} \Big(u_1\frac{\partial(\varphi_1)^*}{\partial t} + u_\lambda\frac{\partial(\varphi_\lambda)^*}{\partial t}\Big)\,\mathrm{d}z.\end{aligned}$$

Note that the functions $(\varphi_1)^*$ and $(\varphi_\lambda)^*$ do not necessarily vanish on the initial boundary $\Omega\times\{t=0\}$, but there are no boundary terms, since u_1 and u_λ have zero initial values.

Since

$$(\varphi_1)^* = (\lambda u_\lambda - \lambda^2 u_1)^* = \lambda(u_\lambda - \lambda u_1)^* = -\lambda(\varphi_\lambda)^*,$$

the terms with the time derivatives produce

$$\begin{aligned}-\int_{\Omega_\infty} &\Big(u_1\frac{\partial(\varphi_1)^*}{\partial t} + u_\lambda\frac{\partial(\varphi_\lambda)^*}{\partial t}\Big)\,\mathrm{d}z \\ &= -\int_{\Omega_\infty} (-\lambda u_1 + u_\lambda)\frac{\partial(\varphi_\lambda)^*}{\partial t}\,\mathrm{d}z \\ &= \int_{\Omega_\infty} \varphi_\lambda\frac{\partial(\varphi_\lambda)^*}{\partial t}\,\mathrm{d}z \\ &= \int_{\Omega_\infty} (\varphi_\lambda)^*\frac{\partial(\varphi_\lambda)^*}{\partial t}\,\mathrm{d}z + \int_{\Omega_\infty} (\varphi_\lambda - (\varphi_\lambda)^*)\frac{\partial(\varphi_\lambda)^*}{\partial t}\,\mathrm{d}z.\end{aligned}$$

Observe, that

$$\begin{aligned}\int_{\Omega_\infty} (\varphi_\lambda)^*\frac{\partial(\varphi_\lambda)^*}{\partial t}\,\mathrm{d}z &= \frac{1}{2}\int_{\Omega_\infty} \frac{\partial((\varphi_\lambda)^*)^2}{\partial t}\,\mathrm{d}z \\ &= -\frac{1}{2}\int_{\Omega} (\varphi_\lambda)^*(x,0)^2\,\mathrm{d}x \le 0,\end{aligned}$$

On the other hand, by (6.2), we have

$$\int_{\Omega_\infty} (\varphi_\lambda - (\varphi_\lambda)^*)\frac{\partial(\varphi_\lambda)^*}{\partial t}\,\mathrm{d}z = -\frac{1}{h}\int_{\Omega_\infty} ((\varphi_\lambda)^* - \varphi_\lambda)^2\,\mathrm{d}z \le 0.$$

It follows that

$$-\int_{\Omega_\infty} \Big(u_1\frac{\partial(\varphi_1)^*}{\partial t} + u_\lambda\frac{\partial(\varphi_\lambda)^*}{\partial t}\Big)\,\mathrm{d}z \le 0$$

and consequently

$$\int_{\Omega_\infty} (\varphi_1)^* \,\mathrm{d}\mu_{u_1} + \int_{\Omega_\infty} (\varphi_\lambda)^* \,\mathrm{d}\mu_{u_\lambda} \\ \leq \int_{\Omega_\infty} \big(\mathcal{A}(\nabla u_1)\cdot\nabla(\varphi_1)^* + \mathcal{A}(\nabla u_\lambda)\cdot\nabla(\varphi_\lambda)^*\big)\,\mathrm{d}z.$$

Then we focus our attention on the source terms and will show that they tend to zero as $h \to 0$. Lemma 6.1 shows that for every $\varepsilon > 0$ there is $h_0 > 0$ such that

$$\sum_{j=1}^N \Big(\int_{Q_j\times[t_{2j}-h_0,t_{2j}]} |(\varphi_1)^*| \,\mathrm{d}\mu_{u_1} + \int_{Q_j\times[t_{2j}-h_0,t_{2j}]} |(\varphi_\lambda)^*| \,\mathrm{d}\mu_{u_\lambda}\Big) \leq \frac{\varepsilon}{2}.$$

