Inverse Problems, Invisibility Cloaking and Wormholes

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based on collaboration with:

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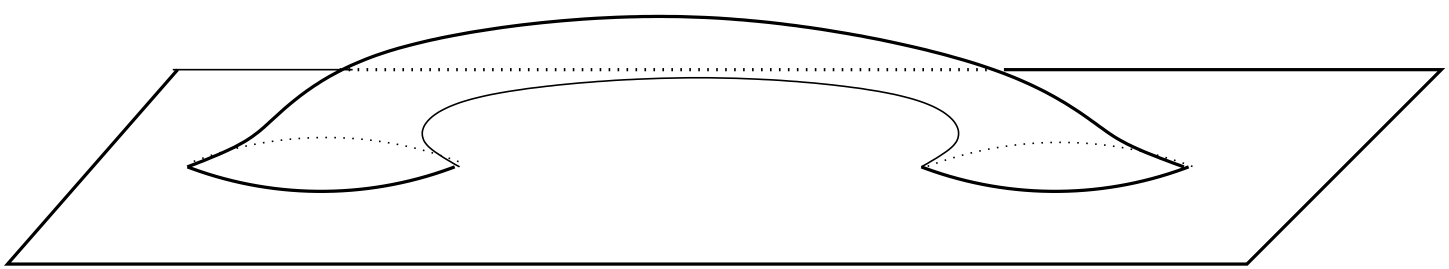
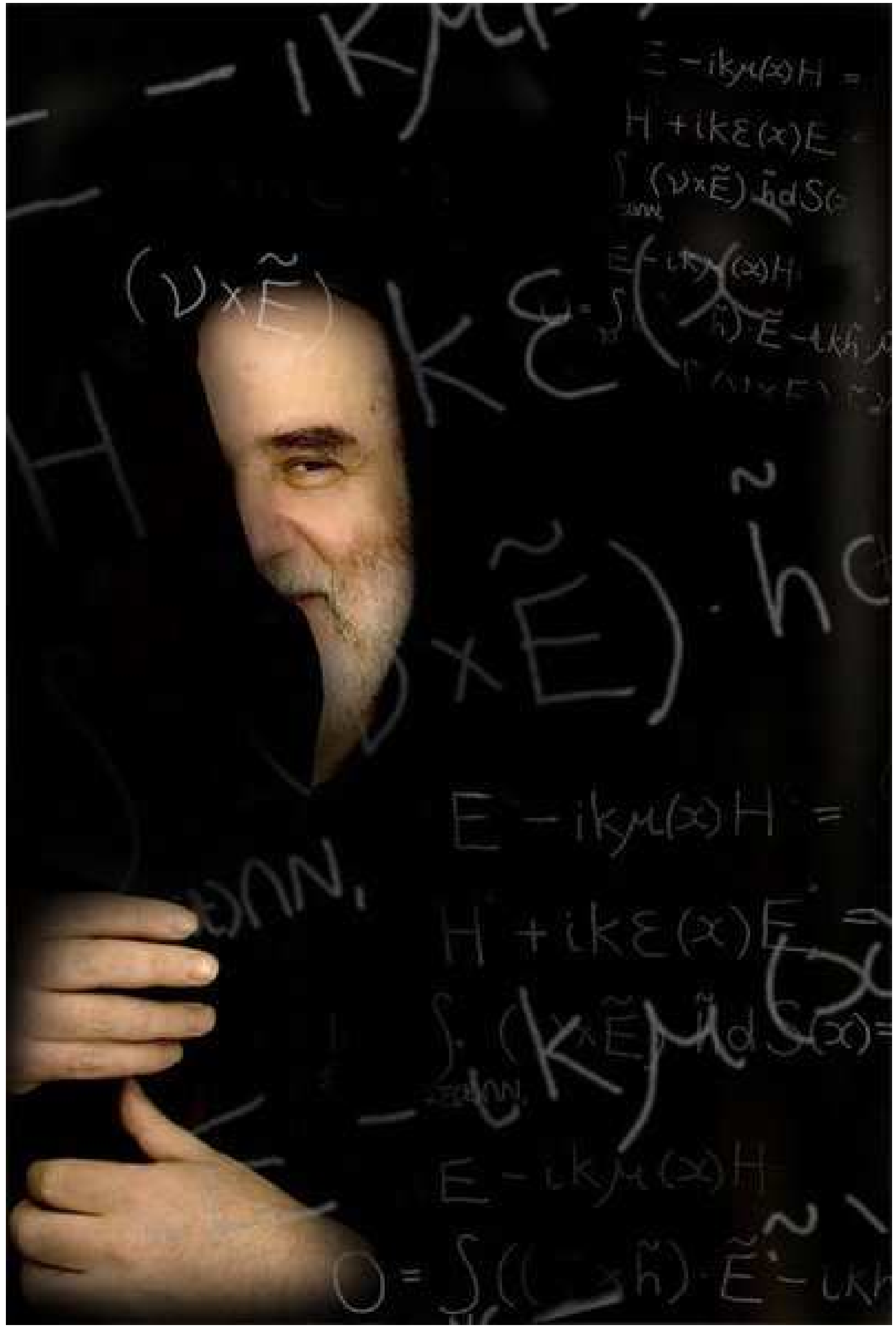
Inverse problem for Maxwell's equation:
Do the electromagnetic measurements on the boundary determine the inside of a body?

Invisibility cloaking:
Can we coat a body with a special material so that it appear like homogeneous material in all measurements?

Wormholes and virtual magnetic monopoles: Can we construct from metamaterials objects that function as invisible tunnels?





Artistic illustration by M. and J. Levin.

1 Inverse conductivity problem

Consider a body $\Omega \subset \mathbb{R}^d$. An electric potential $u(x)$ causes the current

$$J(x) = -\sigma(x)\nabla u(x).$$

If the current has no sources inside the body, we have

$$\nabla \cdot \sigma(x)\nabla u(x) = 0.$$

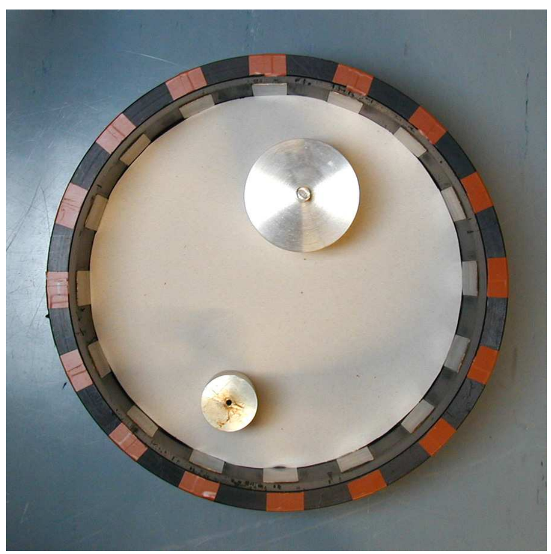
Conductivity equation

$$\nabla\cdot\sigma(x)\nabla u(x)=0 \quad \text{on } \Omega\subset\mathbb{R}^d.$$

Inverse problem: Do the measurements made on the boundary determine the conductivity, that is, does the Dirichlet-to-Neumann operator Λ_σ,

$$\Lambda_\sigma(u|_{\partial\Omega})=\nu\cdot\sigma\nabla u|_{\partial\Omega}$$

determine the conductivity $\sigma(x)$ in Ω?



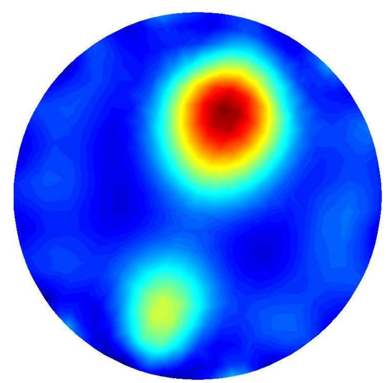


Figure: EIT at University of Kuopio.

Assume that the current $J(x)$ vanishes and $\sigma = 1$ outside $B(0, R)$. Then the problem

$$\nabla\cdot\sigma(x)\nabla u(x) = -\nabla\cdot J(x) \quad \text{on } \mathbb{R}^d,$$
$$u(x) \text{ satisfies radiation condition when } |x| \to \infty$$

is equivalent to

$$\Delta u(x) = -\nabla\cdot J(x) \quad \text{on } \mathbb{R}^d \setminus \Omega,$$
$$\nu\cdot\nabla u|_{\partial\Omega} = \Lambda_\sigma(u|_{\partial\Omega}),$$
$$u(x) \text{ satisfies radiation condition when } |x| \to \infty.$$

For other physical problems, we define Λ as

$$\Lambda(u|_{\partial\Omega}) = \nu\cdot\nabla u|_{\partial\Omega} \quad \text{for} \quad (\Delta + k^2 n(x))u = 0,$$

$$\Lambda(\psi|_{\partial\Omega}) = \nu\cdot\nabla \psi|_{\partial\Omega}, \quad \text{for} \quad (\Delta + q(x) + E)\psi = 0,$$

$$\Lambda(\nu \times E|_{\partial\Omega}) = \nu \times H|_{\partial\Omega}, \quad \text{for Maxwell's equations.}$$

Similar kind of operators can be defined also in time domain, for example

$$\Lambda(\nu \times E|_{\partial\Omega\times\mathbb{R}}) = \nu \times H|_{\partial\Omega\times\mathbb{R}}, \quad \text{for Maxwell's equations.}$$

Some Positive results for inverse conductivity problem:

- Calderón 1980: Solution of the linearized inverse conductivity problem.
- Sylvester-Uhlmann 1987: Uniqueness of inverse conductivity problem in $\mathbb{R}^d$, $d \geq 3$.
- Nachman 1996: Uniqueness inverse conductivity problem problem in $\mathbb{R}^2$
- Astala-Päivärinta 2006: Uniqueness of Calderón's problem in $\mathbb{R}^2$ with L^∞-conductivity.
- Astala-L.-Päivärinta 2005: Inverse problem for anisotropic L^∞-conductivity in $\mathbb{R}^2$.

All these results need assumptions like

$$c_1 I \leq \sigma(x) \leq c_2 I, \quad c_1, c_2 > 0.$$

Next we consider non-uniqueness results.

Invariant formulation. Assume $d \geq 3$ and $\Omega \subset \mathbb{R}^d$. Using the conductivity σ we can define a Riemannian metric

$$g^{jk}(x) = (\det \sigma(x))^{-1/(n-2)} \sigma^{jk}(x).$$

Then conductivity equation is the Laplace-Beltrami equation

$$\Delta_g u = 0 \quad \text{in } \Omega,$$

where

$$\Delta_g u = \sum_{j,k=1}^{d} g^{-1/2} \frac{\partial}{\partial x^j} (g^{1/2} g^{jk} \frac{\partial}{\partial x^k} u)$$

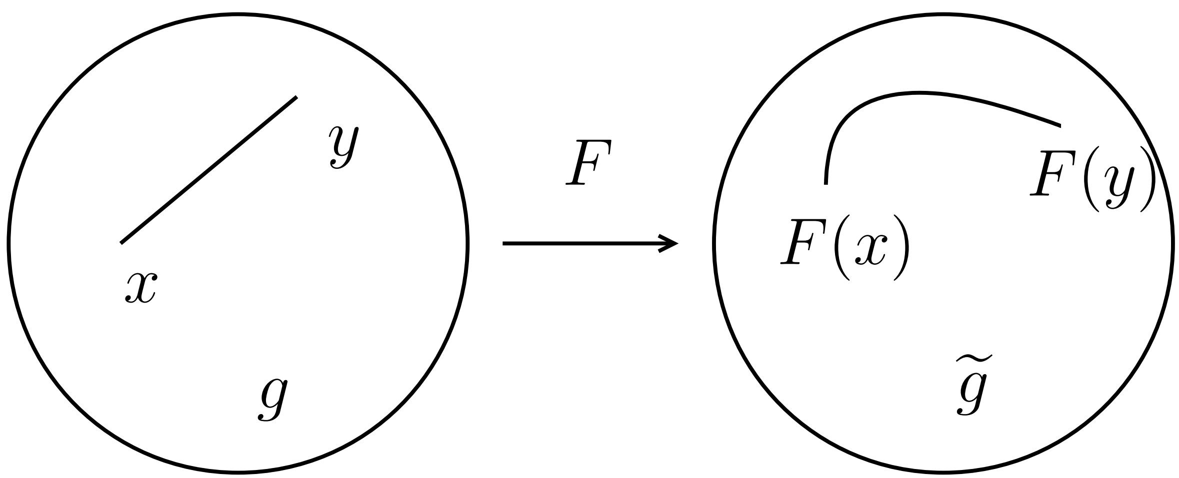
and $g = \det(g_{ij})$, $[g_{ij}] = [g^{jk}]^{-1}$.

