UNIFORMLY QUASIREGULAR MAPPINGS ON ELLIPTIC RIEMANNIAN MANIFOLDS

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Main result

Theorem

If there exists a uniformly quasiregular mapping on a compact Riemannian manifold M^n, there exists a quasiregular mapping $g : \mathbb{R}^n \to M^n$. In other words, the manifold M^n is quasiregularly elliptic.

A three-dimensional compact Riemannian manifold M^3 is elliptic if and only if there exists a uniformly quasiregular mapping on M^3.

Outline

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Quasiregular mappings

Let $D \subset \overline{\mathbb{R}}^n$ be a domain and $f : D \to \overline{\mathbb{R}}^n$ a non-constant mapping of the Sobolev class $W^{1,n}_{loc}(D)$. We consider only orientation-preserving mappings, which means that the Jacobian determinant $J_f(x) \geq 0$ for a.e. $x \in D$. Such a mapping is said to be *K-quasiregular*, where $1 \leq K < \infty$, if

$$\max_{|h|=1} |f'(x)h| \leq K \min_{|h|=1} |f'(x)h|$$

for a.e. $x \in D$, when f' is the formal matrix of weak derivatives.

Quasiregular mappings on manifolds

We generalize the definition to Riemannian manifolds with the help of bilipschitz-continuous coordinate charts:

Let M and N be n-dimensional Riemannian manifolds. A non-constant continuous mapping $f : M \to N$ is K-quasiregular if for every $\varepsilon > 0$ and every $m \in M$ there exists bilipschitz-continuous charts (U, φ), $m \in U$, and (V, ψ), $f(m) \in V$, so that the mapping $\psi \circ f \circ \varphi^{-1}$ is $(K + \varepsilon)$-quasiregular.

A non-constant quasiregular mapping can be redefined in a set of measure zero such that the mapping is made continuous, open and discrete.

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Uniformly quasiregular mappings

Let M be a compact Riemannian manifold. A non-injective mapping f from a domain $D \subset M$ onto itself is called *uniformly quasiregular* (uqr) if there exists a constant $1 \leq K \leq \infty$ such that all the iterates f^k are K-quasiregular.

We will assume our uniformly quasiregular mappings to be non-injective. In other words, we assume the *branch set*

$$B_f = \{x \in M \mid f \text{ is not locally homeomorphic at } x\}$$

to be non-empty.

Fatou and Julia sets

We define the *Fatou set* of a uqr mapping $f : M \to M$ as

$$\mathcal{F}_f = \{x \in M \ : \quad \text{there exists an open set } U \subset M \text{ such that } x \in U \text{ and the family } \{f^k \mid k \in \mathbb{Z}_+\}|U \text{ is normal}\}.$$

The *Julia set* of the mapping f is $\mathcal{J}_f = M \setminus \mathcal{F}_f$.
By the definition, Fatou sets are open, and therefore Julia sets are closed. Both Fatou and Julia set are completely invariant.

Theorem

Let $f : M \to M$ be a uqr mapping, with $\deg(f) \geq 2$, on a compact Riemannian manifold M. Then the Julia set $\mathcal{J}_f$ of the mapping f is non-empty.

Outline

Rescaling principle on a manifold

Theorem

Let $\mathcal{F}$ be a family of K-quasiregular mappings $f : \Omega \to M^n$, where M^n is a closed compact Riemannian manifold, and $\Omega \subset \mathbb{R}^n$ a domain. If the family $\mathcal{F}$ is not equicontinuous at a point $a \in \Omega$, there exists a sequence of real numbers $r_j \searrow 0$, a sequence of points $a_j \to a$, a sequence of mappings $\{f_j\} \subset \mathcal{F}$ and a non-constant K-quasiregular mapping $h : \mathbb{R}^n \to M^n$ such that

$$f_j(r_j x + a_j) \to h(x)$$

locally uniformly in $\mathbb{R}^n$. Especially M^n is K-quasiregularly elliptic.

Manifolds supporting uqr mappings are elliptic I

We use the rescaling principle to prove the first part of our main theorem.

Let $f : M^n \to M^n$ be a K'-quasiregular mapping on a smooth, oriented, compact Riemannian n-manifold M, with a non-empty branch set. Let $x_0 \in \mathcal{J}_f$, and let $\varphi : U \to \mathbb{R}^n$ be such a L-bilipschitz-continuous coordinate mapping in some neighbourhood U of the point x_0 that $\varphi(x_0) = 0$ and $\varphi(U) = \mathbb{B}(0,1)$.