On the other hand, since $\varphi_1 = \varphi_\lambda = 0$ in K, we have

$$\begin{aligned} &|(\varphi_1)^*(x,t) + (\varphi_\lambda)^*(x,t)| \\ &\qquad \leq \frac{1}{h}\int_t^\infty \big(|\lambda u_\lambda - \lambda^2 u_1| + |\lambda u_1 - u_\lambda|\big)(x,s)e^{(t-s)/h}\,\mathrm{d}s \\ &\qquad \leq \frac{\lambda^2+\lambda+1}{h}\int_{t+h_0}^\infty e^{(t-s)/h}\,\mathrm{d}s = (\lambda^2+\lambda+1)e^{-h_0/h} \end{aligned}$$

whenever $x \in \bigcup_{j=1}^\infty Q_j$ and $t_{2j-1} \leq t < t_{2j} - h_0$, $j = 1, 2, \dots, N$. Taking h so small that

$$N(\lambda^2+\lambda+1)e^{-h_0/h}(\mu_{u_1}(\Omega_\infty) + \mu_{u_\lambda}(\Omega_\infty)) \leq \frac{\varepsilon}{2},$$

we obtain

$$\sum_{j=1}^N \Big(\int_{Q_j\times[t_{2j-1},t_{2j}-h_0)} |(\varphi_1)^*| \,\mathrm{d}\mu_{u_1} + \int_{Q_j\times[t_{2j-1},t_{2j}-h_0)} |(\varphi_\lambda)^*| \,\mathrm{d}\mu_{u_\lambda}\Big) \leq \frac{\varepsilon}{2}$$

and consequently

$$\Big|\int_{\Omega_\infty} (\varphi_1)^* \,\mathrm{d}\mu_{u_1} + \int_{\Omega_\infty} (\varphi_\lambda)^* \,\mathrm{d}\mu_{u_\lambda}\Big| \leq \varepsilon$$

for all sufficiently small h. It follows that

$$\Big|\int_{\Omega_\infty} (\varphi_1)^* \,\mathrm{d}\mu_{u_1} + \int_{\Omega_\infty} (\varphi_\lambda)^* \,\mathrm{d}\mu_{u_\lambda}\Big| \to 0 \tag{6.4}$$

as $h \to 0$. Thus we have

$$\int_{\Omega_\infty} (\mathcal{A}(\nabla u_1)\cdot\nabla\varphi_1 + \mathcal{A}(\nabla u_\lambda)\cdot\nabla\varphi_\lambda)\,\mathrm{d}z \geq 0,$$

and hence

$$\begin{aligned} \int_{\Omega_\infty} \Big(&\lambda\mathcal{A}(\nabla u_1)\cdot\nabla u_\lambda - \lambda^2\mathcal{A}(\nabla u_1)\cdot\nabla u_1 \\ &+ \lambda\mathcal{A}(\nabla u_\lambda)\cdot\nabla u_1 - \mathcal{A}(\nabla u_\lambda)\cdot\nabla u_\lambda\Big)\,\mathrm{d}z \geq 0. \end{aligned}$$

By the structure conditions and Young's inequality, we have

$$
\begin{aligned}
\alpha\lambda^2 &\int_{\Omega_\infty} |\nabla u_1|^p \,\mathrm{d}z \\
&\le \int_{\Omega_\infty} \big(\lambda\beta|\nabla u_1|^{p-1}|\nabla u_\lambda| + \lambda\beta|\nabla u_\lambda|^{p-1}|\nabla u_1| - \alpha|\nabla u_\lambda|^p\big)\,\mathrm{d}z \\
&\le \int_{\Omega_\infty} \Big(\frac{\alpha\lambda^2}{3}|\nabla u_1|^p + C\lambda^{2-p}|\nabla u_\lambda|^p \\
&\qquad\qquad + \frac{\alpha\lambda^2}{3}|\nabla u_1|^p + C\lambda^{2-p/(p-1)}|\nabla u_\lambda|^p\Big)\,\mathrm{d}z.
\end{aligned}
$$

