# THE HIGHER INTEGRABILITY OF WEAK SOLUTIONS OF POROUS MEDIUM SYSTEMS

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ABSTRACT. In this paper we establish that the gradient of weak solutions to porous medium type systems admits the self-improving property of higher integrability.

## 1. INTRODUCTION AND RESULTS

In this article, we are interested in the self-improving property of higher integrability of weak solutions to porous medium type systems, whose prototype is

$$\partial_t u - \Delta\big(|u|^{m-1}u\big) = 0.$$

This problem has been open for some time. For non-negative solutions to porous medium type equations it has recently been solved by Gianazza & Schwarzacher [16]. Here, we are able to treat signed solutions and the vectorial case. More precisely, we consider equations (the case $N = 1$) or systems (the case $N \geq 2$) of the form

$$\partial_t u - \operatorname{div} \mathbf{A}(x,t,u,D\boldsymbol{u}^m) = \operatorname{div} F \quad \text{in } \Omega_T, \tag{1.1}$$

with $u\colon \Omega_T \to \mathbb{R}^N$, in a space-time cylinder $\Omega_T := \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open domain, $n \geq 2$, $T > 0$, and we abbreviated $\boldsymbol{u}^m := |u|^{m-1}u$. The assumptions on the vector field $\mathbf{A}\colon \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ are as follows. We assume that $\mathbf{A}$ is measurable with respect to $(x,t) \in \Omega_T$ for all $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$, continuous with respect to $(u,\xi)$ for a.e. $(x,t) \in \Omega_T$, and moreover that $\mathbf{A}$ satisfies for some structural constants $0 < \nu \leq L < \infty$ the following growth and ellipticity conditions

$$\begin{cases} \mathbf{A}(x,t,u,\xi)\cdot\xi \geq \nu|\xi|^2\,, \\ |\mathbf{A}(x,t,u,\xi)| \leq L|\xi|, \end{cases} \tag{1.2}$$

for a.e. $(x,t) \in \Omega_T$ and any $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$. Note that these assumptions are compatible with the ones in [1] and [11, Chapter 3.5] in the scalar case. For the inhomogeneity $F\colon \Omega_T \to \mathbb{R}^{Nn}$ we assume that $F \in L^2(\Omega_T,\mathbb{R}^{Nn})$. As usual, we suppose that the solutions to (1.1) lie in a parabolic Sobolev space; the precise definition will be given below in Definition 1.1.

In the stationary elliptic case it is by now well known that weak solutions to elliptic systems of the type

$$-\operatorname{div} \mathbf{A}(x,t,u,Du) = \operatorname{div} F \quad \text{in } \Omega, \tag{1.3}$$

locally belong to a slightly higher Sobolev space than a priori assumed. The so-called *self-improving property of higher integrability* was first detected by Elcrat & Meyers [25]. Their proof is based, among other things, on a reverse Hölder type inequality – a direct consequence of a Caccioppoli type inequality (also called reverse Poincaré inequality) – and some adaptation of the famous Gehring Lemma [15]; the nowadays standard interior version can be retrieved from [17, Chapter 11, Theorem 1.2], for the boundary version we refer to [22] and [13, Theorem 2.4]. Originally, Gehring's lemma was developed to

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establish the higher integrability of the Jacobian of quasi-conformal mappings. Over time, the self-improving property of higher integrability was first established for solutions of stationary elliptic systems [18] and later for minima of variational integrals [19] by Giaquinta & Modica. A unified treatment in the language of quasi-minima is given in [21, Theorem 6.7]. Corresponding global results for stationary elliptic problems with a Dirichlet boundary condition were established in [21, Section 6.5], [13, Section 3].

The first higher integrability result for vectorial evolutionary problems goes back to Giaquinta & Struwe [20, Theorem 2.1]. More precisely, quasilinear parabolic systems of the type

$$\partial_t u - \operatorname{div}\big(\mathbf{a}(x,t,u)Du\big) = \operatorname{div} F \quad \text{in } \Omega_T, \tag{1.4}$$

whose coefficients $\mathbf{a}$ continuously depend on $(x,t,u)$ have been investigated. The technique of Giaquinta & Struwe does not carry over to the parabolic $p$-Laplacian system

$$\partial_t u - \operatorname{div}\big(|Du|^{p-2}Du\big) = \operatorname{div} F \quad \text{in } \Omega_T,$$

or general parabolic systems with $p$-growth (the growth and coercivity condition from (1.2) have to be replaced by $\mathbf{a}(x,t,u,\xi)\cdot\xi \ge \nu|\xi|^p$ and $|\mathbf{a}(x,t,u,\xi)| \le L(|\xi|^p+1)$). The obstruction relies in the fact that the parabolic $p$-Laplacian equation has a different homogeneity in the time and the diffusion term. In particular, multiples of a solution do not anymore solve the differential equation. This problem has finally been solved by Kinnunen & Lewis [23] who proved the higher integrability result for general parabolic systems with $p$-growth. More precisely, they have shown that weak solutions from the natural energy space $C^0\big([0,T];L^2(\Omega,\mathbb{R}^N)\big) \cap L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big)$ have a more integrable spatial gradient, namely

$$Du \in L^{p+\varepsilon}_{\rm loc}\big(\Omega_T,\mathbb{R}^{Nn}\big) \text{ for some } \varepsilon > 0.$$

This shows that also in the case of parabolic systems with coefficients of $p$-growth and coercivity energy solutions enjoy the self-improving property of higher integrability for the gradient. The key to the result was the use of intrinsic cylinders in the sense of DiBenedetto & Friedman [8, 9, 10, 7], i.e. cylinders of the form $Q_{\varrho,\lambda^{2-p}\varrho^2}$ whose space-time scaling depends on the spatial gradient of the solution via

$$\iint\!\!\!\!\!\!\!-_{Q_{\varrho,\lambda^{2-p}\varrho^2}} |Du|^p \mathrm{d}x\mathrm{d}t \approx \lambda^p.$$

This important result has been generalized over time in various directions. The global result with a Dirichlet boundary condition at the lateral boundary was established by Parviainen [26]. Interior higher integrability for weak solutions of higher order degenerate parabolic systems has been shown by Bögelein [4], while the corresponding global result was established in [6]. The case of parabolic equations with non-standard $p(x,t)$-growth was treated by Antonsev & Zhikov [2], while systems were treated by Zhikov & Pastukhova [29] and independently by Bögelein & Duzaar [5].

For the porous medium equation, the question of higher integrability of the gradient, even for non-negative solutions in the scalar case, remained an open problem for a while. The reason was that when proving regularity of the gradient the degeneracy with respect to $u$ is much more difficult to handle. This difficulty has recently been overcome by Gianazza & Schwarzacher [16] who proved that non-negative weak solutions to porous medium equations of the type (1.1) enjoy the self-improving property of higher integrability. More precisely, this means that the integrability $Du^{\frac{m+1}{2}} \in L^2_{\rm loc}\big(\Omega_T,\mathbb{R}^n\big)$ of weak solutions was improved to

$$Du^{\frac{m+1}{2}} \in L^{2+\varepsilon}_{\rm loc}\big(\Omega_T,\mathbb{R}^n\big), \quad \text{for some } \varepsilon > 0.$$

The main novelty with respect to the proof for the parabolic $p$-Laplacian in [23] is that Gianazza & Schwarzacher work with cylinders which are intrinsically scaled with respect to $u$ rather than the spatial gradient $Du$. This means that they consider cylinders of the type $Q_{\varrho,\theta\varrho^2}$ whose space-time scaling is adapted to the solution $u$ via the coupling

$$\iint_{Q_{\varrho,\theta\varrho^2}} u^{m+1} \mathrm{d}x\mathrm{d}t \approx \theta^{-\frac{m+1}{m-1}}. \tag{1.5}$$

This is exactly the intrinsic scaling which is typically used in the proof of regularity of $u$, as for instance Hölder continuity of $u$, cf. [9]. At first glance it is quite surprising that this approach also yields regularity of the spatial gradient. However, these cylinders are better adapted to the equation and this is crucial for the proof. Nevertheless, the argument becomes much more involved than the one for the parabolic $p$-Laplacian. The overall strategy can be outlined as follows. First, one has to prove a reverse Hölder type inequality on certain intrinsic cylinders. To achieve this, Gianazza & Schwarzacher distinguish whether a cylinder $Q$ belongs to the non-degenerate regime in which the inequality

$$\iint_Q |u-(u)_Q|^{m+1} \mathrm{d}x\mathrm{d}t \le \delta \iint_Q u^{m+1} \mathrm{d}x\mathrm{d}t,$$

holds true for some particular $0<\delta \ll 1$, or $Q$ belongs to the degenerate regime in which the opposite inequality is valid. In the non-degenerate regime they rely on the expansion of positivity in order to guarantee that the solution does not become too small on the cylinder. In a second step, one usually constructs a covering of super-level sets of the spatial gradient with intrinsic cylinders. However, this is not possible for the cylinders which are intrinsically scaled with respect to $u$. Gianazza & Schwarzacher overcame this problem by a very elegant idea. They weakened this property to the so-called sub-intrinsic cylinders for which they succeeded to prove the covering property. Thereby, they call a cylinder sub-intrinsic if (1.5) holds as an inequality, i.e. the mean value integral is bounded from above by the right-hand side.

The methods of proof of this important result are only applicable in the scalar case for non-negative solutions, because tools as the expansion of positivity are neither available in the vectorial case, nor for signed solutions.

The present paper has its origin in the effort to extend the purely scalar result to the vector-valued case. As a by-product of the vectorial case, we are able to deal also with signed solutions in the scalar case. Moreover, contrary to Gianazza & Schwarzacher, we start from the definition of weak (energy) solutions introduced in [28, Theorem 5.5], i.e. we start with solutions satisfying $D\boldsymbol{u}^m \in L^2_{\mathrm{loc}}(\Omega_T,\mathbb{R}^{Nn})$, see (1.8). As main result, we prove that

$$D\boldsymbol{u}^m \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega_T,\mathbb{R}^{Nn}), \quad \text{for some } \varepsilon>0.$$

We note that starting from a vectorial version of the energy estimate used in [16], a modification of our method also applies to the definition of weak solution as considered there. The key to the higher integrability result in the vectorial case is to prove the reverse Hölder type inequality just by the use of an energy estimate and a gluing lemma as stated in Lemmas 3.1 and 3.2. In particular, it is important to omit the use of the expansion of positivity. In fact, for the proof of the Sobolev-Poincaré type inequality in Lemma 4.3 we only use the Gluing Lemma 3.2, the standard Sobolev inequality and some algebraic lemmas. Here, we note that contrary to (1.5) we work with differently scaled cylinders which reflect more clearly the behavior of the porous medium equation and which are adapted to the energy space (1.8) (for the heuristics see also [16, Remark 5.6]). These cylinders are given by $Q_\varrho^{(\theta)} = B_\varrho \times (-\theta^{1-m}\varrho^{\frac{m+1}{m}}, \theta^{1-m}\varrho^{\frac{m+1}{m}})$ with an intrinsic scaling of the form

$$\iint_{Q_\varrho^{(\theta)}} \frac{|\boldsymbol{u}^m|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t \approx \theta^{2m}, \tag{1.6}$$

so that in case that the mean value of $\boldsymbol{u}^m$ on the cylinder $Q_\varrho^{(\theta)}$ is zero, the scaling parameter $\theta$ is comparable to $|D\boldsymbol{u}^m|$. A cylinder $Q_\varrho^{(\theta)}$ is called sub-intrinsic if (1.6) holds as an inequality, where the mean value integral is bounded from above by the right-hand side. Contrary to [16] we present a unified proof of the Sobolev-Poincaré type inequalities on sub-intrinsic cylinders that works likewise in the non-degenerate and degenerate regime. These inequalities are subsequently used to derive reverse Hölder type inequalities on intrinsic cylinders and sub-intrinsic cylinders additionally satisfying

$$\iint_{Q_\varrho^{(\theta)}} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t \gtrsim \theta^{2m}. \tag{1.7}$$

For the final proof of the higher integrability we cover the super-level-sets of $|D\boldsymbol{u}^m|$ by sub-intrinsic cylinders. Here, we rely on the construction by Gianazza & Schwarzacher. The idea is to choose with the help of the intermediate value theorem for a given center $z_o$ and radius $\varrho > 0$ the scaling parameter $\widetilde{\theta}_{z_o;\varrho}$ in such a way that

$$\iint_{Q_\varrho^{(\widetilde{\theta}_{z_o;\varrho})}(z_o)} \frac{|\boldsymbol{u}^m|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t = \widetilde{\theta}_{z_o;\varrho}^{2m}$$

is satisfied, where $Q_\varrho^{(\widetilde{\theta}_{z_o;\varrho})}(z_o) = z_o + Q_\varrho^{(\widetilde{\theta}_{z_o;\varrho})}$. Unfortunately, the mapping $\varrho \mapsto \widetilde{\theta}_{z_o;\varrho}$ is not monotone. Therefore, we modify the parameter $\widetilde{\theta}_{z_o;\varrho}$ by a rising sun type construction, i.e. we define

$$\theta_{z_o;\varrho} := \max_{r\ge\varrho} \widetilde{\theta}_{z_o;r}.$$

Then, the mapping $\varrho \mapsto \theta_{z_o;\varrho}$ is monotonically decreasing and furthermore one can show that the cylinders $Q_\varrho^{(\theta_{z_o;\varrho})}(z_o)$ are still sub-intrinsic. A crucial observation at this point is that by construction either the cylinders are intrinsic or satisfy (1.7). This allows to apply our reverse Hölder inequality. As in [16] our cylinders satisfy a Vitali covering property which allows to cover the super-level-sets of $|D\boldsymbol{u}^m|$ by countably many of these cylinders. In this way, we obtain a reverse Hölder inequality on the super-level-sets of $|D\boldsymbol{u}^m|$. In a standard way, this implies the higher integrability by a Fubini type argument.

### 1.1. **General Setting and Results.**

In this chapter we fix the notations, describe the general setup and present our main result. First, we define what we mean by a weak energy solution to the porous medium type system.

**Definition 1.1.** Assume that the vector field $\mathbf{A}\colon \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ satisfies (1.2) and that $F \in L^2_{\mathrm{loc}}(\Omega_T, \mathbb{R}^{Nn})$. We identify a measurable map $u\colon \Omega_T \to \mathbb{R}^N$ in the class

$$u \in C^0\big((0,T); L^{m+1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\big) \quad \text{with} \quad \boldsymbol{u}^m \in L^2_{\mathrm{loc}}\big(0,T; W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\big) \tag{1.8}$$

as a *weak solution* to the porous medium type system (1.1) if and only if the identity

$$\iint_{\Omega_T} \big[u\cdot\partial_t\varphi - \mathbf{A}(x,t,u,D\boldsymbol{u}^m)\cdot D\varphi\big]\mathrm{d}x\mathrm{d}t = \iint_{\Omega_T} F\cdot D\varphi\,\mathrm{d}x\mathrm{d}t \tag{1.9}$$

holds true, for any testing function $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. □

Existence of weak solutions can be deduced from [1] after the transformation $v = |u|^{m-1}u$; see also [3] for a different approach in the case of non-negative solutions.

Throughout the paper we work with parabolic cylinders of the type

$$Q_R(z_o) = B_R(x_o) \times \big(t_o - R^{\frac{m+1}{m}}, t_o + R^{\frac{m+1}{m}}\big) \Subset \Omega_T,$$

whose associated parabolic dimension is

$$d := n + 1 + \tfrac{1}{m}.$$

Our main result reads now as follows:

**Theorem 1.2.** *Let $m \geq 1$ and $\sigma > 2$. Then, there exist constants $\varepsilon_o = \varepsilon_o(n, m, \nu, L) \in (0, 1]$ and $c = c(n, m, \nu, L) \geq 1$ such that the following holds true: Whenever $F \in L^\sigma_{\rm loc}(\Omega_T, \mathbb{R}^{Nn})$ and*

$$u \in C^0\big((0,T); L^{m+1}_{\rm loc}(\Omega, \mathbb{R}^N)\big) \quad \textit{with} \quad \boldsymbol{u}^m \in L^2_{\rm loc}\big(0,T; W^{1,2}_{\rm loc}(\Omega, \mathbb{R}^N)\big)$$

*is a weak solution of Equation (1.1) under the assumptions (1.2) in the sense of Definition 1.1, there holds*

$$D\boldsymbol{u}^m \in L^{2+\varepsilon_1}_{\rm loc}\big(\Omega_T, \mathbb{R}^{Nn}\big),$$

*where $\varepsilon_1 := \min\{\varepsilon_o, \sigma - 2\}$. Moreover, for every $\varepsilon \in (0, \varepsilon_1]$ and every cylinder $Q_{2R}(z_o) \Subset \Omega_T$, we have the quantitative local higher integrability estimate*

$$\begin{aligned}
&\iint_{Q_R(z_o)} \big|D\boldsymbol{u}^m\big|^{2+\varepsilon} \mathrm{d}x\mathrm{d}t \\
&\qquad \leq c\left[1 + \iint_{Q_{2R}(z_o)} \left[\frac{|u|^{2m}}{R^2} + |F|^2\right] \mathrm{d}x\mathrm{d}t\right]^{\frac{\varepsilon m}{m+1}} \iint_{Q_{2R}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t \\
&\qquad\quad + c\iint_{Q_{2R}} |F|^{2+\varepsilon} \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{1.10}$$

The quantitative local estimate (1.10) can be easily converted into an estimate on the standard parabolic cylinders $C_R(z_o) := B_R(x_o) \times (t_o - R^2, t_o + R^2)$. The precise statement is as follows.