Inverse problem: Can we determine the Riemannian metric g by knowing

$$\Lambda_g : u|_{\partial\Omega} \mapsto \partial_\nu u|_{\partial\Omega}, \quad \Delta_g u = 0 \ ?$$

Consider a diffeomorphism, that is, a smooth transformation $F : \Omega \to \Omega$ fixing the boundary. Then $F : (\Omega, g) \to (\Omega, \widetilde{g})$ is an isometry, where $\widetilde{g} = F_* g$

$$(F_* g)_{jk}(y) = \sum_{p,q=1}^{d} \frac{\partial x^p}{\partial y^j} \frac{\partial x^q}{\partial y^k} \, g_{pq}(x) \Bigg|_{x=F^{-1}(y)} .$$



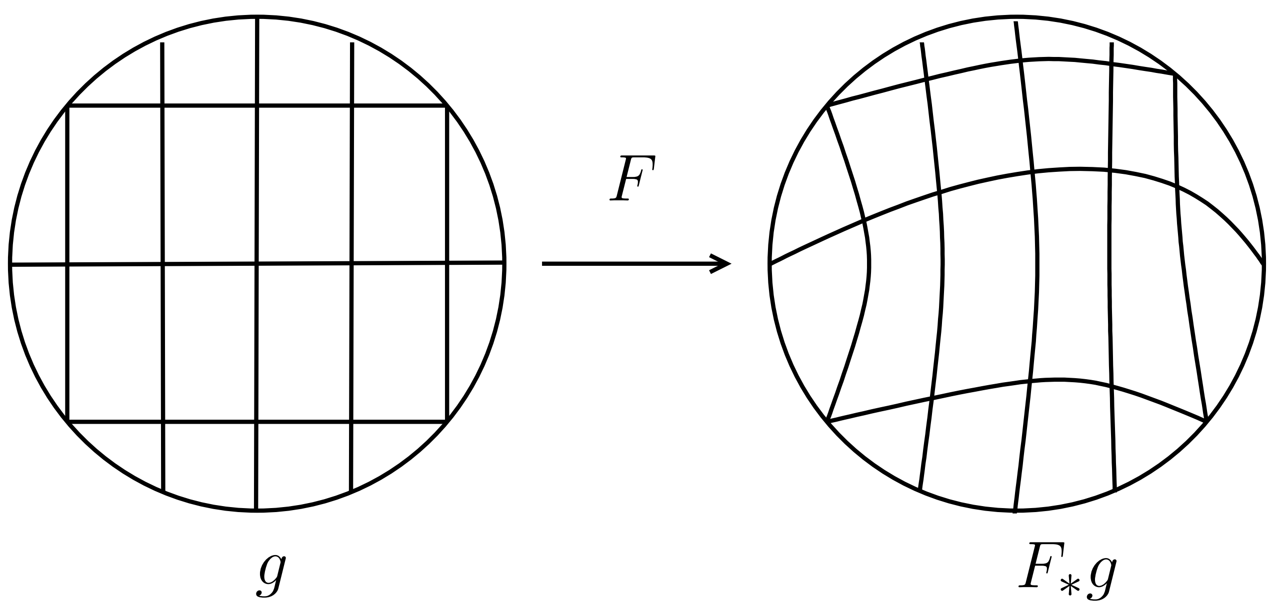
Denoting $\widetilde{u}(y) = u(F^{-1}(y))$ we have

$$\Delta_g u = 0 \quad \text{if and only if} \quad \Delta_{F_* g} \widetilde{u} = 0.$$

We see that

$$\Lambda_g = \Lambda_{F_* g}$$

Metrics g and $F_* g$ define two different conductivities on Ω which appear the same in all boundary measurements.

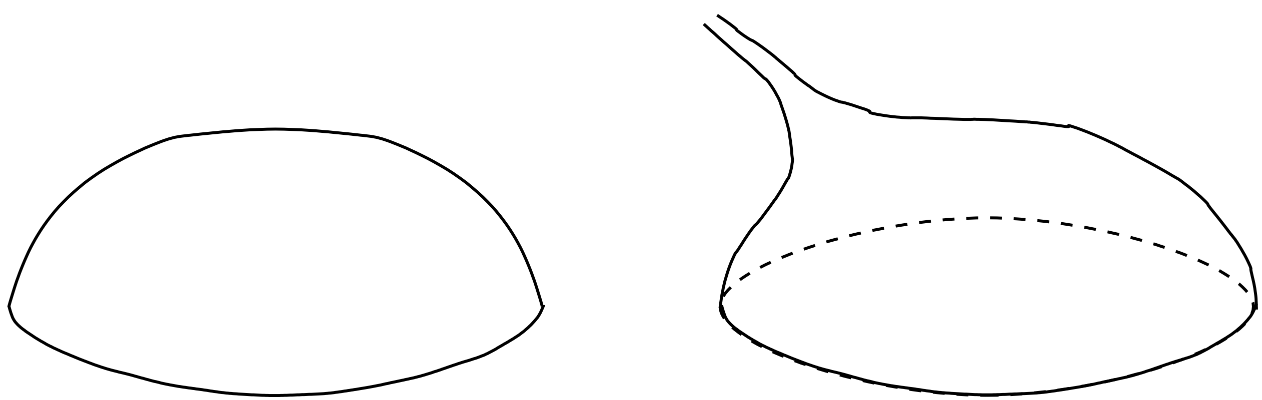


Non-uniqueness results (L.-Taylor-Uhlmann 2003)
Let (M, g) be a compact 2-dimensional manifold. Let $x_0 \in M$, and consider manifold

$$\widetilde{M} = M \setminus \{x_0\}$$

with the metric

$$\widetilde{g}_{ij}(x) = \frac{1}{d_M(x, x_0)^2} g_{ij}(x).$$



Random walk on a Riemannian manifold.
We can interpret the conductivity as a Riemannian metric.

Random walk problem. Let $y \in \partial M$ and let ν be the unit normal vector. Consider a random walk process $B^x(t)$ starting at the point $y + \varepsilon\nu$, where $y \in \partial M$ and $\varepsilon > 0$ is small. Let us measure the probability distribution of the points where $B^x(t)$ hits first time to the boundary. Does these measurement determine the metric on M?

Inverse conductivity problem
is equivalent
to a random walk problem.

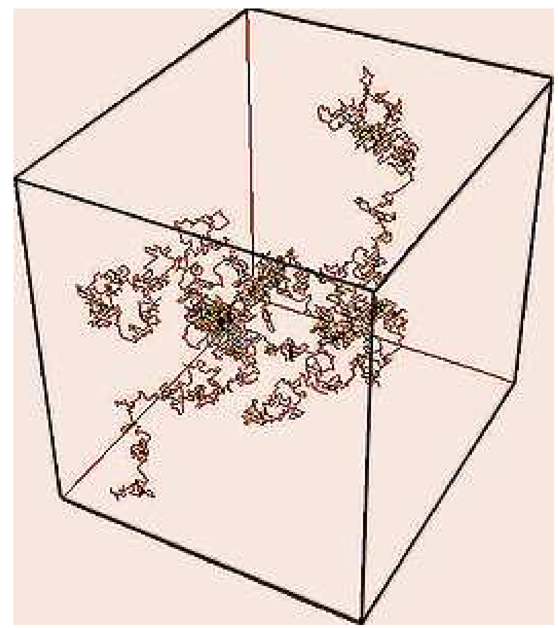


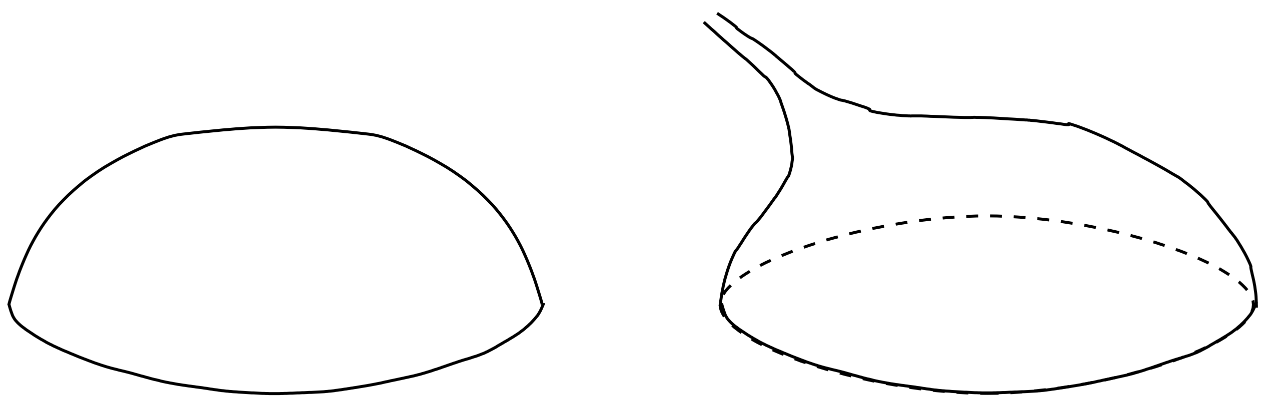
Figure by Z. Ganim (MIT).

Assume $\Delta_g u = 0$, $u|_{\partial M} = h$. Then by the Feynman-Kac formula

$$u(x) = \mathbb{E}(h(B^x(\tau))), \quad h = u|_{\partial M},$$

where $B^x(t)$ is Brownian motion starting at x and τ is hitting time to ∂M.
Normal derivative of u can be found by moving the starting point x to boundary. This yields

$$\Lambda_{M,g} = \Lambda_{\widetilde{M},\widetilde{g}}$$



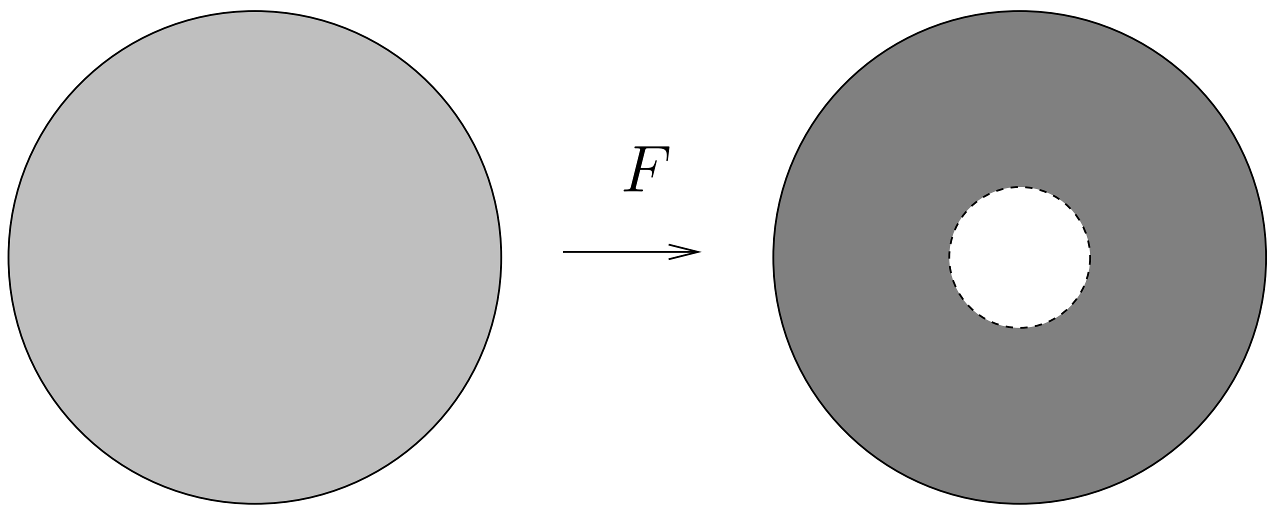
Two manifolds having same boundary measurements

Conductivity equation and a ball with a cloaked pocket.

(Greenleaf-L.-Uhlmann 2003)

Let $B(0,2) \subset \mathbb{R}^3$ be a ball of radius 2 and $B(0,1)$ a ball of radius 1. Consider the map

$$F : B(0,2) \setminus \{0\} \to B(0,2) \setminus \overline{B}(0,1), \quad F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}.$$



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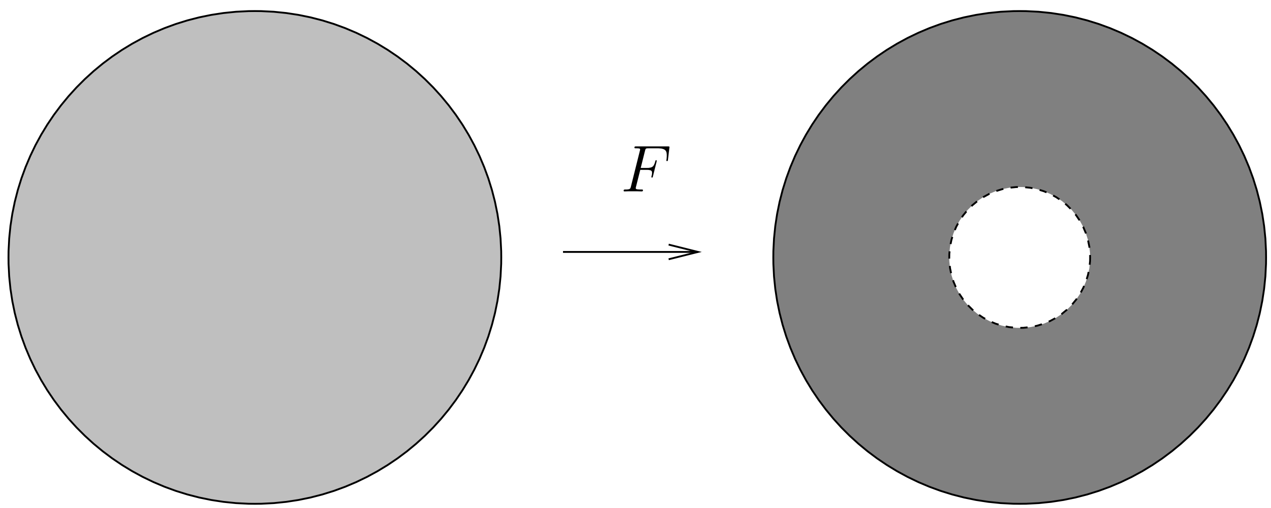


Denote

$$\widetilde g=F_*g,\quad \text{that is,}\quad \widetilde g_{jk}(y)=\sum_{p,q=1}^d \frac{\partial x^p}{\partial y^j}\frac{\partial x^q}{\partial y^k}\,g_{pq}(x).$$

Let $g_{jk} = \delta_{jk}$ be the Euclidian metric in $B(0,2)$ and $\sigma = 1$ be the corresponding conductivity. For $\widetilde{g} = F_* g$ we denote

$$\widetilde{\sigma} = \begin{cases} |\widetilde{g}|^{1/2} \widetilde{g}^{jk} & \text{for } x \in B(0,2) \setminus \overline{B}(0,1) \\ \delta^{jk} & \text{for } x \in B(0,1). \end{cases}$$

Then in the spherical coordinates $(r, \phi, \theta) \mapsto (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta)$ we have

$$\widetilde{\sigma} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}, \quad 1 < |x| \le 2.$$

Theorem 1 *(Greenleaf-L.-Uhlmann 2003) The boundary measurements for $\widetilde{\sigma}$ and σ coincide, that is, $\Lambda_{\widetilde{\sigma}} = \Lambda_\sigma$.*

All boundary measurements for the homogeneous conductivity $\sigma = 1$ and the degenerated conductivity $\widetilde{\sigma}$ are the same.