By absorbing terms to the left side, we arrive at

$$
\int_{\Omega_\infty} |\nabla u_1|^p \,\mathrm{d}z \le C\big(\lambda^{-p} + \lambda^{-p/(p-1)}\big)\int_{\Omega_\infty} |\nabla u_\lambda|^p \,\mathrm{d}z.
$$

Finally, a similar argument shows that

$$
\int_{\Omega_\infty} |\nabla u_\lambda|^p \,\mathrm{d}z \le C\big(\lambda^{p} + \lambda^{p/(p-1)}\big)\int_{\Omega_\infty} |\nabla u_1|^p \,\mathrm{d}z.
$$

This completes the proof. □

Lemma 6.5. *Let u and v be weak supersolutions in Ω_∞ and assume that they are continuous in $\overline{\Omega}_\infty$, outside some compact subset of Ω_∞. If $u \ge v$ in Ω_∞ and $u = v$ on $\partial_p\Omega_\infty$, then*

$$
\mu_v(\Omega_\infty) \le \mu_u(\Omega_\infty).
$$

Here μ_u and μ_v are the Riesz measures of u and v, respectively.

Proof. The proof consists of two steps. First, we use the continuity of u and v near the the parabolic boundary. Let K' be a compact subset of Ω_∞ and, as in the proof of Lemma 5.6, denote

$$
w_\varepsilon = \min\{v+\varepsilon, u\}, \qquad \varepsilon > 0.
$$

By continuity and the fact that $u = v$ on the parabolic boundary, we may choose $\varepsilon > 0$ small enough, so that $w_\varepsilon = u$ in $\Omega_\infty \setminus K'$. Let $\varphi \in C_0^\infty(\Omega_\infty)$ be a cutoff function with the properties $0 \le \varphi \le 1$ and $\varphi = 1$ on K'. Since both w_ε and u are weak supersolutions, it follows that

$$
\begin{aligned}
&\int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_{w_\varepsilon} - \int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_u \\
&\quad = \int_{\Omega_\infty} \Big(\big(\mathcal{A}(\nabla w_\varepsilon) - \mathcal{A}(\nabla u)\big)\cdot\nabla\varphi - (w_\varepsilon - u)\frac{\partial\varphi}{\partial t}\Big)\,\mathrm{d}z = 0,
\end{aligned}
$$

because $w_\varepsilon = u$ on the supports of $\nabla\varphi$ and $\partial\varphi/\partial t$. This implies that

$$
\mu_{w_\varepsilon}(K') \le \int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_{w_\varepsilon} = \int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_u \le \mu_u(\Omega_\infty)
$$

for all small enough $\varepsilon > 0$.

Second, we use the assumption $u \geq v$. It implies that w_ε converges to v pointwise monotonically in Ω_∞ as $\varepsilon \to 0$. By Theorem 3.4, this implies that $\mu_{w_\varepsilon} \to \mu_v$ weakly as $\varepsilon \to 0$. Let $\eta \in C_0^\infty(\Omega_\infty)$ be a cutoff function such that $0 \leq \eta \leq 1$ and $\eta = 1$ in K and let K' be the support of η. The weak convergence implies

$$\begin{aligned}\mu_v(K) &\leq \int_{\Omega_\infty} \eta \,\mathrm{d}\mu_v = \lim_{\varepsilon\to 0}\int_{\Omega_\infty} \eta \,\mathrm{d}\mu_{w_\varepsilon}\\ &\leq \limsup_{\varepsilon\to 0} \mu_{w_\varepsilon}(K') \leq \mu_u(\Omega_\infty).\end{aligned}$$

