THE EULER CHARACTERISTIC FOR SURFACES

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1. Polygons and the Euler Characteristic

Polygons and Tilings. Let us start by defining what we are talking about. What is a polygon (in the plane)? One definition is that a polygon is a collection of *vertices*, *edges* and one *face*. The *vertices* are some collection of points in the plane from each of which leave two line segments (*edges*) to some other vertex such that these edges form a *chain* (a *cycle*) meanining we can move in some order along the edges to get back to the vertex we started from. The area bounded by this chain is the *face* of the polygon. The pictures 1, 2 and 3 show examples of polygons.

Figure 1. Regular polygons

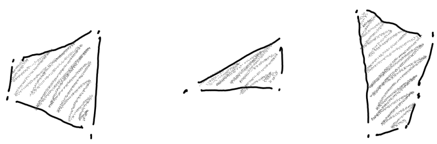

Figure 2. Crooked polygons

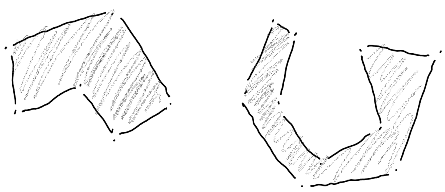

Figure 3. Complicated polygons

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We notice that a polygon can be divided up into multiple smaller polygons. This is called a *tiling* of a polygon.

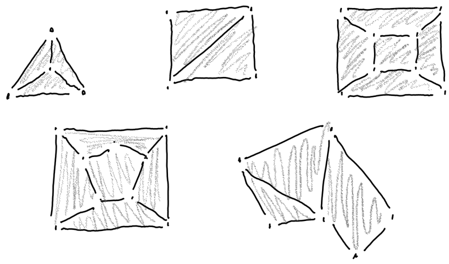

FIGURE 4. Tiled polygons

We notice that every polygon can eventually be subdivided up into triangles. This is due to triangles being the "simplest" polygon in the plane. Why is this? A polygon with only two vertices must necessarily have only one edge. Thus we cannot return to our starting vertex if following the edges and thus do not bound anything "two-dimensional".

About Dimensions.
What do we mean by the word *dimension*? In some sense the dimension of a shape tells us in how many different *independent* directions we can move in *within a shape.* A point is 0-dimensional, because we cannot move in *any* directions within it. Piste on nollaulotteinen, koska emme voi liikkua sen sisällä yhteenkään suuntaan. A line segment is 1-dimensional because we can move in only one direction along it if we consider moving backwards to be "negative" movement. An "area" of the plane is 2-dimensional because we can move in exactly two "independent" directions within it. Any other movement can be expressed as combining movement in these two directions.

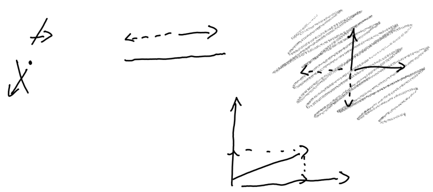

FIGURE 5. Directions of movement within 0-, 1- and 2-dimensional shapes.

Notice that we talk about moving *within* a shape. In this sense dimension is a property "ìnside" the shape itself and independent of the surrounding space in which the shape lives. Usually in mathematics you do not have to think of a shape as living within any outside space.

Itse asiassa, jos käytämme hieman yleisempää määritelmää monikulmioille kuin äsken, niin Indeed, if we use a more general definition for a polygon than before, we get the following.

- A point is a 0-dimensional polygon.

- A line segment between two points is a 1-dimensional polygon.
- For example a cube (with its interior) is a 3-dimensional "polygon", a polyhedron.
- Similarly there exist n-dimensional "polygons", n-polytopes, for all dimensions $n \in \mathbb{N}$ (n a natural number). However when $n \geq 4$ these are not visualisable (by humans).

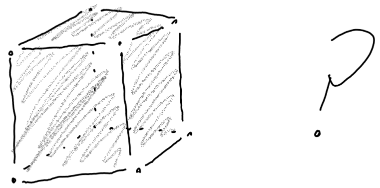

FIGURE 6. A cube, a 3-dimensional polyhedron and a question mark representing an n-dimensional polytope, when $n \geq 4$.