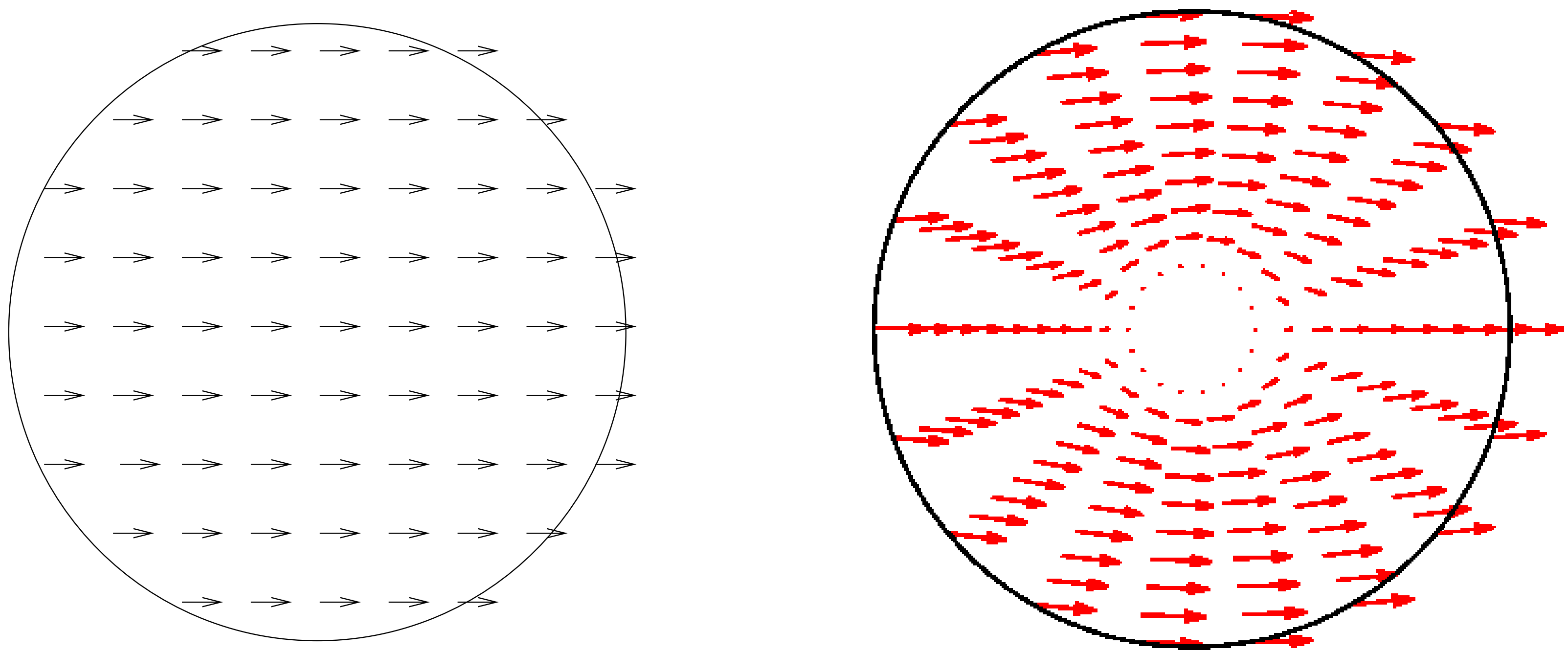
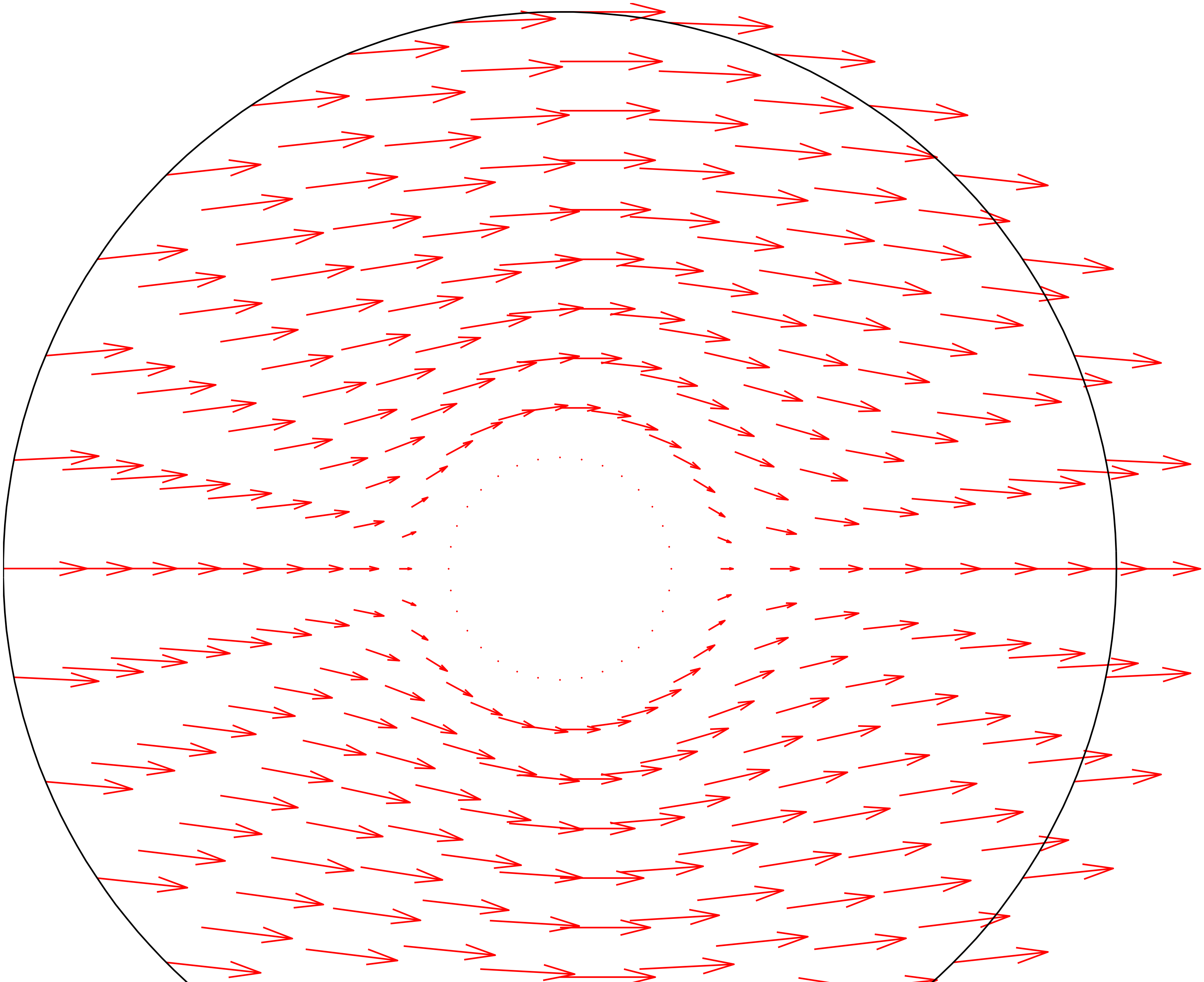


Figure: Analytic solutions for the currents.



2 Invisibility and metamaterials.

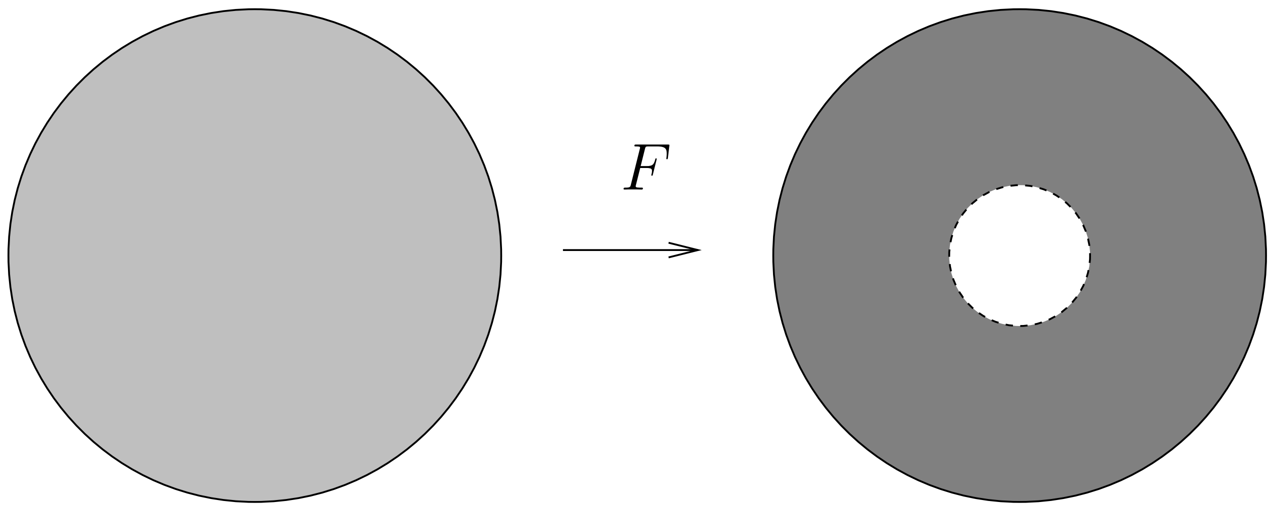
In 2006, metamaterials were proposed and tested to realize invisibility cloaking for the non-zero frequency case.

- *Optical Conformal Mapping* by U. Leonhardt, Science 2006.
- *Controlling electromagnetic fields* by J. Pendry, D. Schurig and D. R. Smith, Science 2006.
- *Metamaterial Electromagnetic Cloak at Microwave Frequencies* by Schurig et al, Science 2006.

How to create material parameters for cloaking?

As before, let $g_{ij} = \delta_{ij}$ and $\widetilde{g} = F_* g$ with

$$F : B(0,2) \setminus \{0\} \to B(0,2) \setminus \overline{B}(0,1), \quad F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}.$$



Define the permittivity $\widetilde{\epsilon}$ and permeability $\widetilde{\mu}$ using the same formula that used for the conductivity,

$$\widetilde{\epsilon} = \widetilde{\mu} = \begin{cases} |\widetilde{g}|^{1/2}\widetilde{g}^{jk} & \text{for } x \in B(0,2) \setminus \overline{B}(0,1) \\ \delta^{jk} & \text{for } x \in B(0,1) \end{cases}$$

Differential forms and transformations of coordinates.
Let $x = (x^1, x^2, x^3)$. We consider electric field E as 1-form

$$E(x) = E_1(x)dx^1 + E_2(x)dx^2 + E_3(x)dx^3.$$

Electric displacement D is 2-form,

$$D(x) = D_{12}(x)dx^1 \wedge dx^2 + D_{13}(x)dx^1 \wedge dx^3 + D_{23}(x)dx^2 \wedge dx^3.$$

When γ is a curve and Σ is a surface, we can define

$$\int_\gamma E \quad \text{and} \quad \int_\Sigma D.$$

Similarly, H is 1-form and B is 2-form. Note that $\nabla \times E = i\omega B$ has a coordinate invariant meaning and can be written as $dE = i\omega B$ where d is the exterior differential.

Let $F:\mathbb{R}^3\to\mathbb{R}^3$ be diffeomorphism. Denote $\widetilde{x}=F(x)$ and function u we define the “push-forward” in F, denoted $F_*u=\widetilde{u}$, as

$$\widetilde{u}(\widetilde{x})=u(F^{-1}(\widetilde{x})).$$

For 1-form $E(x)=E_1(x)dx^1+E_2(x)dx^2+E_3(x)dx^3$ we define $\widetilde{E}=F_*E$, where

$$\begin{aligned}\widetilde{E}(\widetilde{x}) &= \widetilde{E}_1(\widetilde{x})d\widetilde{x}^1+\widetilde{E}_2(\widetilde{x})d\widetilde{x}^2+\widetilde{E}_3(\widetilde{x})d\widetilde{x}^3\\ &= \sum_{j=1}^3\left(\sum_{k=1}^3 (DF^{-1})_j^k(\widetilde{x})\,E_k(F^{-1}(\widetilde{x}))\right)d\widetilde{x}^j.\end{aligned}$$

Similar transformation law is valid for 2-forms.

We consider permittivity $\epsilon(x)$ and permeability $\mu(x)$ as linear operators that map 1-forms to 2-forms,

$$D(x) = \epsilon(x)E(x), \quad B(x) = \mu(x)H(x). \tag{1}$$

The transformation law from ϵ to $\widetilde{\epsilon} = F_*\epsilon$ in diffeomorphism F is given by

$$\widetilde{\epsilon}(\widetilde{x}) = \left. \frac{1}{\det\,(DF(x))} DF(x)\,\epsilon(x)\,DF(x)^t \right|_{x=F^{-1}(\widetilde{x})}$$

The same transformation law is valid of μ and σ.
Then (1) yields that $\widetilde{D} = \widetilde{\epsilon}\widetilde{E}$ and $\widetilde{B} = \widetilde{\mu}\widetilde{H}$.

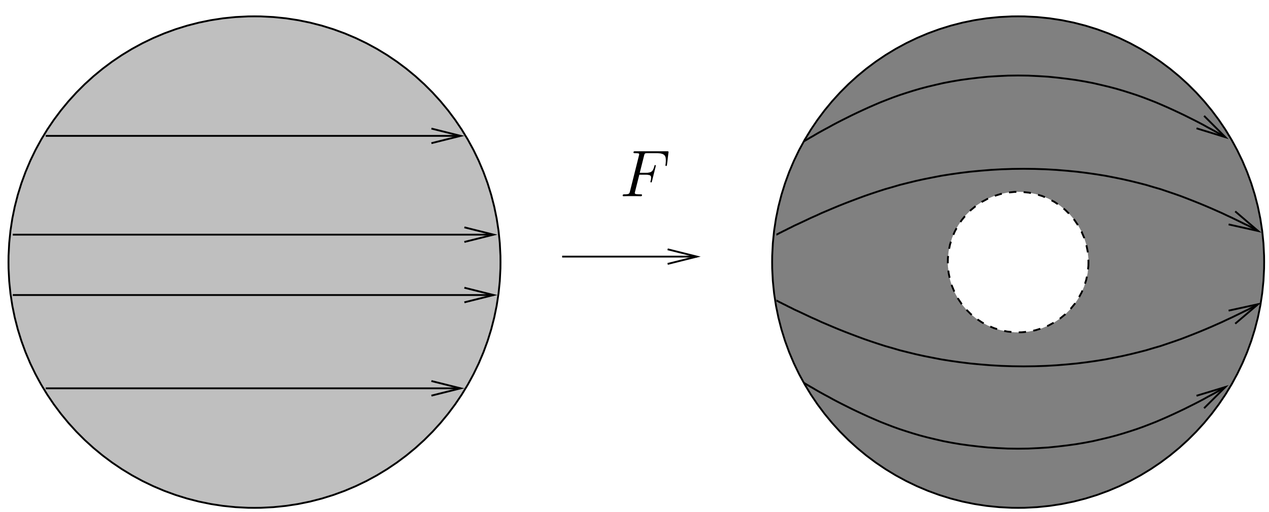
If the 1-forms E and H on $B(0,2)$ satisfy Maxwell's equations

$$\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\epsilon_0 E \quad \text{on } B(0,2)$$

and $\widetilde{\epsilon} = F_*\epsilon_0$ and $\widetilde{\mu} = F_*\mu_0$ then $\widetilde{E} = F_*E$, $\widetilde{H} = F_*H$ satisfy

$$\nabla \times \widetilde{E} = i\omega\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -i\omega\widetilde{\epsilon}(x)\widetilde{E} \quad \text{on } B(0,2) \setminus \overline{B}(0,1).$$

Moreover, the light rays in metric $\widetilde{g}$ go around the ball $B(0,1)$. One possibility is to define $\widetilde{E} = \widetilde{H} = 0$ in $B(0,1)$.