**Corollary 1.3.** *Under the assumptions of Theorem 1.2, the following estimate*

$$\begin{aligned}
&\iint_{C_R(z_o)} \big|D\boldsymbol{u}^m\big|^{2+\varepsilon} \mathrm{d}x\mathrm{d}t \\
&\qquad \leq \frac{c}{R^\varepsilon}\left[1 + \iint_{C_{2R}(z_o)} \big[|u|^{2m} + R^2|F|^2\big] \mathrm{d}x\mathrm{d}t\right]^{\frac{\varepsilon m}{m+1}} \iint_{C_{2R}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t \\
&\qquad\quad + c\iint_{C_{2R}(z_o)} |F|^{2+\varepsilon} \mathrm{d}x\mathrm{d}t
\end{aligned}$$

*holds true on any parabolic cylinder $C_{2R}(z_o) \Subset \Omega_T$ and for every $\varepsilon \in (0, \varepsilon_1]$ and with a constant $c = c(n, m, \nu, L)$.*

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## 2. Preliminaries

2.1. **Notations.** In order not to overburden the notation, we abbreviate in the following the *power of a vector* (or possibly negative number) by

$$\boldsymbol{u}^\alpha := |u|^{\alpha-1}u, \quad \text{for } u \in \mathbb{R}^N \text{ and } \alpha > 0,$$

where we interpret $\boldsymbol{u}^\alpha = 0$ in the case $u = 0$ and $\alpha \in (0, 1)$. Throughout the paper we write $z_o = (x_o, t_o) \in \mathbb{R}^n \times \mathbb{R}$ and use the space-time cylinders

$$Q^{(\theta)}_\varrho(z_o) := B_\varrho(x_o) \times \Lambda^{(\theta)}_\varrho(t_o), \tag{2.1}$$

where

$$\Lambda^{(\theta)}_\varrho(t_o) := \big(t_o - \theta^{1-m}\varrho^{\frac{m+1}{m}}, t_o + \theta^{1-m}\varrho^{\frac{m+1}{m}}\big)$$

with some scaling parameter $\theta > 0$. One of the most important notions for this paper is the notion of *sub-intrinsic cylinders*. We call a cylinder $Q_\varrho^{(\theta)}(z_o)$ sub-intrinsic, if and only if

$$\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t \le \theta^{2m}$$

holds true. If the preceding inequality actually is an equality, we call the cylinder *intrinsic*. In the case $\theta = 1$, we simply omit the parameter in our notation and write

$$Q_\varrho(z_o) := B_\varrho(x_o) \times \big(t_o - \varrho^{\frac{m+1}{m}}, t_o + \varrho^{\frac{m+1}{m}}\big)$$

instead of $Q_\varrho^{(1)}(z_o)$, and, analogously, $\Lambda_\varrho(t_o)$ instead of $\Lambda_\varrho^{(1)}(t_o)$. If $z_o$ is the origin, we write $Q_\varrho$, $B_\varrho$ and $\Lambda_\varrho$ for $Q_\varrho(0)$, $B_\varrho(0)$ and $\Lambda_\varrho(0)$. Moreover, if the center $z_o$ is clear from the context we omit it in our notation.

For a map $u \in L^1(0,T;L^1(\Omega,\mathbb{R}^N))$ and given measurable sets $A \subset \Omega$ and $E \subset \Omega \times (0,T)$ with positive Lebesgue measure the slicewise mean $(u)_A\colon (0,T) \to \mathbb{R}^N$ of $u$ on $A$ is defined by

$$(u)_A(t) := \fint_A u(t)\,\mathrm{d}x, \qquad \text{for a.e. } t \in (0,T), \tag{2.2}$$

whereas the mean value $(u)_E \in \mathbb{R}^N$ of $u$ on $E$ is defined by

$$(u)_E := \iint_E u\,\mathrm{d}x\mathrm{d}t.$$

Note that if $u \in C^0((0,T);L^2(\Omega,\mathbb{R}^N))$ the slicewise means are defined for any $t \in (0,T)$. If the set $A$ is a ball $B_\varrho(x_o)$, then we abbreviate $(u)_{x_o;\varrho}(t) := (u)_{B_\varrho(x_o)}(t)$ and if $E$ is a cylinder of the form $Q_\varrho^{(\theta)}(z_o)$, we use the shorthand notation $(u)_{z_o;\varrho}^{(\theta)} := (u)_{Q_\varrho^{(\theta)}(z_o)}$. Finally, we define the boundary term

$$\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] := \tfrac{m}{m+1}\big(|a|^{m+1} - |u|^{m+1}\big) - u \cdot \big(\boldsymbol{a}^m - \boldsymbol{u}^m\big) \tag{2.3}$$

that will appear in the energy estimate from Lemma 3.1.

2.2. **Auxiliary Material.** In order to “re-absorb” certain terms, we will use the following iteration lemma, which can be retrieved by a change of variable from [21, Lemma 6.1].

**Lemma 2.1.** *Let $0 < \vartheta < 1$, $A, C \ge 0$ and $\alpha, \beta > 0$. Then, there exists a constant $c = c(\beta, \vartheta)$ such that there holds: For any $0 < r < \varrho$ and any non-negative bounded function $\phi\colon [r,\varrho] \to \mathbb{R}_{\ge 0}$ satisfying*

$$\phi(t) \le \vartheta\phi(s) + A\big(s^\alpha - t^\alpha\big)^{-\beta} + C \qquad \textit{for all } r \le t < s \le \varrho.$$

*we have*

$$\phi(r) \le c\big[A(\varrho^\alpha - r^\alpha)^{-\beta} + C\big].$$

**Lemma 2.2.** *For any $\alpha > 1$, there exists a constant $c = c(\alpha)$ such that, for all $a, b \in \mathbb{R}^N$ the following assertions hold true:*

(i) $\frac{1}{c}\big|\boldsymbol{a}^\alpha - \boldsymbol{b}^\alpha\big| \le \big(|a|^{\alpha-1} + |b|^{\alpha-1}\big)|a-b| \le c\big|\boldsymbol{a}^\alpha - \boldsymbol{b}^\alpha\big|$,

(ii) $|a-b|^\alpha \le c\big|\boldsymbol{a}^\alpha - \boldsymbol{b}^\alpha\big|$,

(iii) $\big|\boldsymbol{a}^\alpha - \boldsymbol{b}^\alpha\big|^2 \le c\big(\boldsymbol{a}^{2\alpha-1} - \boldsymbol{b}^{2\alpha-1}\big) \cdot (a-b)$.

The proof of (i) and (ii) can be found in [19, Lemma 2.2]. Inequality (iii) can be derived by combining the proof of [7, Chapter I, Lemma 4.4] with (i). The next lemma provides useful estimates for the boundary term $\mathfrak{b}$ introduced in (2.3).

**Lemma 2.3.** *There exists a constant $c = c(m)$ such that for any $u, a \in \mathbb{R}^N$ the following assertions hold true:*

(i) $\frac{1}{c}\big|\boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{a}^{\frac{m+1}{2}}\big|^2 \le \mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] \le c\big|\boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{a}^{\frac{m+1}{2}}\big|^2$,

(ii) $\frac{1}{c}|\boldsymbol{u}^m - \boldsymbol{a}^m|^2 \le \big[|u|^{m-1} + |a|^{m-1}\big]\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] \le c|\boldsymbol{u}^m - \boldsymbol{a}^m|^2$,

(iii) $\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] \le c|\boldsymbol{u}^m - \boldsymbol{a}^m|^{\frac{m+1}{m}}$.

**Proof.** Using the auxiliary function $\phi \in C^2(\mathbb{R}^N)$, $\phi(x) = \frac{1}{m+1}|x|^{m+1}$, we can re-write the boundary term to

$$\begin{aligned}\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] &= \tfrac{1}{m+1}|u|^{m+1} - \tfrac{1}{m+1}|a|^{m+1} - \boldsymbol{a}^m \cdot (u-a) \\ &= \phi(u) - \phi(a) - \nabla\phi(a)\cdot(u-a).\end{aligned}$$

The Hessian of $\phi$ is given by the matrix

$$H_\phi(x) = |x|^{m-1}\big(\mathbb{I}_N + (m-1)\tfrac{x}{|x|} \otimes \tfrac{x}{|x|}\big),$$

whose eigenvalues are $|x|^{m-1}$ and $m|x|^{m-1}$. Therefore, the integral formula for the remainder in Taylor's expansion yields

$$\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] \ge \int_0^1 |a + t(u-a)|^{m-1}(1-t)\,dt\,|u-a|^2. \tag{2.4}$$

Now, we distinguish between the cases $|u| \ge |a|$ and $|u| < |a|$. In the first case, for any $t \in (\frac{3}{4}, 1)$ we have

$$|a + t(u-a)| \ge t|u| - (1-t)|a| \ge \tfrac{1}{2}|u| \ge \tfrac{1}{4}(|u| + |a|),$$

from which we infer

$$\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] \ge c\int_{\frac{3}{4}}^1 (1-t)\,dt\,(|u|+|a|)^{m-1}|u-a|^2 = c\,(|u|+|a|)^{m-1}|u-a|^2, \tag{2.5}$$

where $c = c(m)$. In the second case $|u| < |a|$, we restrict ourselves to values $t \in (0, \frac{1}{4})$. Interchanging the roles of $u$, $a$ and $t$, $1-t$ we end up with the same estimate for $|a + t(u-a)|$. In view of (2.4), this implies also in the remaining case for $\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m]$ the estimate (2.5). Combining this with Lemma 2.2 (i), we arrive at the first claimed estimate, since

$$\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] \ge c(m)(|u|+|a|)^{m-1}|u-a|^2 \ge c(m)\big|\boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{a}^{\frac{m+1}{2}}\big|^2.$$

For the second asserted estimate, we apply Lagrange's formula for the remainder in Taylor's expansion, which yields

$$\begin{aligned}\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] &\le \tfrac{1}{2}\sup_{t\in(0,1)} (u-a)\cdot H_\phi(a + t(u-a))(u-a) \\ &\le \tfrac{m}{2}|u-a|^2 \sup_{t\in(0,1)} |a + t(u-a)|^{m-1} \\ &\le c(m)(|u|+|a|)^{m-1}|u-a|^2. \end{aligned}\tag{2.6}$$

In view of Lemma 2.2 (i), this yields the second estimate from (i), since

$$\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] \le c(m)(|u|+|a|)^{m-1}|u-a|^2 \le c(m)\big|\boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{a}^{\frac{m+1}{2}}\big|^2.$$

The inequalities in (ii) are a consequence of (i) and Lemma 2.2 (i) applied with $\tilde{u} = \boldsymbol{u}^{\frac{m+1}{2}}$, $\tilde{a} = \boldsymbol{a}^{\frac{m+1}{2}}$ and $\alpha = \frac{2m}{m+1}$, since

$$\begin{aligned}\big[|u|^{m-1} + |a|^{m-1}\big]\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] &\ge c(m)\big[|u|^{m-1} + |a|^{m-1}\big]\big|\boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{a}^{\frac{m+1}{2}}\big|^2 \\ &= c(m)\Big[|\tilde{u}|^{\frac{2(m-1)}{m+1}} + |\tilde{a}|^{\frac{2(m-1)}{m+1}}\Big]|\tilde{u} - \tilde{a}|^2 \\ &\ge c(m)|\boldsymbol{u}^m - \boldsymbol{a}^m|^2.\end{aligned}$$

The reasoning for the second bound in (ii) is similar. The inequality (iii) also follows from (2.6) and Lemma 2.2 (i), (ii), since

$$\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] \le c(m)|\boldsymbol{u}^m - \boldsymbol{a}^m|\,|u-a| \le c(m)|\boldsymbol{u}^m - \boldsymbol{a}^m|^{1+\frac{1}{m}}. \qquad \square$$

The following estimate, which is known as the quasi-minimality of the mean value, can be established by Young's and Hölder's inequality.

**Lemma 2.4.** *Let $\alpha \geq 1$. Then, for any bounded domain $A \subset \mathbb{R}^k$, $k \in \mathbb{N}$, any $u \in L^\alpha(A, \mathbb{R}^N)$, and any $a \in \mathbb{R}^N$ there holds*

$$\fint_A \big|u - (u)_A\big|^\alpha \mathrm{d}x \leq 2^\alpha \fint_A |u - a|^\alpha \mathrm{d}x.$$

The following statement shows that mean values over subsets are still quasi-minimizing. This is well known for $\alpha = 1$. Here, we state the version for powers. As expected, the quasi-minimality constant depends on the ratio of the measures of the set and the subset.

**Lemma 2.5.** *Let $\alpha \geq 1$. Then there exists a universal constant $c = c(\alpha)$ such that whenever $A \subset B \subset \mathbb{R}^k$, $k \in \mathbb{N}$, are two bounded domains and $u \in L^{2\alpha}(B, \mathbb{R}^N)$, there holds*

$$\fint_B \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^2 \mathrm{d}x \leq \frac{c\,|B|}{|A|} \fint_B \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{B}}\big|^2 \mathrm{d}x.$$

**Proof.** We start by estimating the difference $|(\boldsymbol{u})^\alpha_{\boldsymbol{B}} - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}|$. Using Lemma 2.2 (i), (ii), we obtain for a constant $c = c(\alpha)$ that

$$\begin{aligned}
\big|(\boldsymbol{u})^\alpha_{\boldsymbol{B}} - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^2 &\leq c\big[|(u)_B|^{2\alpha-2} + |(u)_A|^{2\alpha-2}\big]\big|(u)_B - (u)_A\big|^2 \\
&\leq c\big[|(u)_B|^{2\alpha-2} + |(u)_A - (u)_B|^{2\alpha-2}\big]\big|(u)_B - (u)_A\big|^2 \\
&\leq c\,|(u)_B|^{2\alpha-2} \fint_A \big|u - (u)_B\big|^2 \mathrm{d}x + c\,\big|(u)_A - (u)_B\big|^{2\alpha} \\
&\leq c \fint_A \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{B}}\big|^2 \mathrm{d}x + c \fint_A \big|u - (u)_B\big|^{2\alpha} \mathrm{d}x \\
&\leq \frac{c\,|B|}{|A|} \fint_B \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{B}}\big|^2 \mathrm{d}x.
\end{aligned}$$

From this estimate we conclude

$$\begin{aligned}
\fint_B \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^2 \mathrm{d}x &\leq 2 \fint_B \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{B}}\big|^2 \mathrm{d}x + 2\big|(\boldsymbol{u})^\alpha_{\boldsymbol{B}} - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^2 \\
&\leq \frac{c\,|B|}{|A|} \fint_B \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{B}}\big|^2 \mathrm{d}x,
\end{aligned}$$

which proves the claim. □

The following lemma is from [12, Lemma 6.2]. For convenience of the reader, we nevertheless include the proof.

**Lemma 2.6.** *Let $\alpha > 1$. Then there exists a universal constant $c = c(\alpha)$ such that for any bounded domain $A \subset \mathbb{R}^n$, any $u \in L^{2\alpha}(A, \mathbb{R}^N)$, and any $a \in \mathbb{R}^N$ there holds*

$$\fint_A \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^2 \mathrm{d}x \leq c \fint_A \big|\boldsymbol{u}^\alpha - \boldsymbol{a}^\alpha\big|^2 \mathrm{d}x.$$

**Proof.** Using Lemma 2.2 (iii), we obtain for a constant $c = c(\alpha)$ that

$$\begin{aligned}
\fint_A \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^2 \mathrm{d}x &\leq c \fint_A \big(u - (u)_A\big) \cdot \big(\boldsymbol{u}^{2\alpha-1} - (\boldsymbol{u})^{2\alpha-1}_{\boldsymbol{A}}\big) \mathrm{d}x \\
&= c \fint_A \big(u - (u)_A\big) \cdot \big(\boldsymbol{u}^{2\alpha-1} - \boldsymbol{a}^{2\alpha-1}\big) \mathrm{d}x \\
&\leq c \fint_A \big|u - (u)_A\big|\big|\boldsymbol{u}^{2\alpha-1} - \boldsymbol{a}^{2\alpha-1}\big| \mathrm{d}x. \qquad (2.7)
\end{aligned}$$

In order to estimate the integrand from above we distinguish between two cases. In the case $|u| \le \frac{1}{2}|a|$, we have

$$|a|^\alpha = \big|\boldsymbol{a}^\alpha - \boldsymbol{u}^\alpha + \boldsymbol{u}^\alpha\big| \le \big|\boldsymbol{a}^\alpha - \boldsymbol{u}^\alpha\big| + 2^{-\alpha}|a|^\alpha$$

and hence $|a|^\alpha \le \frac{2^\alpha}{2^\alpha - 1}|\boldsymbol{u}^\alpha - \boldsymbol{a}^\alpha|$. In turn, this allows us to estimate

$$\big|\boldsymbol{u}^{2\alpha-1} - \boldsymbol{a}^{2\alpha-1}\big| \le 2|a|^{2\alpha-1} \le c(\alpha)\big|\boldsymbol{u}^\alpha - \boldsymbol{a}^\alpha\big|^{\frac{2\alpha-1}{\alpha}},$$

which by Lemma 2.2 (ii) implies

$$\big|u - (u)_A\big|\big|\boldsymbol{u}^{2\alpha-1} - \boldsymbol{a}^{2\alpha-1}\big| \le c(\alpha)\big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^{\frac{1}{\alpha}}\big|\boldsymbol{u}^\alpha - \boldsymbol{a}^\alpha\big|^{\frac{2\alpha-1}{\alpha}}. \tag{2.8}$$

In the remaining case $|a| < 2|u|$, Lemma 2.2 (i) shows

$$\begin{aligned}\big|u - (u)_A\big|\big|\boldsymbol{u}^{2\alpha-1} - \boldsymbol{a}^{2\alpha-1}\big| &\le c(\alpha)\,\big|u - (u)_A\big|\big(|u|^{2\alpha-2} + |a|^{2\alpha-2}\big)|u - a| \\ &\le c(\alpha)\,|u|^{2\alpha-2}\big|u - (u)_A\big||u - a| \\ &= c(\alpha)\,|u|^{\alpha-1}\big|u - (u)_A\big||u|^{\alpha-1}|u - a|.\end{aligned}$$

An application of Lemma 2.2 (i) therefore yields

$$\big|u - (u)_A\big|\big|\boldsymbol{u}^{2\alpha-1} - \boldsymbol{a}^{2\alpha-1}\big| \le c\,\big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|\big|\boldsymbol{u}^\alpha - \boldsymbol{a}^\alpha\big|. \tag{2.9}$$

Combining (2.8) and (2.9), we infer that in any case the estimate

$$\begin{aligned}&\big|u - (u)_A\big|\big|\boldsymbol{u}^{2\alpha-1} - \boldsymbol{a}^{2\alpha-1}\big| \\ &\qquad \le c\,\big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^{\frac{1}{\alpha}}\big|\boldsymbol{u}^\alpha - \boldsymbol{a}^\alpha\big|^{\frac{2\alpha-1}{\alpha}} + c\,\big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|\big|\boldsymbol{u}^\alpha - \boldsymbol{a}^\alpha\big|\end{aligned}$$

holds true for a constant $c = c(\alpha)$. We insert this into (2.7) and apply Young's inequality twice. This leads to

$$⨍_A \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^2 \mathrm{d}x \le \tfrac{1}{2} ⨍_A \big|\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}\big|^2 \mathrm{d}x + c ⨍_A \big|\boldsymbol{u}^\alpha - \boldsymbol{a}^\alpha\big|^2 \mathrm{d}x.$$

Here we re-absorb the term $\frac{1}{2}⨍_A |\boldsymbol{u}^\alpha - (\boldsymbol{u})^\alpha_{\boldsymbol{A}}|^2 \mathrm{d}x$ into the left-hand side and obtain the asserted inequality. □

Finally, we ensure that the mean value is also a quasi-minimizer of $a \mapsto ⨍_A \mathfrak{b}[u, a]\mathrm{d}x$.

