Aalto University

Groups acting on hyperbolic buildings

Aalto University, Analysis and Geometry Seminar

Riikka Kangaslampi

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Abstract

We construct and classify all groups, given by triangular presentations associated to the smallest thick generalized quadrangle, that act simply transitively on the vertices of hyperbolic triangular buildings of the smallest non-trivial thickness. In analogy with the $\widetilde{A}_2$ case, we find both torsion and torsion free groups acting on the same building.

These groups are the first examples of cocompact lattices acting simply transitively on vertices of hyperbolic triangular Kac-Moody buildings that are not right-angled.

Collaborators

Joint work with

- Dr. Alina Vdovina, Newcastle University, UK
- Prof. Lisa Carbone, Rutgers SUNJ, USA
- Dr. Frédéric Haglund, Université Paris-Sud 11, France

Preliminaries
Definitions

A graph is *bipartite*, if its set of vertices can be partitioned into two disjoint subsets P and Q (“black” and “white” vertices) such that no vertices in the same subset lie on common edge.

A *generalized m-gon* is a connected, bipartite graph of diameter m and girth (length of the smallest circuit) $2m$, in which each vertex lies on at least two edges.

A generalized m-gon is *thick* if all vertices lie on at least three edges.

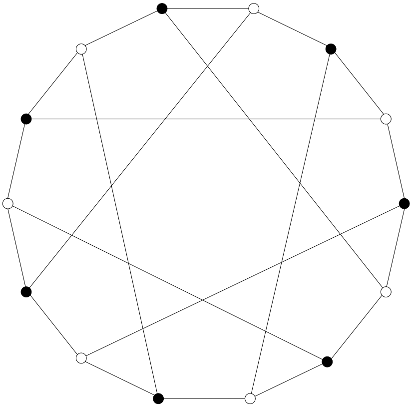

Figure: Generalized 3-gon: bipartite graph with diameter 3, girth 6.

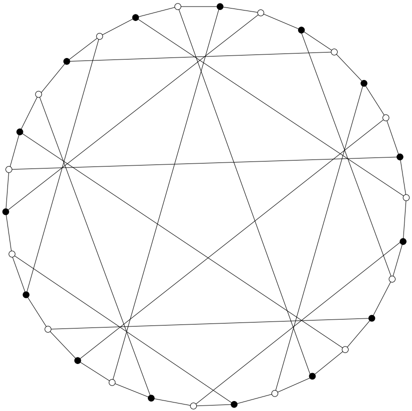

Figure: Generalized 4-gon: bipartite graph with diameter 4, girth 8.

A *polyhedron* is a two-dimensional complex, which is obtained from several oriented m-gons with words on their boundary, by identifying sides with the same letters, respecting orientation.

A *link* is a graph, obtained as the intersection of a polyhedron and a small sphere centered at a vertex.

Preliminaries II
Hyperbolic buildings

Let $\mathcal{P}(p, m)$ be a tesselation of the hyperbolic plane by regular polygons with p sides, with angles π/m, $m \in \mathbb{Z}_+$, in each vertex. A *hyperbolic building* is a polygonal complex X, which can be expressed as the union of subcomplexes called apartments, such that

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1. Every apartment is isomorphic to $\mathcal{P}(p, m)$.
2. For any two polygons of X, there is an apartment containing both of them.
3. For any two apartments $A_1, A_2 \in X$ containing same polygon, there exists an isomorphism $A_1 \to A_2$ fixing $A_1 \cap A_2$.

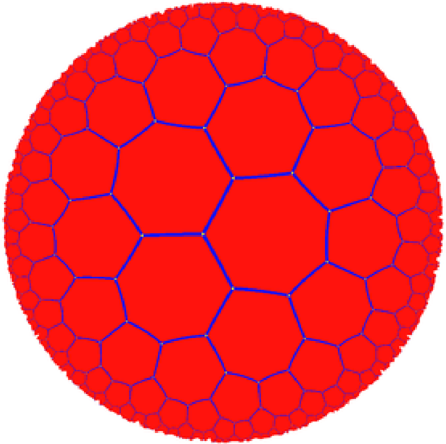

Figure: Example of a tiling of the hyperbolic plane with hexagons.