If the 1-forms E and H on $B(0,2)$ satisfy Maxwell's equations

$$\nabla\times E = i\omega\mu_0 H, \quad \nabla\times H = -i\omega\epsilon_0 E \quad \text{on } B(0,2)$$

and $\widetilde{\epsilon} = F_*\epsilon_0$ and $\widetilde{\mu} = F_*\mu_0$ then $\widetilde{E} = F_*E$, $\widetilde{H} = F_*H$ satisfy

$$\nabla\times \widetilde{E} = i\omega\widetilde{\mu}(x)\widetilde{H}, \quad \nabla\times \widetilde{H} = -i\omega\widetilde{\epsilon}(x)\widetilde{E} \quad \text{on } B(0,2)\setminus\overline{B}(0,1).$$

Moreover, the light rays in metric $\widetilde{g}$ go around the ball $B(0,1)$. One possibility is to define $\widetilde{E} = \widetilde{H} = 0$ in $B(0,1)$.

We consider the question: What happens on $\partial B(0,1)$?

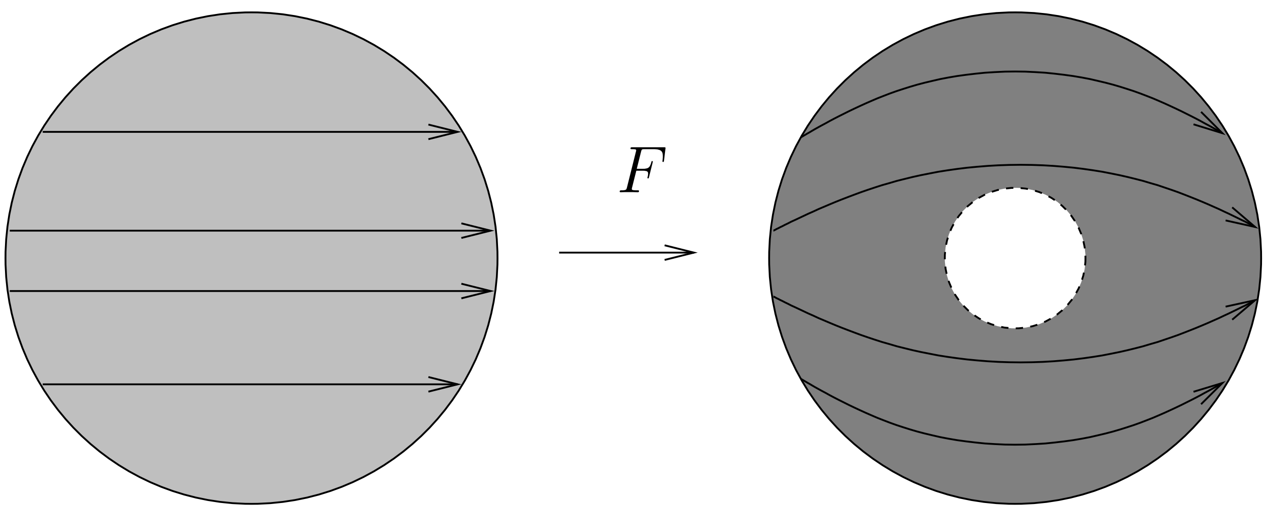
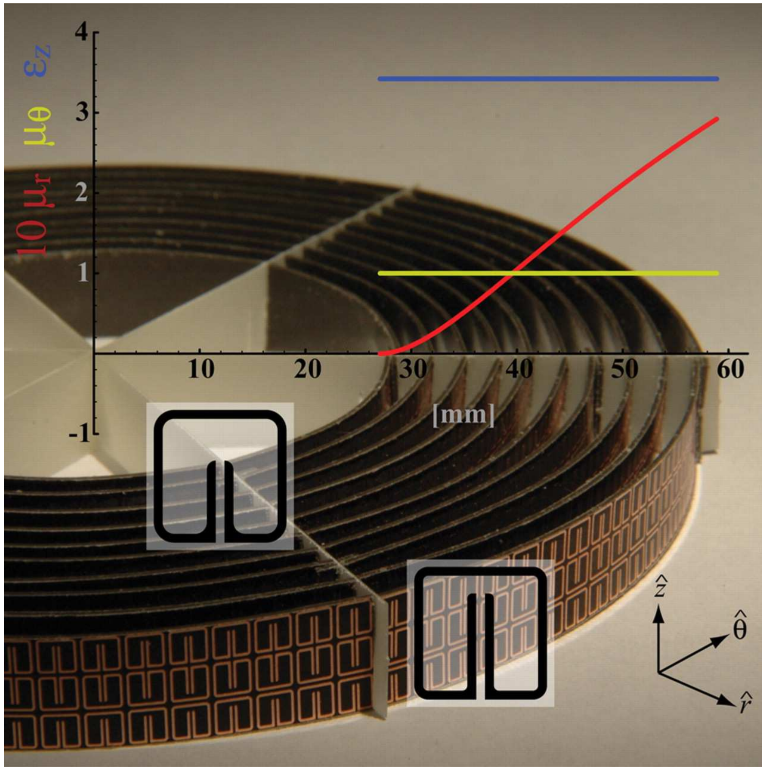
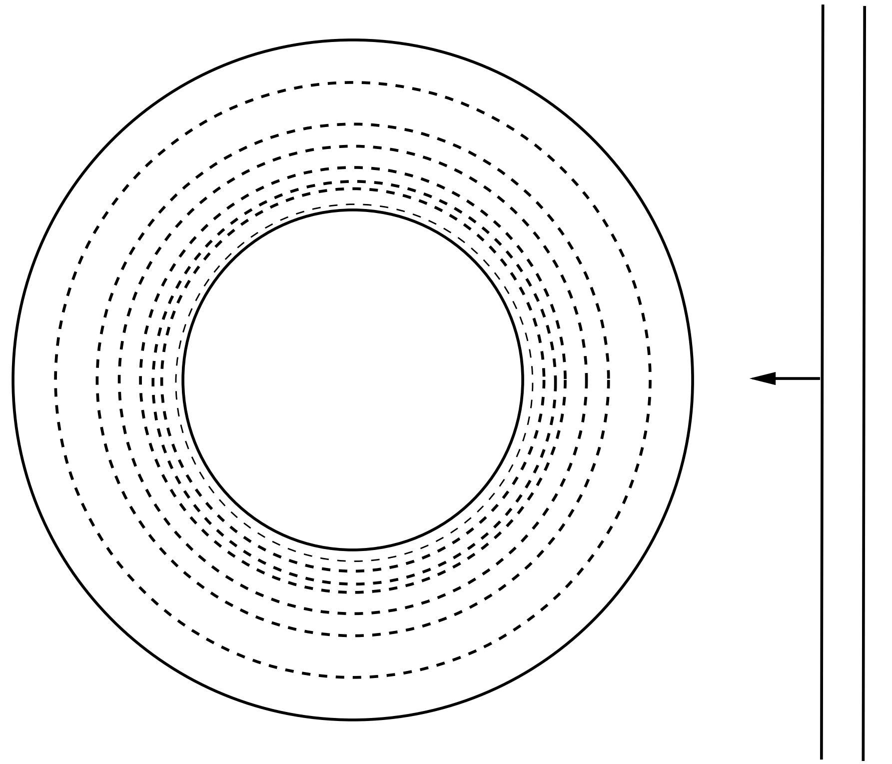


Figure: Invisibility cloak for 4 cm waves build using metamaterials, Schurig et al, Science 2006.





3 Distributional solutions for differential equations

Consider relations

$$\widetilde{D} = \widetilde{\epsilon}\widetilde{E}, \quad \widetilde{B} = \widetilde{\mu}\widetilde{H}.$$

Later, the elements of matrices $\widetilde{\mu}$ and $\widetilde{\epsilon}$ will have values ∞ and 0 on $\partial B(0,1)$.
There are two alternatives:

1. We need to consider undefined products like $0 \cdot \infty$, or
2. Consider equations in sense of distributions.

Notations

Let $x = (x^1, x^2, \dots, x^d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$.
We denote

$$x^\alpha = (x^1)^{\alpha_1}(x^2)^{\alpha_2}\dots(x^d)^{\alpha_d}$$

and

$$\partial^\alpha = \nabla^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x^2}\right)^{\alpha_2}\dots\left(\frac{\partial}{\partial x^d}\right)^{\alpha_d}.$$

Definition of distributions. Let $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$ be the space of the test functions,

$$\mathcal{D} = \{\phi : \mathbb{R}^d \to \mathbb{C} \quad : \quad \partial^\alpha \phi \text{ are continuous for all } \alpha, \\ \phi(x) = 0 \text{ if } |x| > s \text{ for some } s > 0\}.$$

We say that a linear operator $\lambda : \mathcal{D} \to \mathbb{C}$ is a distribution if for any $s > 0$ there are N and C such that

$$|\lambda(\phi)| \leq C \sum_{|\alpha| \leq N} \max |\partial^\alpha \phi|, \quad \text{for all } \phi \in \mathcal{D} \text{ satisfying } \phi|_{|x|>s}=0.$$

We denote $\langle \lambda, \phi \rangle = \lambda(\phi)$. All continuous functions $h : \mathbb{R}^d \to \mathbb{C}$ define a distribution

$$\langle h, \phi \rangle = \int_{\mathbb{R}^d} h(x)\phi(x)\, dx.$$

There are also other distributions, like $\langle \delta, \phi \rangle = \phi(0)$.

If h is a continuously differentiable function

$$\int_{\mathbb{R}} \frac{\partial}{\partial x} h(x)\, \phi(x)\, dx = \int_{\mathbb{R}} h(x)(-\frac{\partial}{\partial x}\phi(x))\, dx, \quad \phi \in \mathcal{D}.$$

Imitating this, we define that a distribution λ has the derivative $\lambda' = \frac{\partial}{\partial x}\lambda$ that is given by

$$\langle \lambda', \phi \rangle = \langle \lambda, (-\frac{\partial}{\partial x}\phi) \rangle, \quad \phi \in \mathcal{D}.$$

Consider now examples:

$$\frac{\partial}{\partial x} H(x) = \delta(x), \qquad H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

This means that for all functions $\phi \in \mathcal{D}$ we have

$$\langle \frac{\partial}{\partial x} H, \phi \rangle = \int_{\mathbb{R}} H(x)(-\frac{\partial \phi}{\partial x}(x))\, dx = \phi(0) = \langle \delta, \phi \rangle.$$

Let $P(y) = \sum_{|\alpha| \leq m} a_\alpha y^\alpha$. Distributions λ and f satisfy $P(\partial)\lambda = F$ if

$$\langle \lambda, P(-\partial)\phi \rangle = \langle f, \phi \rangle, \quad \phi \in \mathcal{D}.$$

Example: Integrable functions E, F, and G satisfy

$$\nabla \times E = F, \quad \nabla \cdot E = G$$

in distributional sense if

$$\int_{\mathbb{R}^3} E \cdot \nabla \times \phi \, dx = \int_{\mathbb{R}^3} F \cdot \phi \, dx \quad \phi \in \mathcal{D}^3$$

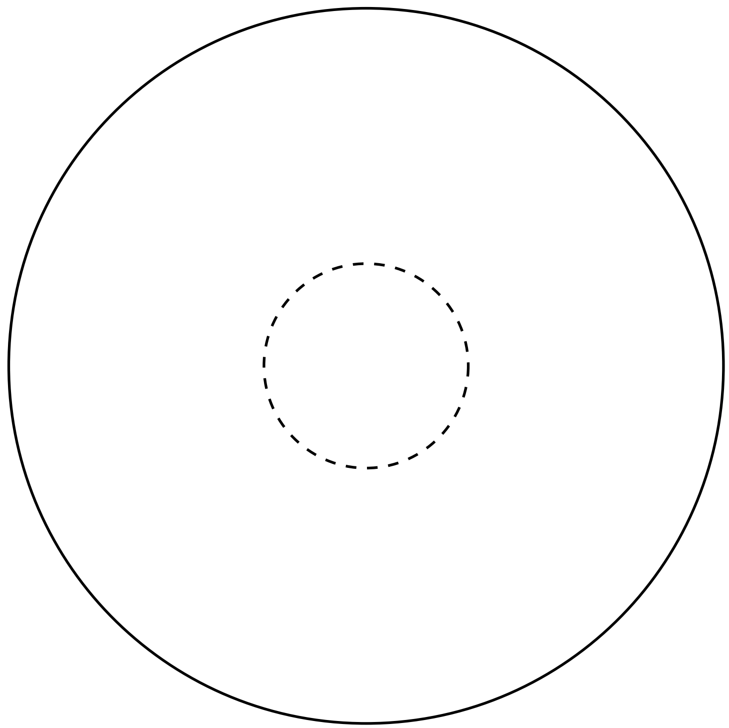
and

$$-\int_{\mathbb{R}^3} E \cdot \nabla \psi \, dx = \int_{\mathbb{R}^3} G \, \phi \, dx \quad \psi \in \mathcal{D}.$$

4 Mathematical theory of cloaking for Maxwell's equation

We consider

$\nabla \times \widetilde{E} = i\omega\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -i\omega\widetilde{\epsilon}(x)\widetilde{E} + \widetilde{J}$ on $B(0,2)$



Definition 1 *We say that* $(\widetilde{E},\widetilde{H})$ *is a finite energy solution for Maxwell's equations on* $N=B(0,2)$ *and denote*

$$\nabla\times\widetilde{E}=i\omega\widetilde{\mu}(x)\widetilde{H},\quad \nabla\times\widetilde{H}=-i\omega\widetilde{\epsilon}(x)\widetilde{E}+\widetilde{J}\quad \text{on } N,$$

if $\widetilde{E}$, $\widetilde{H}$, $\widetilde{D}=\widetilde{\epsilon}\,\widetilde{E}$ *and* $\widetilde{B}=\widetilde{\mu}\,\widetilde{H}$ *are forms in* N *which coefficients are integrable functions,*

$$\int_N \widetilde{\epsilon}^{jk}\,\widetilde{E}_j\,\overline{\widetilde{E}_k}\,dx<\infty,\quad \int_N \widetilde{\mu}^{jk}\,\widetilde{H}_j\,\overline{\widetilde{H}_k}\,dx<\infty,$$

Maxwell's equations hold in a neighbourhood of ∂N*, and*

$$\int_N((\nabla\times\psi)\cdot\widetilde{E}-\psi\cdot i\omega\widetilde{\mu}(x)\widetilde{H})\,dx=0,$$
$$\int_N((\nabla\times\phi)\cdot\widetilde{H}+\phi\cdot(i\omega\widetilde{\epsilon}(x)\widetilde{E}-\widetilde{J}))\,dx=0$$

for all ψ,ϕ *having coefficients in* $\mathcal{D}=C_0^\infty(N)$.