**Lemma 2.7.** *There exists a universal constant $c = c(m)$ such that for any bounded domain $A \subset \mathbb{R}^n$, any non-negative $u \in L^{m+1}(A, \mathbb{R}^N)$, and any $a \in \mathbb{R}^N$ there holds*

$$⨍_A \mathfrak{b}\big[u, (u)_A\big]\mathrm{d}x \le c ⨍_A \mathfrak{b}[u, a]\mathrm{d}x.$$

**Proof.** Due to Lemmas 2.3 (i) and 2.6 we obtain

$$\begin{aligned}⨍_A \mathfrak{b}\big[u, (u)_A\big]\mathrm{d}x &\le c ⨍_A \big|\boldsymbol{u}^{\frac{m+1}{2}} - (\boldsymbol{u})_{\boldsymbol{A}}^{\frac{m+1}{2}}\big|^2 \mathrm{d}x \\ &\le c ⨍_A \big|\boldsymbol{u}^{\frac{m+1}{2}} - \boldsymbol{a}^{\frac{m+1}{2}}\big|^2 \mathrm{d}x \le c ⨍_A \mathfrak{b}[u, a]\mathrm{d}x.\end{aligned}$$

This proves the asserted inequality. □

## 3. Energy bounds

In this section we derive an energy inequality and a gluing lemma which follow from the weak formulation (1.9) of the differential equation by testing with suitable testing functions. Later on, they will be used in order to prove Sobolev-Poincaré and reverse Hölder type inequalities.

**Lemma 3.1.** *Let $m \ge 1$ and $u$ be a weak solution to (1.1) in $\Omega_T$ in the sense of Definition 1.1, where the vector-field $\mathbf{A}$ fulfills the growth and ellipticity assumptions (1.2). Then, there exists a constant $c = c(m,\nu,L)$ such that on any cylinder $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ with $0 < \varrho \le 1$ and $\theta > 0$, and for any $r \in [\varrho/2, \varrho)$ and any $a \in \mathbb{R}^N$ the following energy estimate*

$$\sup_{t\in\Lambda_r^{(\theta)}(t_o)} \fint_{B_r(x_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),\boldsymbol{a}^m\big]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x + \iint_{Q_r^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t$$
$$\le c \iint_{Q_\varrho^{(\theta)}(z_o)} \bigg[\frac{\big|\boldsymbol{u}^m - \boldsymbol{a}^m\big|^2}{(\varrho-r)^2} + \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m,\boldsymbol{a}^m\big]}{\varrho^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}}\bigg]\mathrm{d}x\mathrm{d}t + c\iint_{Q_\varrho^{(\theta)}(z_o)} |F|^2\mathrm{d}x\mathrm{d}t,$$

*holds true, where $\mathfrak{b}$ has been defined in (2.3).*

*Proof.* For $v \in L^1(\Omega_T,\mathbb{R}^N)$, we define the following mollification in time

$$[\![v]\!]_h(x,t) := \tfrac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}} v(x,s)\,ds.$$

From the weak form (1.9) of the differential equation we deduce the mollified version (without loss of generality we may assume that $u \in C^0([0,T); L^2_{\rm loc}(\Omega,\mathbb{R}^N))$)

$$\begin{aligned}\iint_{\Omega_T} \Big[\partial_t[\![u]\!]_h\cdot\varphi + [\![\mathbf{A}(x,t,u,D\boldsymbol{u}^m)]\!]_h\cdot D\varphi\Big]\mathrm{d}x\mathrm{d}t \\ = -\iint_{\Omega_T}[\![F]\!]_h\cdot D\varphi\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{h}\int_\Omega u(0)\cdot\int_0^T \mathrm{e}^{-\frac{s}{h}}\varphi\mathrm{d}s\,\mathrm{d}x,\end{aligned}\tag{3.1}$$

for any $\varphi \in L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^N))$. Let $\eta \in C_0^1\big(B_\varrho(x_o),[0,1]\big)$ be the standard cut off function with $\eta \equiv 1$ in $B_r(x_o)$ and $|D\eta| \le \frac{2}{\varrho-r}$ and $\zeta \in W^{1,\infty}\big(\Lambda_\varrho^{(\theta)}(t_o),[0,1]\big)$ defined by

$$\zeta(t) := \begin{cases} 1, & \text{for } t \ge t_o - \theta^{1-m}r^{\frac{m+1}{m}},\\ \frac{(t-t_o)\theta^{m-1}+\varrho^{\frac{m+1}{m}}}{\varrho^{\frac{m+1}{m}}-r^{\frac{m+1}{m}}}, & \text{for } t \in \big(t_o-\theta^{1-m}\varrho^{\frac{m+1}{m}}, t_o-\theta^{1-m}r^{\frac{m+1}{m}}\big).\end{cases}$$

Furthermore, for given $\varepsilon > 0$ and $t_1 \in \Lambda_r^{(\theta)}(t_o)$ we define the cut-off function $\psi_\varepsilon \in W^{1,\infty}\big(\Lambda_\varrho^{(\theta)}(t_o),[0,1]\big)$ by

$$\psi_\varepsilon(t) := \begin{cases} 1, & \text{for } t \in \big(t_o-\theta^{1-m}\varrho^{\frac{m+1}{m}}, t_1\big],\\ 1-\frac{1}{\varepsilon}(t-t_1), & \text{for } t\in(t_1,t_1+\varepsilon),\\ 0, & \text{for } t\in[t_1+\varepsilon,t_o).\end{cases}$$

We choose

$$\varphi(x,t) = \eta^2(x)\zeta(t)\psi_\varepsilon(t)\big(\boldsymbol{u}^m(x,t)-\boldsymbol{a}^m\big)$$

as testing function in the mollified version (3.1) of the differential equation. For the integral containing the time derivative we compute

$$\begin{aligned}\iint_{Q_\varrho^{(\theta)}(z_o)} \partial_t[\![u]\!]_h\cdot\varphi\,\mathrm{d}x\mathrm{d}t &= \iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2\zeta\psi_\varepsilon\partial_t[\![u]\!]_h\cdot\big([\![\boldsymbol{u}]\!]_{\boldsymbol{h}}^m-\boldsymbol{a}^m\big)\mathrm{d}x\mathrm{d}t\\ &\quad+\iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2\zeta\psi_\varepsilon\partial_t[\![u]\!]_h\cdot\big(\boldsymbol{u}^m-[\![\boldsymbol{u}]\!]_{\boldsymbol{h}}^m\big)\mathrm{d}x\mathrm{d}t\\ &\ge \iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2\zeta\psi_\varepsilon\partial_t\Big(\tfrac{1}{m+1}\big|[\![u]\!]_h\big|^{m+1}-\boldsymbol{a}^m\cdot[\![u]\!]_h\Big)\mathrm{d}x\mathrm{d}t\\ &= \iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2\zeta\psi_\varepsilon\partial_t\big(\mathfrak{b}\big[[\![\boldsymbol{u}]\!]_{\boldsymbol{h}}^m,\boldsymbol{a}^m\big]\big)\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$= - \iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2 \big(\zeta\psi_\varepsilon' + \psi_\varepsilon\zeta'\big) \mathfrak{b}\big[[\![\boldsymbol{u}]\!]_{\boldsymbol{h}}^m, \boldsymbol{a}^m\big] \mathrm{d}x\mathrm{d}t,$$

where we also used the identity $\partial_t[\![u]\!]_h = -\frac{1}{h}([\![u]\!]_h - u)$, cf. [24, Chapter 2]. Since $[\![u]\!]_h \to u$ in $L^{m+1}_{\mathrm{loc}}(\Omega_T)$ we may pass to the limit $h \downarrow 0$ in the integral on the right-hand side and therefore find that

$$\begin{aligned}\liminf_{h\downarrow 0} &\iint_{Q_\varrho^{(\theta)}(z_o)} \partial_t[\![u]\!]_h \cdot \varphi \, \mathrm{d}x\mathrm{d}t \\ &\geq - \iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2 \big(\zeta\psi_\varepsilon' + \psi_\varepsilon\zeta'\big) \mathfrak{b}\big[\boldsymbol{u}^m, \boldsymbol{a}^m\big] \mathrm{d}x\mathrm{d}t =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon.\end{aligned}$$

At this point, we pass to the limit $\varepsilon \downarrow 0$ and obtain for the first term

$$\lim_{\varepsilon\downarrow 0} \mathrm{I}_\varepsilon = \int_{B_\varrho(x_o)} \eta^2 \mathfrak{b}\big[\boldsymbol{u}^m(\cdot, t_1), \boldsymbol{a}^m\big] \mathrm{d}x,$$

for any $t_1 \in \Lambda_r^{(\theta)}(t_o)$, whereas the term $\mathrm{II}_\varepsilon$ can be estimated in the following way (observe that the boundary term is non-negative)

$$|\mathrm{II}_\varepsilon| \leq \iint_{Q_\varrho^{(\theta)}(z_o)} \zeta' \mathfrak{b}\big[\boldsymbol{u}^m, \boldsymbol{a}^m\big] \mathrm{d}x\mathrm{d}t \leq \iint_{Q_\varrho^{(\theta)}(z_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, \boldsymbol{a}^m\big]}{\varrho^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}} \mathrm{d}x\mathrm{d}t.$$

Next, we consider the diffusion term in (3.1). After passing to the limit $h \downarrow 0$, we use the ellipticity and growth assumption (1.2), and later on Young's inequality. In this way, we obtain

$$\begin{aligned}\iint_{Q_\varrho^{(\theta)}(z_o)} &\mathbf{A}(x,t,u,D\boldsymbol{u}^m) \cdot D\varphi \, \mathrm{d}x\mathrm{d}t \\ &= \iint_{Q_\varrho^{(\theta)}(z_o)} \mathbf{A}(x,t,u,D\boldsymbol{u}^m) \cdot \big[\eta^2\zeta\psi_\varepsilon D\boldsymbol{u}^m + 2\eta\zeta\psi_\varepsilon(\boldsymbol{u}^m - \boldsymbol{a}^m) \otimes D\eta\big] \mathrm{d}x\mathrm{d}t \\ &\geq \nu \iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2\zeta\psi_\varepsilon \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t - 2L \iint_{Q_\varrho^{(\theta)}(z_o)} \eta\zeta\psi_\varepsilon |D\eta| \big|\boldsymbol{u}^m - \boldsymbol{a}^m\big| \big|D\boldsymbol{u}^m\big| \mathrm{d}x\mathrm{d}t \\ &\geq \tfrac{\nu}{2} \iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2\zeta\psi_\varepsilon \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t - c \iint_{Q_\varrho^{(\theta)}(z_o)} \zeta\psi_\varepsilon |D\eta|^2 \big|\boldsymbol{u}^m - \boldsymbol{a}^m\big|^2 \mathrm{d}x\mathrm{d}t \\ &\geq \tfrac{\nu}{2} \iint_{Q_\varrho^{(\theta)}(z_o)} \eta^2\zeta\psi_\varepsilon \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t - c \iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - \boldsymbol{a}^m\big|^2}{(\varrho - r)^2} \mathrm{d}x\mathrm{d}t,\end{aligned}$$

for a constant $c = c(m, \nu, L)$. Finally, we consider the right-hand side integrals in (3.1). The second integral disappears in the limit $h \downarrow 0$, since $\varphi(0) = 0$. In the integral containing the inhomogeneity $F$ we pass to the limit $h \downarrow 0$ and subsequently apply Hölder's inequality. In this way, we obtain

$$\begin{aligned}\iint_{Q_\varrho^{(\theta)}(z_o)} &F \cdot D\varphi \, \mathrm{d}x\mathrm{d}t \\ &= \iint_{Q_\varrho^{(\theta)}(z_o)} \Big[\eta^2\zeta\psi_\varepsilon F \cdot D\boldsymbol{u}^m + 2\eta\zeta\psi_\varepsilon F \cdot (\boldsymbol{u}^m - \boldsymbol{a}^m) \otimes D\eta\Big] \mathrm{d}x\mathrm{d}t \\ &\leq \tfrac{\nu}{4} \iint_{Q_\varrho^{(\theta)}(z_o)} \bigg[\eta^2\zeta\psi_\varepsilon \big|D\boldsymbol{u}^m\big|^2 + \frac{\big|\boldsymbol{u}^m - \boldsymbol{a}^m\big|^2}{(\varrho - r)^2}\bigg] \mathrm{d}x\mathrm{d}t + c \iint_{Q_\varrho^{(\theta)}(z_o)} |F|^2 \mathrm{d}x\mathrm{d}t.\end{aligned}$$

We combine these estimates and then pass to the limit $\varepsilon \downarrow 0$. This leads to

$$\int_{B_r(x_o)} \mathfrak{b}\big[\boldsymbol{u}^m(\cdot, t_1), \boldsymbol{a}^m\big] \mathrm{d}x + \tfrac{\nu}{4} \int_{t_o - \theta^{1-m} r^{\frac{m+1}{m}}}^{t_1} \int_{B_\varrho(x_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t$$

$$\le c \iint_{Q_\varrho^{(\theta)}(z_o)} \bigg[\frac{\big|\boldsymbol{u}^m - \boldsymbol{a}^m\big|^2}{(\varrho-r)^2} + \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m, \boldsymbol{a}^m\big]}{\varrho^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}}\bigg]\mathrm{d}x\mathrm{d}t + c\iint_{Q_\varrho^{(\theta)}(z_o)} |F|^2 \mathrm{d}x\mathrm{d}t$$

for any $t_1 \in \Lambda_r^{(\theta)}(t_o)$, with a constant $c = c(m,\nu,L)$. In the preceding inequality we take in the first term on the left-hand side the supremum over $t_1 \in \Lambda_\varrho^{(\theta)}(t_o)$, and then pass to the limit $t_1 \uparrow t_o + \theta^{1-m} r^{\frac{m+1}{m}}$. Finally, we take means on both sides. This procedure leads to the claimed inequality. □

The following lemma serves to compare the slice-wise mean values at different times. This is necessary since Poincaré's and Sobolev's inequality can only be applied slice-wise. Such a result, which connects means on different time slices, is termed *gluing Lemma*.

**Lemma 3.2.** *Let $m \ge 1$ and $u$ be a weak solution to (1.1) in $\Omega_T$ in the sense of Definition 1.1, where the vector-field $\mathbf{A}$ fulfills the growth and ellipticity assumptions (1.2). Then, for any cylinder $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ with $0 < \varrho \le 1$ and $\theta > 0$ there exists $\hat\varrho \in [\frac{\varrho}{2}, \varrho]$ such that for all $t_1, t_2 \in \Lambda_\varrho^{(\theta)}(t_o)$ there holds*

$$\big|(u)_{x_o;\hat\varrho}(t_2) - (u)_{x_o;\hat\varrho}(t_1)\big| \le \frac{c\,\varrho^{\frac{1}{m}}}{\theta^{m-1}} \fint\!\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \Big[\big|D\boldsymbol{u}^m\big| + |F|\Big]\mathrm{d}x\mathrm{d}t,$$

*for a constant $c = c(L)$.*

**Proof.** Let $t_1, t_2 \in \Lambda_\varrho^{(\theta)}(t_o)$ with $t_1 < t_2$ and assume that $r \in [\frac{\varrho}{2}, \varrho]$. For $\delta > 0$ and $0 < \varepsilon \ll 1$, we define $\xi_\varepsilon \in W_0^{1,\infty}(t_1 - \varepsilon, t_2 + \varepsilon)$ by

$$\xi_\varepsilon(t) := \begin{cases} 0, & \text{for } t_o - \theta^{1-m}\varrho^{\frac{m+1}{m}} \le t \le t_1 - \varepsilon,\\ \frac{t - t_1 + \varepsilon}{\varepsilon}, & \text{for } t_1 - \varepsilon < t < t_1,\\ 1, & \text{for } t_1 \le t \le t_2,\\ \frac{t_2 + \varepsilon - t}{\varepsilon}, & \text{for } t_2 < t < t_2 + \varepsilon,\\ 0, & \text{for } t_2 + \varepsilon \le t \le t_o, \end{cases}$$

and a radial function $\Psi_\delta \in W_0^{1,\infty}(B_{r+\delta}(x_o))$ by $\Psi_\delta(x) := \psi_\delta(|x - x_o|)$, where

$$\psi_\delta(s) := \begin{cases} 1, & \text{for } 0 \le s \le r,\\ \frac{r + \delta - s}{\delta}, & \text{for } r < s < r + \delta,\\ 0, & \text{for } r + \delta \le s \le \varrho, \end{cases}$$

for $s \in [0, \varrho]$. For fixed $i \in \{1, \dots, N\}$ we choose $\varphi_{\varepsilon,\delta} = \xi_\varepsilon \Psi_\delta e_i$ as testing function in the weak formulation (1.9), where $e_i$ denotes the $i$-th canonical basis vector in $\mathbb{R}^N$. In the limit $\varepsilon, \delta \downarrow 0$ we obtain

$$\begin{aligned}
&\int_{B_r(x_o)} \big[u(\cdot, t_2) - u(\cdot, t_1)\big] \cdot e_i \,\mathrm{d}x\\
&\qquad = \int_{t_1}^{t_2} \int_{\partial B_r(x_o)} \big[\mathbf{A}(x,t,u,D\boldsymbol{u}^m) + F\big] \cdot e_i \otimes \frac{x - x_o}{|x - x_o|}\,\mathrm{d}\mathcal{H}^{n-1}(x)\mathrm{d}t.
\end{aligned}$$

We multiply the preceding inequality by $e_i$ and sum over $i = 1, \dots, N$. This yields

$$\begin{aligned}
&\int_{B_r(x_o)} \big[u(\cdot, t_2) - u(\cdot, t_1)\big]\mathrm{d}x\\
&\qquad = \int_{t_1}^{t_2} \int_{\partial B_r(x_o)} \big[\mathbf{A}(x,t,u,D\boldsymbol{u}^m) + F\big]\frac{x - x_o}{|x - x_o|}\mathrm{d}\mathcal{H}^{n-1}(x)\mathrm{d}t.
\end{aligned}$$

Here, we use the growth condition $(1.2)_2$ and immediately get for any $t_1, t_2 \in \Lambda_\varrho^{(\theta)}(t_o)$ and any $r \in [\frac{\varrho}{2}, \varrho]$ that there holds

$$\bigg| \int_{B_r(x_o)} \big[u(\cdot, t_2) - u(\cdot, t_1)\big] \mathrm{d}x \bigg| \le \int_{t_1}^{t_2} \int_{\partial B_r(x_o)} \big[L\big|D\boldsymbol{u}^m\big| + |F|\big] \,\mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t.$$

