802656S ALGEBRALLISET LUVUT

Harjoituksia 2020

1. Näytä, että/Show that

   (a)
   $$2^{1/2}, \quad 3^{1/2}, \quad 2^{1/3};$$

   (b)
   $$2^{1/2} + 3^{1/2};$$

   (c)
   $$2^{1/3} + 3^{1/2};$$

   (d)
   $$e^{i\pi/m}, \quad m \in \mathbb{Z} \setminus \{0\};$$

   (e)
   $$\sin(\pi/m),\ \cos(\pi/m),\ \tan(\pi/m), \quad m \in \mathbb{Z} \setminus \{0\};$$

   ovat algebrallisia lukuja/are algebraic numbers.
   Solutions.
   1a: $p_1(x) = x^2 - 2, \quad p_2(x) = x^2 - 3, \quad p_3(x) = x^3 - 2.$
   1b: $p(x) = (x - (2^{1/2} + 3^{1/2}))(x - (-2^{1/2} + 3^{1/2}))(x - (2^{1/2} - 3^{1/2}))$
   $(x - (-2^{1/2} - 3^{1/2})) = x^4 - 10x^2 + 1.$

2. Olkoon K kokonaisalue/Let K be an integral domain ja $P(x), Q(x) \in K[x]$.

   (a) Todista, että/Prove that
   $$\deg P(x)Q(x) = \deg P(x) + \deg Q(x).$$

   (b) Osoita, että jos nolla-polynomille pätisi/Show that, if the zero-polynomial would satisfy
   $$\deg 0(x) \in \mathbb{Z},$$
   niin a) kohdan astekaava ei toimisi/then the degree formula would not work.

3. Näytä, että renkaat/Show that the rings

   (a)
   $$\mathbb{Z}_{10} = \{\overline{0}, \overline{1}, ..., \overline{9}\};$$

   (b)
   $$\mathbb{Z}_{10}[x];$$

   eivät ole kokonaisalueita/are not integral domains.

Solutions.
3a: $\overline{2}\cdot\overline{5}=\overline{10}=\overline{0}$, where $\overline{2}\neq\overline{0},\overline{5}\neq\overline{0}$.
3b: $\mathbb{Z}_{10}\subseteq\mathbb{Z}_{10}[x]$, thus the proof of case 3a works.
Or, take e.g. polynomials $\overline{2}x^2,\overline{5}x^3$.

4. Olkoon R kommutatiivinen ykkösellinen rengas/Let R be a commutative ring with unity.

   (a) Näytä, että yksikköjen joukko/Show that the set of units

   $$R^*=\{u\in R\mid \exists\, u^{-1}\in R:\ uu^{-1}=1\}$$

   on ryhmä kertolaskun suhteen/is a group with respect to multiplication.

   (b) Osoita, että relaatio

   $$a\sim b\quad\Leftrightarrow\quad\exists\quad u\in R^*:\ b=ua$$

   on ekvivalenssirelaatio.

   (c) Määrää/Determine

   $$[1]=\{b\sim 1|\ b\in R\}.$$

5. Määrää

   (a)

   $$\mathbb{Z}_{10}^*=\{\overline{1},\overline{3},\overline{7},\overline{9}\},\quad\text{e.g.}\quad\overline{3}\cdot\overline{7}=\overline{1};$$

   (b)

   $$\mathbb{Z}_{10}[x]^*;$$

   (c)

   $$m\in\mathbb{Z}_{m\geq 2},\quad\mathbb{Z}_m^*=\{\overline{k}\mid 1\leq k\leq m-1,\ \gcd(k,m)=1\};$$

   see, Basics of Number Theory course.

   (d)

   $$\mathbb{Z}_m[x]^*,\quad m\in\mathbb{Z}_{m\geq 2};$$

   (e)

   $$\mathbb{Z}[i]^*=\{1,-1,i.-i\};$$

   Solution: Assume $(a+ib)(c+id)=1$. Take complex conjugates to get $(a-ib)(c-id)=1$. Then by multipling $(a^2+b^2)(c^2+d^2)=1$, where a^2+b^2 and c^2+d^2 are positive integers dividing 1. Therefore $a^2+b^2=1$ implying $a+ib=1,-1,i$ or $-i$.