Theorem 2 *Consider Maxwell's equations in $N = B(0,2)$,*

$$\nabla \times \widetilde{E} = i\omega\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -i\omega\widetilde{\epsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N,$$

where $\widetilde{J}$ vanishes near $\partial B(0,1)$. Then

- *If $\widetilde{J} = 0$, the boundary measurements coincide with the boundary measurements in a homogeneous ball $B(0,2)$.*
- *If a solution exists for a source $\widetilde{J}$, then $\widetilde{E}$ and $\widetilde{H}$ vanish in a domain $r_0 < |x| < 1$.*
- *Finite energy solutions do not exist for generic $\widetilde{J}$.*

Theorem 3 *Consider Maxwell's equations in* $N = B(0,2)$,

$$\nabla \times \widetilde{E} = i\omega\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -i\omega\widetilde{\epsilon}(x)\widetilde{E} + \widetilde{J} \quad \textit{on } N,$$

where $\widetilde{J}$ *vanishes near* $\partial B(0,1)$. *Then*

- *If* $\widetilde{J} = 0$, *the boundary measurements coincide with the boundary measurements in a homogeneous ball* $B(0,2)$.
- *If a solution exists for a source* $\widetilde{J}$, *then* $\widetilde{E}$ *and* $\widetilde{H}$ *vanish in a domain* $r_0 < |x| < 1$.
- *Finite energy solutions do not exist for generic* $\widetilde{J}$.

Thus passive objects in $B(0,1)$ can be cloaked. Cloaking of active objects is challenging.

Theorem 4 *Consider Maxwell's equations in $N = B(0,2)$,*

$$\nabla \times \widetilde{E} = i\omega\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -i\omega\widetilde{\epsilon}(x)\widetilde{E} + \widetilde{J} \quad \textit{on } N,$$

where $\widetilde{J}$ vanishes near $\partial B(0,1)$. Then

- *If $\widetilde{J} = 0$, the boundary measurements coincide with the boundary measurements in a homogeneous ball $B(0,2)$.*
- *If a solution exists for a source $\widetilde{J}$, then $\widetilde{E}$ and $\widetilde{H}$ vanish in a domain $r_0 < |x| < 1$.*
- *Finite energy solutions do not exist for generic $\widetilde{J}$.*

Thus passive objects in $B(0,1)$ can be cloaked. Cloaking of active objects is challenging.
Remedy: We coat both sides of $\partial B(0,1)$ with metamaterial, or put a perfectly conducting lining on $\partial B(0,1)$.

Idea of the proof: We consider the map

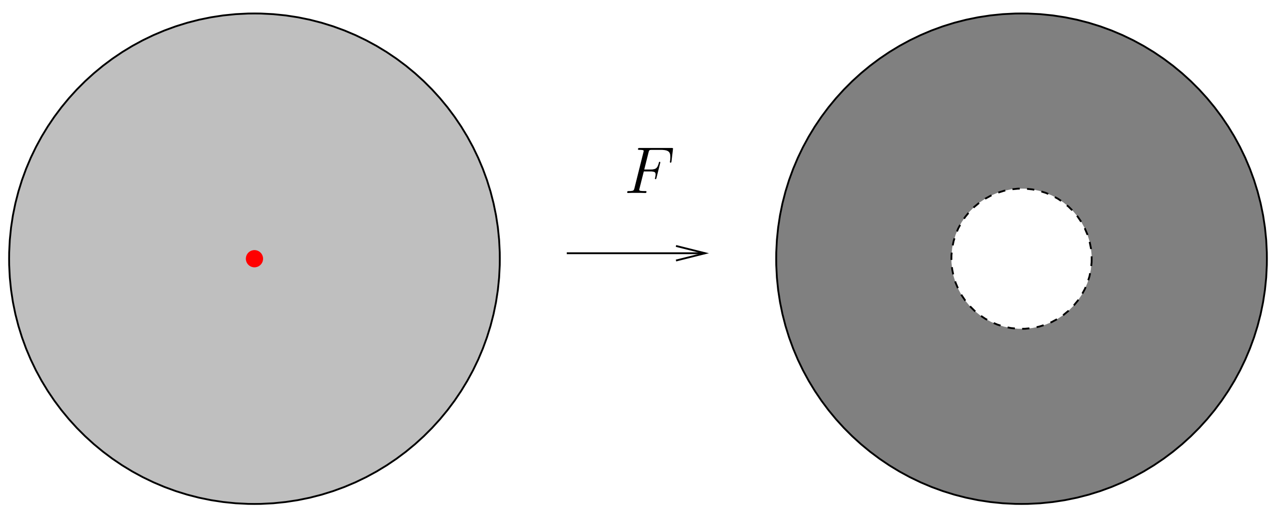
$$F : B(0,2) \setminus \{0\} \to B(0,2) \setminus \overline{B}(0,1) \subset N = B(0,2).$$

If $\widetilde{E}$ and $\widetilde{H}$ are solutions on N with $\widetilde{\epsilon}$ and $\widetilde{\mu}$, then

$$E = (F^{-1})_* \widetilde{E}, \quad H = (F^{-1})_* \widetilde{H}, \quad J = (F^{-1})_* \widetilde{J} \quad \text{on } B(0,2) \setminus \{0\}$$

satisfy

$$\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\epsilon_0 E + J \quad \text{on } B(0,2) \setminus \{0\}.$$



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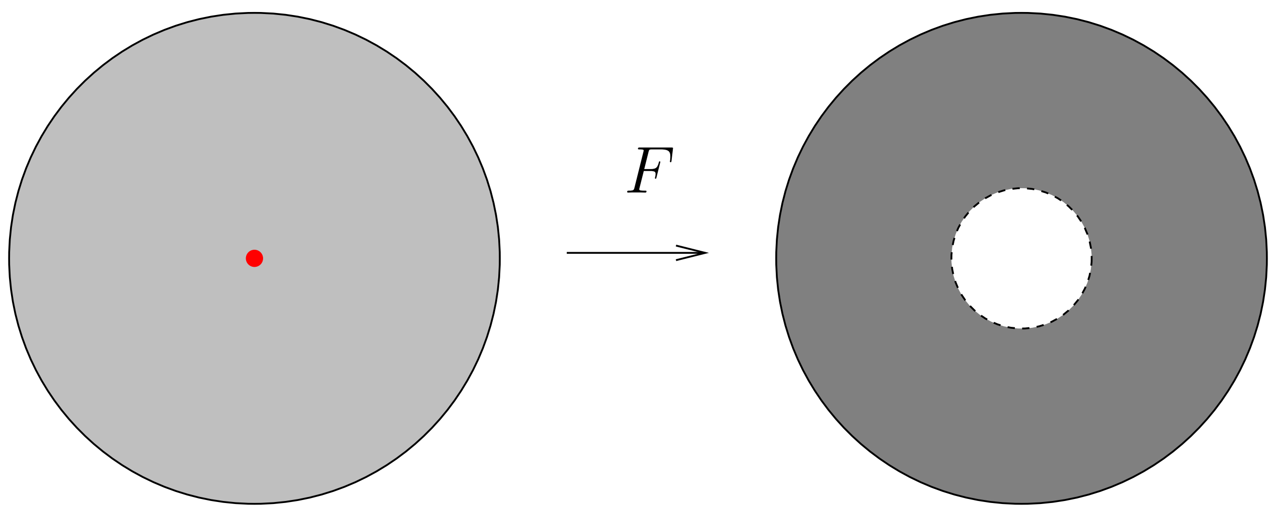
satisfy

$$\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\epsilon_0 E + J \quad \text{on } B(0,2) \setminus \{0\}.$$

Since the fields E and H have finite energy and the set $\{0\}$ has a Hausdorff dimension $d \leq 1$, the solutions can be extended to solutions on whole $B(0,2)$.

Consider next the boundary conditions on $\Sigma = \partial B(0,1)$.
Next, assume that $\widetilde{E}$ and $\widetilde{H}$ are solutions on N with $\widetilde{\epsilon}$ and $\widetilde{\mu}$.
If $\widetilde{E}$ is a piecewise smooth function having jump across $\Sigma = \partial B(0,1)$,

$$\begin{aligned}\nabla \times \widetilde{E} &= \{\nabla \times \widetilde{E}\} + \nu \times (\widetilde{E}^{+} - \widetilde{E}^{-})\delta_{\Sigma}(x) \\ &= \text{function part + delta distribution on } \Sigma.\end{aligned}$$



Consider next the boundary conditions on $\Sigma = \partial B(0,1)$. Next, assume that $\widetilde{E}$ and $\widetilde{H}$ are solutions on N with $\widetilde{\epsilon}$ and $\widetilde{\mu}$. If $\widetilde{E}$ is a piecewise smooth function having jump across $\Sigma = \partial B(0,1)$,

$$\begin{aligned} \nabla \times \widetilde{E} &= \{\nabla \times \widetilde{E}\} + \nu \times (\widetilde{E}^{+} - \widetilde{E}^{-})\delta_{\Sigma}(x) \\ &= \text{function part + delta distribution on } \Sigma. \end{aligned}$$

Since $\nabla \times \widetilde{E} = i\omega\widetilde{B}$ is a function, the delta distribution part of $\nabla \times \widetilde{E}$ has to vanish. Hence

$$\nu \times \widetilde{E}|_{\Sigma+} - \nu \times \widetilde{E}|_{\Sigma-} = 0.$$

Recall that the transformation law $\widetilde{E} = F_* E$ in local coordinates is

$$\widetilde{E}(\widetilde{x}) = \sum_{j=1}^{3} \left(\sum_{k=1}^{3} (DF^{-1})_j^k(\widetilde{x})\, E_k(F^{-1}(\widetilde{x})) \right) d\widetilde{x}^j.$$

Using this we see that in the exterior domain

$$\nu \times \widetilde{E}|_{\Sigma+} = 0.$$

Since the jump across Σ is zero,

$$\nu \times \widetilde{E}|_{\Sigma-} = 0.$$

In this way we see that both $\nu \times \widetilde{E}|_{\Sigma-} = 0$ and $\nu \times \widetilde{H}|_{\Sigma-} = 0$. These boundary conditions are overdetermined and lead to a non-existence result.

Cloaking of an infinite cylinder. Let

$$N = B_2(0,2) \times \mathbb{R} \subset \mathbb{R}^3, \quad D = \overline{B}_2(0,1) \times \mathbb{R}.$$

We use cylindrical coordinates (r, θ, z) and the transformation

$$\begin{aligned} F &: N \setminus (\{0,0\} \times \mathbb{R}) \to N \setminus \overline{D}, \\ & F(r,\theta,z) = (\frac{1+r}{2}, \theta, z). \end{aligned}$$

In $N \setminus \overline{D}$ we define the metric $\widetilde{g} = F_* g$, $g = \delta_{jk}$ and

$$\widetilde{\epsilon} = \widetilde{\mu} = \begin{cases} |\widetilde{g}|^{1/2} \widetilde{g}^{jk} & \text{for } x \in B(0,2) \setminus \overline{B}(0,1) \\ \delta^{jk} & \text{for } x \in B(0,1) \end{cases}$$

In D metric $\widetilde{g}_{jk} = \delta_{jk}$ is Euclidian and $\widetilde{\epsilon} = \widetilde{\mu} = 1$.

In the cloaking of an infinite cylinder the cylindrical coordinates are

$$(r, \theta, z) \mapsto (r\cos\theta, r\sin\theta, z).$$

In these coordinates, $\widetilde{\epsilon}$ and $\widetilde{\mu}$ are

$$\widetilde{\epsilon} = \widetilde{\mu} = \begin{pmatrix} (r-1) & 0 & 0 \\ 0 & (r-1)^{-1} & 0 \\ 0 & 0 & 4(r-1) \end{pmatrix}, \quad 1 < r \leq 2.$$

Theorem 5 *Let $D = \overline{B}_2(0,1) \times \mathbb{R} \subset \mathbb{R}^3$ be an infinite cylinder and $N = B_2(0,2) \times \mathbb{R}$. Consider*

$$\nabla \times \widetilde{E} = i\omega\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -i\omega\widetilde{\epsilon}(x)\widetilde{E} + \widetilde{J} \quad \textit{on } N.$$

- *The finite energy solutions do not exist generally.*
- *The finite energy solutions exists if on ∂D we put a surface with the Soft-and-Hard boundary condition,*

$$e_\theta \cdot E|_{\partial D} = 0 \quad \textit{and} \quad e_\theta \cdot H|_{\partial D} = 0.$$

Here e_θ is the unit vector to θ-direction and $B_2(0,1) \subset \mathbb{R}^2$ is the two-dimensional unit disc.

