GEOMETRY OF NUMBERS

Exercises 2022

Problems denoted by $\diamondsuit$ may be challenging.

First set: problems 1–7.

Second set: problems 8–13.

Third set: problems 14–19.

Fourth set: problems 20–25.

Fifth set: problems 26–30.

1. Let R be a commutative ring with unity and $n \in \mathbb{Z}^+$. In the Cartesian product
$$R^n := R \times \ldots \times R = \{\overline{x} = (x_1, \ldots, x_n) |\ x_1, \ldots, x_n \in R\}$$
we set standard identity relation, addition and scalar product by
$$\begin{aligned} &\overline{x} = \overline{y} \quad \Leftrightarrow \quad x_i = y_i \quad \forall\ i = 1, ..., n; \\ &\overline{x} + \overline{y} = (x_1 + y_1, ..., x_n + y_n); \\ &r \cdot \overline{x} = (rx_1, ..., rx_n) \end{aligned}$$
for $\overline{x} = (x_1, \ldots, x_n), \overline{y} = (y_1, \ldots, y_n) \in R^n$ and $r \in R$. Show that
$$(R^n, +, \cdot)$$
is an R-module and $\operatorname{rank}_R R^n = n$.

2. (a) Show that
$$\big\langle (1, 0), (1, 1) \big\rangle_{\mathbb{Z}} = \big\langle (-1, 0), (0, 1) \big\rangle_{\mathbb{Z}}.$$
(b) Show that
$$\big\langle (-2, 0), (1, 1) \big\rangle_{\mathbb{Z}} \cong \mathbb{Z}^2.$$
(c) Let $e, \pi \in \mathbb{R}^+$. Show that
$$\big\langle (e, 0)^t, (0, \pi)^t \big\rangle_{\mathbb{Z}}$$
is a full lattice in $\mathbb{R}^2$.

3. Prove, that
$$\|\overline{x}\|_1 = \sum_{k=1}^{n} |x_k|,$$
for $\overline{x} = (x_1, ..., x_n)^t \in \mathbb{C}^n$, determines a norm.

4. Let $\lambda \in \mathbb{R}_{\geq 0}$ and assume that $\mathcal{C} \subseteq \mathbb{R}^n$ is a central symmetric convex body. Prove that

$$\lambda\mathcal{C} := \{\lambda\bar{a} |\ \bar{a} \in \mathcal{C}\}$$

is also a central symmetric convex body.

5. (a) Show that $\mathcal{B}_1^n(1)$ is a central symmetric convex body.
   (b) Show that $\mathcal{B}_{1/2}^2(1)$ is not a convex body.

6. Let $L = [\bar{l}_1, ..., \bar{l}_r]$, where $\bar{l}_1, ..., \bar{l}_r \in \mathbb{R}^n$. Prove that

$$\det(L^t L) = \det[\bar{l}_i \cdot \bar{l}_j]_{1 \leq i,j \leq r},$$

where $\cdot$ is the standard inner product in $\mathbb{R}^n$.

7. $\diamondsuit$ Let $L = [\bar{l}_1, ..., \bar{l}_r]$, where $\bar{l}_1, ..., \bar{l}_r \in \mathbb{R}^n$. Prove

$$\det[\bar{l}_i \cdot \bar{l}_j]_{1 \leq i,j \leq r} \geq 0.$$

8. Let $\lambda \in \mathbb{R}_{\geq 0}$, $\mathcal{C} \subseteq \mathbb{R}^n$ and assume that $\operatorname{vol}\mathcal{C}$ exists. Show that

$$\operatorname{vol}\lambda\mathcal{C} = \lambda^n \operatorname{vol}\mathcal{C}.$$

9. Let $r \in \mathbb{R}_{\geq 0}$.
   (a) $\diamondsuit$ Compute the volume of the 4-ball $\mathcal{B}_2^4(r)$.
   (b) $\diamondsuit$ Compute the volume of the n-octahedron $\mathcal{B}_1^n(r)$, $n \in \mathbb{Z}_{\geq 1}$.

10. (a) Show that $(\sqrt{3}, \sqrt{3})$ and $(1,1)$ are linearly independent over $\mathbb{Z}$.
    (b) Show that $(\sqrt{3}, \sqrt{3})$ and $(1,1)$ are linearly dependent over $\mathbb{R}$.

11. (a) Compute the determinant of the integer lattice

$$\mathbb{Z}\bar{e}_1 + ... + \mathbb{Z}\bar{e}_n.$$

    (b) Compute the determinant of the lattice

$$\left\langle (2,-1,1)^t, (1,3,-2)^t \right\rangle_{\mathbb{Z}}.$$

12. Define a lattice

$$\Lambda := \left\langle \bar{e}_1 + \bar{e}_2, \bar{e}_2 + \bar{e}_3, \bar{e}_3 + \bar{e}_1 \right\rangle_{\mathbb{Z}}$$

in $\mathbb{R}^3$.
    (a) Is Λ a full lattice?
    (b) Compute the determinant of the lattice Λ.
    (c) Determine the matrix of the linear map L associated to Λ.

13. Prove the equality

$$\mathcal{D} = L\mathcal{E},$$

formula (41) announced in Lecture notes C.

