CONTINUED FRACTIONS

Exercises 2019

1. Todista/Prove
   (a) Lause 2.2. käyttäen Lausetta 2.3./by using Theorem 2.3.
   (b) Lause 2.4. käyttäen Lausetta 2.3./by using Theorem 2.3.
2. Determine Cantor expansions for the numbers
   (a)
   $$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 23,\ 117;$$
   (b)
   $$(m+1)!-1, \quad m \in \mathbb{N}.$$
3. Let $b \in \mathbb{Z}_{\geq 2}$. Determine b-base expansions for the numbers
   (a)
   $$\frac{1}{b-1};$$
   (b)
   $$\frac{1}{b+1};$$
   (c)
   $$\frac{1}{b-1} - \frac{1}{b+1}.$$
4. Determine binary and 3-adic expansions of the numbers
   (a) 8/3;
   (b) 1/22.
5. Express the numbers
   (a) $(0,123)_7$;
   (b) $(0,\overline{123})_7$;
   (c) $(0;0\overline{13})_6$
   as rational numbers
6. Determine desimal (base 10) expansions of the numbers
   $$1/7, \quad 2/7, \quad 3/7, \quad 4/7, \quad 5/7, \quad 6/7$$
   by using the algorithm of Theorem 2.5 i.e. by using the recurrences (2.11) ja (2.12). Note that the expansions of the other numbers are coming from the algorithm of the number 1/7.

7. Determine lenghts of the initial terms and periods of the numbers

$$\frac{1}{2\cdot 7},\quad \frac{1}{4\cdot 7},\quad \frac{1}{8\cdot 7},\quad \frac{1}{5\cdot 7}$$

by using heorem 2.9.

8. (a) Determine $\operatorname{ord}_{13}10$.
   (b) Determine lenght of the initial term and period of the number $1/13$ by using Theorem 2.9.
   (c) Määrää $\operatorname{ord}_{17}10$.
   (d) Määrää luvun

$$\frac{1}{2^a5^c\cdot 7},\quad a,c\in\mathbb{N}$$

   desimaaliesityksen alkutermin ja jakson pituudet käyttäen Lausetta 2.9.

9. Let $b\in\mathbb{Z}_{\geq 3}$. Show that
   (a)

$$\operatorname{ord}_{(b-1)^2} b = b-1.$$

   (b)

$$\frac{1}{(b-1)^2} = (0,\overline{0123...b-3\ b-1})_b.$$

10. Let $p\in\mathbb{P}$, $m\in\mathbb{Z}$, $1\leq m\leq p-1$ and suppose that

$$\frac{1}{p} = (0,\overline{c_1...c_{p-1}})_b,$$

where $p-1$ is the minimum period. Show that

$$\frac{m}{p} = (0,\overline{c_{k+1}...c_{p-1}c_1...c_k})_b,$$

with some $k\in\mathbb{Z}$, $0\leq k\leq p-1$.

11. Let $b\in\mathbb{Z}_{\geq 2}$. show that

$$\sum_{n=0}^{\infty}\frac{1}{b^{n^2}}\notin\mathbb{Q}.$$

12. Osoita, että

$$0,123456789\ 10\ 11\ 12\ 13...\notin\mathbb{Q}.$$

13. Determine n. convergents of the following continued fractions, when $n\leq 6$.
   (a)

$$[1;1,1,...];$$

   (b)

$$[b_0;b_1,...]_\pi = [3;7,15,1,292,1,1,1,2,...].$$

(c) Based on the case b) show that

$$\left|\pi - \frac{22}{7}\right| < \frac{1}{15 \cdot 7^2};$$

$$\left|\pi - \frac{355}{113}\right| < \frac{1}{292 \cdot 113^2}.$$

14. Determine simple continued fraction expansions of the following numbers.

(a)

$$\sqrt{6};$$

(b)

$$\sqrt{43}.$$

15. Determine the values of the following continued fractions.

(a)

$$[\overline{2, 4}];$$

(b)

$$[2, \overline{2, 4}];$$

(c)

$$[4, 3, 2, 1, \overline{2, 4}];$$

(d)

$$[5, \overline{1, 1, 1, 10}];$$

(e)

$$[2^{2^{n-1}}, \overline{2^{2^{n-1}+1}}].$$

16. Let $\alpha = [b_0, b_1, ...] > 1$ be a simple continued fraction and $A_n/B_n = [b_0, b_1, ..., b_n]$ its n. convergent.

(a) Show that

$$\frac{1}{\alpha} = [0, b_0, b_1, ...];$$

(b) Deduce the identity

$$\frac{A_{k+1}}{A_k} = [b_{k+1}, b_k, ..., b_0] \quad \forall\ k \in \mathbb{N};$$

(c) Deduce the identity

$$\frac{B_{k+1}}{B_k} = [b_{k+1}, b_k, ..., b_1] \quad \forall\ k \in \mathbb{N}.$$

17. Show that, for the convergents of the infinite simple continued fraction $\tau = [b_0; b_1, ...]$ hold

(a)
$$A_{n+2}B_n - A_nB_{n+2} = b_{n+2}(-1)^n \quad \forall\, n \in \mathbb{N}.$$

(b)
$$B_n \geq F_{n+1} \geq \left(\frac{\sqrt{5}+1}{2}\right)^{n-1} \qquad \forall\, n \in \mathbb{Z}^+,$$

where (F_n) is the Fibonacci sequence.