Theorem 6 *Let $D = \overline{B}_2(0,1) \times \mathbb{R} \subset \mathbb{R}^3$ be an infinite cylinder and $N = B_2(0,2) \times \mathbb{R}$. Consider*

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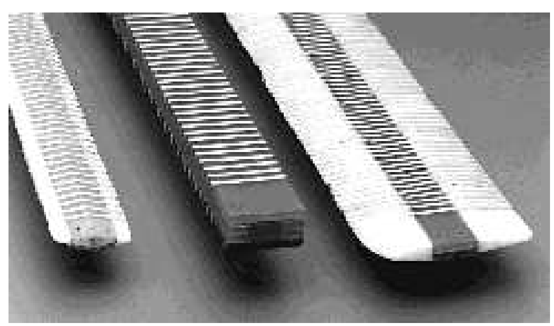
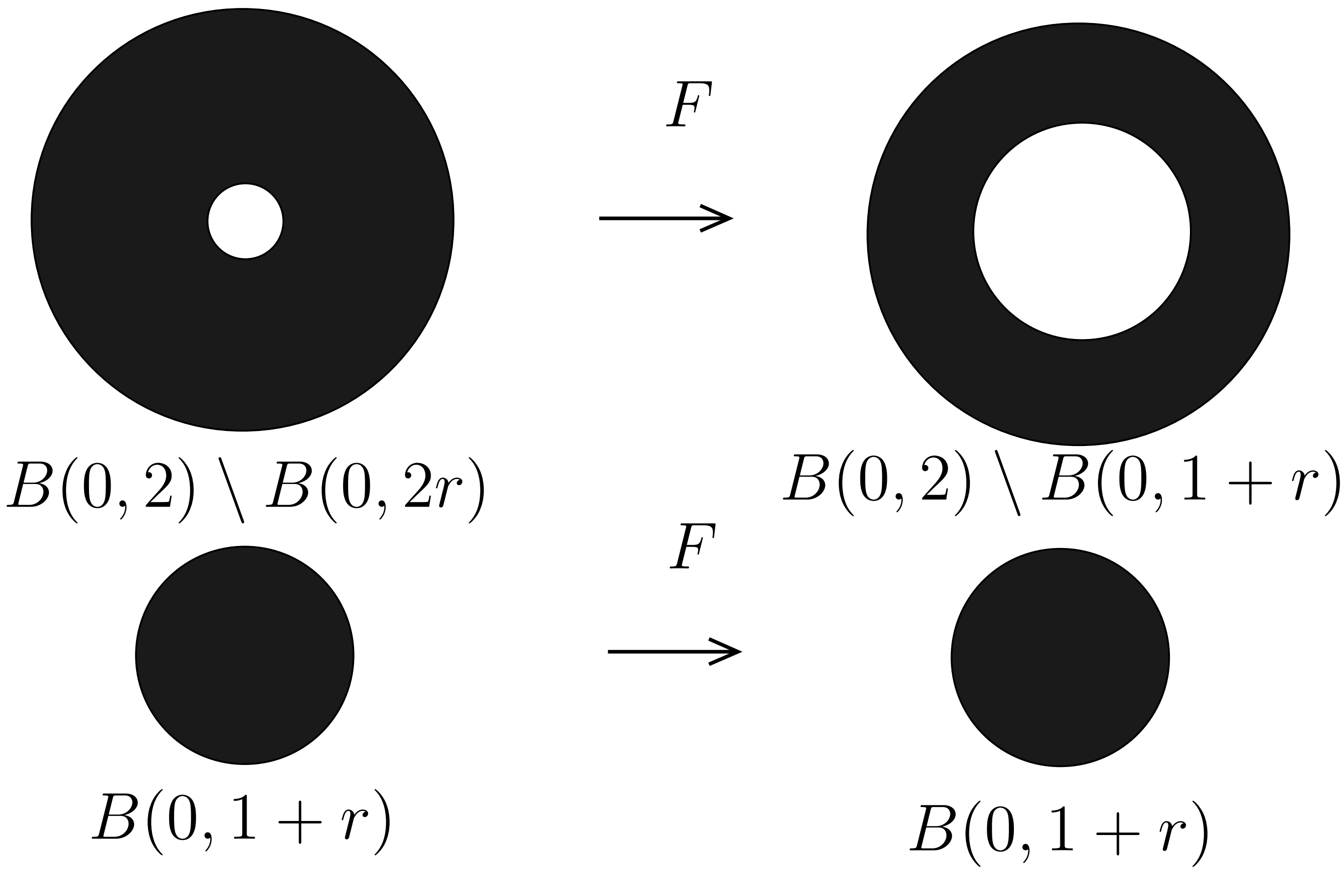


Figure: SH surfaces created by Kildal et al.

What happens for solutions without the SH lining?

Consider an approximative invisibility cloak built using materials that have uniformly bounded anisotropy ratio L, that is, $\max \widetilde{\epsilon}_{\theta\theta}/\widetilde{\epsilon}_{rr} \leq L$ and $\max \widetilde{\mu}_{\theta\theta}/\widetilde{\mu}_{rr} \leq L$. For this end, consider the map



Denote $M_1 = B(0,2) \setminus B(0,2r)$ and $M_2 = B(0,1+r)$. Consider in the cylindrical coordinates a vertically polarized $\widetilde{E}$ field that is independent of z-variable. Let $E = (F^{-1})_* \widetilde{E}$, $E = u_j(r,\theta)\vec{e}_z$ in M_j, $j = 1, 2$. Then $(\Delta + k^2)u_j = 0$ and

$$u_1(r,\theta) = \sum_{n=-\infty}^{\infty} (a_n J_{|n|}(kr) + b_n H^{(1)}_{|n|}(kr))e^{in\theta},$$

$$u_2(r,\theta) = \sum_{n=-\infty}^{\infty} c_n J_{|n|}(kr)e^{in\theta}.$$

When $L \to \infty$ or equivalently $r \to 0$, the fields $\widetilde{E}$ and $\widetilde{H}$ converge to the solutions of

$$\nabla \times \widetilde{E} = i\omega \widetilde{B} + K_{new},$$
$$\nabla \times \widetilde{H} = -i\omega \widetilde{D} + J_{new}.$$

Here the coefficients of $\widetilde{B} = \widetilde{\mu}\widetilde{H}$ and $\widetilde{D} = \widetilde{\epsilon}\widetilde{E}$ are smooth functions and

$$K_{new} = s_e \delta_\Sigma, \quad J_{new} = s_h \delta_\Sigma,$$

where δ_Σ is a measure supported on $\Sigma = \partial B(0,1)$, and s_e and s_h are smooth functions.

An alternative formulation without the SH lining.
Let us write

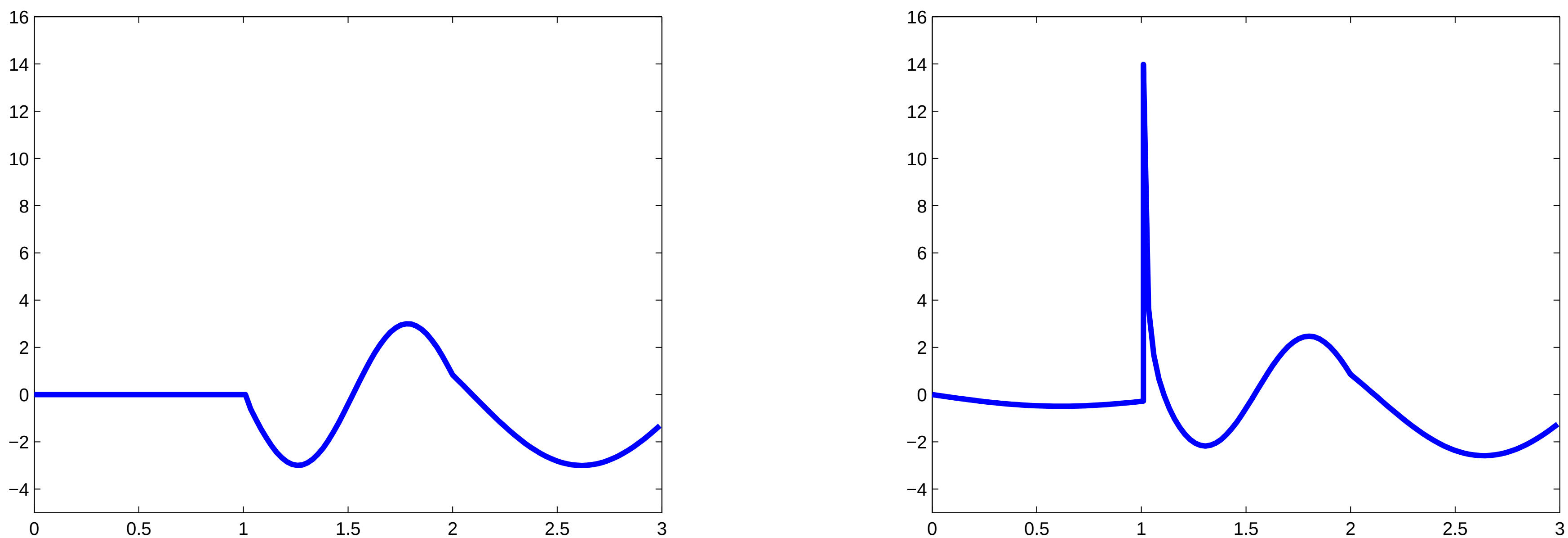
$$\widetilde{B}_{new} = \widetilde{B} + (i\omega)^{-1} s_e \delta_\Sigma, \quad \widetilde{D}_{new} = \widetilde{D} - (i\omega)^{-1} s_h \delta_\Sigma.$$

We can write

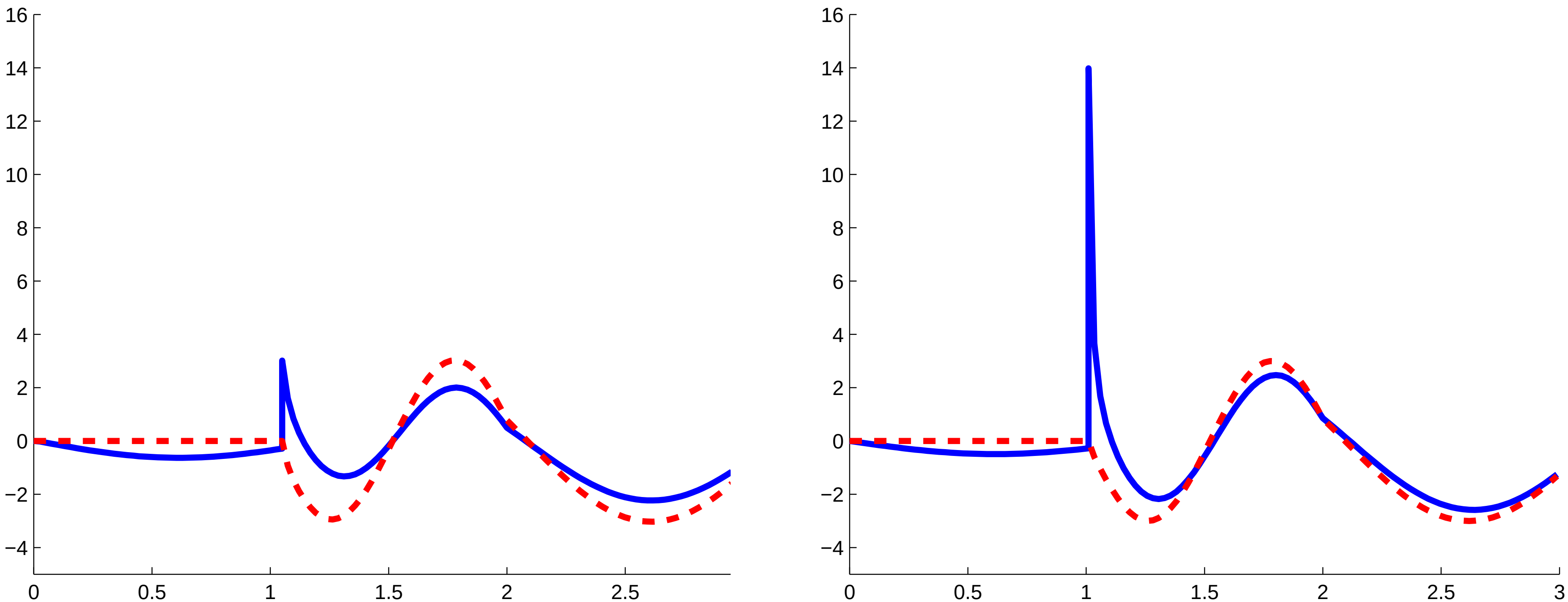
$$\nabla \times \widetilde{E} = i\omega \widetilde{B}_{new},$$
$$\nabla \times \widetilde{H} = -i\omega \widetilde{D}_{new}$$

but there are some problems: Formulation of the constitutive laws is unclear and the effective medium theory for composite medium should be developed for distributional fields.