14. Find a subset $\mathcal{S}_1$ in $\mathbb{R}^2$ which is convex and whose area $\operatorname{vol}\mathcal{S}_1 = 10$, but $\mathcal{S}_1$ does not contain any integer points.

15. Find a subset $\mathcal{S}_2$ in $\mathbb{R}^2$ which is symmetric with respect to the origin and whose area $\operatorname{vol}\mathcal{S}_2 = 10$, but $\mathcal{S}_2$ does not contain any integer points.

16. Sketch a picture of the subset $\mathcal{S}_3$ in $\mathbb{R}^2$ defined by

$$\mathcal{S}_3 := \{(x_1, x_2) \in \mathbb{R}^2 \big| \, |x_1^2 - 2x_2^2| < 1\}.$$

    (a) Is the set $\mathcal{S}_3$ symmetric with respect to the origin?

    (b) Is the set $\mathcal{S}_3$ convex?

17. (a) Find the area of the set $\mathcal{S}_3$.

    (b) How many integer points does the set $\mathcal{S}_3$ contain?

18. Let α_1 and α_2 be real numbers and n a positive integer. Prove that there are integers p_1, p_2, q such that $1 \le q \le n$ and

$$\left(\alpha_1 - \frac{p_1}{q}\right)^2 + \left(\alpha_2 - \frac{p_2}{q}\right)^2 \le \frac{4}{\pi n q^2}.$$

19. Let $\alpha \in \mathbb{R}, h \in \mathbb{Z}^+$ be given. Prove by Minkowski's convex body theorem that there exist $p, q \in \mathbb{Z}$, such that $1 \le q \le h, \quad p \perp q$, and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qh}.$$

20. Determine the area of the ellipse

$$\{(x, y) \in \mathbb{R}^2 \big| \, 5x^2 + 16xy + 13y^2 < 2\}.$$

    (See B:(42)–(47)).

21. Prove by Minkowski's convex body theorem that the Diophantine equation

$$5x^2 + 16xy + 13y^2 = 1$$

    has a solution $(x, y) \in \mathbb{Z}^2$.

22. Consider the lattice

$$\Lambda = \left\langle (1, 0)^t, (-1, 1/2)^t \right\rangle_{\mathbb{Z}}.$$

    (a) Find all shortest vectors in Λ.

    (b) Estimate the length σ of the shortest vector by formula E(40):

$$\sigma \le \frac{2}{\sqrt{\pi}} \Gamma(1 + n/2)^{1/n} (\det \Lambda)^{1/n} .$$

23. Consider the lattice

$$\Lambda = \left\langle (1,0,0)^t, (-1,1/2,1)^t, (0,-1/2,1/2)^t \right\rangle_{\mathbb{Z}}.$$

(a) Find one of the shortest vectors in Λ.

(b) Estimate the length σ of the shortest vector by formula E(40).

24. Consider the lattice

$$\Lambda_\pi := \left\langle (\pi, 1/7)^t, (1, 1/22)^t \right\rangle_{\mathbb{Z}}.$$

(a) Find one of the shortest vectors in Λ_π.

(b) Estimate the length σ of the shortest vector by formula E(40).

25. Determine the successive minima of the rectangle

$$\mathcal{K} = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, |x| \leq \frac{1}{3},\ |y| \leq \frac{1}{5} \right\}$$

with respect to the lattice

$$\Lambda = \left\langle \bar{e}_1, \bar{e}_2 \right\rangle_{\mathbb{Z}}.$$

26. Determine the sphere lattice packing density

$$\Delta_2(\mathcal{B}^2(\sigma_{\Lambda_\pi}/2), \Lambda_\pi)$$

of the lattice Λ_π defined above.

27. Let $\Lambda_n \subseteq \mathbb{R}^n$ be a full lattice. By using the estimate

$$\sigma_{\Lambda_n} \leq \frac{2}{\sqrt{\pi}} \Gamma(1+n/2)^{1/n} \left(\det \Lambda_n\right)^{1/n} \left(\Delta_{\mathcal{B},n}\right)^{1/n}$$

and results from the lectures show either (a) or (b):

(a)

$$\sigma_{\Lambda_2} \leq \left(\frac{2}{\sqrt{3}}\right)^{1/2} \left(\det \Lambda_2\right)^{1/2}.$$

(b)

$$\sigma_{\Lambda_3} \leq 2^{1/6} \left(\det \Lambda_3\right)^{1/3}.$$

28. In the lattices of problems 22,23,24 estimate the lengths of minimal vectors by using above formulae in problem 27.

29. Consider the face-centered cubic lattice

$$\Lambda_{fcc} := \left\langle (1,1,0)^t, (1,-1,0)^t, (0,1,-1)^t \right\rangle_{\mathbb{Z}}.$$

(a) Find one of the shortest vectors in Λ_{fcc}.

(b) Estimate the length $\sigma_{\Lambda_{fcc}}$ of the shortest vector by a formula in problem 27.

30. Determine the sphere lattice packing density

$$\Delta_3(\mathcal{B}^3(\sigma_{\Lambda_{fcc}}/2), \Lambda_{fcc})$$

of the face-centered cubic lattice Λ_{fcc}.