Lattès-type uniformly quasiregular mappings and their Julia sets

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Abstract

A uniformly quasiregular mapping acting on a compact Riemannian manifold distorts the metric by a bounded amount, independently of the number of iterates. There is a Fatou-Julia type theory associated with the dynamical system obtained by iterating these mappings.

We study a rich subclass of uniformly quasiregular mappings that can be produced using an analogy of classical Lattès' construction of chaotic rational functions acting on the extended complex plane (see [4]). We construct several essentially different examples of Lattès-type uniformly quasiregular mappings on three and higher dimensional compact Riemannian manifolds with a variety of Julia sets.

Definitions

Analytic mappings acting on complex plane have a natural generalization called quasiregular maps (QR) to higher dimensional euclidean space:

Definition.
Let $D \subset \overline{\mathbb{R}}^n$ be a domain and $f : D \to \overline{\mathbb{R}}^n$ a non-constant mapping of the Sobolev class $W^{1,n}_{loc}(D)$. We consider only orientation-preserving mappings, which means that the Jacobian determinant $J_f(x) \geq 0$ for a.e. $x \in D$. Such a mapping is said to be *K-quasiregular*, where $1 \leq K < \infty$, if

$$\max_{|h|=1} |f'(x)h| \leq K \min_{|h|=1} |f'(x)h|$$

for a.e. $x \in D$, when f' is the formal matrix of weak derivatives.

This local definition can be generalized to maps $f : M \to N$ between arbitrary riemannian manifolds M and N of same dimension. In the UQR setting we study noninjective maps $f : M \to M$ whose all iterates f^n satisfy the above QR condition for fixed $K \geq 1$ independently of the number of iterates. Such maps distort the given metric by a bounded amount.

In 1997, V. Mayer discovered an important family of examples of uqr mappings, the so called Lattès-type uqr mappings (see [4]). They are analogues of the rational functions that are called critically finite with parabolic orbifold. V. Mayer generalized S. Lattès' construction of so-called chaotic rational maps [3]. We have the following existence theorem by T. Iwaniec and G. Martin [2].

Theorem.
Let Γ be a discrete group such that $h : \mathbb{R}^n \to M$ is automorphic with respect to Γ in the strong sense. If there is a similarity $A = \lambda\mathcal{O}$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $\mathcal{O}$ an orthogonal transformation, such that

$$A\Gamma A^{-1} \subset \Gamma,$$

then there is a unique solution $f : h(\mathbb{R}^n) \to h(\mathbb{R}^n)$ to the Schröder functional equation

$$f \circ h = h \circ A, \tag{1}$$

and f is a uniformly quasiregular mapping, if h is quasiregular.