Define a composite mapping

$$f_\nu := f^\nu \circ \varphi^{-1}|_{\mathbb{B}(0,1)} : \mathbb{B}(0,1) \to M^n$$

from the iterates f^ν of f and the coordinate mapping φ.

Manifolds supporting uqr mappings are elliptic II

All the mappings f_ν, $\nu \in \mathbb{N}$, are K-quasiregular with the same constant $K = K(K', L)$.

The family of mappings $\mathcal{F} = \{f_\nu \mid \nu \in \mathbb{N}\}$ is not normal, since $x_0 = \varphi^{-1}(0) \in \mathcal{J}_f$, meaning that $0 \in \mathcal{J}_{\mathcal{F}}$.

Starting with a uqr mapping $f : M \to M$ we have thus constructed a family $\mathcal{F} = \{f_\nu \mid \nu \in \mathbb{N}\}$ of K-quasiregular mappings

$$f_\nu = f^\nu \circ \varphi^{-1}|_{\mathbb{B}(0,1)} : \mathbb{B}(0,1) \to M,$$

and the family $\mathcal{F}$ is not normal at the origin. Now we can use the rescaling principle, and as a limit mapping we get a K-quasiregular mapping $g : \mathbb{R}^n \to M$.

Outline

3-dimensional elliptic manifolds

Thurston's conjecture states that after a three-manifold is splitten into its connected sum and the Jaco-Shalen-Johannson torus decomposition, the remaining components each admit exactly one of the following model geometries: $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathcal{SL}}_2(\mathbb{R})$, Nil, or Sol. (G. Perelman 2002)

By J. Jormakka (1988), the only closed elliptic 3-manifolds are those manifolds which are covered by $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$.

We already know that manifolds covered by $\mathbb{S}^3$ support uqr mappings (K. Peltonen 1999). The Euclidean space forms and manifolds covered by $\mathbb{S}^2 \times \mathbb{R}$ are considered in the thesis.

Lattès-type uqr mappings I

Let Υ be a discrete group of isometries of $\mathbb{R}^n$. A mapping $h : \mathbb{R}^n \to M$ is *automorphic* with respect to Υ in the *strong sense* if

1. $h \circ \gamma = h$ for any $\gamma \in \Upsilon$,
2. Υ acts transitively on the fibres $\mathcal{O}_y = h^{-1}(y)$.

By the latter condition we mean that for any two points x_1, x_2 with $h(x_1) = h(x_2)$ there is an isometry $\gamma \in \Upsilon$ such that $x_2 = \gamma(x_1)$.

Lattès-type uqr mappings II

Theorem

Let Υ be a discrete group such that $h : \mathbb{R}^n \to M$ is automorphic with respect to Υ in the strong sense. If there is a similarity $A = \lambda\mathcal{O}$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $\mathcal{O}$ an orthogonal transformation, such that

$$A\Upsilon A^{-1} \subset \Upsilon,$$

then there is a unique solution $f : h(\mathbb{R}^n) \to h(\mathbb{R}^n)$ to the Schröder functional equation

$$f \circ h = h \circ A$$

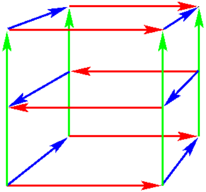
and f is a uniformly quasiregular mapping.

Note that $f^k \circ h = h \circ A^k$ for all k.

3-dimensional Euclidean space forms I

By J. Wolf (1984) there are just 6 affine diffeomorphism classes of compact connected flat 3-dimensional Riemannian manifolds. They are represented by the manifolds $\mathbb{R}^3/\Gamma$, where Γ is one of 6 groups. We consider here the case G_2 and the corresponding manifold M_2 from the thesis as an example.

In the polyhedron schema for M_2 we have a cube where we identify opposite vertical faces and glue the top to the bottom of the cube with a twist of angle π.