The Euler Characteristic for shapes in the plane. Over time geometry has moved to consider very, very complicated and abstract shapes/objects (/spaces!). These are usually not graspable visually on a any level higher than coarse intuition. Thus the question arises of how we know that these shapes are "different" (on some coarseness of classification). To answer this question mathematicians have realised that we can "calculate" other mathematical objects, for example numbers, out of these geometric shapes that depend only on the (on some level of coarseness) shape of the geometric shape. These are called *invariants* and the most classical of these is the **Euler(-Poincaré) characteristic**.

Definition 1.1 (The Euler(-Poincaré) characteristic)**.** *Let S be a tiling of a shape in the plane. Then the* ***Euler(-Poincaré) characteristic*** *of S, $\chi(S)$ (or just χ if S is clear from context) is*

$$\chi(S) = \textit{Number of vertices } - \textit{ Number of edges } + \textit{ Number of faces.}$$

Esimerkiksi:

- A square has 4 vertices, 4 edges and 1 face. Thus
$$\chi = 4 - 4 + 1 = 1.$$
- A square divided up into two triangles has 4 vertices, 5 edges and 2 faces. Thus
$$\chi = 4 - 5 + 2 = 1.$$
- The square in picture 7 with a square hole has 8 vertices, 12 edges and 4 faces. Thus
$$\chi = 8 - 12 + 4 = 0.$$
- The square in picture 7 with two square holes has 12 vertices, 15 edges and 2 faces. Thus
$$\chi = 12 - 15 + 2 = -1.$$

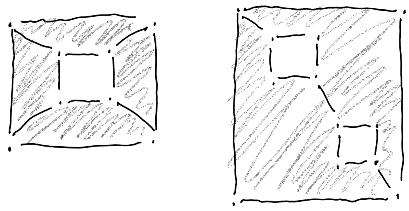

Figure 7. More complicated shapes in the plane and their tilings.

Usually we write

$$\begin{aligned} V &= \text{Number of vertices} \\ E &= \text{Number of edges} \\ F &= \text{Number of faces} \end{aligned}$$

meaning

$$\chi = V - E + F.$$

Based on the previous conversation we notice that

$$\begin{aligned} V &= \text{Number of 0-dimensional polygons} \\ E &= \text{Number of 1-dimensional polygons} \\ F &= \text{Number of 2-dimensional polygons} \end{aligned}$$

and thus

$$\begin{aligned} \chi =& V - E + F = 1 \cdot V + (-1) \cdot E + (-1) \cdot (-1) \cdot F \\ =& (-1)^0 \cdot V + (-1)^1 \cdot E + (-1)^2 \cdot F \\ =& \sum_{k=0}^{2} (-1)^k \cdot \text{Number of } k\text{-dimensional polygons.} \end{aligned}$$

2. Surfaces and the Euler characteristic

Topology. Topology is the most flexible (or correspondingly the coarsest) sub-(or sup)field of geometry. Shapes/objects are differentiated only by how they differ from each other by "non-continuous" transformations. Examples of continuous transformations are stretching, twisting or crunching. Examples of non-continuous transformations are making holes, separating or gluing.

Examples of topological shapes. In figures 8, 9, 10, 11, 12 and 13.

Polygons from the perspective of topology. Because of the flexibility of the topological perspective, the polygons talked about previously are (from the perspective of topology) all the exact same. All are incarnations or ways to represent the disk. Thus in our tilings we are no longer restricted to using straight lines. The most important thing is that every "tile" is a topological disk. The topological perspective also means that the triangle is no longer the simplest "polygon". Edges can curve and one edge can even curve all the way back to its starting vertex. In topology the "n-

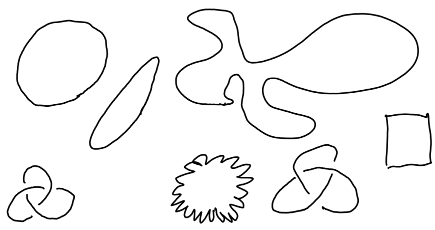

FIGURE 8. Topological circles. Notice the two differently "oriented" trefoil-knots, which live in 3-dimensional space instead of the plane.