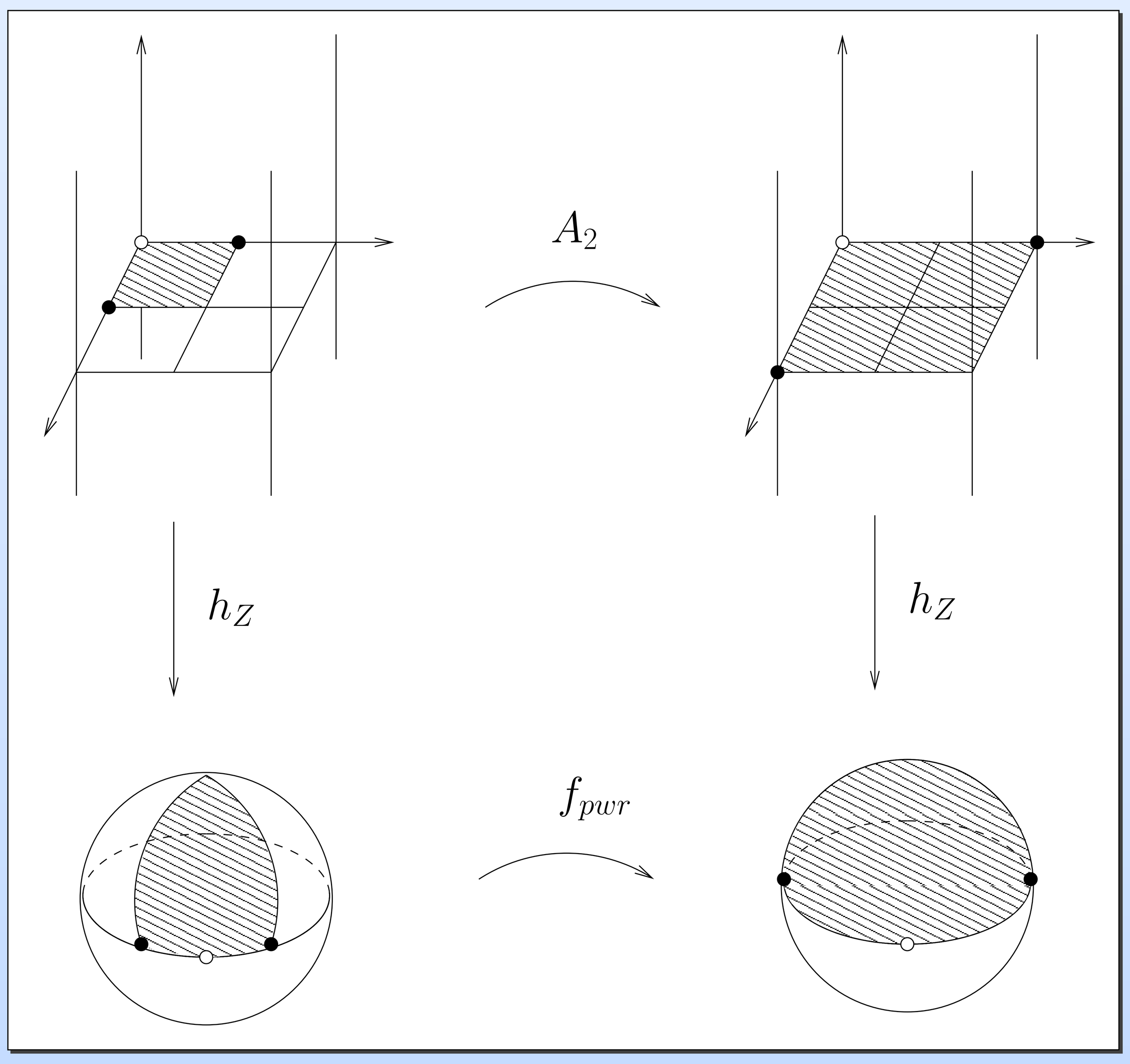
Note that following from (1) we have the equation $f^k \circ h = h \circ A^k$ for all k. Thus the dilatation of the uqr mapping f^k is exactly the dilatation of h^2 for all k.

Lattès-mappings on 3-manifolds

We present here two examples of Lattès-type uqr maps on 3-manifolds. For more, see [1]. Let us first show how a 4 to 1 uqr mapping arises on $\mathbb{S}^3$ via Zorich mapping, the higher dimensional counterpart of the planar exponential function, as presented by Rickman [5, p.15]) and later Mayer [4].

Denote the Zorich mapping by $h_Z : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$. This mapping is constructed by first subdividing space into infinite cylinders with integer lattice base in the x_1x_2-plane. Take a slice $Z = \{(x_1, x_2, x_3) \mid 0 < x_1, x_2 \leq 1, x_3 \in \mathbb{R}\}$ in $\mathbb{R}^3$ first to an infinite cylinder of radius 1 by a bilipschitz radial stretching, and then map the round cylinder quasiconformally onto the upper half-space $\mathbb{H}$ by the mapping $(r, \varphi, x_3) \mapsto (\exp(x_3), \varphi, \frac{\pi r}{2})$. The mapping h_Z is obtained by extending this map to a qr map of $\mathbb{R}^3$ to $\mathbb{R}^3 \setminus \{0\}$ by using reflections on faces of Z and $\partial\mathbb{H}$. So h_Z alternately maps neighbouring cylinders to the upper and lower half-space. The branch set of h_Z consists of the edges of Z and the reflected cyliders, i.e. $B_h = \mathbb{Z}^2 \times \mathbb{R}$.