This proves the result by the inner regularity of μ_v. □

Lemma 6.6. *Assume that K is a compact subset of Ω_∞, let u_1 be the solution of the obstacle problem in Ω_∞ with the obstacle χ_K and let u_λ be the solution of the corresponding problem with $\lambda\chi_K$, $\lambda > 0$. Then*

$$\mu_{u_1}(\Omega_\infty) \leq C(\lambda^{1-p} + \lambda^{-1/(p-1)})\mu_{u_\lambda}(\Omega_\infty) \tag{6.7}$$

and

$$\mu_{u_\lambda}(\Omega_\infty) \leq C(\lambda^{p-1} + \lambda^{1/(p-1)})\mu_{u_1}(\Omega_\infty). \tag{6.8}$$

Here μ_{u_1} and μ_{u_λ} are the Riesz measures of u_1 and u_λ, respectively.

Proof. First assume that K is a finite union of closed boxes as in Lemma 6.1 and in Lemma 6.3. Note carefully that for the double mollification, we have

$$\frac{1}{4} \leq ((u_1)_h)_h \leq 1 \quad \text{and} \quad \frac{\lambda}{4} \leq ((u_\lambda)_h)_h \leq \lambda$$

on K, if $h > 0$ is small enough. Here u_h stands for the standard mollification in the time direction. The function $((u_1)_h)_h$ is admissible. From this we conclude that

$$\begin{aligned}\frac{1}{4}\mu_{u_1}(\Omega_\infty) &\leq \int_{\Omega_\infty} ((u_1)_h)_h \,\mathrm{d}\mu_{u_1}\\ &= \int_{\Omega_\infty} \Big(\mathcal{A}(\nabla u_1)\cdot\nabla((u_1)_h)_h - u_1\frac{\partial((u_1)_h)_h}{\partial t}\Big)\,\mathrm{d}z\\ &\leq \beta\Big(\int_{\Omega_\infty}|\nabla u_1|^p\,\mathrm{d}z\Big)^{(p-1)/p}\Big(\int_{\Omega_\infty}|\nabla((u_1)_h)_h|^p\,\mathrm{d}z\Big)^{1/p}.\end{aligned}$$

Here we used the fact that

$$\begin{aligned}\int_{\Omega_\infty} u_1\frac{\partial((u_1)_h)_h}{\partial t}\,\mathrm{d}z &= \int_{\Omega_\infty} u_1\Big(\frac{\partial(u_1)_h}{\partial t}\Big)_h\,\mathrm{d}z\\ &= \int_{\Omega_\infty}(u_1)_h\frac{\partial(u_1)_h}{\partial t}\,\mathrm{d}z = \frac{1}{2}\int_{\Omega_\infty}\frac{\partial((u_1)_h)^2}{\partial t}\,\mathrm{d}z = 0.\end{aligned}$$

We let $h \to 0$ and obtain

$$\frac{1}{4}\mu_{u_1}(\Omega_\infty) \leq \beta\int_{\Omega_\infty}|\nabla u_1|^p\,\mathrm{d}z.$$

A similar argument shows that

$$\mu_{u_\lambda}(\Omega_\infty) \geq \frac{\alpha}{\lambda} \int_{\Omega_\infty} |\nabla u_\lambda|^p \, \mathrm{d}z.$$

A combination of the estimates above together with Lemma 6.3 implies that

$$\mu_{u_1}(\Omega_\infty) \leq C(\lambda^{1-p} + \lambda^{-1/(p-1)})\mu_{u_\lambda}(\Omega_\infty). \tag{6.9}$$

This proves (6.7) in the case when K is a finite union of boxes.