3-dimensional Euclidean space forms II

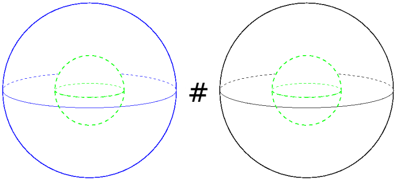
We obtain a Lattès-type uqr mapping on M_2 as follows:

- Construct a covering map $g_2 : T^3 \to M_2$ such that T^3 covers the manifold twice.
- Define mapping $F_2 : \mathbb{R}^3 \to \mathbb{R}^3$ as $F_2 : x \mapsto 3x$.
- F_2 and covering map π_1 induce a mapping F_2' on the torus.
- The mapping F_2 descends to a Lattès-type uqr mapping $f_2 : M_2 \to M_2$.

$$
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{F_2} & \mathbb{R}^3 \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
T^3 & \xrightarrow{F_2'} & T^3 \\
{\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle g_2} \\
M_2 & \xrightarrow{f_2} & M_2
\end{array}
$$

Manifolds covered by $\mathbb{S}^2 \times \mathbb{R}$

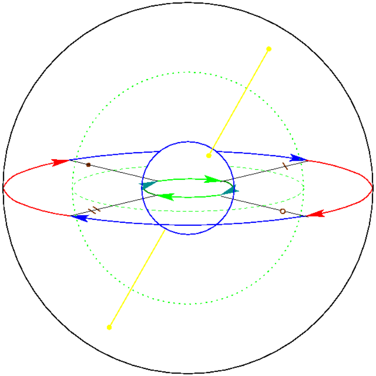
There are only two orientable compact 3-manifolds which have $\mathbb{S}^2 \times \mathbb{R}$ as the Riemannian covering space: the sphere bundle $\mathbb{S}^2 \times \mathbb{S}$ and the connected sum of two projective 3-spaces $\mathbb{P}^3 \# \mathbb{P}^3$. We take here $\mathbb{P}^3 \# \mathbb{P}^3$ as an example.



The manifold $\mathbb{P}^3 \# \mathbb{P}^3$ is obtained by identifying diametrical points of the boundary spheres K_1 and K_2 of $\mathbb{S}^2 \times I$.

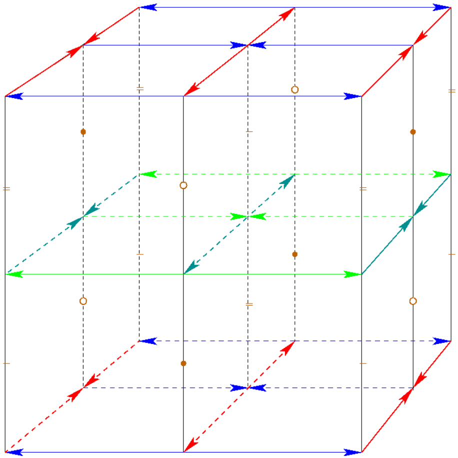
Manifold $\mathbb{P}^3 \# \mathbb{P}^3$ I

The dotted 2-sphere separates this manifold into two punctured projective spaces. The fibres are the radii of $\mathbb{S}^2 \times I$; any two diametrical radii form one fibre.



Manifold $\mathbb{P}^3 \# \mathbb{P}^3$ II

Let us look at the projective space $\mathbb{P}^3 \# \mathbb{P}^3$ as a block in $\mathbb{R}^3$:



Manifold $\mathbb{P}^3 \# \mathbb{P}^3$ III

We define a covering map $g : \mathbb{R}^3 \to \mathbb{P}^3 \# \mathbb{P}^3$ which takes each one-by-one cube in $\mathbb{R}^3$ to $\mathbb{P}^3 \# \mathbb{P}^3$ according to the previous picture. We use the mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$, where $F : x \mapsto 2x$ for any $x \in \mathbb{R}^3$. The mappings F and g again induce a mapping f to the manifold:

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "F"] \arrow[d, "g"'] & \mathbb{R}^3 \arrow[d, "g"] \\ \mathbb{P}^3 \# \mathbb{P}^3 \arrow[r, "f"] & \mathbb{P}^3 \# \mathbb{P}^3 \end{tikzcd}$$

The mapping f is a well-defined and uniformly quasiregular mapping of Lattès type.

Outline

References and Follow-up

Kangaslampi, Riikka
Uniformly quasiregular mappings on elliptic Riemannian manifolds.
Ann. Acad. Sci. Fenn. Math. Diss. 151 (2008)

Since the thesis was published already in 2008, some progress has been made after that:

- Astola, L., Kangaslampi, R., and Peltonen, K.
  Lattés-type mappings on compact manifolds
  (preprint)
- Kangaslampi, R., and Peltonen, K.
  Uniformly quasiregular mappings on lens spaces
  (in preparation)

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