Approximate invisibility coating. Build an approximative invisibility cloak using materials that have the maximal anisotropy ratio $4 \cdot 10^4$.
Let the incoming wave be a TE-polarized plane wave with $k = 1$. The θ-component of B-field of the total wave is:

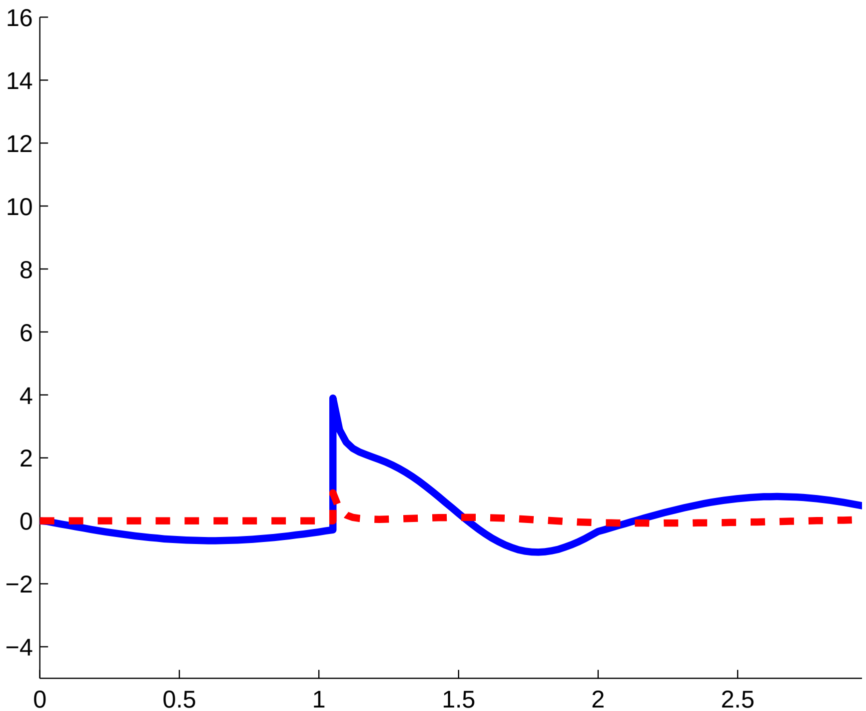


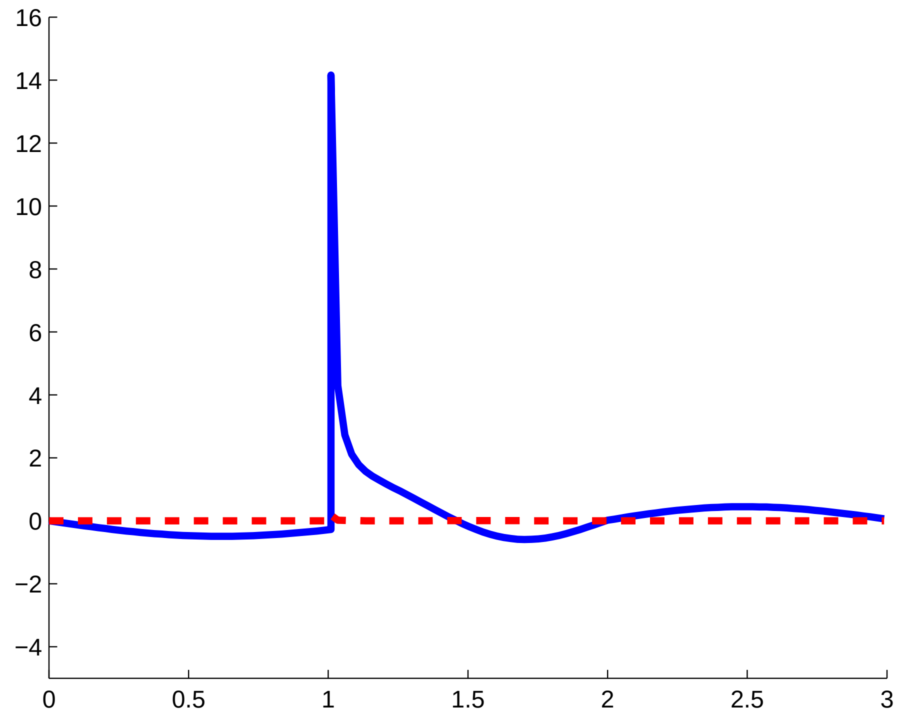
With SH lining. Without SH lining.



L=1600 L=40000.

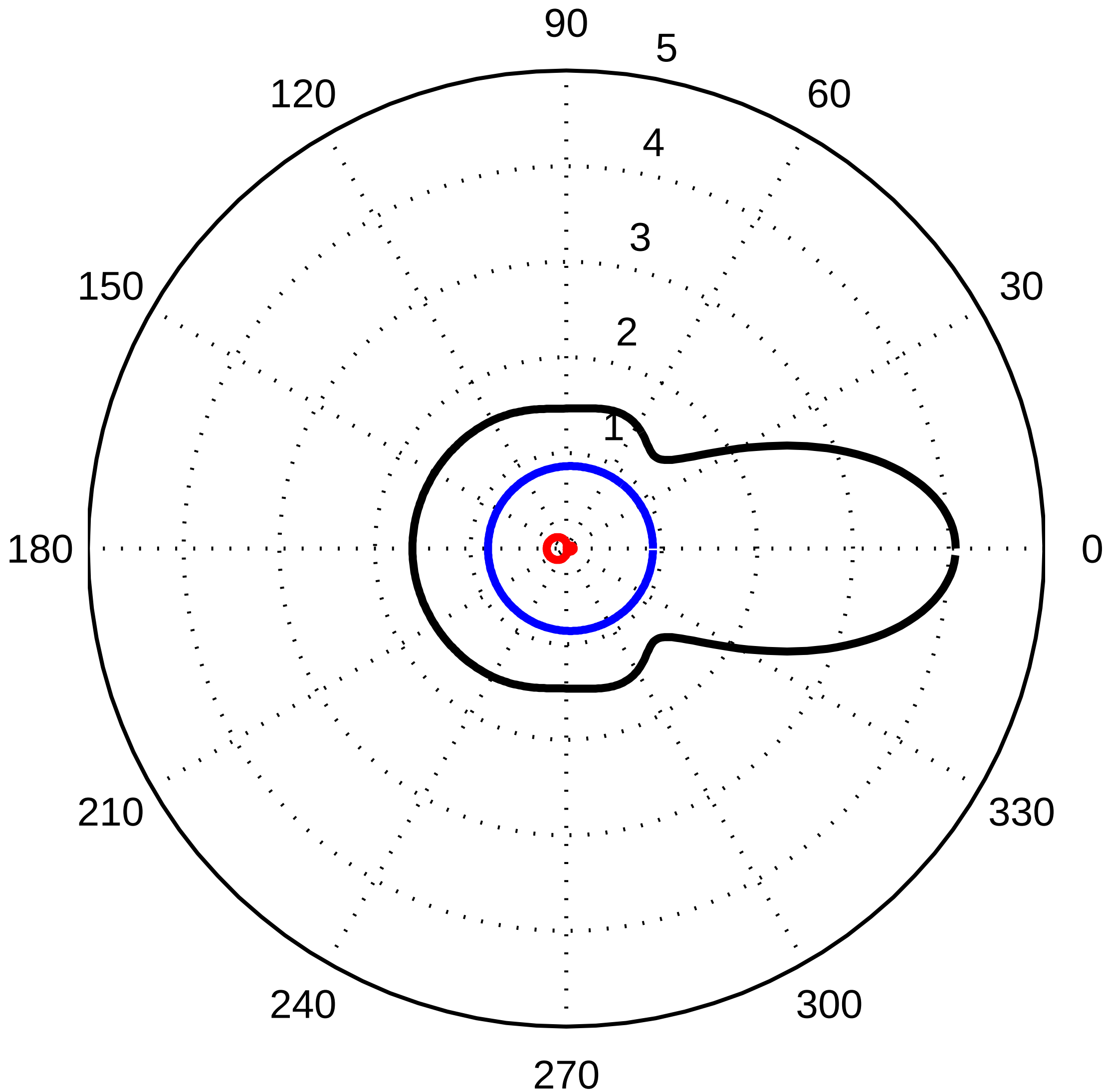
The real part of the y-component of the **total** B-field on the line $\{(x,0,0):\ x\in[0,3]\}$. Blue solid curve is the field with no physical lining. Red dashed curve is the field with SHS lining.





L=1600 L=40000.

The real part of the y-component of the **scattered** B-field on the line $\{(x, 0, 0) : \; x \in [0, 3]\}$. Blue solid curve is the field with no physical lining. Red dashed line is the field with Soft-and-Hard lining.


90
5
120
60
4
3
150
30
2
1
180
0
210
330
240
300
270

Benefits of SH lining:
The L^2-norm of the far field pattern of the scattered field with SH lining is only 2% of the scattered field without SH lining.

Summary: When cloaking active objects in a ball or any objects in a cylinder, we have to use a PEC lining or a Soft-and-Hard lining. Otherwise, the fields blow up.

Summary on existence and blow up of waves:
In cloaking of 3D ball with an approximative cloak:

Passive objects cloaked: Solutions exist on the limit, no blow up on limit.

Active objects cloaked: Solutions do not exist on the limit, we have not yet studied the possible blow up. (See also Zhang-Chen-Wu-Kong, Arxiv 19. Oct. 2007 on boundary currents)

In cloaking of a cylinder with an approximative cloak:

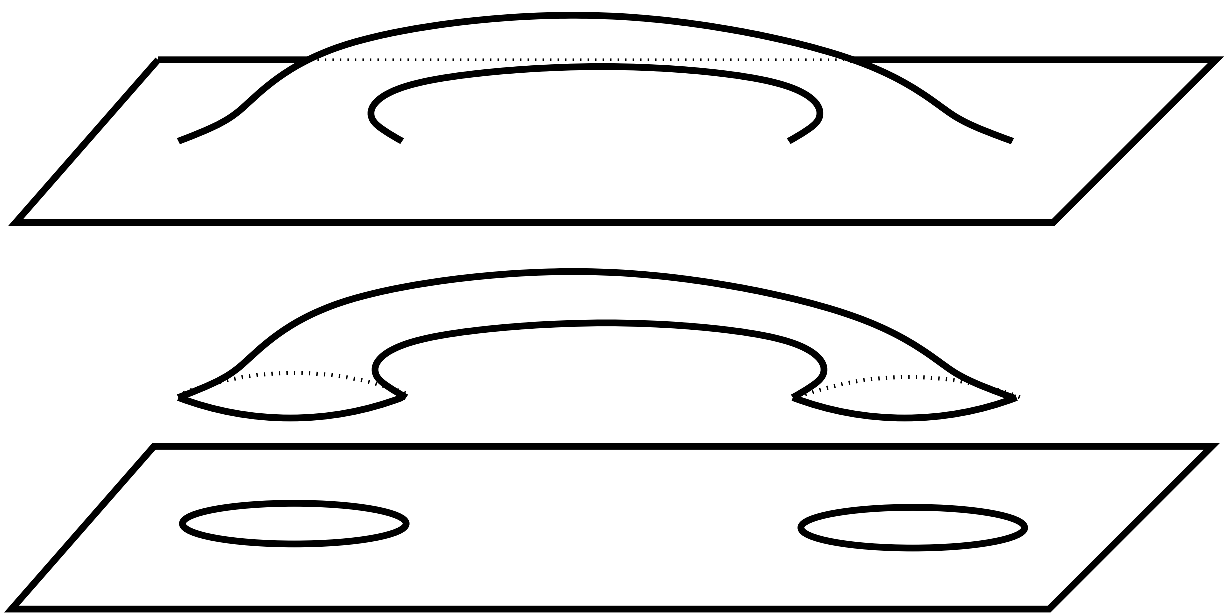
Vacuum inside: Solutions do not exist on the limit. It is shown that the blow up happens on the limit.

Soft-and-Hard surface inside: Solutions exist on the limit, no blow up on the limit.

5 Electromagnetic wormhole device

Let $M = M_1 \# M_2$ where

$$M_1 = \mathbb{R}^3 \setminus (B(O,1) \cup B(P,1)),$$
$$M_2 = \mathbb{S}^2 \times [0,1].$$

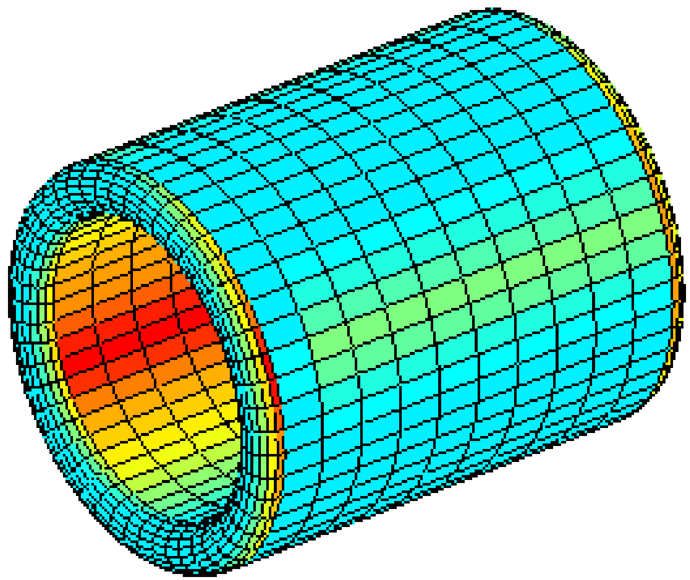


Building a wormhole for electromagnetic waves.

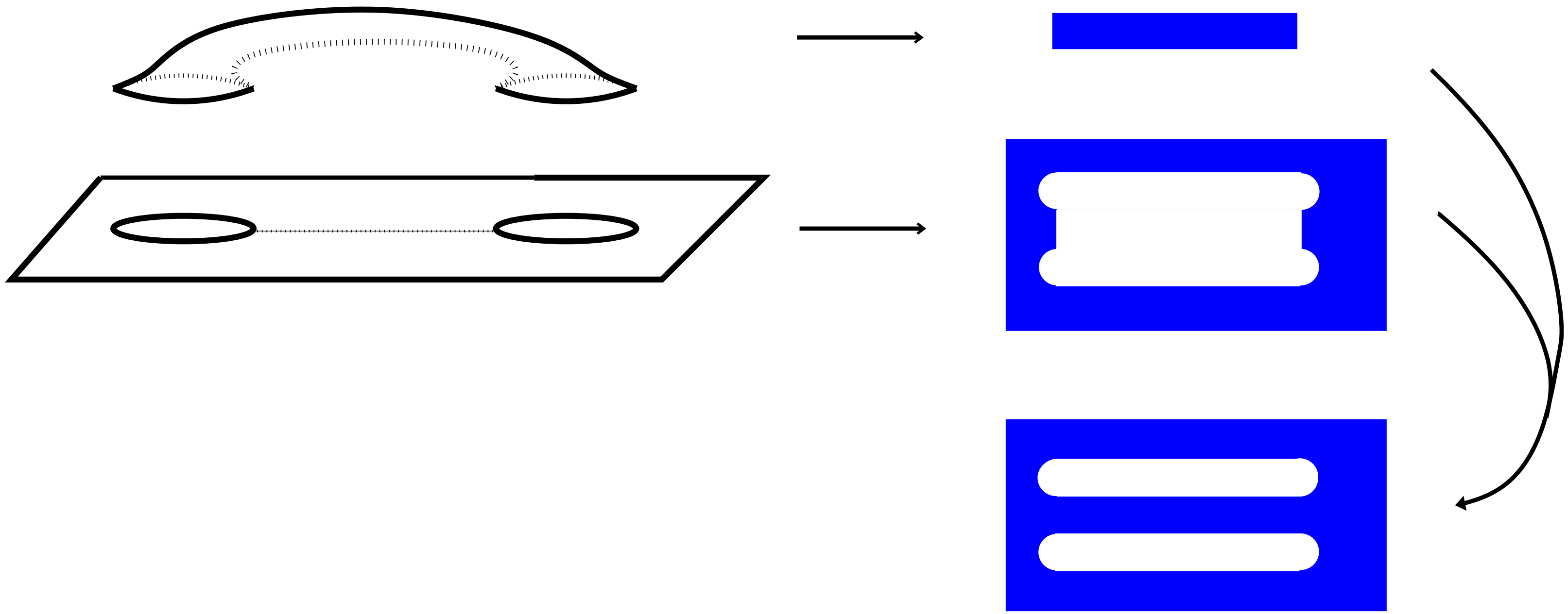
Let T be the two-dimensional finite cylinder

$$T = \mathbb{S}^1 \times [0, L] \subset \mathbb{R}^3.$$

Let $K = \{x \in \mathbb{R}^3 : \ \text{dist}\,(x, T) < \rho\}$ and $N = \mathbb{R}^3 \setminus K$.