(c)
$$0 < \frac{A_{2k}}{B_{2k}} < \tau < \frac{A_{2k+1}}{B_{2k+1}}, \quad \forall\, k \in \mathbb{Z}^+.$$

(d)
$$(B_{n+1}\tau - A_{n+1})(B_n\tau - A_n) < 0, \quad \forall\, n \in \mathbb{N}.$$

18. Show that 377/233 is a convergent of the continued fraction expansion of the number
$$\frac{\sqrt{5}+1}{2}$$
(seek a proper result from the lectures).

19. Determine the value of Napier's constant e by 10 decimals by using the expansion

(a)
$$e = [2, \overline{1, 2k, 1}]_{k=1}^{\infty}.$$

(b)
$$e = 1 + 2[0, 1, \overline{4k+2}]_{k=1}^{\infty}.$$

20. Olkoon $d \in \mathbb{Z}^+$.

(a) Find out the simple continued fraction expansion for the number
$$\frac{1+\sqrt{4d^2+1}}{2}.$$

(b) Determine the value of the expansion
$$[d, \overline{1, 1, 2d-1}].$$

(c) Johda luvulle
$$\sqrt{d^2+1}$$
yksinkertainen ketjumurtokehitelmä.

(d) Määrää kehitelmän
$$[d, \overline{2d}]$$
arvo.

21. Olkoon $d \in \mathbb{Z}^+$. Show (by computing the value), that

(a)
$$\sqrt{d^2+2} = [d, \overline{d, 2d}].$$

(b)
$$\sqrt{d^2+4} = [d, \overline{(d-1)/2, 1, 1, (d-1)/2, 2d}], \quad 2 \not| d \geq 3.$$

(c)
$$\sqrt{d^2-1} = [d-1, \overline{1, 2d-2}], \quad d \geq 2.$$

(d)
$$\sqrt{d^2-2} = [d-1, \overline{1, d-2, 1, 2d-2}], \quad d \geq 3.$$

(e)
$$\sqrt{d^2-4} = [d-1, \overline{1, (d-3)/2, 2, (d-3)/2, 1, 2d-2}], \quad 2 \not| d \geq 5.$$

22. What are the lenghts of the periods of the continued fractions in the previous problem?

23. Show that for the positive solutions $x, y \in \mathbb{Z}^+$ of the Diophantine equation
$$x^2 - 7y^2 = 2$$
holds
$$\frac{x}{y} = \frac{A_k}{B_k},$$
for some of the convergents A_k/B_k of the number $\sqrt{7} = [2, \overline{1, 1, 1, 4}]$. Find at least two solutions.

24. Let $d \in \mathbb{Z}^+$, $\sqrt{d} = [b_0, b_1, ...]$ be irrational and $A_n/B_n = [b_0, b_1, ..., b_n]$ its nth convergent. Let us use the notations of Theorem 4.16.

(a) Show that
$$A_k^2 - dB_k^2 = (-1)^{k+1} Q_{k+1}.$$

(b) By using case a) deduce that
$$1 \leq Q_k \leq d, \quad |P_k| \leq d-1.$$

(c) Solve the Diophantine equation
$$x^2 - 6y^2 = -2.$$

(d) Ratkaise Diofantoksen yhtälö
$$x^2 - 7y^2 = 2.$$

25. Show that
$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 + \ldots} = b_0 - \frac{a_1}{-b_1 +} \frac{a_2}{-b_2 +} \frac{a_3}{-b_3 + \ldots}.$$

26. Determine value of

(a)

$$\mathbb{K}_{k=1}^{\infty}\left(\frac{1}{1+i}\right);$$

(b)

$$\mathbb{K}_{k=1}^{\infty}\left(\frac{i}{2}\right);$$

by solving the recurrences of A_n and B_n and computing limit of the convergent sequence.

27. Determine the value of

(a)

$$b+\frac{a}{b}\,\frac{a}{+b}\,\frac{}{+\dots}$$

when $a=-1, b=3$.

(b)

$$b+\frac{a}{c}\,\frac{d}{+b}\,\frac{a}{+c}\,\frac{d}{+b}\,\frac{}{+\dots}$$

arvo, kun $a=3, b=1, c=5, d=2$.

28. (a) Solve the recurrence

$$a_{n+2}-(n+3)a_{n+1}+(n+1)a_n=0.$$

(b) Let $f_n=n!$ ja $e_n=n!\sum_{k=0}^{n}\frac{1}{k!}$. Show that $\{(e_n),(f_n)\}$ is a solution base for case a).