Next we consider a general compact set $K \subset \Omega_\infty$. We can exhaust $\Omega_\infty \setminus K$ with an expanding sequence of polygonal sets. Hence there are sets K_j, $j = 1, 2, \ldots$, consisting of finite unions of boxes, so that

$$K_1 \supset K_2 \supset \ldots \quad \text{and} \quad K = \bigcap_{j=1}^{\infty} K_j.$$

We denote by $u_{1,j}$ and $u_{\lambda,j}$ the solutions of the obstacle problems in Ω_∞, related to the obstacles χ_{K_j} and $\lambda\chi_{K_j}$, respectively. On the one hand, since $u_{1,j} \geq u_1$ and $u_{1,j} = u_1$ on $\partial_p\Omega_\infty$, Lemma 6.5 gives

$$\mu_{u_1}(\Omega_\infty) \leq \mu_{u_{1,j}}(\Omega_\infty).$$

Moreover, Lemma 6.5 implies that

$$\mu_{u_{1,j+1}}(\Omega_\infty) \leq \mu_{u_{1,j}}(\Omega_\infty)$$

for every $j = 1, 2, \ldots$, and consequently, we have

$$\mu_{u_1}(\Omega_\infty) \leq \lim_{j\to\infty} \mu_{u_{1,j}}(\Omega_\infty).$$

On the other hand, $u_{\lambda,j}$, $j = 1, 2, \ldots$, is a decreasing sequence in j and it converges pointwise to u_λ in Ω_∞. Thus Theorem 3.4 implies that $\mu_{u_{\lambda,j}} \to \mu_{u_\lambda}$ weakly as $j \to \infty$. From this we conclude that

$$\limsup_{j\to\infty} \mu_{u_{\lambda,j}}(K_1) \leq \mu_{u_\lambda}(K_1) = \mu_{u_\lambda}(\Omega_\infty).$$

These inequalities together with (6.9), applied to $u_{1,j}$ and $u_{\lambda,j}$, give

$$\begin{aligned}\mu_{u_1}(\Omega_\infty) &\leq \lim_{j\to\infty} \mu_{u_{1,j}}(\Omega_\infty)\\ &\leq C(\lambda^{1-p} + \lambda^{-1/(p-1)}) \lim_{j\to\infty} \mu_{u_{\lambda,j}}(\Omega_\infty)\\ &\leq C(\lambda^{1-p} + \lambda^{-1/(p-1)})\mu_{u_\lambda}(\Omega_\infty).\end{aligned}$$

This proves (6.7). The proof of (6.8) is analogous and we leave the details to the reader. □

Lemma 6.10. *Let v be a weak supersolution with zero boundary data in Ω_∞. Then*

$$\mu_v(\Omega_\infty) = \mu_{\min\{v,\lambda\}}(\Omega_\infty)$$

for every $\lambda > 0$.

Proof. Let $\varphi \in C_0^\infty(\Omega_\infty)$ such that $\varphi = 1$ in $\{v > \lambda\}$. Then

$$\begin{aligned}
&\int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_v - \int_{\Omega_\infty} \varphi \,\mathrm{d}\mu_{\min\{v,\lambda\}} \\
&\qquad = \int_{\Omega_\infty} \Big(\mathcal{A}(\nabla v) \cdot \nabla \varphi - v \frac{\partial \varphi}{\partial t}\Big) \,\mathrm{d}z \\
&\qquad\qquad - \int_{\Omega_\infty} \Big(\mathcal{A}(\nabla \min\{v,\lambda\}) \cdot \nabla \varphi - \min\{v,\lambda\} \frac{\partial \varphi}{\partial t}\Big) \,\mathrm{d}z \\
&\qquad = \int_{\{v>\lambda\}} \Big(\mathcal{A}(\nabla v) \cdot \nabla \varphi - (v-\lambda) \frac{\partial \varphi}{\partial t}\Big) \,\mathrm{d}z = 0,
\end{aligned}$$

since $\nabla \varphi = 0$ and $\frac{\partial \varphi}{\partial t} = 0$ in $\{v > \lambda\}$. This implies the result. $\square$

The following result gives an estimate for the capacity of the distribution set of a superparabolic function. For clarity, we denote the solution of the obstacle problem with the obstacle $u\chi_K$ by R_K^u.