Since

$$\begin{aligned}\int_{t_1}^{t_2} \int_{B_\varrho(x_o)} \big[L\big|D\boldsymbol{u}^m\big| + |F|\big] \mathrm{d}x\mathrm{d}t &= \int_0^\varrho \int_{t_1}^{t_2} \int_{\partial B_r(x_o)} \big[L\big|D\boldsymbol{u}^m\big| + |F|\big] \mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t\mathrm{d}r \\ &\ge \int_{\varrho/2}^\varrho \int_{t_1}^{t_2} \int_{\partial B_r(x_o)} \big[L\big|D\boldsymbol{u}^m\big| + |F|\big] \mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t\mathrm{d}r,\end{aligned}$$

there exists a radius $\hat\varrho \in [\frac{\varrho}{2}, \varrho)$ with

$$\int_{t_1}^{t_2} \int_{\partial B_{\hat\varrho}(x_o)} \big[L\big|D\boldsymbol{u}^m\big| + |F|\big] \mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t \le \tfrac{2}{\varrho} \int_{t_1}^{t_2} \int_{B_\varrho(x_o)} \big[L\big|D\boldsymbol{u}^m\big| + |F|\big] \mathrm{d}x\mathrm{d}t.$$

Therefore, we choose in the above inequality $r = \hat\varrho$ and then take means on both sides of the resulting inequality. This implies

$$\begin{aligned}\big|(u)_{x_o;\hat\varrho}(t_2) - (u)_{x_o;\hat\varrho}(t_1)\big| &\le \frac{c}{\varrho} \int_{\Lambda_\varrho^{(\theta)}(t_o)} \fint_{B_\varrho(x_o)} \Big[\big|D\boldsymbol{u}^m\big| + |F|\Big] \mathrm{d}x\mathrm{d}t \\ &= \frac{c\,\varrho^{\frac{1}{m}}}{\theta^{m-1}} \fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \Big[\big|D\boldsymbol{u}^m\big| + |F|\Big] \mathrm{d}x\mathrm{d}t\end{aligned}$$

for any $t_1, t_2 \in \Lambda_\varrho^{(\theta)}(t_o)$ and with a constant $c = c(L)$. □

## 4. Parabolic Sobolev-Poincaré type inequalities

Throughout this section we consider so-called *sub-intrinsic cylinders*. These cylinders are characterized as follows: On the scaled cylinder $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ with $0 < \varrho \le 1$ and $\theta > 0$ the following coupling between the mean of $|u|^{2m}/\varrho^2$ on $Q_\varrho^{(\theta)}(z_o)$ and $\theta$ holds true:

$$\fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t \le 2^{d+2}\theta^{2m}. \tag{4.1}$$

The following Lemma is the first step towards a Poincaré type inequality for weak solutions to the porous medium system. This is necessary because the standard Poincaré inequality in $\mathbb{R}^n \times \mathbb{R}$ cannot be applied directly, since weak solutions $u$ a priori do not possess the necessary regularity with respect to time; note that we only assume for the spatial derivative $D\boldsymbol{u}^m \in L^2_{\mathrm{loc}}\big(\Omega_T, \mathbb{R}^{Nn}\big)$, while no regularity assumption with respect to time is incorporated in the definition of weak solutions. Nevertheless, we are able to prove some sort of Poincaré inequality. This is achieved by considering the space and time direction separately. In $x$-direction we can apply the Poincaré inequality on $\mathbb{R}^n$, while in $t$-direction the needed regularity is gained from the gluing lemma.

**Lemma 4.1.** *Let $m \ge 1$ and $u$ be a weak solution to (1.1) in $\Omega_T$ in the sense of Definition 1.1, where the vector-field $\mathbf{A}$ fulfills the growth and ellipticity assumptions (1.2). Then, on any cylinder $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ satisfying the sub-intrinsic coupling (4.1) for some $0 < \varrho \le 1$ and some $\theta > 0$, the inequality*

$$\begin{aligned}&\fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\big|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t \\ &\le c \fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{z_o;\varrho}(t)\big|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t + c\bigg[\fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \Big[\big|D\boldsymbol{u}^m\big| + |F|\Big] \mathrm{d}x\mathrm{d}t\bigg]^2\end{aligned} \tag{4.2}$$

*holds true with a universal constant $c = c(n, m, L)$.*

**Proof.** In the following we shall again omit for simplification the reference point $z_o$ in our notation. Moreover, we let $\hat\varrho \in [\frac{\varrho}{2}, \varrho]$ be the radius from Lemma 3.2. By adding and subtracting the slice-wise means $(\boldsymbol{u})^m_{\hat\varrho}(t)$ as defined in (2.2), we obtain the inequality

$$\begin{aligned}\iint_{Q_\varrho^{(\theta)}} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_\varrho\big|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t &\le 3\Bigg[\iint_{Q_\varrho^{(\theta)}} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u})^m_{\hat\varrho}(t)\big|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t \\ &\quad + \frac{1}{\varrho^2}\fint_{\Lambda_\varrho^{(\theta)}}\bigg|\fint_{\Lambda_\varrho^{(\theta)}}\Big[(\boldsymbol{u})^m_{\hat\varrho}(t) - (\boldsymbol{u})^m_{\hat\varrho}(\tau)\Big]\mathrm{d}\tau\bigg|^2\mathrm{d}t \\ &\quad + \frac{1}{\varrho^2}\bigg|\fint_{\Lambda_\varrho^{(\theta)}}(\boldsymbol{u})^m_{\hat\varrho}(\tau)\mathrm{d}\tau - (\boldsymbol{u}^m)^{(\theta)}_\varrho\bigg|^2\Bigg] \\ &=: 3\big[\mathrm{I} + \mathrm{II} + \mathrm{III}\big], \end{aligned} \tag{4.3}$$

with the obvious meaning of I – III. In the following, we treat the terms of the right side in order. We start with the term I. Using the fact that $\hat\varrho \in [\frac{\varrho}{2}, \varrho]$, we can first replace the slice-wise means $(\boldsymbol{u})^m_{\hat\varrho}(t)$ by $(\boldsymbol{u})^m_{\varrho}(t)$ with the help of Lemma 2.5, and afterwards apply Lemma 2.6, to obtain

$$\mathrm{I} \le c\iint_{Q_\varrho^{(\theta)}} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u})^m_{\varrho}(t)\big|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t \le c\iint_{Q_\varrho^{(\theta)}} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{\varrho}(t)\big|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t,$$

where $c = c(m,n)$. Since III $\le$ I, it remains to treat the term II. In turn, we apply Lemma 2.2 (i) and Lemma 3.2 to infer that for any $t, \tau \in \Lambda_\varrho^{(\theta)}$ there holds

$$\begin{aligned}&\big|(\boldsymbol{u})^m_{\hat\varrho}(t) - (\boldsymbol{u})^m_{\hat\varrho}(\tau)\big| \\ &\qquad\le c\Big[|(u)_{\hat\varrho}(t)|^{m-1} + |(u)_{\hat\varrho}(\tau)|^{m-1}\Big]\big|(u)_{\hat\varrho}(t) - (u)_{\hat\varrho}(\tau)\big| \\ &\qquad\le \frac{c\,\varrho^{\frac{1}{m}}}{\theta^{m-1}}\iint_{Q_\varrho^{(\theta)}}\Big[\big|D\boldsymbol{u}^m\big| + |F|\Big]\mathrm{d}x\mathrm{d}t\Big[|(u)_{\hat\varrho}(t)|^{m-1} + |(u)_{\hat\varrho}(\tau)|^{m-1}\Big],\end{aligned}$$

where $c = c(m, L)$. Taking squares on both sides, integrating with respect to $t$ and $\tau$ over $\Lambda_\varrho^{(\theta)}$ and applying Hölder's inequality and the sub-intrinsic coupling (4.1), we infer

$$\begin{aligned}\mathrm{II} &\le \frac{c}{\varrho^{\frac{2(m-1)}{m}}\theta^{2(m-1)}}\bigg[\iint_{Q_\varrho^{(\theta)}}|u|^{2m}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{m-1}{m}}\bigg[\iint_{Q_\varrho^{(\theta)}}\Big[\big|D\boldsymbol{u}^m\big| + |F|\Big]\mathrm{d}x\mathrm{d}t\bigg]^2 \\ &\le c\bigg[\iint_{Q_\varrho^{(\theta)}}\Big[\big|D\boldsymbol{u}^m\big| + |F|\Big]\mathrm{d}x\mathrm{d}t\bigg]^2,\end{aligned}$$

for a constant $c$ depending only on $n, m$, and $L$. At this point, we use the estimates for I – III in (4.3) and obtain the claimed inequality. □

With the help of Lemma 4.1 we can now easily deduce a Poincaré type inequality. Later on, Lemma 4.1 will also be the starting point for the proof of a Sobolev-Poincaré type inequality; see Lemma 4.3.

**Lemma 4.2.** *Let $m \ge 1$ and $u$ be a weak solution to (1.1) in $\Omega_T$ in the sense of Definition 1.1, where the vector-field $\mathbf{A}$ fulfills the growth and ellipticity assumptions (1.2). Then, on any cylinder $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ satisfying the sub-intrinsic coupling (4.1) for some $0 < \varrho \le 1$ and some $\theta > 0$, the following Poincaré type inequality*

$$\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\big|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t \le c\iint_{Q_\varrho^{(\theta)}(z_o)}\Big[\big|D\boldsymbol{u}^m\big|^2 + |F|^2\Big]\mathrm{d}x\mathrm{d}t,$$

*holds true with a universal constant $c = c(n, m, L)$.*

**Proof.** In the following we shall again omit for simplification the reference point $z_o$ in our notation. We will take estimate (4.2) from Lemma 4.1 as starting point for our considerations. To the first integral on the right-hand side we apply Poincaré's inequality slice wise for a.e. $t \in \Lambda_\varrho^{(\theta)}$. In this way, we obtain

$$\iint_{Q_\varrho^{(\theta)}} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_\varrho(t)\big|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t \le c \iint_{Q_\varrho^{(\theta)}} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t,$$

where $c = c(n,m)$. Applying Hölder's inequality to the second integral on the right-hand side of (4.2) yields the claimed Poincaré type inequality on sub-intrinsic cylinders. □

The next statement can be interpreted as some sort of Sobolev-Poincaré inequality for the $L^2$-deviation of $\boldsymbol{u}^m$ from its mean value on the sub-intrinsic cylinder $Q_\varrho^{(\theta)}(z_o)$. Later on, we shall use this inequality to estimate the right-hand side in the energy inequality from Lemma 3.1. As usual, this leads to a reduction in the integration exponent of the energy term of the right-hand side, i.e. the integral containing $D\boldsymbol{u}^m$. Similar to Lemma 4.2, we take Lemma 4.1 as starting point in the proof. Then, the idea is to extract a part of the integration exponent from the $L^2$-oscillation integral by the sup-term (occurring in the left-hand side of the energy estimate) and then to apply Sobolev's inequality to the remainder.

**Lemma 4.3.** *Let $m \ge 1$ and $u$ be a weak solution to (1.1) in $\Omega_T$ in the sense of Definition 1.1, where the vector-field $\mathbf{A}$ fulfills the growth and ellipticity assumptions (1.2). Then, on any sub-cylinder $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ as in (4.1) for some $0 < \varrho \le 1$ and some $\theta > 0$, and for any given $\varepsilon \in (0,1]$ the following Sobolev-type inequality holds*

$$\begin{aligned}
&\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\big|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t \\
&\qquad \le \varepsilon \sup_{t \in \Lambda_\varrho^{(\theta)}(t_o)} \int_{B_\varrho(x_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t), (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\big]}{\varrho^{\frac{m+1}{m}}} \mathrm{d}x \\
&\qquad\quad + \frac{c}{\varepsilon^{\frac{2}{n}}} \bigg[\iint_{Q_\varrho^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^{2q} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{q}} + c \iint_{Q_\varrho^{(\theta)}(z_o)} |F|^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

*for a universal constant $c = c(n,m,L)$ and $q := \frac{n}{d} < 1$.*

**Proof.** In the following, we shall again omit the reference point $z_o$ in our notation. As in the proof of Lemma 4.2 we take inequality (4.2) from Lemma 4.1 as starting point. Moreover, we abbreviate $(\boldsymbol{u}^m)_\varrho(t)$ by $(\boldsymbol{u}^m)_\varrho$. From the context, it is clear that $(\boldsymbol{u}^m)_\varrho$ is to be interpreted as a function of $t$. To the first integral on the right-hand side we apply the lower bound for the boundary term from Lemma 2.3 (ii) and Hölder's inequality with exponents $\frac{m(n+2)}{m-1}$, $\frac{n+2}{d}$. In this way, we obtain

$$\begin{aligned}
&\iint_{Q_\varrho^{(\theta)}} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_\varrho(t)\big|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t \\
&= \frac{c}{\varrho^2} \iint_{Q_\varrho^{(\theta)}} \big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_\varrho\big|^{\frac{4}{n+2}} \big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_\varrho\big|^{\frac{2n}{n+2}} \mathrm{d}x\mathrm{d}t \\
&\le \frac{c}{\varrho^2} \iint_{Q_\varrho^{(\theta)}} \Big[|\boldsymbol{u}^m|^{\frac{1}{m}} + |(\boldsymbol{u}^m)_\varrho|^{\frac{1}{m}}\Big]^{\frac{2(m-1)}{n+2}} \mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_\varrho\big]^{\frac{2}{n+2}} \big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_\varrho\big|^{\frac{2n}{n+2}} \mathrm{d}x\mathrm{d}t \\
&\le \frac{c}{\varrho^2} \bigg[\iint_{Q_\varrho^{(\theta)}} \Big[|\boldsymbol{u}^m|^{\frac{1}{m}} + |(\boldsymbol{u}^m)_\varrho|^{\frac{1}{m}}\Big]^{2m} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{m-1}{m(n+2)}} \\
&\qquad\qquad \cdot \bigg[\iint_{Q_\varrho^{(\theta)}} \mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_\varrho\big]^{\frac{2}{d}} \big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_\varrho\big|^{\frac{2n}{d}} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{d}{n+2}}
\end{aligned}$$

$$\leq \frac{c}{\varrho^2}\bigg[\iint_{Q_\varrho^{(\theta)}} |u|^{2m}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{m-1}{m(n+2)}}\bigg[\iint_{Q_\varrho^{(\theta)}} \mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)_\varrho\big]^{\frac{2}{d}}\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_\varrho\big|^{\frac{2n}{d}}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{d}{n+2}}.$$

Now, we use the sub-intrinsic coupling (4.1), Hölder's inequality with exponents $\frac{d}{2}$, $\frac{d}{d-2}$ and for a.e. $t\in\Lambda_\varrho^{(\theta)}$ Sobolev's inequality slicewise (note that $\frac{2n}{d}\geq 1$, since $n\geq 2$). This yields

$$\begin{aligned}
&\iint_{Q_\varrho^{(\theta)}} \frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_\varrho(t)\big|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t\\
&\quad\leq \frac{c\,\theta^{\frac{2(m-1)}{n+2}}}{\varrho^{\frac{2d}{n+2}}}\bigg[\iint_{Q_\varrho^{(\theta)}} \mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)_\varrho\big]^{\frac{2}{d}}\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_\varrho\big|^{\frac{2n}{d}}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{d}{n+2}}\\
&\quad= c\Bigg[\iint_{Q_\varrho^{(\theta)}}\bigg[\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)_\varrho\big]}{\varrho^{\frac{m+1}{m}}}\bigg]^{\frac{2}{d}}\frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_\varrho\big|^{\frac{2n}{d}}}{\varrho^{\frac{2n}{d}}}\mathrm{d}x\mathrm{d}t\Bigg]^{\frac{d}{n+2}}\\
&\quad\leq c\Bigg[\fint_{\Lambda_\varrho^{(\theta)}}\bigg[\fint_{B_\varrho}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)_\varrho\big]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x\bigg]^{\frac{2}{d}}\bigg[\fint_{B_\varrho}\frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_\varrho\big|^{\frac{2n}{d-2}}}{\varrho^{\frac{2n}{d-2}}}\mathrm{d}x\bigg]^{\frac{d-2}{d}}\mathrm{d}t\Bigg]^{\frac{d}{n+2}}\\
&\quad\leq c\sup_{t\in\Lambda_\varrho^{(\theta)}}\bigg[\fint_{B_\varrho}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_\varrho(t)\big]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x\bigg]^{\frac{2}{n+2}}\\
&\qquad\qquad\cdot\Bigg[\fint_{\Lambda_\varrho^{(\theta)}}\bigg[\fint_{B_\varrho}\frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_\varrho\big|^{\frac{2n}{d-2}}}{\varrho^{\frac{2n}{d-2}}}\mathrm{d}x\bigg]^{\frac{d-2}{d}}\mathrm{d}t\Bigg]^{\frac{d}{n+2}}\\
&\quad\leq c\sup_{t\in\Lambda_\varrho^{(\theta)}}\bigg[\fint_{B_\varrho}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_\varrho^{(\theta)}\big]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x\bigg]^{\frac{2}{n+2}}\bigg[\iint_{Q_\varrho^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{\frac{2n}{d}}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{d}{n+2}},
\end{aligned}$$

with a universal constant $c=c(n,m)$. In the last line we have used Lemma 2.7 in order to replace in the boundary term $\mathfrak{b}$ the slice wise mean $(\boldsymbol{u}^m)_\varrho(t)$ by the mean $(\boldsymbol{u}^m)_\varrho^{(\theta)}$. Inserting this inequality into (4.2) and applying Young's and Hölder's inequality, this results for any $\varepsilon\in(0,1]$ in

$$\begin{aligned}
&\iint_{Q_\varrho^{(\theta)}} \frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_\varrho^{(\theta)}\big|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t\\
&\qquad\leq c\sup_{t\in\Lambda_\varrho^{(\theta)}}\bigg[\fint_{B_\varrho}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_\varrho^{(\theta)}\big]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x\bigg]^{\frac{2}{n+2}}\bigg[\iint_{Q_\varrho^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{\frac{2n}{d}}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{d}{n+2}}\\
&\qquad\quad+c\bigg[\iint_{Q_\varrho^{(\theta)}}\Big[\big|D\boldsymbol{u}^m\big|+|F|\Big]\mathrm{d}x\mathrm{d}t\bigg]^2\\
&\qquad\leq \varepsilon\sup_{t\in\Lambda_\varrho^{(\theta)}}\fint_{B_\varrho}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_\varrho^{(\theta)}\big]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x+\frac{c}{\varepsilon^{\frac{2}{n}}}\bigg[\iint_{Q_\varrho^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{\frac{2n}{d}}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{d}{n}}\\
&\qquad\quad+c\iint_{Q_\varrho^{(\theta)}}|F|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

This completes the proof of the Sobolev-Poincaré type inequality. □

## 5. Reverse Hölder inequality

As it is well known, the core of each higher-integrability result is a so-called *reverse Hölder inequality* for the quantity in question, which in our case is the gradient $D\boldsymbol{u}^m$. These reverse Hölder inequalities result in a certain way from the previously established

Caccioppoli-type estimate and Sobolev-Poincaré-type inequalities. In principle, the right-hand side integrals of the Caccioppoli inequality are estimated by applying the Sobolev-Poincaré inequalities. However, the proof turns out to be more subtle than originally expected. The assumption that a sub-intrinsic coupling assumption must be imposed for the cylinder $Q^{(\theta)}_{2\varrho}(z_o)$ is obvious, since this was presupposed in Lemma 4.3. However, this is not sufficient because the factor $\theta^{m-1}$ in the energy estimate has to be converted into an $L^{m+1}$-oscillation integral of $u$. This is done by a *super-intrinsic coupling* on the cylinder $Q^{(\theta)}_{\varrho}(z_o)$; see the assumption $(5.1)_2$. Both assumptions together, i.e. $(5.1)_1$ and $(5.1)_2$, mean that the cylinder $Q^{(\theta)}_{2\varrho}(z_o)$ is intrinsic in some sense. On such an intrinsic cylinder the oscillations of $u$ are small compared to the mean value of $u$. This case could be called the *non-degenerate case*.