(c) Determine a solution base for

$$(n+2)b_{n+2}-(n+3)b_{n+1}+b_n=0.$$

(d) Ratkaise rekursio

$$a_{n+2}-(n+1)a_{n+1}-(n+1)a_n=0.$$

(e) Olkoon $g_n=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$. Osoita, että $\{(f_n),(g_n)\}$ on (d)-kohdan ratkaisukanta.

29. Todista Lause 5.2.

30. Todista Lause 5.4.

31. Olkoon

$$x_n+y_n\sqrt{2}=(3+2\sqrt{2})^n \quad \forall\, n\in\mathbb{Z}^+.$$

Määrää palautuskaavat luvuille x_n ja y_n (Katso: Esimerkki 15).

32. Näytä, että

(a)

$$\sin z = z\ {}_0F_1\left(\begin{matrix} * \\ 3/2 \end{matrix}\ \middle|\ -\frac{z^2}{4}\right);$$

(b)

$$\cosh z = {}_0F_1\left(\begin{matrix} * \\ 1/2 \end{matrix}\ \middle|\ \frac{z^2}{4}\right);$$

(c)

$$\frac{\arctan t}{t} = {}_2F_1\left(\begin{matrix} 1, 1/2 \\ 3/2 \end{matrix}\ \middle|\ -t^2\right).$$

33. Osoita, että sarjalle

$$f(c) = {}_0F_1\left(\begin{matrix} * \\ c \end{matrix}\ \middle|\ t\right) = \sum_{n=0}^{\infty} \frac{1}{n!(c)_n} t^n$$

pätee palautuskaava

$$f(c) = f(c+1) + \frac{t}{c(c+1)} f(c+2).$$

34. By using the result

$$\frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{z}{1+}\frac{z^2}{3+}\frac{z^2}{5+}\frac{z^2}{7+}\ldots$$

show that

(a)

$$e^{\frac{r}{s}} \notin \mathbb{Q} \quad \forall\ r/s \in \mathbb{Q}^*.$$

(b)

$$\log\frac{a}{b} \notin \mathbb{Q} \quad \forall\ a/b \in \mathbb{Q}^+ \setminus \{1\}.$$

(c)

$$\log 2 \notin \mathbb{Q}.$$

35. Määrää luvun e^2 yksinkertainen ketjumurtokehitelmä

$$e^2 = [7, \overline{3k-1, 1, 1, 3k, 12k+6}]_{k=1}^{\infty}.$$

36. Näytä, että

$$I_n(\pi) = \frac{1}{2n!}\int_0^{\pi} x^n(\pi - x)^n \sin x\ dx = \sum_{n \le 2l \le 2n} (-1)^{n+l}\frac{(2l)!}{n!}\binom{n}{2l-n}\pi^{2n-2l}.$$

37. Osoita, että

$$1 + \cfrac{\frac{t}{(c)(c+1)}}{1 + \cfrac{\frac{t}{(c+1)(c+2)}}{1+\ldots}} = 1 + \cfrac{t/c}{c+1+\cfrac{t}{c+2+\cfrac{t}{c+3+\ldots}}}.$$

38. Osoita, että

$$\frac{z}{1+\frac{z^2/2}{3/2+\frac{z^2/4}{5/2+\frac{z^2/4}{7/2+\dots}}}} = \frac{z}{1+\frac{z^2}{3+\frac{z^2}{5+\frac{z^2}{7+\dots}}}}.$$

39. Todista

(a)

$$\alpha \in \mathbb{Q} \quad \Leftrightarrow \quad \exists\, a, b \in \mathbb{Z},\ a \neq 0 :\ a\alpha + b = 0.$$

(b)

$$\alpha \notin \mathbb{Q} \quad \Leftrightarrow \quad \forall\, a, b \in \mathbb{Z},\ a \neq 0 :\ a\alpha + b \neq 0.$$

(c)

$$\alpha \notin \mathbb{Q} \quad \Leftrightarrow \quad 1, \alpha \quad \text{lin. vapaita}/\mathbb{Q}$$

(d)

$$\alpha \notin \mathbb{Q} \quad \Leftrightarrow \quad \dim_{\mathbb{Q}} \{\mathbb{Q} + \alpha\mathbb{Q}\} = 2.$$

(e)

$$\alpha \in \mathbb{Q} \quad \Leftrightarrow \quad \dim_{\mathbb{Q}} \{\mathbb{Q} + \alpha\mathbb{Q}\} = 1.$$

40. Todista

(a)

$$e \notin \mathbb{Q} \quad \Leftrightarrow \quad 1, e \quad \text{lin. vapaita}/\mathbb{Q}$$

(b)

$$\pi \notin \mathbb{Q} \quad \Leftrightarrow \quad \dim_{\mathbb{Q}} \{\mathbb{Q} + \pi\mathbb{Q}\} = 2.$$