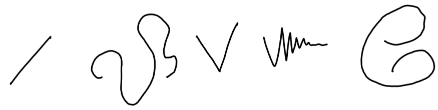

FIGURE 9. Topological line segments or 1-dimensional polygons.

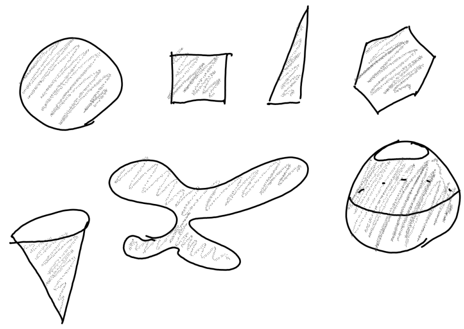

FIGURE 10. Topological disks. Notice the sphere with a hole and the cone.

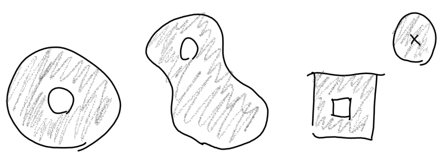

FIGURE 11. Topological annuli (an annulus). The cross represents a hole the size of a single point.

dimensional polygons" are called n-cells and "tilings" created from them cell decompositions.

- Points (*vertices*) are still 0-dimensional cells.
- Line segments and circles (*edges*) are 1-dimensional cells.
- Disks (*faces*) are 2-dimensional cells.

There exist such weird and pathological topological shapes that they do not necessarily have cell decompositions.

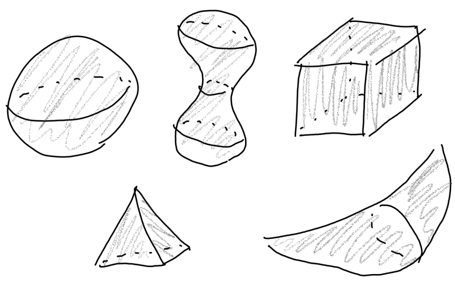

FIGURE 12. Topological spheres.

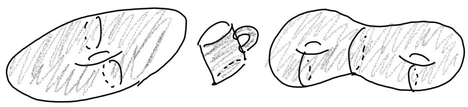

FIGURE 13. Two topological "donuts" (tori (a torus)) and one "genus 2 surface".

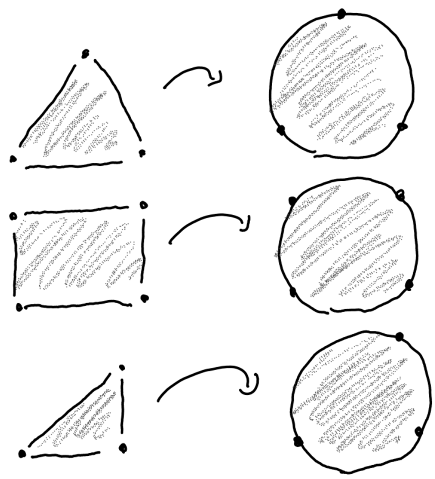

FIGURE 14. Polygons stretched into disks.

Let us redefine the Euler characteristic.

Definition 2.1 (Euler(-Poincaré) characteristic)**.** *Let S be a cell decomposition of a shape. Then the* ***Euler(-Poincaré) characteristic*** *of the shape S, $\chi(S)$ (or just χ if S is clear from context) is*

$$\chi(S) = \textit{Number of } 0\textit{-cells} \; - \; \textit{Number of } 1\textit{-cells} \; + \; \textit{Number of } 2\textit{-cells}.$$

The following is a fundamental fact.

Theorem 2.2. *If two shapes are the same from the perspective of topology then their Euler characteristics χ also agree. Thus the Euler characteristic is a* ***topological invariant****.*

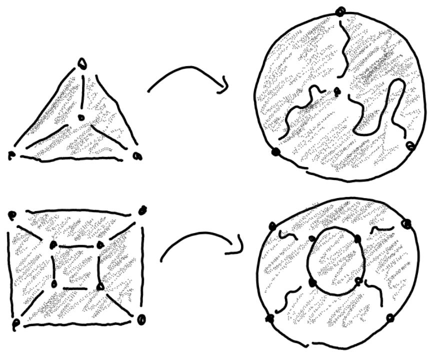

FIGURE 15. Curved tilings for the disk.