**Proposition 5.1.** *Let $m \geq 1$ and $u$ be a weak solution to (1.1) in $\Omega_T$ in the sense of Definition 1.1, where the vector-field $\mathbf{A}$ fulfills the structural assumptions (1.2). Then, on any cylinder $Q^{(\theta)}_{2\varrho}(z_o) \Subset \Omega_T$ with an intrinsic coupling of the form*

$$\iint_{Q^{(\theta)}_{2\varrho}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2}\mathrm{d}x\mathrm{d}t \leq \theta^{2m} \leq \iint_{Q^{(\theta)}_{\varrho}(z_o)} \frac{|u|^{2m}}{\varrho^2}\mathrm{d}x\mathrm{d}t \tag{5.1}$$

*for some $0 < \varrho \leq 1$ and $\theta > 0$, the following reverse Hölder type inequality holds true*

$$\iint_{Q^{(\theta)}_{\varrho}(z_o)} \big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t \leq c\bigg[\iint_{Q^{(\theta)}_{2\varrho}(z_o)} \big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{q}} + c\iint_{Q^{(\theta)}_{2\varrho}(z_o)} |F|^2\mathrm{d}x\mathrm{d}t,$$

*for some universal constant $c = c(n, m, \nu, L)$ and where $q := \frac{n}{d} < 1$.*

**Proof.** Once again, we omit the reference to the center $z_o$ in the notation. We consider radii $r, s$ with $\varrho \leq r < s \leq 2\varrho$. From the energy estimate in Lemma 3.1, we obtain

$$\begin{aligned}
&\sup_{t\in\Lambda^{(\theta)}_r} \int_{B_r} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)^{(\theta)}_r\big]}{r^{\frac{m+1}{m}}}\mathrm{d}x + \iint_{Q^{(\theta)}_r} \big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t \\
&\qquad\leq c\iint_{Q^{(\theta)}_s} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_r\big|^2}{(s-r)^2}\mathrm{d}x\mathrm{d}t + c\iint_{Q^{(\theta)}_s} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)^{(\theta)}_r\big]}{s^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}}\mathrm{d}x\mathrm{d}t \\
&\qquad\quad + c\iint_{Q^{(\theta)}_s} |F|^2\mathrm{d}x\mathrm{d}t \\
&\qquad=: \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}\tag{5.2}$$

with the obvious meaning of I–III. We abbreviate

$$\mathcal{R}_{r,s} := \frac{s^{\frac{m+1}{2m}}}{s^{\frac{m+1}{2m}} - r^{\frac{m+1}{2m}}},\tag{5.3}$$

and observe that $s^{\frac{m+1}{2m}} - r^{\frac{m+1}{2m}} \leq (s-r)^{\frac{m+1}{2m}}$. This together with Lemma 2.5 yields for the first term

$$\mathrm{I} \leq c\,\mathcal{R}^{\frac{4m}{m+1}}_{r,s}\iint_{Q^{(\theta)}_s} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_s\big|^2}{s^2}\mathrm{d}x\mathrm{d}t.\tag{5.4}$$

For the second term we use the intrinsic coupling $(5.1)_2$, Lemma 2.3 (ii), (iii), Hölder's inequality and Lemma 2.5 to infer that

$$\begin{aligned}
\mathrm{II} &\leq c\,\mathcal{R}^2_{r,s}\iint_{Q^{(\theta)}_s} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)^{(\theta)}_r\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x\mathrm{d}t \\
&\leq c\,\mathcal{R}^2_{r,s}\bigg[\iint_{Q^{(\theta)}_s} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_r\big|^2}{s^2}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{m-1}{2m}}\iint_{Q^{(\theta)}_s} \frac{\mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)^{(\theta)}_r\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+c\,\mathcal{R}_{r,s}^2\big|(\boldsymbol{u}^m)_r^{(\theta)}\big|^{\frac{m-1}{m}}\iint_{Q_s^{(\theta)}}\frac{\mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)_r^{(\theta)}\big]}{s^2}\mathrm{d}x\mathrm{d}t\\
&\le c\,\mathcal{R}_{r,s}^2\iint_{Q_s^{(\theta)}}\frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_r^{(\theta)}\big|^2}{s^2}\mathrm{d}x\mathrm{d}t\\
&\le c\,\mathcal{R}_{r,s}^2\iint_{Q_s^{(\theta)}}\frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)_s^{(\theta)}\big|^2}{s^2}\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Inserting the estimates for I and II above and applying Lemma 4.3, we find for any $\varepsilon\in(0,1]$ that

$$
\begin{aligned}
&\sup_{t\in\Lambda_r^{(\theta)}}\fint_{B_r}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_r^{(\theta)}\big]}{r^{\frac{m+1}{m}}}\mathrm{d}x+\iint_{Q_r^{(\theta)}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t\\
&\quad\le c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}}\Bigg[\varepsilon\sup_{t\in\Lambda_s^{(\theta)}}\fint_{B_s}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_s^{(\theta)}\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x\\
&\qquad\qquad+\frac{1}{\varepsilon^{\frac{2}{n}}}\bigg[\iint_{Q_s^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac1q}+\iint_{Q_s^{(\theta)}}|F|^2\mathrm{d}x\mathrm{d}t\Bigg].
\end{aligned}
$$

With the choice $\varepsilon=\frac{1}{2c\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}$, this yields

$$
\begin{aligned}
&\sup_{t\in\Lambda_r^{(\theta)}}\fint_{B_r}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_r^{(\theta)}\big]}{r^{\frac{m+1}{m}}}\mathrm{d}x+\iint_{Q_r^{(\theta)}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t\\
&\quad\le\tfrac12\sup_{t\in\Lambda_s^{(\theta)}}\fint_{B_s}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_s^{(\theta)}\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x\\
&\qquad+c\,\mathcal{R}_{r,s}^{\frac{4m(n+2)}{n(m+1)}}\bigg[\iint_{Q_{2\varrho}^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac1q}+c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}}\iint_{Q_{2\varrho}^{(\theta)}}|F|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

for a constant $c=c(n,m,\nu,L)$. To re-absorb the term $\frac12[\dots]$ from the right-hand side into the left-hand side, we apply the Iteration Lemma 2.1. This leads to the claimed reverse Hölder type inequality, i.e. to

$$
\begin{aligned}
&\sup_{t\in\Lambda_\varrho^{(\theta)}}\fint_{B_\varrho}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_\varrho^{(\theta)}\big]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x+\iint_{Q_\varrho^{(\theta)}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t\\
&\quad\le c\bigg[\iint_{Q_{2\varrho}^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac1q}+c\iint_{Q_{2\varrho}^{(\theta)}}|F|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

This finishes the proof of Proposition 5.1. □

The next lemma deals with the *degenerate case* which is characterized by the fact that $u$ is small compared to the oscillations of $u$. In terms of integral quantities this means that on the one hand $Q_{2\varrho}^{(\theta)}(z_o)$ is sub-intrinsic, and on the other hand the scaling parameter $\theta^{2m}$ is smaller than the mean of $|D\boldsymbol{u}^m|^2$ on $Q_\varrho^{(\theta)}(z_o)$. As in the non-degenerate case, we need the assumption $(5.5)_1$, i.e. that $Q_{2\varrho}^{(\theta)}(z_o)$ is sub-intrinsic, as a prerequisite for the application of Lemma 4.3, which serves to deal with some of the right-hand side integrals of the Caccioppoli type estimate. However, during this procedure, a term of the order of magnitude $\delta\theta^{2m}$ appears, and it is precisely there where we need assumption $(5.5)_2$, which converts this term into the oscillation term that can be re-absorbed into the left-hand side of Caccioppoli's inequality.

**Proposition 5.2.** *Let $m \ge 1$ and $u$ be a weak solution to (1.1) in $\Omega_T$ in the sense of Definition 1.1, where the vector-field $\mathbf{A}$ fulfills the structure assumptions (1.2). Then, on any cylinder $Q_{2\varrho}^{(\theta)}(z_o) \Subset \Omega_T$ satisfying a coupling of the form*

$$\tag{5.5} \iint_{Q_{2\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2}\mathrm{d}x\mathrm{d}t \le \theta^{2m} \le K \iint_{Q_{\varrho}^{(\theta)}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2 + |F|^2\Big]\mathrm{d}x\mathrm{d}t$$

*for some scaling parameter $\theta > 0$ and some constant $K \ge 1$, the following reverse Hölder type inequality holds true*

$$\iint_{Q_{\varrho}^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t \le c\bigg[\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac1q} + c\iint_{Q_{2\varrho}^{(\theta)}(z_o)} |F|^2\mathrm{d}x\mathrm{d}t,$$

*with a constant $c = c(n,m,\nu,L)K^{\frac{(n+2)(m-1)}{n(m+1)}}$ and $q := \frac{n}{d} < 1$.*

**Proof.** We omit in our notation the reference to the center $z_o$. Furthermore, we consider radii $r, s$ with $\varrho \le r < s \le 2\varrho$. As in the proof of Proposition 5.1 we start from inequality (5.2) which follows from the energy estimate in Lemma 3.1 and we recall the abbreviation (5.3). The estimate (5.4) for I is the same as in the proof of Proposition 5.1. This is clear, since we did not use hypothesis $(5.1)_2$ for their proof. Therefore, it remains to consider the term II. Applying Young's inequality, Lemma 2.3 (iii), and Lemma 2.5, we infer for any $\delta \in (0,1]$ that

$$\begin{aligned}
\mathrm{II} &\le \mathcal{R}_{r,s}^2 \iint_{Q_s^{(\theta)}} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_r^{(\theta)}\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x\mathrm{d}t \\
&\le \delta\theta^{2m} + \frac{\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}{\delta^{\frac{m-1}{m+1}}} \iint_{Q_s^{(\theta)}} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_r^{(\theta)}\big]^{\frac{2m}{m+1}}}{s^2}\mathrm{d}x\mathrm{d}t \\
&\le \delta\theta^{2m} + \frac{c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}{\delta^{\frac{m-1}{m+1}}} \iint_{Q_s^{(\theta)}} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_r^{(\theta)}\big|^2}{s^2}\mathrm{d}x\mathrm{d}t \\
&\le \delta\theta^{2m} + \frac{c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}{\delta^{\frac{m-1}{m+1}}} \iint_{Q_s^{(\theta)}} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_s^{(\theta)}\big|^2}{s^2}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

From (5.4), the preceding estimate and Lemma 4.3 we obtain for $\delta, \varepsilon \in (0,1]$ that

$$\begin{aligned}
\mathrm{I} + \mathrm{II} \le \delta\theta^{2m} + \frac{c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}{\delta^{\frac{m-1}{m+1}}}\Bigg[&\varepsilon \sup_{t\in\Lambda_s^{(\theta)}} \int_{B_s} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t), (\boldsymbol{u}^m)_s^{(\theta)}\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x \\
&+ \frac{1}{\varepsilon^{\frac{2}{n}}}\bigg[\iint_{Q_s^{(\theta)}} \big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac1q} + \iint_{Q_s^{(\theta)}} |F|^2\mathrm{d}x\mathrm{d}t\Bigg].
\end{aligned}$$

Moreover, from the coupling $(5.5)_2$ we infer that

$$\theta^{2m} \le 2^d K \iint_{Q_r^{(\theta)}} \big[|D\boldsymbol{u}^m|^2 + |F|^2\big]\mathrm{d}x\mathrm{d}t.$$

We insert the estimates for I and II into (5.2) and choose $\delta = 2^{-(d+1)}K^{-1}$. This allows us to re-absorb the integral of $|D\boldsymbol{u}^m|^2$ into the left-hand side. Proceeding in this way, we obtain

$$\begin{aligned}
\sup_{t\in\Lambda_r^{(\theta)}} \int_{B_r} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t), (\boldsymbol{u}^m)_r^{(\theta)}\big]}{r^{\frac{m+1}{m}}}\mathrm{d}x &+ \iint_{Q_r^{(\theta)}} \big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t \\
&\le c\,\varepsilon K^{\frac{m-1}{m+1}}\mathcal{R}_{r,s}^{\frac{4m}{m+1}} \sup_{t\in\Lambda_s^{(\theta)}} \int_{B_s} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t), (\boldsymbol{u}^m)_s^{(\theta)}\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x
\end{aligned}$$

$$+ cK^{\frac{m-1}{m+1}}\mathcal{R}_{r,s}^{\frac{4m}{m+1}}\Bigg[\frac{1}{\varepsilon^{\frac{2}{n}}}\bigg[\iint_{Q_s^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{q}} + \iint_{Q_s^{(\theta)}}\big[|F|^2\mathrm{d}x\mathrm{d}t\Bigg].$$

At this stage the choice $\varepsilon = \frac{1}{2cK^{\frac{m-1}{m+1}}\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}$ yields

$$\begin{aligned}
&\sup_{t\in\Lambda_r^{(\theta)}}\fint_{B_r}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_r^{(\theta)}\big]}{r^{\frac{m+1}{m}}}\mathrm{d}x + \iint_{Q_r^{(\theta)}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t\\
&\quad\le \tfrac12\sup_{t\in\Lambda_s^{(\theta)}}\fint_{B_s}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_s^{(\theta)}\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x\\
&\qquad + cK^{\frac{(m-1)(n+2)}{(m+1)n}}\mathcal{R}_{r,s}^{\frac{4m(n+2)}{(m+1)n}}\Bigg[\bigg[\iint_{Q_{2\varrho}^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{q}} + \iint_{Q_{2\varrho}^{(\theta)}}|F|^2\mathrm{d}x\mathrm{d}t\Bigg].
\end{aligned}$$

Now, we apply the Iteration Lemma 2.1 to re-absorb the sup-term from the right-hand side into the left. This leads us to

$$\begin{aligned}
&\sup_{t\in\Lambda_\varrho^{(\theta)}}\fint_{B_\varrho}\theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_\varrho^{(\theta)}\big]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x + \iint_{Q_\varrho^{(\theta)}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t\\
&\quad\le c\bigg[\iint_{Q_{2\varrho}^{(\theta)}}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{q}} + c\iint_{Q_{2\varrho}^{(\theta)}}|F|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where the constant $c$ is of the form $c(n,m,\nu,L)K^{\frac{(m-1)(n+2)}{(m+1)n}}$. This finishes the proof of the Proposition. $\square$

## 6. Proof of the higher integrability

As we have seen in the last chapter, one can establish reverse Hölder inequalities in both the degenerate and the non-degenerate regime. It should be recalled, however, that the cylinders on which these reverse Hölder inequalities are valid, are essentially scaled by the solution $u$. More precisely, the relationship between $\iint_{Q_\varrho^{(\theta)}(z_o)}\frac{|u|^{2m}}{\varrho^2}\mathrm{d}x\mathrm{d}t$, the scaling parameter $\theta$ and $\iint_{Q_\varrho^{(\theta)}(z_o)}|D\boldsymbol{u}^m|^2\mathrm{d}x\mathrm{d}t$ plays the decisive role. Therefore, the main objective in the proof of the higher integrability theorem is to find parabolic cylinders covering the super-level set of the spatial gradient of $\boldsymbol{u}^m$ in the sense of a Vitali-type covering, such that on each cylinder either a coupling in the form of (5.1) or in the form of (5.5) holds true. These cylinders will be constructed by some sort of *stopping time argument*, combined with a *rising sun type construction*. This very nice idea, which has already been explained in the introduction, goes back to [16]. Once the covering has been constructed by means of such cylinders, the application of the reverse Hölder inequalities leads to a quantitative estimate of $|D\boldsymbol{u}^m|^2$ on the super-level sets in terms of $|D\boldsymbol{u}^m|^{2q}$, for $q=\frac{n}{d}<1$. The decay in terms of the super-level sets can then be converted into the higher integrability of $D\boldsymbol{u}^m$.

Before we start the construction of the system of non-uniform cylinders reflecting the character of the porous medium system as explained above, we fix the setup. We consider a fixed cylinder $Q_{8R}(y_o,\tau_o)\equiv B_{8R}(y_o)\times\big(\tau_o-(8R)^{\frac{m+1}{m}},\tau_o+(8R)^{\frac{m+1}{m}}\big)\Subset\Omega_T$ with $R\in(0,1]$. In the following, we abbreviate $Q_\varrho:=Q_\varrho(y_o,\tau_o)$ for $\varrho\in(0,8R]$ and define

$$\lambda_o := 1+\Bigg[\iint_{Q_{4R}}\bigg[\frac{|u|^{2m}}{(4R)^2}+\big|D\boldsymbol{u}^m\big|^2+|F|^2\bigg]\mathrm{d}x\mathrm{d}t\Bigg]^{\frac{1}{m+1}}.$$

At this point, we recall the notation for space-time cylinders $Q_\varrho^{(\theta)}(z_o)$ from (2.1), which will be used in the following construction. Moreover, we observe that $Q_\varrho^{(\theta)}(z_o)\subset Q_{4R}$ whenever $z_o\in Q_{2R}$, $\varrho\in(0,R]$ and $\theta\ge1$.