Now if we semiconjugate the mapping $A_2 : x \mapsto 2x$ in $\mathbb{R}^3$ with the Zorich mapping h_Z, a 4 to 1 mapping $f_{pwr} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ is induced via equation 1. The mapping is further extended continuously to $\mathbb{S}^3$ by defining $f_{pwr}(0) = 0$ and $f_{pwr}(\infty) = \infty$. By theorem the mapping $f_{pwr} : \mathbb{S}^3 \to \mathbb{S}^3$ is uniformly quasiregular. The Julia set is the image of the x_1x_2-plane in $\mathbb{R}^3$ under the Zorich mapping, that is, a 2-sphere of radius 1, centered at the origin. The point $h_Z(0)$ is a repelling fixed point of f_{pwr}. The mapping f_{pwr} has two basins of attraction, the inside and the outside of the Julia set, the origin and infinity being superattractive fixed points. The branch set consists of six half-lines that meet at the origin and infinity.



Construction of a uqr mapping f_{pwr} on $\mathbb{S}^3$

As another example, we construct a uqr mapping on $\mathbb{P}^3$ with Julia set $\mathbb{P}^2$ as follows. We use the generalization of the power mapping on $\mathbb{S}^3$ and the 2 to 1 covering map $\pi : \mathbb{S}^3 \to \mathbb{P}^3$.

$$\begin{array}{ccc} \mathbb{S}^3 & \xrightarrow{f_{pwr}} & \mathbb{S}^3 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^3 & \xrightarrow{f_{\mathbb{P}^3}} & \mathbb{P}^3 \end{array} \tag{2}$$

Thus a mapping $f_{\mathbb{P}^3}$ is induced between the to projective spaces on the bottom of the diagram. The mapping $f_{\mathbb{P}^3}$ is uniformly quasiregular by the Lattès theorem . The Julia set of the mapping $f_{\mathbb{P}^3}$ is the set corresponding to the sphere $\mathbb{S}^2$ (radius 1, centered at the origin) in $\mathbb{S}^3$. That is, the Julia set is two-dimensional projective space $\mathbb{P}^2$ in the three-dimensional projective space $\mathbb{P}^3$. The mapping $f_{\mathbb{P}^3}$ has only one basin of attraction, since the two basins of attraction on $\mathbb{S}^3$ are identified by the covering map π.

Higher dimensions

In higher dimensions we can construct uqr mappings with a variety of Julia sets with the help of the generalized power mapping, chaotic uqr mapping f_{cube} from cubical decomposition, the generalized Tchebychev mapping f_T etc (see [1] for constructions). We obtain various interesting examples of uqr mappings of Lattès type.

Manifold	Mapping	Julia set
$\mathbb{S}^n$	f_{cube}	$\mathbb{S}^n$
$\mathbb{S}^n$	f_{pwr}	$\mathbb{S}^{n-1}$
$\mathbb{S}^n$	f_T	$\bar{\mathbb{D}}^{n-1}$
$\mathbb{P}^{2n+1}$	$f_{chaotic}$	$\mathbb{P}^{2n+1}$
$\mathbb{P}^{2n+1}$	$f_{\mathbb{P}^{2n+1}}$	$\mathbb{P}^{2n}$
T^n/Γ	$f_{quotient}$	T^n/Γ
$\mathbb{S}^n \times \mathbb{S}^m$	$f_{cube} \times f_{cube}$	$\mathbb{S}^n \times \mathbb{S}^m$

References

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This research has been partly funded by Magnus Ehrnrooth Foundation.