Theorem 6.11. *Let u be $\mathcal{A}$-superparabolic in Ω_∞ and $\lambda > 1$. Then there is a constant C, independent of K, such that*

$$\operatorname{cap}(\{u > \lambda\} \cap K) \le C \mu_{R_K^u}(\Omega_\infty)\big(\lambda^{1-p} + \lambda^{-1/(p-1)}\big)$$

for all compact sets K in Ω_∞.

Proof. There is a compact set K' in Ω_∞ such that R_K^u is $\mathcal{A}$-parabolic and $R_K^u \le \lambda$ in $\Omega_\infty \setminus K'$. Let ψ_i, $i = 1, 2, \dots$, be an increasing sequence of continuous functions such that $\psi_i = R_K^u$ in $\Omega_\infty \setminus K'$ and ψ_i converges to $\widehat{R}_K^u$ pointwise in Ω_∞ as $i \to \infty$. Let u_i be the solution of the obstacle problem with the obstacle ψ_i, $i = 1, 2, \dots$ Observe that u_i, $i = 1, 2, \dots$, is an increasing sequence of continuous weak supersolutions in Ω_∞. Since u_i is a weak solution in $\Omega_\infty \setminus K'$, the Riesz measure of u_i is supported in K' for every $i = 1, 2, \dots$ The distribution sets $\{u_i \ge \lambda\}$ are compact and

$$\{u_i \ge \lambda\} \subset \{u_{i+1} \ge \lambda\}$$

for $i = 1, 2, \dots$ Since

$$(\{u > \lambda\} \cap K) \subset \{\widehat{R}_K^u > \lambda\} \subset \bigcup_{i=1}^\infty \{u_i \ge \lambda\},$$

it follows by Lemma 5.4 that

$$\operatorname{cap}(\{u > \lambda\} \cap K) \le \operatorname{cap}\Big(\bigcup_{i=1}^\infty \{u_i \ge \lambda\}\Big) = \lim_{i\to\infty} \operatorname{cap}(\{u_i \ge \lambda\}).$$

Furthermore, let $u_{1,i}$ be the solution to the obstacle problem with the obstacle $\chi_{\{u_i \ge \lambda\}}$, and $u_{\lambda,i}$ be the solution to the obstacle problem with the obstacle $\lambda\chi_{\{u_i\geq\lambda\}}$. According to Theorem 5.7 and Lemma 6.6, we obtain

$$\begin{aligned}\operatorname{cap}(\{u_i\geq\lambda\}) &= \mu_{u_{1,i}}(\Omega_\infty)\\ &\leq C(\lambda^{1-p}+\lambda^{-1/(p-1)})\mu_{u_{\lambda,i}}(\Omega_\infty).\end{aligned}$$

On the other hand, weak supersolutions u_i and $u_{\lambda,i}$ are continuous near the parabolic boundary $\partial_p\Omega_\infty$, $u_i\geq u_{\lambda,i}$, and $u_i=u_{\lambda,i}$ on $\partial_p\Omega_\infty$. Hence Lemma 6.5 implies

$$\mu_{u_{\lambda,i}}(\Omega_\infty)\leq\mu_{\min\{u_i,\lambda\}}(\Omega_\infty).$$

Using the fact that u_i is a weak solution in $\Omega_\infty\setminus K'$, we have

$$\mu_{\min\{u_i,\lambda\}}(\Omega_\infty)=\mu_{\min\{u_i,\lambda\}}(K').$$

Since $\min\{u_i,\lambda\}$ converges to $\min\{R_K^u,\lambda\}$ pointwise in Ω_∞ as $i\to\infty$, Theorem 3.4 implies the weak convergence of the corresponding Riesz measures and

$$\limsup_{i\to\infty}\mu_{\min\{u_i,\lambda\}}(K')\leq\mu_{\min\{R_K^u,\lambda\}}(K').$$