FIGURE 16. “Polygons” that are simpler than a triangle.

Gluing. Gluing is not a continuous transformation. Thus the sum of the Euler characteristics χ is not necessarily the same as the Euler characteristic χ of the glued together shape.

Lemma 2.3. *Let S be two shapes S_1 and S_2 though of as a single disconnected shape. Then for the Euler characteristic it holds that*

$$\chi(S) = \chi(S_1) + \chi(S_2).$$

FIGURE 17. n triangles. According to lemma 2.3 $\chi = \sum_{k=1}^{n}(3-3+1) = n \cdot 1 = n$.

Lemma 2.4. *Let the cell decomposition (or tiling) of a shape S_{old} (connected or not) have two different 1-cells each of which have two 0-cells at their ends (meaning two line segments with a total of four different vertices). If the shape S is glued together by these cells then the glued together shape S_{new} has*

$$\chi(S_{new}) = \chi(S_{old}) - 1.$$

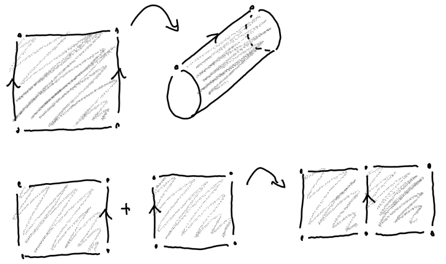

Figure 18. Gluings using lemma 2.4.

Lemma 2.5. *Let the cell decomposition (or tiling) of a shape S_{old} (connected or not) have two different topological circles (meaning a chain of edges and vertices) which have the same cell decomposition (meaning the chain has the same number of edges and vertices). If the shape S is glued together by these circles then the glued together shape S_{new} has*

$$\chi(S_{new}) = \chi(S_{old}).$$

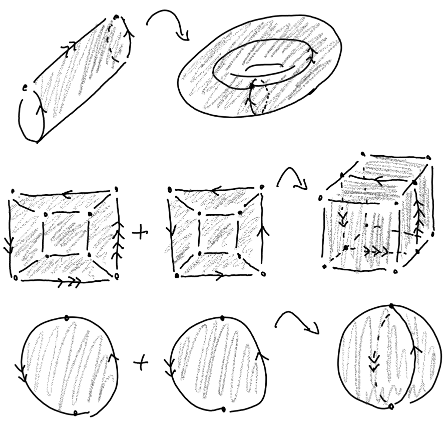

Figure 19. Gluings using lemma 2.5.

Lemma 2.6. *Let the cell decomposition (or tiling) of a shape S_{old} (connected or not) have two different topological disks (meaning faces) the bounding circles of which have the same cell decomposition (meaning the chain has the same number of edges and vertices). If the shape S is glued together by these disks then the glued together shape S_{new} has*

$$\chi(S_{new}) = \chi(S_{old}) - 2.$$

Using these results we can build surfaces step-by-step out of polygons. However we also have the following result.

Theorem 2.7. *Every surface "of finite size" can be tiled using "triangles".*

Using this we can also cut up every surface back into polygons. This leads to the following results.

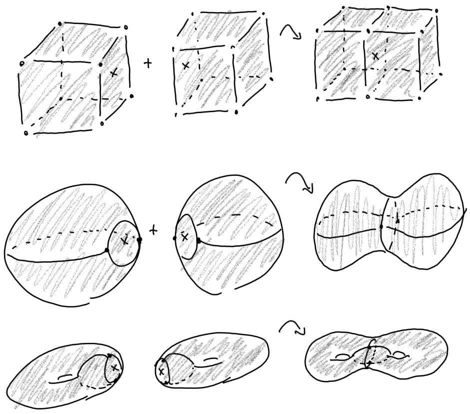

FIGURE 20. Gluings using lemma 2.6.

Theorem 2.8. *Let S_1 and S_2 be* "completely 2-dimensional" *surfaces* "of finite size" *"without a boundary". Then both S_1 and S_2 are the same shape from the perspective of topology if and only if $\chi(S_1) = \chi(S_2)$* ***and*** *they're both either* "orientable" *or* "nonorientable".