6.1. **Construction of a non-uniform system of cylinders.** The following construction of a non-uniform system of cylinders is similar to the one in [16, 27]. Let $z_o \in Q_{2R}$. For a radius $\varrho \in (0, R]$ we define

$$\widetilde{\theta}_\varrho \equiv \widetilde{\theta}_{z_o;\varrho} := \inf \left\{ \theta \in [\lambda_o, \infty) : \frac{1}{|Q_\varrho|} \iint_{Q_\varrho^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t \le \theta^{m+1} \right\}.$$

Note that $\widetilde{\theta}_\varrho$ is well defined, since the set of those $\theta \ge \lambda_o$ for which the integral condition is satisfied, is non-empty. In fact, in the limit $\theta \to \infty$ the integral on the left-hand side converges to zero, while the right-hand side blows up with speed $\theta^{m+1}$. Note also that the condition in the infimum above can be rewritten as

$$\iint\!\!\!\!\!\!-_{Q_\varrho^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t \le \theta^{2m}.$$

Therefore, we either have that

$$\widetilde{\theta}_\varrho = \lambda_o \quad \text{and} \quad \iint\!\!\!\!\!\!-_{Q_\varrho^{(\widetilde{\theta}_\varrho)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t \le \widetilde{\theta}_\varrho^{2m} = \lambda_o^{2m},$$

or that

$$\widetilde{\theta}_\varrho > \lambda_o \quad \text{and} \quad \iint\!\!\!\!\!\!-_{Q_\varrho^{(\widetilde{\theta}_\varrho)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t = \widetilde{\theta}_\varrho^{2m} \tag{6.1}$$

holds true. In any case we have $\widetilde{\theta}_R \ge \lambda_o \ge 1$. On the other hand, if $\lambda_o < \widetilde{\theta}_R$ then (again by definition and the fact that $Q_R^{(\widetilde{\theta}_R)}(z_o) \subset Q_{4R}$), we have

$$\widetilde{\theta}_R^{m+1} = \frac{1}{|Q_R|} \iint_{Q_R^{(\widetilde{\theta}_R)}(z_o)} \frac{|u|^{2m}}{R^2} \mathrm{d}x\mathrm{d}t \le \frac{4^2}{|Q_R|} \iint_{Q_{4R}} \frac{|u|^{2m}}{(4R)^2} \mathrm{d}x\mathrm{d}t \le 4^{d+2} \lambda_o^{m+1}.$$

Therefore, we end up with the bound

$$\widetilde{\theta}_R \le 4^{\frac{d+2}{m+1}} \lambda_o. \tag{6.2}$$

Next, we establish that the mapping $(0, R] \ni \varrho \mapsto \widetilde{\theta}_\varrho$ is continuous. To this end, consider $\varrho \in (0, R]$ and $\varepsilon > 0$, and define $\theta_+ := \widetilde{\theta}_\varrho + \varepsilon$. Then, there exists $\delta = \delta(\varepsilon, \varrho) > 0$ such that

$$\frac{1}{|Q_r|} \iint_{Q_r^{(\theta_+)}(z_o)} \frac{|u|^{2m}}{r^2} \mathrm{d}x\mathrm{d}t < \theta_+^{m+1}$$

for all radii $r \in (0, R]$ with $|r - \varrho| < \delta$. Indeed, the preceding strict inequality holds by the very definition of $\widetilde{\theta}_\varrho$ with $r = \varrho$, since the integral on the left-hand side decreases with the replacement of $\widetilde{\theta}_\varrho$ by $\theta_+$ (note that the domain of integration shrinks), while the right-hand side strictly increases. The claim now follows, since both, i.e. the integral on the right and the left hand side, are continuous with respect to the radius. With other words, we have shown that $\widetilde{\theta}_r \le \theta_+ = \widetilde{\theta}_\varrho + \varepsilon$ for $r$ sufficiently close to $\varrho$. Therefore, it remains to prove $\widetilde{\theta}_r \ge \theta_- := \widetilde{\theta}_\varrho - \varepsilon$ for $r$ close to $\varrho$. This is clear from the construction if $\theta_- \le \lambda_o$, since $\widetilde{\theta}_r \ge \lambda_o$ for any $r$. In the other case, after diminishing $\delta = \delta(\varepsilon, \varrho) > 0$ if necessary, we get

$$\frac{1}{|Q_r|} \iint_{Q_r^{(\theta_-)}(z_o)} \frac{|u|^{2m}}{r^2} \mathrm{d}x\mathrm{d}t > \theta_-^{m+1} \tag{6.3}$$

for all $r \in (0, R]$ with $|r - \varrho| < \delta$. For $r = \varrho$, this is a direct consequence of the definition of $\widetilde{\theta}_\varrho$, since otherwise, we would have $\widetilde{\theta}_\varrho \le \theta_-$, which is a contradiction. For $r$ with $|r - \varrho| < \delta$ the claim follows from the continuity of both sides as a function of $r$. By definition of $\widetilde{\theta}_r$, the preceding inequality implies $\widetilde{\theta}_r \ge \theta_- = \widetilde{\theta}_\varrho - \varepsilon$, as claimed. This completes the proof of the continuity of $(0, R] \ni \varrho \mapsto \widetilde{\theta}_\varrho$.

Unfortunately, the mapping $(0,R]\ni\varrho\to\widetilde\theta_\varrho$ might not be monotone. For this reason we modify $\widetilde\theta_\varrho$ in a way, such that the modification – denoted by $\theta_\varrho$ – becomes monotone. The precise construction is as follows: We define

$$\theta_\varrho\equiv\theta_{z_o;\varrho}:=\max_{r\in[\varrho,R]}\widetilde\theta_{z_o;r}\,.$$

This construction can be viewed as a rising sun construction, because on those intervals $(\varrho,\bar r)$ on which $\widetilde\theta_r<\widetilde\theta_{\bar r}$, for $r\in(\varrho,\bar r)$, one replaces $\widetilde\theta_r$ by $\widetilde\theta_{\bar r}$. Then, by construction the mapping $(0,R]\ni\varrho\mapsto\theta_\varrho$ is continuous and monotonically decreasing; see Figure 1 for an illustration of the construction.

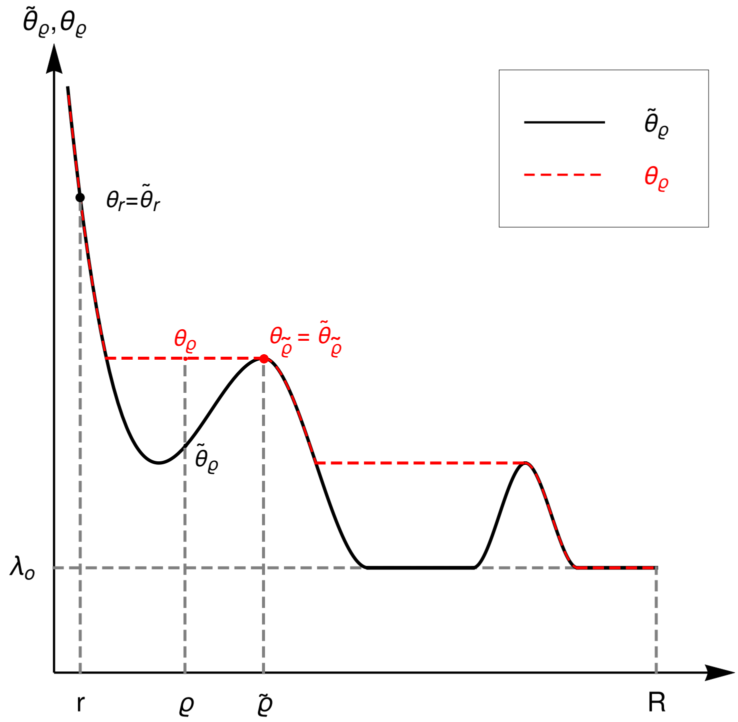


FIGURE 1. Illustration of the rising sun construction

Moreover, the cylinders $Q_s^{(\theta_\varrho)}(z_o)$ are sub-intrinsic whenever $\varrho\le s$. More specifically, we have

$$\iint_{Q_s^{(\theta_\varrho)}(z_o)}\!\!\!\!\!\!\!\!\!\!\!\!-\quad\frac{|u|^{2m}}{s^2}\mathrm{d}x\mathrm{d}t\le\theta_\varrho^{2m}\quad\text{for any }0<\varrho\le s\le R.\tag{6.4}$$

In fact, the definition of $\theta_s$ and its monotonicity imply $\widetilde\theta_s\le\theta_s\le\theta_\varrho$, so that $Q_s^{(\theta_\varrho)}(z_o)\subset Q_s^{(\widetilde\theta_s)}(z_o)$. Therefore, we have

$$\begin{aligned}\iint_{Q_s^{(\theta_\varrho)}(z_o)}\!\!\!\!\!\!\!\!\!\!\!\!-\quad\frac{|u|^{2m}}{s^2}\mathrm{d}x\mathrm{d}t&\le\Big(\frac{\theta_\varrho}{\widetilde\theta_s}\Big)^{m-1}\iint_{Q_s^{(\widetilde\theta_s)}(z_o)}\!\!\!\!\!\!\!\!\!\!\!\!-\quad\frac{|u|^{2m}}{s^2}\mathrm{d}x\mathrm{d}t\\&\le\Big(\frac{\theta_\varrho}{\widetilde\theta_s}\Big)^{m-1}\widetilde\theta_s^{2m}=\theta_\varrho^{m-1}\widetilde\theta_s^{m+1}\le\theta_\varrho^{2m}.\end{aligned}$$

We now define

$$\widetilde\varrho:=\begin{cases}R & \text{if }\theta_\varrho=\lambda_o,\\ \min\big\{s\in[\varrho,R]:\theta_s=\widetilde\theta_s\big\} & \text{if }\theta_\varrho>\lambda_o.\end{cases}\tag{6.5}$$

In particular, we have $\theta_r = \widetilde{\theta}_{\widetilde{\varrho}}$ for any $r \in [\varrho, \widetilde{\varrho}]$; see again Figure 1. Next, we claim that

$$\theta_\varrho \le \Big(\frac{s}{\varrho}\Big)^{\frac{d+2}{m+1}} \theta_s \quad \text{for any } s \in (\varrho, R]. \tag{6.6}$$

In the case that $\theta_\varrho = \lambda_o$ we know that also $\theta_s = \lambda_o$, so that (6.6) trivially holds. Therefore, it remains to consider the case $\theta_\varrho > \lambda_o$. If $s \in (\varrho, \widetilde{\varrho}]$, then $\theta_\varrho = \theta_s$, and the claim (6.6) follows again. Finally, if $s \in (\widetilde{\varrho}, R]$, then the monotonicity of $\varrho \mapsto \theta_\varrho$, (6.1) and (6.4) imply

$$\begin{aligned}
\theta_\varrho = \widetilde{\theta}_{\widetilde{\varrho}} &= \bigg[\frac{1}{|Q_{\widetilde{\varrho}}|} \iint_{Q_{\widetilde{\varrho}}^{(\theta_{\widetilde{\varrho}})}(z_o)} \frac{|u|^{2m}}{\widetilde{\varrho}^2} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{m+1}} \\
&\le \Big(\frac{s}{\widetilde{\varrho}}\Big)^{\frac{d+2}{m+1}} \bigg[\frac{1}{|Q_s|} \iint_{Q_s^{(\theta_s)}(z_o)} \frac{|u|^{2m}}{s^2} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{m+1}} \le \Big(\frac{s}{\varrho}\Big)^{\frac{d+2}{m+1}} \theta_s.
\end{aligned}$$

We now apply (6.6) with $s = R$. Since $\theta_R = \widetilde{\theta}_R$ the estimate (6.2) for $\widetilde{\theta}_R$ yields

$$\theta_\varrho \le \Big(\frac{R}{\varrho}\Big)^{\frac{d+2}{m+1}} \theta_R \le \Big(\frac{4R}{\varrho}\Big)^{\frac{d+2}{m+1}} \lambda_o. \tag{6.7}$$

In the following, we consider the system of concentric cylinders $Q_\varrho^{(\theta_{z_o;\varrho})}(z_o)$ with radii $\varrho \in (0, R]$ and $z_o \in Q_{2R}$. Note that the cylinders are nested in the sense that

$$Q_r^{(\theta_{z_o;r})}(z_o) \subset Q_s^{(\theta_{z_o;s})}(z_o) \text{ whenever } 0 < r < s \le R.$$

The inclusion holds true due to the monotonicity of the mapping $\varrho \mapsto \theta_{z_o;\varrho}$. The disadvantage of this system of nested cylinders is, that in general the cylinders only fulfill a sub-intrinsic coupling condition.

6.2. **Covering property.** Here, we will prove a Vitali type covering property for the cylinders constructed in the last section. The precise result is the following

**Lemma 6.1.** *There exists a constant $\hat{c} = \hat{c}(n, m) \ge 20$ such that the following holds true: Let $\mathcal{F}$ be any collection of cylinders $Q_{4r}^{(\theta_{z;r})}(z)$, where $Q_r^{(\theta_{z;r})}(z)$ is a cylinder of the form constructed in Section 6.1 with radius $r \in (0, \frac{R}{\hat{c}})$. Then there exists a countable subfamily $\mathcal{G}$ of disjoint cylinders in $\mathcal{F}$ such that*

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{Q \in \mathcal{G}} \widehat{Q}, \tag{6.8}$$

*where $\widehat{Q}$ denotes the $\frac{\hat{c}}{4}$-times enlarged cylinder $Q$, i.e. if $Q = Q_{4r}^{(\theta_{z;r})}(z)$, then $\widehat{Q} = Q_{\hat{c}r}^{(\theta_{z;r})}(z)$.*

**Proof.** For $j \in \mathbb{N}$ we consider the sub-collection

$$\mathcal{F}_j := \big\{Q_{4r}^{(\theta_{z;r})}(z) \in \mathcal{F} : \tfrac{R}{2^j \hat{c}} < r \le \tfrac{R}{2^{j-1}\hat{c}}\big\}$$

and choose $\mathcal{G}_j \subset \mathcal{F}_j$ as follows: We let $\mathcal{G}_1$ be any maximal disjoint collection of cylinders in $\mathcal{F}_1$. Note that $\mathcal{G}_1$ is finite, since by (6.7) and the definition of $\mathcal{F}_1$ the $\mathcal{L}^{n+1}$-measure of each cylinder $Q \in \mathcal{G}_1$ is bounded from below. Now, assume that $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_{k-1}$ have already been selected for some integer $k \ge 2$. Then, we choose $\mathcal{G}_k$ to be any maximal disjoint subcollection of

$$\bigg\{Q \in \mathcal{F}_k : Q \cap Q^* = \emptyset \text{ for any } Q^* \in \bigcup_{j=1}^{k-1} \mathcal{G}_j\bigg\}.$$

Note again that also $\mathcal{G}_k$ is finite. Finally, we define

$$\mathcal{G} := \bigcup_{j=1}^{\infty} \mathcal{G}_j.$$

Then, $\mathcal{G}$ is a countable collection of disjoint cylinders and $\mathcal{G} \subset \mathcal{F}$. At this point it remains to prove that for each $Q \in \mathcal{F}$ there exists a cylinder $Q^* \in \mathcal{G}$ with $Q \cap Q^* \neq \emptyset$, and that this implies $Q \subset \widehat{Q}^*$.

To this aim fix $Q = Q_{4r}^{(\theta_{z;r})}(z) \in \mathcal{F}$. Then, there exists $j \in \mathbb{N}$ such that $Q \in \mathcal{F}_j$. By the maximality of $\mathcal{G}_j$, there exists a cylinder $Q^* = Q_{4r_*}^{(\theta_{z_*;r_*})}(z_*) \in \bigcup_{i=1}^j \mathcal{G}_i$ with $Q \cap Q^* \neq \emptyset$. We know that $r \leq \frac{R}{2^{j-1}\hat{c}}$ and $r_* > \frac{R}{2^j \hat{c}}$, so that $r \leq 2r_*$. This ensures that $B_{4r}(x) \subset B_{20r_*}(x_*)$. In the following we shall prove

$$\theta_{z_*;r_*} \leq 64^{\frac{d+2}{m+1}} \theta_{z;r} . \tag{6.9}$$

By $\widetilde{r}_* \in [r_*, R]$ we denote the radius from (6.5) associated to the cylinder $Q_{r_*}^{(\theta_{z_*;r_*})}(z_*)$. Recall that either $Q_{\widetilde{r}_*}^{(\theta_{z_*;r_*})}(z_*)$ is intrinsic or $\widetilde{r}_* = R$ and $\theta_{z_*;r_*} = \lambda_o$. In the latter case we have due to the definition of $\theta_{z;r}$ that

$$\theta_{z_*;r_*} = \lambda_o \leq \theta_{z;r} .$$

Therefore, we may assume that $Q_{\widetilde{r}_*}^{(\theta_{z_*;r_*})}(z_*)$ is intrinsic, which means

$$\theta_{z_*;r_*}^{m+1} = \frac{1}{|Q_{\widetilde{r}_*}|} \iint_{Q_{\widetilde{r}_*}^{(\theta_{z_*;r_*})}(z_*)} \frac{|u|^{2m}}{\widetilde{r}_*^2} \mathrm{d}y\mathrm{d}\tau. \tag{6.10}$$

In the following, we distinguish between the cases $\widetilde{r}_* \leq \frac{R}{\eta}$ and $\widetilde{r}_* > \frac{R}{\mu}$, where $\mu := 16$. In the latter case we exploit (6.10) and the definition of $\lambda_o$ and $\theta_{z;r}$ to obtain

$$\theta_{z_*;r_*}^{m+1} \leq \Big(\frac{4R}{\widetilde{r}_*}\Big)^2 \frac{1}{|Q_{\widetilde{r}_*}|} \iint_{Q_{4R}} \frac{|u|^{2m}}{(4R)^2} \mathrm{d}y\mathrm{d}\tau \leq \Big(\frac{4R}{\widetilde{r}_*}\Big)^{d+2} \lambda_o^{m+1} \leq (4\mu)^{d+2} \theta_{z;r}^{m+1}.$$

This shows that

$$\theta_{z_*;r_*} \leq (4\mu)^{\frac{d+2}{m+1}} \theta_{z;r} .$$

Therefore, it suffices to consider the case $\widetilde{r}_* \leq \frac{R}{\mu}$. Since $\widetilde{r}_* \geq r_*$ and $|x - x_*| < 4r + 4r_* \leq 12r_*$, we know that $B_{4\widetilde{r}_*}(x_*) \subset B_{\mu\widetilde{r}_*}(x)$. In addition, we have

$$|t - t_*| \leq \theta_{z;r}^{1-m}(4r)^{\frac{m+1}{m}} + \theta_{z_*;r_*}^{1-m}(4r_*)^{\frac{m+1}{m}} . \tag{6.11}$$

Without restriction one can now assume $\theta_{z;r} \leq \theta_{z_*;r_*}$, because otherwise (6.9) trivially holds. Now, the monotonicity of $\varrho \mapsto \theta_{z;\varrho}$ and $r \leq 2r_* \leq 2\widetilde{r}_* \leq \mu\widetilde{r}_*$ yield

$$\theta_{z_*;r_*} \geq \theta_{z;r} \geq \theta_{z;\mu\widetilde{r}_*} ,$$

so that

$$\begin{aligned}
\theta_{z_*;r_*}^{1-m}(4\widetilde{r}_*)^{\frac{m+1}{m}} + |t - t_*| &\leq 2\theta_{z_*;r_*}^{1-m}(4\widetilde{r}_*)^{\frac{m+1}{m}} + \theta_{z;r}^{1-m}(4r)^{\frac{m+1}{m}} \\
&\leq 2 \cdot 8^{\frac{m+1}{m}} \theta_{z;\mu\widetilde{r}_*}^{1-m} \widetilde{r}_*^{\frac{m+1}{m}} \leq \theta_{z;\mu\widetilde{r}_*}^{1-m} (\mu\widetilde{r}_*)^{\frac{m+1}{m}} .
\end{aligned}$$