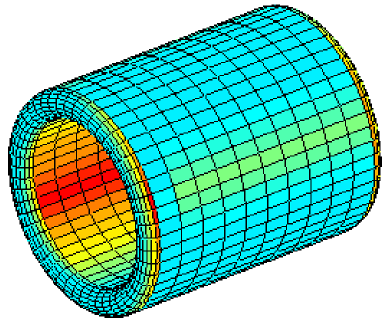
We put on ∂K the soft-and-hard boundary condition and cover K with "invisibility cloaking material".

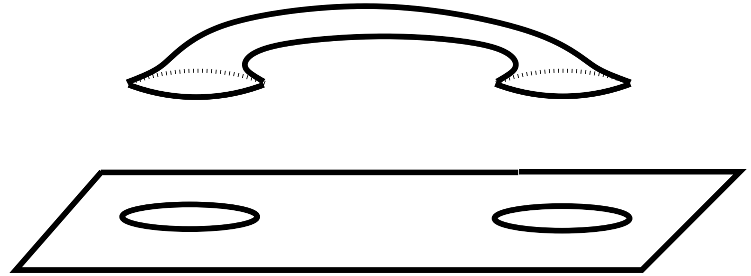


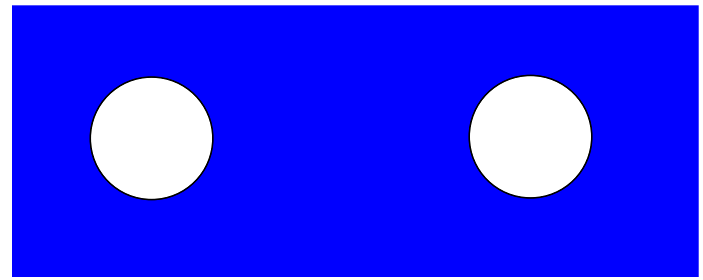
Let g be a smooth metric on the wormhole manifold M.
Using a closed path $\gamma \subset M$ and a diffeomorphism

$$F : M \setminus \gamma \to N \setminus \partial K$$

we define $\widetilde{g} = F_* g$ and choose $\widetilde{\epsilon}$ and $\widetilde{\mu}$ in N to correspond $\widetilde{g}$.







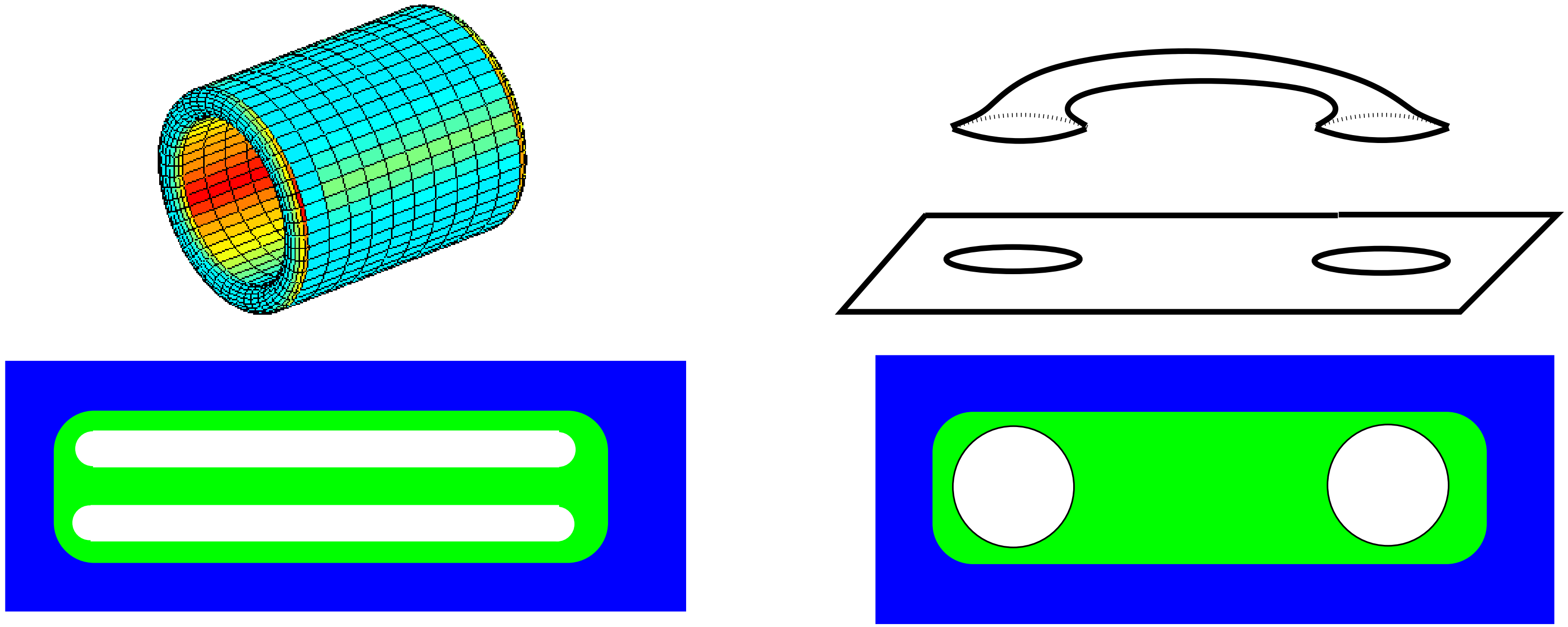
The set

$$U = \{x : \text{ dist}\,(x, K) > 1\} \subset \mathbb{R}^3$$

can be considered both as a subset of N and M.

Theorem 7 *All measurements of fields E and H in $U \subset M$ and $U \subset N$ coincide with currents that are supported in U.*

Thus $(N, \widetilde{\epsilon}, \widetilde{\mu})$ behaves as the wormhole M in all external measurements.



The set

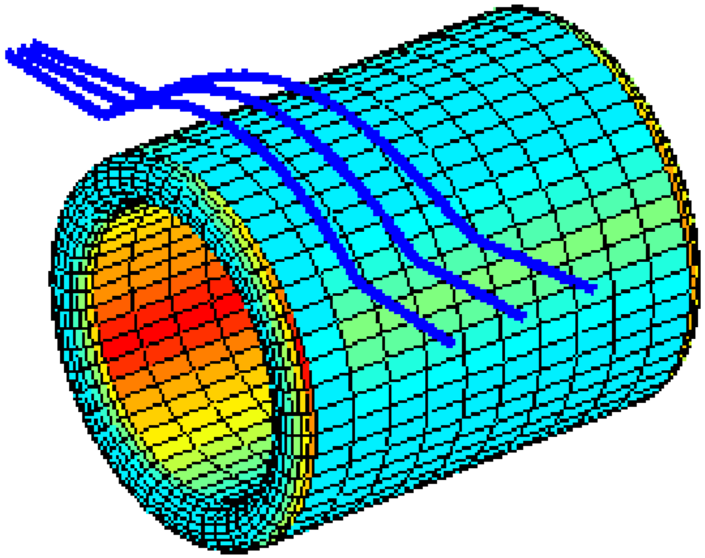
$$U = \{x : \text{ dist}\,(x, K) > 1\} \subset \mathbb{R}^3$$

can be considered both as a subset of N and M.

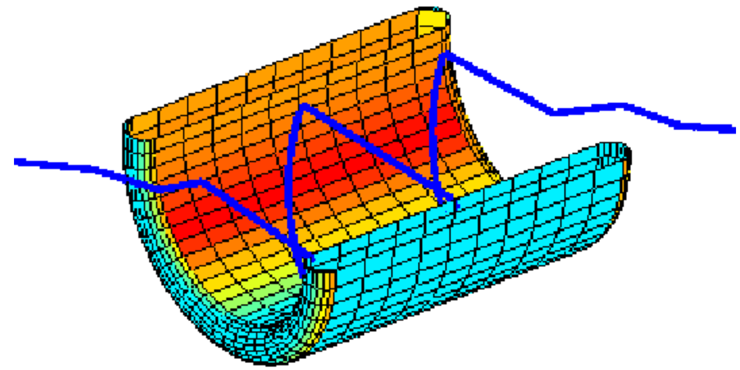
Theorem 8 *All measurements of fields E and H in $U \subset M$ and $U \subset N$ coincide with currents that are supported in U.*

Thus $(N, \widetilde{\epsilon}, \widetilde{\mu})$ behaves as the wormhole M in all external measurements.

Ray tracing simulations:

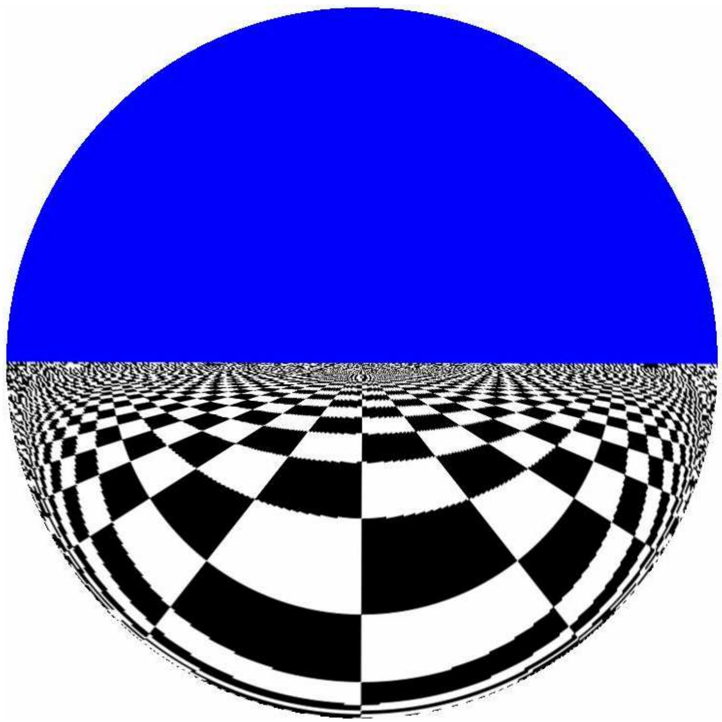


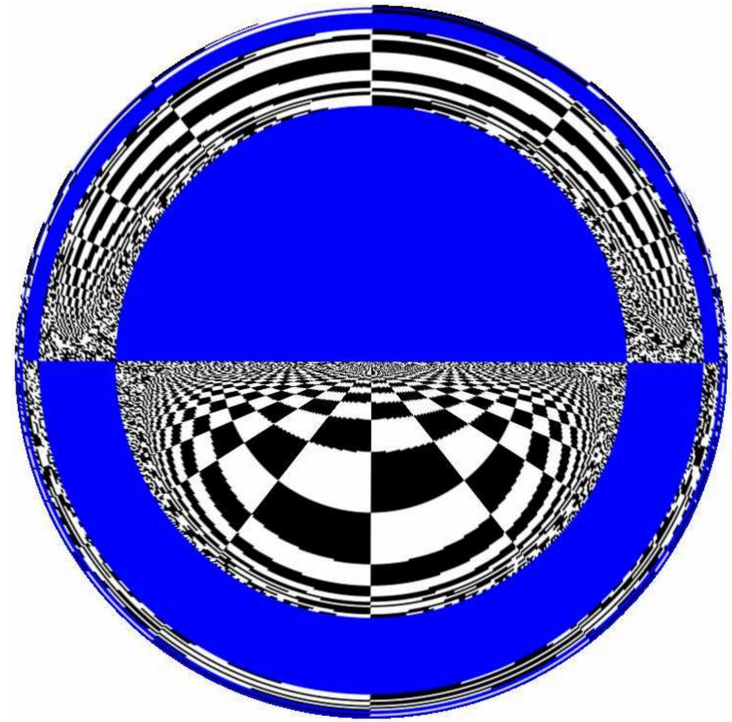
Rays travelling outside.



A ray travelling inside.

Ray tracing simulations:





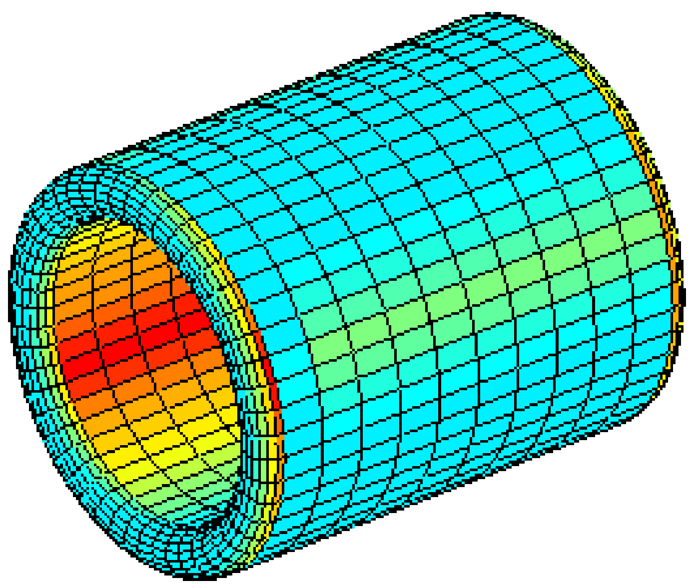
Length of handle $<< 1$.

Length of handle ≈ 1.

Pictures how an end of the wormhole looks like when the other end is over an infinite chess board and under the blue sky.

Possible applications in future:

- Invisible optical fibers.
- Components for optical computers.
- 3D video displays: ends of invisible tunnels work as light source in 3D voxels.
- Light beam collimation.
- Virtual magnetic monopoles.
- Scopes for Magnetic Resonance Imaging devices.



Thank you!