Theorem 2.9. *Let S_1 and S_2 be* "completely 2-dimensional" *surfaces* "of finite size" *"with 1-dimensional boundary". Then both S_1 and S_2 are the same shape from the perspective of topology if and only if $\chi(S_1) = \chi(S_2)$, they're both either* "orientable" *or* "nonorientable" ***and*** *they have the same number of components in their boundary.*

The exercises show that for "orientable" surfaces the condition on the Euler characteristic χ means the number of "donut holes".

FIGURE 21. A "nonorientable" surface, with one "nonorientable donut hole", a Klein bottle embedded into 3-dimensional space. Picture by en:User:Lethe on Wikipedia. Licensed under CC-BY-SA 3.0 Unported.

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3. Problems

(1) **Untiled polygons**
   a) Calculate the Euler characteristic χ for the untiled polygons in figures 1 and 3 (as many as you feel like).
   b) **Bonus:** Prove what the Euler characteristic for an untiled polygon is in general.

(2) **Euler characteristics of other dimensional polygons**
   a) Calculate the Euler characteristic χ for a point, a 0-dimensional polygon.
   b) Calculate the Euler characteristic χ for a line segment, a 1-dimensional polygon.
   c) Calculate the Euler characteristic χ for a cube (with interior included), a 3-dimensional polygon, by first calculating the characteristic for its surface and then substracting 1 from the result, meaning
   $$\chi = V - E + F - 1.$$
   This -1 is $(-1)^3 \cdot$ 3-dimensional polygons.
   d) Guess what the Euler characteristic χ is for an n-dimensional polygon.

(3) **Tiling invariance of the Euler characteristic**
   a) Calculate the Euler characteristic χ for the tiled polygons in figure 4 (as many as you feel like).

b) In some tiling add a new vertex onto an edge. How does this affect the Euler characteristic χ.
c) In some tiling and some nontriangular polygon add a new edge between two vertices. How does this affect the Euler characteristic χ?
d) In some tiling remove an edge from in between two polygons such that the resulting combined area of the plane is still a polygon. How does this affect the Euler characteristic χ?
e) In some tiling straighten the edges of some polygon by removing a vertex from between them. How does this affect the Euler characteristic χ?
f) **Bonus:** Deduce (or finish the proof) that every tiling of a certain figure in the plane has the same Euler characteristic meaning it is **an invariant of the polygon**.

(4) **The effect of holes on the Euler characteristic**

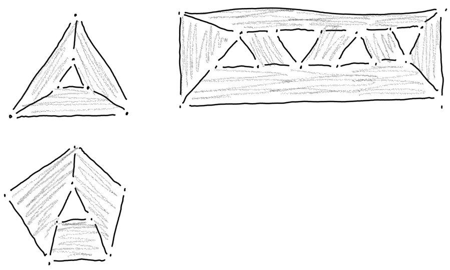

FIGURE 22. Polygons with holes

a) Calculate the Euler characteristic χ for the polygons with holes in figure 22.
b) Guess (**Bonus:** or prove) the formula for the Euler characteristic χ of a polygon with n holes. You can use the result in problem 3.f) to help yourself.

(5) **Gluing by an edge**
a) Calculate the Euler characteristic χ for the shapes in figure 18.
b) Find a shape so that if you glue it together with something using lemma 2.4, the Euler characteristic does not change?
c) **Bonus:** Prove lemma 2.4.

(6) **Gluing by a circle**
a) Calculate the Euler characteristic χ for the shapes in figure 19.
b) Find a shape so that if you glue it together with something using lemma 2.5, the Euler characteristic does not change?
c) **Bonus:** Prove lemma 2.5.

(7) **Gluing by a disk**
a) Calculate the Euler characteristic χ for the shapes in figure 20.
b) Find a shape so that if you glue it together with something using lemma 2.6, the Euler characteristic does not change?
c) **Bonus:** Prove lemma 2.6.

(8) **Donut holes** Calculate the Euler characteristic χ for a surface with n "donut holes".

a) Using the result in problem 4.b) and lemma 2.5.
b) Using the result in problem 6.a) and lemma 2.6.