But this means

$$\Lambda_{4\widetilde{r}_*}^{(\theta_{z_*;r_*})}(t_*) \subset \Lambda_{\mu\widetilde{r}_*}^{(\theta_{z;\mu\widetilde{r}_*})}(t).$$

Therefore, from (6.10) and (6.4) with $\varrho = s = \mu\widetilde{r}_*$, we obtain

$$\theta_{z_*;r_*}^{m+1} \leq \frac{\mu^2}{|Q_{\widetilde{r}_*}|} \iint_{Q_{\mu\widetilde{r}_*}^{(\theta_{z;\mu\widetilde{r}_*})}(z)} \frac{|u|^{2m}}{(\mu\widetilde{r}_*)^2} \mathrm{d}y\mathrm{d}\tau \leq \mu^{d+2} \theta_{z;r}^{m+1}.$$

This implies that

$$\theta_{z_*;r_*} \leq \mu^{\frac{d+2}{m+1}} \theta_{z;r}.$$

This finishes the proof of (6.9). With (6.11), $r \leq 2r_*$, and (6.9) we conclude

$$\begin{aligned}
\theta_{z;r}^{1-m}(4r)^{\frac{m+1}{m}} + |t - t_*| &\leq 2\theta_{z;r}^{1-m}(4r)^{\frac{m+1}{m}} + \theta_{z_*;r_*}^{1-m}(4r_*)^{\frac{m+1}{m}} \\
&\leq 4^{\frac{m+1}{m}} \Big[ 1 + 2 \cdot 2^{\frac{m+1}{m}} \cdot 64^{\frac{(m-1)(d+2)}{m+1}} \Big] \theta_{z_*;r_*}^{1-m} r_*^{\frac{m+1}{m}}
\end{aligned}$$

$$\leq \theta_{z_*;r_*}^{1-m}(\hat{c}r_*)^{\frac{m+1}{m}},$$

for a constant $\hat{c} = \hat{c}(n,m) > 4$. This yields the inclusion $\Lambda_{4r}^{(\theta_{z;r})}(t) \subset \Lambda_{\hat{c}r_*}^{(\theta_{z_*;r_*})}(t_*)$. After possibly enlarging $\hat{c}$, so that $\hat{c} \geq 20$, this implies $Q \subset \widehat{Q}^* = Q_{\hat{c}r_*}^{(\theta_{z_*;r_*})}(z_*)$. This establishes (6.8) and completes the proof of the Vitali covering type Lemma. □

6.3. **Stopping time argument.** For $\lambda > \lambda_o$ and $r \in (0, 2R]$, we define the super-level set of the function $|D\boldsymbol{u}^m|$ by

$$\boldsymbol{E}(r,\lambda) := \Big\{ z \in Q_r : z \text{ is a Lebesgue point of } \big|D\boldsymbol{u}^m\big| \text{ and } \big|D\boldsymbol{u}^m\big|(z) > \lambda^m \Big\}.$$

The Lebesgue points are to be understood with regard to the cylinders constructed in Section 6.1. Note that $\mathcal{L}^{n+1}$ a.e. point is a Lebesgue point with respect to these cylinders; cf. [14, 2.9.1] and the Vitali type covering Lemma 6.1. For fixed radii $R \leq R_1 < R_2 \leq 2R$, we consider the concentric parabolic cylinders

$$Q_R \subseteq Q_{R_1} \subset Q_{R_2} \subseteq Q_{2R}.$$

Note that the inclusion $Q_\varrho^{(\kappa)}(z_o) = B_\varrho(x_o) \times \big(t_o - \kappa^{1-m}\varrho^{\frac{m+1}{m}}, t_o + \kappa^{1-m}\varrho^{\frac{m+1}{m}}\big) \subset Q_{R_2}$ holds true, whenever $z_o \in Q_{R_1}$, $\kappa \in [\lambda_o, \infty)$ and $\varrho \in (0, R_2 - R_1]$. We fix $z_o \in \boldsymbol{E}(R_1, \lambda)$ and abbreviate $\theta_s \equiv \theta_{z_o;s}$ for $s \in (0, R]$ throughout this section. By Lebesgue's Differentiation Theorem, cf. [14, 2.9.1] we have that

$$\lim_{s\downarrow 0} \iint\!\!\!\!\!\!\!-_{Q_s^{(\theta_s)}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2 + |F|^2\Big] \mathrm{d}x\mathrm{d}t \geq \big|D\boldsymbol{u}^m\big|^2(z_o) > \lambda^{2m}. \tag{6.12}$$

In the following, we consider values of $\lambda$ satisfying

$$\lambda > B\lambda_o, \qquad \text{where} \quad B := \Big(\frac{4\hat{c}R}{R_2 - R_1}\Big)^{\frac{n+2}{m+1}} > 1, \tag{6.13}$$

where $\hat{c} = \hat{c}(n,m)$ denotes the constant from the Vitali-type covering Lemma 6.1. For radii $s$ with

$$\frac{R_2 - R_1}{\hat{c}} \leq s \leq R \tag{6.14}$$

we have, by the definition of $\lambda_o$, for any $s$ as in (6.14) that

$$\begin{aligned}
\iint\!\!\!\!\!\!\!-_{Q_s^{(\theta_s)}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2 + |F|^2\Big] \mathrm{d}x\mathrm{d}t &\leq \frac{|Q_{4R}|}{|Q_s^{(\theta_s)}|} \iint\!\!\!\!\!\!\!-_{Q_{4R}} \Big[\big|D\boldsymbol{u}^m\big|^2 + |F|^2\Big] \mathrm{d}x\mathrm{d}t \\
&\leq \frac{|Q_{4R}|}{|Q_s|} \theta_s^{m-1} \lambda_o^{m+1} \\
&\leq \Big(\frac{4R}{s}\Big)^{d + \frac{(d+2)(m-1)}{m+1}} \lambda_o^{2m} \\
&\leq \Big(\frac{4\hat{c}R}{R_2 - R_1}\Big)^{d + \frac{(d+2)(m-1)}{m+1}} \lambda_o^{2m} \\
&= B^{2m}\lambda_o^{2m} < \lambda^{2m}.
\end{aligned}$$

In the last chain of inequalities we used (6.7), (6.14) and $d + \frac{(d+2)(m-1)}{m+1} = \frac{2m(n+2)}{m+1}$. On the other hand, on behalf of (6.12) we find a sufficiently small radius $0 < s < \frac{R_2 - R_1}{\hat{c}}$ such that the above integral with $Q_s^{(\theta_s)}(z_o)$ as domain of integration, possesses a value larger than $\lambda^{2m}$. Consequently, by the absolute continuity of the integral there exists a maximal radius $0 < \varrho_{z_o} < \frac{R_2 - R_1}{\hat{c}}$ such that

$$\iint\!\!\!\!\!\!\!-_{Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2 + |F|^2\Big] \mathrm{d}x\mathrm{d}t = \lambda^{2m}. \tag{6.15}$$

The maximality of the radius $\varrho_{z_o}$ implies in particular that

$$\tag{6.16} \iint_{Q_s^{(\theta_s)}(z_o)} \Big[|D\boldsymbol{u}^m|^2 + |F|^2\Big] \mathrm{d}x\mathrm{d}t < \lambda^{2m} \qquad \text{for any } s \in (\varrho_{z_o}, R].$$

Finally, we know from the construction that $Q_{\hat{c}\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$ is contained in $Q_{\hat{c}\varrho_{z_o}}(z_o)$, which in turn is contained in $Q_{R_2}$.

6.4. **A Reverse Hölder Inequality.** As before, we consider $z_o \in \boldsymbol{E}(r_1, \lambda)$ with $\lambda$ as in (6.13) and abbreviate $\theta_{\varrho_{z_o}} \equiv \theta_{z_o;\varrho_{z_o}}$. As in (6.5) we construct the radius $\widetilde{\varrho}_{z_o} \in [\varrho_{z_o}, R]$. Exactly at this point, we pass from the possibly sub-intrinsic cylinder $Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$ to the intrinsic cylinder $Q_{\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$. Observe that $\theta_s = \theta_{\varrho_{z_o}}$ for any $s \in [\varrho_{z_o}, \widetilde{\varrho}_{z_o}]$, and, in particular, $\theta_{\widetilde{\varrho}_{z_o}} = \theta_{\varrho_{z_o}}$. Our aim now is to prove the following reverse Hölder inequality

$$\tag{6.17} \begin{aligned} &\iint_{Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |D\boldsymbol{u}^m|^2 \mathrm{d}x\mathrm{d}t \\ &\quad\le c\bigg[\iint_{Q_{4\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |D\boldsymbol{u}^m|^{2q} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{q}} + c\iint_{Q_{4\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |F|^2 \mathrm{d}x\mathrm{d}t, \end{aligned}$$

with $q := \frac{n}{d} < 1$ and $c = c(n, m, \nu, L)$. We distinguish between the cases in which $\widetilde{\varrho}_{z_o} \le 2\varrho_{z_o}$ or $\widetilde{\varrho}_{z_o} > 2\varrho_{z_o}$. In the case $\widetilde{\varrho}_{z_o} \le 2\varrho_{z_o}$ we apply Proposition 5.1 on the intrinsic cylinder $Q_{\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$ (note that $Q_{\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$ is intrinsic and, thanks to (6.4), $Q_{2\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$ is sub-intrinsic) and obtain

$$\begin{aligned} &\iint_{Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |D\boldsymbol{u}^m|^2 \mathrm{d}x\mathrm{d}t \\ &\quad\le 2^d \iint_{Q_{\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |D\boldsymbol{u}^m|^2 \mathrm{d}x\mathrm{d}t \\ &\quad\le c\bigg[\iint_{Q_{2\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |D\boldsymbol{u}^m|^{2q} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{q}} + c\iint_{Q_{2\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |F|^2 \mathrm{d}x\mathrm{d}t \\ &\quad\le c\bigg[\iint_{Q_{4\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |D\boldsymbol{u}^m|^{2q} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{q}} + c\iint_{Q_{4\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} |F|^2 \mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $c = c(n, m, \nu, L)$. In the other case $\tilde{\varrho}_{z_o} > 2\varrho_{z_o}$, we want to apply Proposition 5.2 on the cylinder $Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$. However, this is only permitted if the hypothesis (5.5) is satisfied. First, we notice that $(5.5)_1$ is an immediate consequence of (6.4), and therefore we only need to verify $(5.5)_2$. To this aim, we consider two cases. If $\theta_{\varrho_{z_o}} = \lambda_o$, we obtain $(5.5)_2$ by the following computation

$$\theta_{\varrho_{z_o}}^{2m} = \lambda_o^{2m} < \lambda^{2m} = \iint_{Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} \Big[|D\boldsymbol{u}^m|^2 + |F|^2\Big] \mathrm{d}x\mathrm{d}t.$$

Here we used (6.15) for the last identity. If $\theta_{\varrho_{z_o}} > \lambda_o$, then by construction $Q_{\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$ is intrinsic. Moreover, since $\frac{1}{2}\widetilde{\varrho}_{z_o} > \varrho_{z_o}$, we can apply (6.4) with $(\varrho, s)$ replaced by $(\varrho_{z_o}, \frac{1}{2}\widetilde{\varrho}_{z_o})$. This together with Lemma 4.2 and (6.16) (applied with $s = \widetilde{\varrho}_{z_o} \in (\varrho_{z_o}, R]$) ensures that

$$\begin{aligned} \theta_{\varrho_{z_o}} &= \bigg[\iint_{Q_{\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} \frac{|u|^{2m}}{\widetilde{\varrho}_{z_o}^2} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{2m}} \\ &\le \bigg[\iint_{Q_{\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{z_o;\frac{1}{2}\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}\big|^2}{\widetilde{\varrho}_{z_o}^2} \mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{2m}} + \frac{\big|(\boldsymbol{u}^m)_{z_o;\frac{1}{2}\widetilde{\varrho}_{z_o}}^{(\theta_{\varrho_{z_o}})}\big|^{\frac{1}{m}}}{\widetilde{\varrho}_{z_o}^{\frac{1}{m}}} \end{aligned}$$

$$
\begin{aligned}
&\le c\bigg[\iint_{Q^{(\theta_{\varrho_{z_o}})}_{\widetilde\varrho_{z_o}}(z_o)} \frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)^{(\theta_{\varrho_{z_o}})}_{z_o;\widetilde\varrho_{z_o}}\big|^2}{\widetilde\varrho_{z_o}^2}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{2m}} \\
&\qquad\qquad + 2^{-\frac1m}\bigg[\iint_{Q^{(\theta_{\varrho_{z_o}})}_{\frac12\widetilde\varrho_{z_o}}(z_o)} \frac{|u|^{2m}}{(\frac12\widetilde\varrho_{z_o})^2}\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{2m}} \\
&\le c\bigg[\iint_{Q^{(\theta_{\varrho_{z_o}})}_{\widetilde\varrho_{z_o}}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2+|F|^2\Big]\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{2m}} + 2^{-\frac1m}\theta_{\varrho_{z_o}} \\
&\le c\,\lambda + 2^{-\frac1m}\theta_{\varrho_{z_o}},
\end{aligned}
$$

for a constant $c=c(n,m,L)$. Re-absorbing $2^{-\frac1m}\theta_{\varrho_{z_o}}$ into the left-hand side and using (6.15), we find that

$$
\theta_{\varrho_{z_o}} \le c\,\lambda = c\bigg[\iint_{Q^{(\theta_{\varrho_{z_o}})}_{\varrho_{z_o}}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2+|F|^2\Big]\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{2m}},
$$

for a constant $c=c(n,m,L)\ge 1$. This yields $(5.5)_2$ in the second case with $K=c^{2m}\ge 1$. Therefore, we are allowed to apply Proposition 5.2 on the cylinder $Q^{(\theta_{\varrho_{z_o}})}_{\varrho_{z_o}}(z_o)$, thereby obtaining that

$$
\begin{aligned}
&\iint_{Q^{(\theta_{\varrho_{z_o}})}_{\varrho_{z_o}}(z_o)} |D\boldsymbol{u}^m|^2\mathrm{d}x\mathrm{d}t \\
&\qquad\qquad \le c\bigg[\iint_{Q^{(\theta_{\varrho_{z_o}})}_{2\varrho_{z_o}}(z_o)} \big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac1q} + c\iint_{Q^{(\theta_{\varrho_{z_o}})}_{2\varrho_{z_o}}(z_o)} |F|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

In conclusion, we have shown that in any case the claimed reverse Hölder inequality (6.17) holds true.

6.5. **Estimate on super-level sets.** So far we have shown that if $\lambda$ satisfies (6.13), then for every $z_o\in\boldsymbol{E}(R_1,\lambda)$ there exists a cylinder $Q^{(\theta_{z_o;\varrho_{z_o}})}_{\varrho_{z_o}}(z_o)$ with $Q^{(\theta_{z_o;\varrho_{z_o}})}_{\hat c\varrho_{z_o}}(z_o)\subset Q_{R_2}$ such that (6.15), (6.16) and (6.17) hold true on this specific cylinder. As before, we abbreviate $\theta_{\varrho_{z_o}}\equiv\theta_{z_o;\varrho_{z_o}}$. We define the super-level set of the inhomogeneity $F$ by

$$
\boldsymbol{F}(r,\lambda) := \Big\{z\in Q_r : z \text{ is a Lebesgue point of } F \text{ and } |F|>\lambda^m\Big\}.
$$

As for the super-level set $\boldsymbol{E}(r,\lambda)$ the Lebesgue points have to be understood with regard to the cylinders constructed in Section 6.1. Using (6.15) and (6.17) we obtain for $\eta\in(0,1]$ (to be specified later in a universal way) that

$$
\begin{aligned}
\lambda^{2m} &= \iint_{Q^{(\theta_{\varrho_{z_o}})}_{\varrho_{z_o}}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2+|F|^2\Big]\mathrm{d}x\mathrm{d}t \\
&\le c\,\bigg[\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)} \big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac1q} + c\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)} |F|^2\mathrm{d}x\mathrm{d}t \\
&\le c\,\eta^{2m}\lambda^{2m} + c\left[\frac{1}{\big|Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\big|}\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\cap\boldsymbol{E}(R_2,\eta\lambda)} \big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\right]^{\frac1q} \\
&\quad + \frac{c}{\big|Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\big|}\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\cap\boldsymbol{F}(R_2,\eta\lambda)} |F|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

for a constant $c = c(n, m, \nu, L)$. In the preceding inequality we choose the $\eta$ in the form $\eta^{2m} = \frac{1}{2c}$. This choice allows the re-absorption of $\frac12\lambda^{2m}$ into the left-hand side. Furthermore, we use Hölder's inequality and (6.16) to estimate

$$\begin{aligned}\bigg[\frac{1}{\big|Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\big|}\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\cap \boldsymbol{E}(R_2,\eta\lambda)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]^{\frac1q-1}\\ \le \bigg[\Yint\longdash_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)}\big|D\boldsymbol{u}^m\big|^{2}\mathrm{d}x\mathrm{d}t\bigg]^{1-q}\le\lambda^{2m(1-q)}.\end{aligned}$$

We insert this above, and multiply the result, i.e. the inequality where we already fixed $\eta$ and re-absorbed $\frac12\lambda^{2m}$, by $\big|Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\big|$. This leads to the inequality

$$\begin{aligned}\lambda^{2m}\Big|Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\Big|\le c\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\cap\boldsymbol{E}(R_2,\eta\lambda)}\lambda^{2m(1-q)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\\ + c\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\cap\boldsymbol{F}(R_2,\eta\lambda)}|F|^2\mathrm{d}x\mathrm{d}t,\end{aligned}$$

again with $c = c(n, m, \nu, L)$. Now, (6.16) with the choice $s = \hat c\varrho_{z_o}$ allows us to estimate $\lambda^{2m}$ from below. The precise argument is as follows: Using in turn (6.16), the monotonicity of $\varrho\mapsto\theta_\varrho$ and (6.6), i.e. that $\theta_{\hat c\varrho_{z_o}}\le\theta_{\varrho_{z_o}}\le\hat c^{\frac{d+2}{m+1}}\theta_{\hat c\varrho_{z_o}}$, we obtain that

$$\lambda^{2m} > \Yint\longdash_{Q^{(\theta_{\hat c\varrho_{z_o}})}_{\hat c\varrho_{z_o}}(z_o)}\big|D\boldsymbol{u}^m\big|^{2}\mathrm{d}x\mathrm{d}t\ge\frac{1}{\hat c^{\frac{(m-1)(d+2)}{m+1}}}\Yint\longdash_{Q^{(\theta_{\varrho_{z_o}})}_{\hat c\varrho_{z_o}}(z_o)}\big|D\boldsymbol{u}^m\big|^{2}\mathrm{d}x\mathrm{d}t.$$

Inserting this above and keeping in mind that $\hat c$ depends only on $n$ and $m$, we deduce

$$\begin{aligned}\iint_{Q^{(\theta_{\varrho_{z_o}})}_{\hat c\varrho_{z_o}}(z_o)}\big|D\boldsymbol{u}^m\big|^{2}\mathrm{d}x\mathrm{d}t\le c\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\cap\boldsymbol{E}(R_2,\eta\lambda)}\lambda^{2m(1-q)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\\ + c\iint_{Q^{(\theta_{\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\cap\boldsymbol{F}(R_2,\eta\lambda)}|F|^2\mathrm{d}x\mathrm{d}t\end{aligned}\tag{6.18}$$

with $c = c(n, m, \nu, L)$.