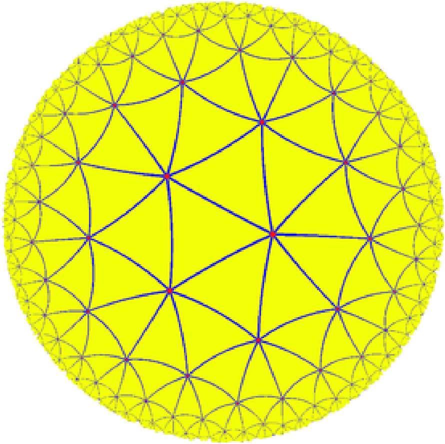

Figure: Example of a tiling of the hyperbolic plane with triangles.

Let C_p be a polyhedron whose faces are p-gons and links are generalized m-gons with $mp > 2m + p$. We equip every face of C_p with the hyperbolic metric such that all sides of the polygons are geodesics and all angles are π/m. Then the universal covering of such a polyhedron is a hyperbolic building. (Gaboriau & Paulin 2001)

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$\Rightarrow$ To construct hyperbolic buildings with cocompact group actions, it is sufficient to construct finite polyhedra with appropriate links.

Polygonal presentations

Let $G_1, \ldots, G_n$ be disjoint connected bipartite graphs. Let P_i and Q_i be the sets of black and white vertices respectively in G_i. Let $P = \bigcup P_i$, $Q = \bigcup Q_i$, $P_i \cap P_j = \emptyset$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to Q$.

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2. given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for some $x_3, \ldots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident in some G_i;

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3. given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

We can associate a polyhedron K on n vertices with each polygonal presentation $\mathcal{K}$ as follows: for every cyclic k-tuple $(x_1, x_2, x_3, \ldots, x_k)$ we take an oriented k-gon with the word $x_1 x_2 x_3 \ldots x_k$ written on the boundary. To obtain the polyhedron we identify the corresponding sides of the polygons, respecting orientation.

A polyhedron K which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \ldots, G_n$ as vertex-links. (Vdovina 2002)

Triagonal presentations for generalized 4-gon

We construct all polygonal presentations with $k = 3$ and $n = 1$ and for which the graph G_1 is a generalized 4-gon.

The smallest thick generalized 4-gon can be presented in the following way:

- "points" in P are pairs (i, j), where $i, j = 1, ..., 6$, $i \neq j$
- "lines" in Q are triples $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ of those pairs, where i_1, i_2, i_3, j_1, j_2 and j_3 are all different.

(Tits & Weiss 2002)

(12)	(34)	(56)
(12)	(35)	(46)
(12)	(36)	(45)
(13)	(24)	(56)
(13)	(25)	(46)
(13)	(26)	(45)
(14)	(23)	(56)
(14)	(25)	(36)
(14)	(26)	(35)
(15)	(23)	(46)
(15)	(24)	(36)
(15)	(26)	(34)
(16)	(23)	(45)
(16)	(24)	(35)
(16)	(25)	(34)

We denote the elements of P by x_i and the elements of Q by y_i, $i = 1, 2, \ldots, 15$. In all cases we define the basic bijection $\lambda : P \to Q$ by $\lambda(x_i) = y_i$.

We build a tableau as follows: For each row take three pairs $(i_1, j_1), (i_2, j_2)$, and (i_3, j_3), where i_1, i_2, i_3, j_1, j_2 and j_3 are all different and in $1, 2, \ldots, 6$. These are our points: $x_1 = (1, 2)$, $x_2 = (1, 3)$,..., $x_{15} = (5, 6)$.

Then we label the rows by $y_1, \ldots, y_{15}$ in such a way that the result is an incidence tableau that gives a triagonal presentation with the basic bijection λ.

$$
\begin{array}{cccccccc}
(12) & (34) & (56) & \Rightarrow & x_1 & x_{10} & x_{15} \\
(12) & (35) & (46) & & x_1 & x_{11} & x_{14} \\
(12) & (36) & (45) & & x_1 & x_{12} & x_{13} \\
(13) & (24) & (56) & & x_2 & x_7 & x_{15} \\
(13) & (25) & (46) & & x_2 & x_8 & x_{14} \\
(13) & (26) & (45) & & x_2 & x_9 & x_{13} \\
(14) & (23) & (56) & & x_3 & x_6 & x_{15} \\
(14) & (25) & (36) & & x_3 & x_8 & x_{12} \\
(14) & (26) & (35) & & x_3 & x_9 & x_{11} \\
(15) & (23) & (46) & & x_4 & x_6 & x_{14} \\
(15) & (24) & (36) & & x_4 & x_7 & x_{12} \\
(15) & (26) & (34) & & x_4 & x_9 & x_{10} \\
(16) & (23) & (45) & & x_5 & x_6 & x_{13} \\
(16) & (24) & (35) & & x_5 & x_7 & x_{11} \\
(16) & (25) & (34) & & x_5 & x_8 & x_{10}
\end{array}
$$