Combining the estimates together with

$$\mu_{\min\{R_K^u,\lambda\}}(K')=\mu_{R_K^u}(K'),$$

see Lemma 6.10, we conclude that

$$\begin{aligned}\operatorname{cap}(\{u>\lambda\}\cap K) &\leq C(\lambda^{1-p}+\lambda^{-1/(p-1)})\limsup_{i\to\infty}\mu_{u_{\lambda,i}}(K')\\ &\leq C(\lambda^{1-p}+\lambda^{-1/(p-1)})\limsup_{i\to\infty}\mu_{\min\{u_i,\lambda\}}(K')\\ &\leq C(\lambda^{1-p}+\lambda^{-1/(p-1)})\mu_{R_K^u}(\Omega_\infty).\end{aligned}$$

This completes the proof. □

The previous theorem immediately gives the following result for polar sets.

Corollary 6.12. *Let u be $\mathcal{A}$-superparabolic in Ω_∞. Then the parabolic capacity of the polar set $\{z\in\Omega_\infty : u(z)=\infty\}$ is zero.*

References

[1] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.

[2] D. G. Aronson. Removable singularities for linear parabolic equations. *Arch. Rat. Mech. Anal.*, 17:79–84, 1964.

[3] L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87(1):149–169, 1989.

[4] L. Boccardo, A. Dall'Aglio, T. Gallouët, and L. Orsina. Nonlinear parabolic equations with measure data. *J. Funct. Anal.*, 147(1):237–258, 1997.

[5] V. Bögelein, F. Duzaar, and G. Mingione. Degenerate problems with irregular obstacles. *J. Reine Angew. Math.*, 650:107–160, 2011.

[6] E. DiBenedetto. *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York, 1993.

[7] E. DiBenedetto, U. Gianazza, and V. Vespri. Harnack estimates for quasi-linear degenerate parabolic differential equations. *Acta Math.*, 200(2):181–209, 2008.

[8] E. DiBenedetto, U. Gianazza, and V. Vespri. *Harnack's inequality for degenerate and singular parabolic equations.* Springer Monographs in Mathematics. Springer, New York, 2012.

[9] J. L. Doob. *Classical potential theory and its probabilistic couterpart.* Springer-Verlag, 1984.

[10] J. Droniou, A. Porretta, and A. Prignet. Parabolic capacity and soft measures for nonlinear equations. *Potential Anal.*, 19:99–161, 2003.

[11] D. E. Edmunds, and L. A. Peletier. Removable singularities of solutions of quasilinear parabolic equations. *J. London Math. Soc.*, 2(2):273–283, 1970.

[12] L. C. Evans and R. F. Gariepy. Wiener's test for the heat equation. *Arch. Rational Mech. Anal.*, 78:293–314, 1982.

[13] R. Gariepy and W. P. Ziemer. Removable sets for quasilinear parabolic equations. *J. London Math. Soc.*, 21(2):311–318, 1980.

[14] R. Gariepy and W. P. Ziemer. Thermal capacity and boundary regularity. *J. Differential Equations.*, 45:374–388, 1982.

[15] R. Harvey and J. Polking. A notion of capacity which characterizes removable singularities. *Trans. Amer. Math. Soc.*, 1669:183–195, 1972.

[16] J. Heinonen and T. Kilpeläinen. Polar sets for supersolutions of degenerate elliptic equations. *Math. Scand.*, 63:136–150, 1988.

[17] J. Heinonen, T. Kilpeläinen and O. Martio. *Nonliear Potential Theory of Degenerate Elliptic Equations.* Oxford University Press, Oxford 1993.

[18] T. Kilpeläinen and P. Lindqvist. On the Dirichlet boundary value problem for a degenerate parabolic equation. *SIAM J. Math. Anal.*, 27(3):661–683, 1996.

[19] T. Kilpeläinen and J. Malý. Degenerate elliptic equations with measure data and nonlinear potentials. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(4):591–613, 1992.

[20] J. Kinnunen and P. Lindqvist. Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)*, 185(3):411–435, 2006.

[21] J. Kinnunen and P. Lindqvist. Summability of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(1):59–78, 2005.