So far, we showed that for any value $\lambda > B\lambda_o$ the super-level set $\boldsymbol{E}(R_1, \lambda)$ can be covered by a family $\mathcal F\equiv\big\{Q^{(\theta_{z_o;\varrho_{z_o}})}_{4\varrho_{z_o}}(z_o)\big\}$ of parabolic cylinders with center $z_o\in\boldsymbol{E}(R_1,\lambda)$, which are contained in $Q_{R_2}$, and such that on each cylinder the estimate (6.18) holds true. At this point, we use the Vitali type Covering Lemma 6.1 and gain a countable subfamily

$$\Big\{Q^{(\theta_{z_i;\varrho_{z_i}})}_{4\varrho_{z_i}}(z_i)\Big\}_{i\in\mathbb N}\subset\mathcal F$$

consisting of pairwise disjoint cylinders, such that the $\frac{\hat c}{4}$-times enlarged cylinders $Q^{(\theta_{z_i;\varrho_{z_i}})}_{\hat c\varrho_{z_i}}(z_i)$ are contained in $Q_{R_2}$ and cover the super-level set $\boldsymbol{E}(R_1,\lambda)$, i.e.

$$\boldsymbol{E}(R_1,\lambda)\subset\bigcup_{i=1}^\infty Q^{(\theta_{z_i;\varrho_{z_i}})}_{\hat c\varrho_{z_i}}(z_i)\subset Q_{R_2}.$$

Since the cylinders $Q^{(\theta_{z_i;\varrho_{z_i}})}_{4\varrho_{z_i}}(z_i)$ are pairwise disjoint we obtain from (6.18) that

$$\begin{aligned}\iint_{\boldsymbol{E}(R_1,\lambda)}\big|D\boldsymbol{u}^m\big|^{2}\mathrm{d}x\mathrm{d}t&\le\sum_{i=1}^\infty\iint_{Q^{(\theta_{z_i;\varrho_{z_i}})}_{\hat c\varrho_{z_i}}(z_i)}\big|D\boldsymbol{u}^m\big|^{2}\mathrm{d}x\mathrm{d}t\\&\le c\sum_{i=1}^\infty\iint_{Q^{(\theta_{z_i;\varrho_{z_i}})}_{4\varrho_{z_i}}(z_i)\cap\boldsymbol{E}(R_2,\eta\lambda)}\lambda^{2m(1-q)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+ c\sum_{i=1}^{\infty} \iint_{Q_{4\varrho_{z_i}}^{(\theta_{z_i};\varrho_{z_i})}(z_i)\cap \boldsymbol{F}(R_2,\eta\lambda)} |F|^2 \mathrm{d}x\mathrm{d}t \\
&\le c\iint_{\boldsymbol{E}(R_2,\eta\lambda)} \lambda^{2m(1-q)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t + c\iint_{\boldsymbol{F}(R_2,\eta\lambda)} |F|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where the constant $c$ depends only on $n$, $m$, $\nu$, and $L$. On $\boldsymbol{E}(R_1,\eta\lambda)\setminus\boldsymbol{E}(R_1,\lambda)$ we have the pointwise bound $\big|D\boldsymbol{u}^m\big|^2 \le \lambda^{2m}$ and therefore

$$
\iint_{\boldsymbol{E}(R_1,\eta\lambda)\setminus\boldsymbol{E}(R_1,\lambda)} \big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t \le \iint_{\boldsymbol{E}(R_2,\eta\lambda)} \lambda^{2m(1-q)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t.
$$

We combine the last two inequalities and get the following reverse Hölder inequality on super-level sets

$$
\begin{aligned}
&\iint_{\boldsymbol{E}(R_1,\eta\lambda)} \big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t \\
&\qquad \le c\iint_{\boldsymbol{E}(R_2,\eta\lambda)} \lambda^{2m(1-q)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t + c\iint_{\boldsymbol{F}(R_2,\eta\lambda)} |F|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Here, we replace $\eta\lambda$ by $\lambda$ and recall that $\eta < 1$ depends only on $n, m, \nu$, and $L$. With this replacement we obtain for any $\lambda \ge \eta B\lambda_o =: \lambda_1$ that

$$
\begin{aligned}
&\iint_{\boldsymbol{E}(R_1,\lambda)} \big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t \\
&\qquad \le c\iint_{\boldsymbol{E}(R_2,\lambda)} \lambda^{2m(1-q)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t + c\iint_{\boldsymbol{F}(R_2,\lambda)} |F|^2\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{6.19}
$$

holds true with a constant $c = c(n,m,\nu,L)$. This is the desired estimate on super-level sets.

6.6. **Proof of the gradient estimate.** For $k > \lambda_1$ we define the *truncation* of $|D\boldsymbol{u}^m|$ by

$$
\big|D\boldsymbol{u}^m\big|_k := \min\Big\{\big|D\boldsymbol{u}^m\big|, k^m\Big\},
$$

and for $r \in (0, 2R]$ the corresponding super-level set

$$
\boldsymbol{E}_k(r,\lambda) := \Big\{z \in Q_r : \big|D\boldsymbol{u}^m\big|_k > \lambda^m\Big\}.
$$

Note that $|D\boldsymbol{u}^m|_k \le |D\boldsymbol{u}^m|$ a.e., as well as $\boldsymbol{E}_k(r,\lambda) = \emptyset$ for $k \le \lambda$ and $\boldsymbol{E}_k(r,\lambda) = \boldsymbol{E}(r,\lambda)$ for $k > \lambda$. Therefore, it follows from (6.19) that

$$
\begin{aligned}
&\iint_{\boldsymbol{E}_k(R_1,\lambda)} \big|D\boldsymbol{u}^m\big|_k^{2-2q}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t \\
&\qquad \le c\iint_{\boldsymbol{E}_k(R_2,\lambda)} \lambda^{2m(1-q)}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t + c\iint_{\boldsymbol{F}(R_2,\lambda)} |F|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

whenever $k > \lambda \ge \lambda_1$. Since $\boldsymbol{E}_k(r,\lambda) = \emptyset$ for $k \le \lambda$, the last inequality also holds in this case. Now, we multiply the preceding inequality by $\lambda^{\varepsilon m-1}$, where $\varepsilon \in (0,1]$ will be chosen later in a universal way, and integrate the result with respect to $\lambda$ over the interval $(\lambda_1,\infty)$. This gives

$$
\begin{aligned}
&\int_{\lambda_1}^{\infty} \lambda^{\varepsilon m-1}\bigg[\iint_{\boldsymbol{E}_k(R_1,\lambda)} \big|D\boldsymbol{u}^m\big|_k^{2-2q}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]\mathrm{d}\lambda \\
&\qquad \le c\int_{\lambda_1}^{\infty} \lambda^{m(2-2q+\varepsilon)-1}\bigg[\iint_{\boldsymbol{E}_k(R_2,\lambda)} \big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]\mathrm{d}\lambda \\
&\qquad\quad + c\int_{\lambda_1}^{\infty} \lambda^{\varepsilon m-1}\bigg[\iint_{\boldsymbol{F}(R_2,\lambda)} |F|^2\mathrm{d}x\mathrm{d}t\bigg]\mathrm{d}\lambda.
\end{aligned} \tag{6.20}
$$

Here we exchange the order of integration with the help of Fubini's theorem. For the integral on the left-hand side Fubini's theorem implies

$$\begin{aligned}
&\int_{\lambda_1}^{\infty} \lambda^{\varepsilon m-1}\bigg[\iint_{\boldsymbol{E}_k(R_1,\lambda)} \big|D\boldsymbol{u}^m\big|_k^{2-2q}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]\mathrm{d}\lambda \\
&\quad= \iint_{\boldsymbol{E}_k(R_1,\lambda_1)} \big|D\boldsymbol{u}^m\big|_k^{2-2q}\big|D\boldsymbol{u}^m\big|^{2q}\bigg[\int_{\lambda_1}^{|D\boldsymbol{u}^m|_k^{\frac{1}{m}}} \lambda^{\varepsilon m-1}\mathrm{d}\lambda\bigg]\mathrm{d}x\mathrm{d}t \\
&\quad= \tfrac{1}{\varepsilon m}\iint_{\boldsymbol{E}_k(R_1,\lambda_1)} \Big[\big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q} - \lambda_1^{\varepsilon m}\big|D\boldsymbol{u}^m\big|_k^{2-2q}\big|D\boldsymbol{u}^m\big|^{2q}\Big]\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

while for the first integral on the right-hand side we find that

$$\begin{aligned}
&\int_{\lambda_1}^{\infty} \lambda^{m(2-2q+\varepsilon)-1}\bigg[\iint_{\boldsymbol{E}_k(R_2,\lambda)} \big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\bigg]\mathrm{d}\lambda \\
&\quad= \iint_{\boldsymbol{E}_k(R_2,\lambda_1)} \big|D\boldsymbol{u}^m\big|^{2q}\bigg[\int_{\lambda_1}^{|D\boldsymbol{u}^m|_k^{\frac{1}{m}}} \lambda^{m(2-2q+\varepsilon)-1}\mathrm{d}\lambda\bigg]\mathrm{d}x\mathrm{d}t \\
&\quad\le \tfrac{1}{m(2-2q+\varepsilon)}\iint_{\boldsymbol{E}_k(R_2,\lambda_1)} \big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t \\
&\quad\le \tfrac{1}{2m(1-q)}\iint_{\boldsymbol{E}_k(R_2,\lambda_1)} \big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Finally, for the last integral in (6.20) we obtain

$$\begin{aligned}
&\int_{\lambda_1}^{\infty} \lambda^{\varepsilon m-1}\bigg[\iint_{\boldsymbol{F}(R_2,\lambda)} |F|^2\mathrm{d}x\mathrm{d}t\bigg]\mathrm{d}\lambda \\
&\quad= \iint_{\boldsymbol{F}(R_2,\lambda_1)} |F|^2\bigg[\int_{\lambda_1}^{|F|^{\frac{1}{m}}} \lambda^{\varepsilon m-1}\mathrm{d}\lambda\bigg]\mathrm{d}x\mathrm{d}t \\
&\quad\le \tfrac{1}{\varepsilon m}\iint_{\boldsymbol{F}(R_2,\lambda_1)} |F|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t \\
&\quad\le \tfrac{1}{\varepsilon m}\iint_{Q_{2R}} |F|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

We insert these estimates into (6.20) and multiply by $\varepsilon m$. This leads to

$$\begin{aligned}
&\iint_{\boldsymbol{E}_k(R_1,\lambda_1)} \big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t \\
&\qquad\le \lambda_1^{\varepsilon m}\iint_{\boldsymbol{E}_k(R_1,\lambda_1)} \big|D\boldsymbol{u}^m\big|_k^{2-2q}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t \\
&\qquad\quad + \frac{c\,\varepsilon}{1-q}\iint_{\boldsymbol{E}_k(R_2,\lambda_1)} \big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t \\
&\qquad\quad + c\iint_{Q_{2R}} |F|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

The last inequality is now combined with the corresponding inequality on the complement $Q_{R_1} \setminus \boldsymbol{E}_k(R_1,\lambda_1)$, i.e. with the inequality

$$\begin{aligned}
&\iint_{Q_{R_1}\setminus\boldsymbol{E}_k(R_1,\lambda_1)} \big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t \\
&\qquad\le \lambda_1^{\varepsilon m}\iint_{Q_{R_1}\setminus\boldsymbol{E}_k(R_1,\lambda_1)} \big|D\boldsymbol{u}^m\big|_k^{2-2q}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

We also take into account that $|D\boldsymbol{u}^m|_k \le |D\boldsymbol{u}^m|$. All together this gives the inequality

$$\begin{aligned}\iint_{Q_{R_1}} &\big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\\ &\le \frac{c_*\varepsilon}{1-q}\iint_{Q_{R_2}} \big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\\ &\quad+\lambda_1^{\varepsilon m}\iint_{Q_{2R}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t + c\iint_{Q_{2R}}|F|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $c_* = c_*(n,m,\nu,L)\ge 1$. Now, we choose

$$0<\varepsilon\le\min\{\varepsilon_o,\sigma-2\},\quad \text{where } \varepsilon_o := \frac{1-q}{2c_*}<1.$$

Note that $\varepsilon_o$ depends only on $n, m, \nu$, and $L$. Moreover, observe that $\lambda_1^\varepsilon \equiv (\eta B\lambda_o)^\varepsilon \le B\lambda_o^\varepsilon$, since $\eta\le 1$, $B\ge 1$ and $0<\varepsilon\le 1$. Therefore, from the previous inequality we conclude that for any pair of radii $R_1, R_2$ with $R\le R_1<R_2\le 2R$ there holds

$$\begin{aligned}\iint_{Q_{R_1}} &\big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\\ &\le \tfrac12\iint_{Q_{R_2}} \big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t\\ &\quad+c\Big(\frac{R}{R_2-R_1}\Big)^{\frac{m(n+2)}{m+1}}\lambda_o^{\varepsilon m}\iint_{Q_{2R}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t + c\iint_{Q_{2R}}|F|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We can now apply the Iteration Lemma 2.1 to the last inequality, which yields

$$\iint_{Q_R}\big|D\boldsymbol{u}^m\big|_k^{2-2q+\varepsilon}\big|D\boldsymbol{u}^m\big|^{2q}\mathrm{d}x\mathrm{d}t \le c\,\lambda_o^{\varepsilon m}\iint_{Q_{2R}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t + c\iint_{Q_{2R}}|F|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t.$$

On the left side we apply Fatou's Lemma and pass to the limit $k\to\infty$. In the result, we go over to means on both sides. This gives

$$⨏\!\!\!⨏_{Q_R}\big|D\boldsymbol{u}^m\big|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t \le c\,\lambda_o^{\varepsilon m}⨏\!\!\!⨏_{Q_{2R}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t + c⨏\!\!\!⨏_{Q_{2R}}|F|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t.$$

At this point, we estimate $\lambda_o$ with the help of the energy estimate from Lemma 3.1 applied with $\theta=1$ and $a=0$ and Hölder's inequality. This leads to the bound

$$\lambda_o \le c\bigg[1+⨏\!\!\!⨏_{Q_{8R}}\bigg[\frac{|u|^{2m}}{R^2}+|F|^2\bigg]\mathrm{d}x\mathrm{d}t\bigg]^{\frac{1}{m+1}},$$

where $c=c(m,\nu,L)$. Inserting this above, we deduce

$$\begin{aligned}⨏\!\!\!⨏_{Q_R}&\big|D\boldsymbol{u}^m\big|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t\\ &\le c\bigg[1+⨏\!\!\!⨏_{Q_{8R}}\bigg[\frac{|u|^{2m}}{R^2}+|F|^2\bigg]\mathrm{d}x\mathrm{d}t\bigg]^{\frac{\varepsilon m}{m+1}}⨏\!\!\!⨏_{Q_{2R}}\big|D\boldsymbol{u}^m\big|^2\mathrm{d}x\mathrm{d}t + c⨏\!\!\!⨏_{Q_{2R}}|F|^{2+\varepsilon}\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $c=c(n,m,\nu,L)$. The claimed estimate (1.10) involving the cylinders $Q_R$ and $Q_{2R}$ now follows by a covering argument. This completes the proof of Theorem 1.2. $\square$

6.7. **Proof of Corollary 1.3.** It remains to deduce a corresponding estimate on a standard parabolic cylinder $C_{2R}(z_o) := B_{2R}(x_o) \times (t_o - (2R)^2, t_o + (2R)^2) \Subset \Omega_T$. To this end, we rescale the solution $u$, the vector-field $\mathbf{A}$, and the right-hand side $F$ via

$$\begin{cases} v(x,t) := u(x_o + Rx, t_o + R^2 t) \\ \mathbf{B}(x,t,u,\xi) := R\,\mathbf{A}\big(x_o + Rx, t_o + R^2 t, u, \tfrac{1}{R}\xi\big) \\ G(x,t) := R\,F(x_o + Rx, t_o + R^2 t), \end{cases}$$

whenever $(x,t) \in C_2$ and $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$. Then $v$ is a weak solution of the differential equation

$$\partial_t v - \operatorname{div} \mathbf{B}(x,t,v,D\boldsymbol{v}^m) = \operatorname{div} G \qquad \text{in } Q_2 \subset C_2,$$

in the sense of Definition 1.1. Moreover, the assumptions (1.2) are satisfied for the rescaled vector-field $\mathbf{B}$ in place of $\mathbf{A}$. Therefore, the estimate (1.10) is applicable to $v$ on the cylinder $Q_2$, which yields

$$\begin{aligned} &\iint_{Q_1} \big|D\boldsymbol{v}^m\big|^{2+\varepsilon} \mathrm{d}x\mathrm{d}t \\ &\quad \le c\bigg[1 + \iint_{Q_2} \big[|v|^{2m} + |G|^2\big] \mathrm{d}x\mathrm{d}t\bigg]^{\frac{\varepsilon m}{m+1}} \iint_{Q_2} \big|D\boldsymbol{v}^m\big|^2 \mathrm{d}x\mathrm{d}t + c \iint_{Q_2} |G|^{2+\varepsilon} \mathrm{d}x\mathrm{d}t, \end{aligned}$$

for every $\varepsilon \in (0, \varepsilon_o]$, with a constant $c = c(n, m, \nu, L)$. Scaling back and recalling that $Q_2 \subset C_2$, we arrive at the estimate

$$\begin{aligned} &R^{2+\varepsilon} \iint_{C_R(z_o)} \big|D\boldsymbol{u}^m\big|^{2+\varepsilon} \mathrm{d}x\mathrm{d}t \\ &\quad \le c\,R^2 \bigg[1 + \iint_{C_{2R}(z_o)} \big[|u|^{2m} + R^2|F|^2\big] \mathrm{d}x\mathrm{d}t\bigg]^{\frac{\varepsilon m}{m+1}} \iint_{C_{2R}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t \\ &\qquad + c\,R^{2+\varepsilon} \iint_{C_{2R}(z_o)} |F|^{2+\varepsilon} \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Dividing both sides by $R^{2+\varepsilon}$ yields the assertion of Corollary 1.3. □

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