y_1 :	x_1	x_{10}	x_{15}
y_2 :	x_1	x_{11}	x_{14}
y_{10} :	x_1	x_{12}	x_{13}
y_3 :	x_2	x_7	x_{15}
y_9 :	x_2	x_8	x_{14}
y_{15} :	x_2	x_9	x_{13}
y_{14} :	x_3	x_6	x_{15}
y_4 :	x_3	x_8	x_{12}
y_{13} :	x_3	x_9	x_{11}
y_6 :	x_4	x_6	x_{14}
y_7 :	x_4	x_7	x_{12}
y_{11} :	x_4	x_9	x_{10}
y_8 :	x_5	x_6	x_{13}
y_{12} :	x_5	x_7	x_{11}
y_5 :	x_5	x_8	x_{10}

y_1 :	$\mathbf{x_1}$	$\mathbf{x_{10}}$	x_{15}
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y_{12} :	x_5	x_7	x_{11}
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$$
\begin{array}{llll}
y_1 : & \mathbf{x_1} & \mathbf{x_{10}} & \mathbf{x_{15}} \\
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y_9 : & x_2 & x_8 & x_{14} \\
y_{15} : & \mathbf{x_2} & x_9 & x_{13} \\
y_{14} : & x_3 & x_6 & x_{15} \\
y_4 : & x_3 & x_8 & x_{12} \\
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y_9 :	$\mathbf{x_2}$	x_8	x_{14}
y_{15} :	$\mathbf{x_2}$	x_9	x_{13}
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y_{11} :	x_4	$\mathbf{x_9}$	x_{10}
y_8 :	x_5	x_6	x_{13}
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y_{11} :	x_4	x_9	x_{10}
y_8 :	x_5	x_6	x_{13}
y_{12} :	x_5	x_7	x_{11}
y_5 :	x_5	x_8	x_{10}

This corresponds to the triplets

$$
\begin{array}{lll lll}
(x_1, & x_1, & x_{10}), & (x_5, & x_5, & x_8) \\
(x_1, & x_{15}, & x_2), & (x_5, & x_{10}, & x_{12}) \\
(x_2, & x_{11}, & x_9), & (x_6, & x_6, & x_{14}) \\
(x_2, & x_{14}, & x_3), & (x_7, & x_7, & x_{12}) \\
(x_3, & x_7, & x_4), & (x_8, & x_{13}, & x_9) \\
(x_3, & x_{15}, & x_{13}), & (x_9, & x_{14}, & x_{15}) \\
(x_4, & x_8, & x_6), & (x_{10}, & x_{13}, & x_{11}) \\
(x_4, & x_{12}, & x_{11}) & & &
\end{array}
$$

As a result, we obtain 21196 different incidence tableaus, which can be divided into 45 different equivalence classes of triagonal presentations.

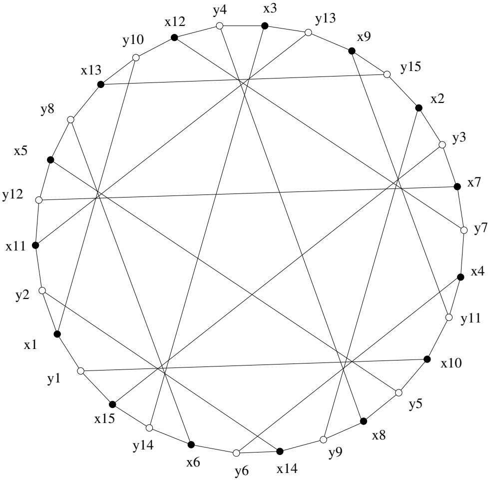

Figure: Graph G_1 for the obtained presentation T_1 with $\lambda(x_i) = y_i$.

For a polygonal presentation $T_i, i = 1, \ldots, 45$, take 15 oriented regular hyperbolic triangles with angles $\pi/4$, write words from the presentation on their boundaries and glue together sides with the same letters, respecting orientation.

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The result is a hyperbolic polyhedron with one vertex and 15 triagonal faces, and its universal covering is a triangular hyperbolic building. The fundamental group Γ_i, $i = 1, \ldots, 45$ of the polyhedron acts simply transitively on vertices of the building. The group Γ_i has 15 generators and 15 relations (from T_i).