[22] J. Kinnunen, T. Lukkari, and M. Parviainen. An existence result for superparabolic functions. *J. Funct. Anal.*, 258:713–728, 2010.

[23] R. Korte, T. Kuusi, and M. Parviainen. A connection between a general class of superparabolic functions and supersolutions. *J. Evol. Eq*, 10(1):1–20, 2010.

[24] R. Korte, T. Kuusi and J. Siljander. Obstacle problem for nonlinear parabolic equations. *J. Differential Equations*, 246(9), 3668–3680, 2009.

[25] T. Kuusi. Lower semicontinuity of weak supersolutions to nonlinear parabolic equations. *Differential Integral Equations*, 22(11-12), 1211–1222, 2009.

[26] R. Kuusi and M. Parviainen. Existence for a degenerate Cauchy problem. *Manuscripta Math.*, 128(2):213–249, 2009.

[27] E. Lanconelli. Sul problema di Dirichlet per l'equazione del calore. *Ann. Mat. Pura Appl.*, 97:83–114, 1973.

[28] E. Lanconelli. Sul problema di Dirichlet per equazione paraboliche del secondo ordine a coefficiente discontinui. *Ann. Mat. Pura Appl.*, 106:11–38, 1975.

[29] E. M. Landis. Necessary and sufficient conditions for the regularity of a boundary point for the Dirichlet problem for the heat equation. *Soviet Math. Dokl.*, 10:380–384, 1969.

[30] P. Lindqvist and J. J. Manfredi. Viscosity supersolutions of the evolutionary p-Laplace equation. *Differential Integral Equations*, 20(11): 1303–1319, 2007.
[31] P. Lindqvist and O. Martio. Regularity and polar sets of supersolutions of certain degenerate elliptic equations. *J. Anal. Math.*, 50:1–17, 1988.
[32] P. Lindqvist and M. Parviainen. Irregular time dependent obstacles. *Submitted.*
[33] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaire*. Dunord et Gauthier-Villars, 1969.
[34] M. Pierre. Parabolic capacity and Sobolev spaces. *SIAM J. Math. Anal.*, 14(3):522–533, 1983.
[35] L. M. Saraiva. Removable singularities and quasilinear parabolic equations. *Proc. London Math. Soc.*, 48:385–400, 1984.
[36] L. M. Saraiva. Removable singularities of solutions of quasilinear parabolic equations. *Ann. Mat. Pura Appl.*, (IV) CXLI:187–221.
[37] S. J. Taylor and N. A. Watson. A Hausdorff measure classification of polar sets for the heat equation. *Math. Proc. Camb. Phil. Soc.*, 97:325–344, 1985.
[38] N. S. Trundinger and X.-J. Wang. On the weak continuity of elliptic operators and applications to potential theory. *Amer. J. Math.*, 124(2):369–410, 2002.
[39] N. A. Watson. A theory of subtemperatures in several variables. *Proc. London Math. Soc.*, 26:385–417, 1973.
[40] N. A. Watson. Green functions, potential, and the Dirichlet problem for the heat equation. *Proc. London Math. Soc.*, 33:251–298; Corrigendum, ibid., 37:32-34, 1978.
[41] N. A. Watson. Thermal capacity. *Proc. London Math. Soc.*, 37:342–362, 1978.
[42] W. P. Ziemer. Behavior at the boundary of solutions of quasilinear parabolic equations. *J. Diff. Eq.*, 35:291–305, 1980.
[43] Z. Wu, J. Zhao, J. Yin and H. Li. *Nonlinear Diffusion Equations*. World Scientific, Singapore, 2001.

J.K., T.K., Department of Mathematics, P.O. Box 11100, FI-00076 Aalto University, Finland
juha.kinnunen@tkk.fi, tuomo.kuusi@tkk.fi

R.K., Department of Mathematics and Statistics, P.O. Box 68, FI-00014 University of Helsinki, Finland
riikka.korte@helsinki.fi

M.P., Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyväskylä, Finland
mikko.j.parviainen@jyu.fi