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To distinguish groups Γ_i it is sufficient to distinguish the isometry classes of polyhedra, according to the Mostow-type rigidity for hyperbolic buildings (Xie 2006).

Dual graphs

We define dual graphs with 30 vertices: first 15 correspond to the edges of the triangles in a triagonal presentation, and the second 15 correspond to the faces of the triangles.

There is an edge between vertices i (from $1 - 15$) and j (from $16 - 30$), if edge i is on the boundary of the face j in the triagonal presentation. Thus we obtain bipartite trivalent graphs with 30 vertices.

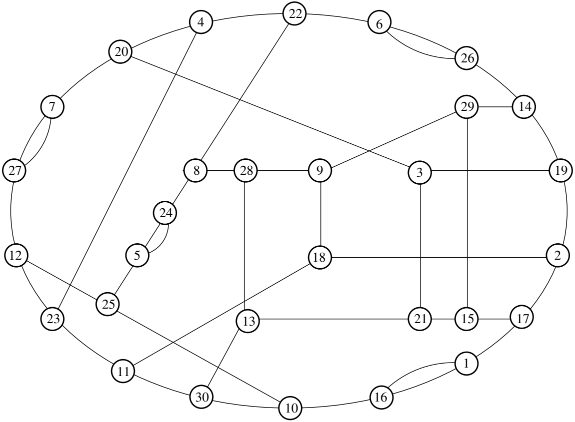

Figure: Dual graph of G_1

The presentations occur pair-wise: in the set of the 45 non-equivalent triagonal representations we have 22 pairs of isomorphic dual graphs. Therefore, when we take the dual graphs into account as invariants to distinguish the presentations, we finally have 23 non-isomorphic groups.

Classification

We obtain a complete classification of groups acting simply transitively on the vertices of hyperbolic triangular buildings of the smallest non-trivial thickness, since simply-transitive action on vertices is an analogue of a triangle presentation. (Cartwright-Mantero-Steger-Zappa 1993)

“Proof“: It is enough to consider quadrangles, since there are no three-valent generalized hexagons or octagons. Minimal non-trivial thickness is obviously 3.

Result 1. There are 45 non-equivalent torsion free triangle presentations associated to the smallest thick generalized quadrangle. These give rise to 23 non-isomorphic torsion free groups, acting simply transitively on vertices of triangular hyperbolic buildings of smallest non-trivial thickness.

If we allow torsion, that is triangles of the type (x_i, x_i, x_i), we obtain many more presentations:

Result 2. There are 7159 non-equivalent triangle presentations corresponding to groups with torsion associated to the smallest generalized quadrangle. These give rise to 168 non-isomorphic groups, acting on vertices of a triangular hyperbolic buildings with the smallest thick generalized quadrangle as the link of each vertex.

It is known (Swiatkowski 1998) that up to isomorphism, there are at most two triangular hyperbolic buildings with the smallest generalized quadrangle as the link of each vertex, admitting a simply transitive action.

Comparing the links of order 2 in our polyhedra, we can divide the presentations into two sets regarding whether the obtained group acts on building 1 or 2.

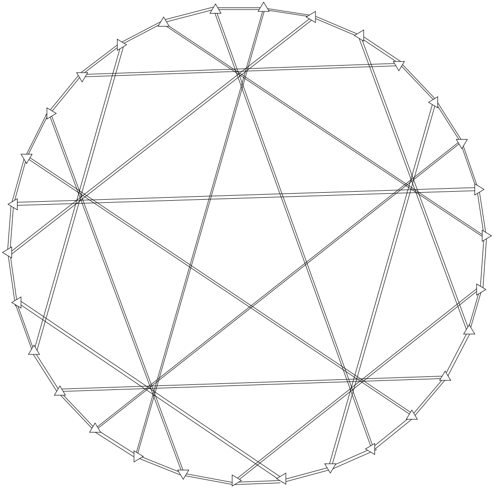

Figure: A 2-link from far, details missing

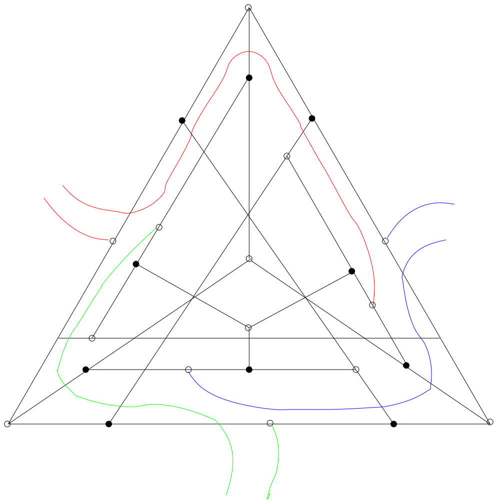

Figure: A part of the 2-link more closely

Torsion-free groups:

(1)	$T_1, T_4, T_5, T_8, T_9, T_{11}, T_{13}, T_{15}, T_{17}, T_{18}, T_{19}, T_{23}$
(2)	$T_2, T_3, T_6, T_7, T_{10}, T_{12}, T_{14}, T_{16}, T_{20}, T_{21}, T_{22}$

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(1)	T_1, T_4, T_5, T_8, T_9, T_{11}, T_{13}, T_{15}, T_{17}, T_{18}, T_{19}, T_{23}
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Torsion groups:

(1)	$T_{24}, T_{27}, T_{29}, T_{33}, \ldots, T_{189}$
(2)	$T_{25}, T_{26}, T_{28}, T_{30}, \ldots, T_{191}$

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In analogy with the $\widetilde{A}_2$ case, we find both torsion and torsion free groups acting on the same building.

The building number (2) coincides with the Kac-Moody building with the minimal generalized quadrangle as the link of each vertex and equilateral triangular chambers.

Construction of polyhedra with n-gonal faces

Given a generalized quadrangle G we shall denote by G' the graph arising by calling black (respectively white) vertices of G black (respectively white) vertices of G'.

Starting from on of our previous torsion-free triagonal presentations, we construct a polyhedron, whose faces are m-gons and whose m vertices have links G or G'.

Let $w = z_1 \dots z_m$ be a word of length m in three letters a, b and c such that $z_1 = a$, $z_2 = b$, $z_3 = c$, $z_m \neq a$, and $z_t \neq z_{t+1}$ for all $t = 1, \dots, m-1$.

Let $w = z_1 \dots z_m$ be a word of length m in three letters a, b and c such that $z_1 = a$, $z_2 = b$, $z_3 = c$, $z_m \neq a$, and $z_t \neq z_{t+1}$ for all $t = 1, \dots, m-1$.

Define 45 such words w: For each of the 15 triples (x_i, x_j, x_k) in K take a 3-cover (x_i^1, x_j^2, x_k^3), (x_k^1, x_i^2, x_j^3) and (x_j^1, x_k^2, x_i^3).

(By glueing together triangles with these words on the boundary, we would obtain a polyhedron with 45 triagonal faces and 3 vertices, each of them with the group G as the link.)

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(By glueing together triangles with these words on the boundary, we would obtain a polyhedron with 45 triagonal faces and 3 vertices, each of them with the group G as the link.)

Then we construct 45 m-tuples: for each triple $(x_\alpha^1, x_\beta^2, x_\gamma^3)$ we define an m-tuple, which corresponds a word w with $a = x_\alpha^1$, $b = x_\beta^2$ and $c = x_\gamma^3$.

If we glue together the m-gones with these words on the boundary, we obtain a polyhedron with 45 m-gonal faces and m vertices, wich all have the link G or G'.

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The type of the link can be seen from the letters of the edges meeting at that vertex. Set

$$Sign(ab) = Sign(bc) = Sign(ca) = 1$$

and

$$Sign(ba) = Sign(cb) = Sign(ac) = -1.$$

Then for vertex $t = 1, \ldots, m-1$ the group G_t of the link is G if $Sign(z_t, z_{t+1}) = 1$ and G' if $Sign(z_t, z_{t+1}) = -1$. For the last vertex we have $G_m = G$ if $Sign(z_m, a) = 1$ and G' if $Sign(z_m, a) = -1$.

Let us denote the set of m-tuples by T_m. Then we have the obtained following:

Result 3. The above constructed subset $T_m \subset P \times \cdots \times P$ is a polygonal presentation. It defines a polyhedron X whose faces are m-gones and whose m vertices have links G or G'.

Corollary: The universal covering of X is a hyperbolic building with m-gonal chambers and links G and G'.

The story continues...

Surface subgroups? (already found in some of these groups)
Residual finiteness?
Kazdan's property T?
Buildings with different links?
Geometric or combinatorial construction instead of a computer search?

Some references

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