   (f)

   $$\mathbb{Z}_5[\sqrt{2}]^*;$$

6. Olkoon D kokonaisalue. Näytä, että

   (a)

   $$0|0;$$

(b)

$$a|a \quad \forall\, a \in D.$$

(c)

$$1|a \quad \forall\, a \in D.$$

7. Olkoon $R = \mathbb{Z}[\sqrt{-5}]$.

   (a) Onko R kokonaisalue? Yes. Solution: Assume $(a + b\sqrt{-5})(c + d\sqrt{-5}) = 0$. Take complex conjugates to get $(a - b\sqrt{-5})(c - d\sqrt{-5}) = 0$. Then by multipling $(a^2 + 5b^2)(c^2 + 5d^2) = 0$. Thus $a^2 + 5b^2 = 0$ or $c^2 + 5d^2 = 0$ implying $a + b\sqrt{-5} = 0$ or $c + d\sqrt{-5} = 0$.

   (b) Määrää R:n yksikköryhmä R^*. Answer: $R^* = \{1, -1\}$, compare 5e.

   (c) Määrää lukujen 3 ja $2 + \sqrt{-5}$ liitännäisalkiot/Determine the associates of the numbers.

   (d) Ovatko 3 ja $2 + \sqrt{-5}$ jaottomia? V: Ovat.

   (e) Ovatko 3 ja $2+\sqrt{-5}$ alkualkioita? (Tutki yhtälöä $3 \cdot 3 = (2+\sqrt{-5})(2-\sqrt{-5})$.) Eivät.

   (f) Onko R UFD? V: Ei.

   (g) Onko R Eukleideen alue? V: Ei.

8. Tutki polynomien

$$a(x) = x^4 + x + 1, \quad b(x) = x^4 + 1$$

jaottomuutta polynomirenkaassa/investigate the irreducibility of the polynomials in the ring $K[x]$, kun

   (a) $K = \mathbb{Q}$;

   (b) $K = \mathbb{R}$;

   (c) $K = \mathbb{C}$;

   (d) $K = \mathbb{Z}_2$;

   (e) $K = \mathbb{Z}_3$.

Solutions:

Case 8d, $a(x) \in \mathbb{Z}_2[x]$. Because $a(0) = a(1) = 1 \neq 0$, there are no 1st degree factors. If $x^4 + x + 1 = (bx^2 + cx + d)(ex^2 + fx + g)$, then $be = 1 = dg$ implying $b = e = d = g = 1$. Further $x^4+x+1 = x^4+(f+c)x^3+(1+cf+1)x^2+(c+f)x+1$. No solutions. Hence no 2nd degree factors. Irreducible in $\mathbb{Z}_2[x]$.

Case 8a, $a(x) \in \mathbb{Q}[x]$. Use reduction $\pmod 2$, then $\overline{a}(x) = x^4 + x + 1 \in \mathbb{Z}_2[x]$. But $\overline{a}(x)$ is irreducible in $\mathbb{Z}_2[x]$ by 8d. Then by Theorem 25 polynomial $a(x)$ is irreducible in $\mathbb{Q}[x]$.

Case 8b, $a(x) \in \mathbb{R}[x]$. No real zeros, so no first degree factors in $\mathbb{R}[x]$. But $a(x) = (x^2 + cx + d)(x^2 + fx + g)$, where $x^2 + cx + d, x^2 + fx + g \in \mathbb{R}[x]$, see Chapter 6.5.1.

Case 8c, $a(x) \in \mathbb{C}[x]$. Now $a(x) = (x - \alpha_1) \cdots (x - \alpha_4)$, where $(x - \alpha_k) \in \mathbb{C}[x]$, see Chapter 6.5.1.

Case 8e, $a(x) \in \mathbb{Z}_3[x]$. Irreducible in $\mathbb{Z}_3[x]$.

9. Determine $\gcd(a(x), b(x))$ in the polynomial ring $\mathbb{Q}[x]$, when
   (a) $a(x) = x^4 + x + 1, \quad b(x) = x^4 + 1$; Answer: 1.
   (b) $a(x) = x^4 - 2x^3 + 3x^2 - 2x + 1, \quad b(x) = 4x^3 - 6x^2 + 6x - 2$.
   V: $x^2 - x + 1$.

10. Määrää polynomin $a(x)$ ja sen derivaatan $Da(x)$ syt$(a(x), Da(x))$ polynomirenkaassa $\mathbb{Q}[x]$, kun
    (a) $a(x) = x^5 - x^4 + 2x^3 - 2x^2 - 1$;
    (b) $a(x) = x^4 - 2x^3 + 3x^2 - 2x + 1$.
    Mitä huomaat? What do you notice?
    Answer: $x^2 - x + 1$, see 9b. Further, by the proof of Theorem 15 this means that $(x^2 - x + 1)^2 | a(x)$. So $a(x) = (x^2 - x + 1)^2$. (Or if you study your Euclidean algorithm in 9b, you see $a(x) = (x^2 - x + 1)^2$ and $b(x) = (4x - 2)(x^2 - x + 1)$.)

11. Tutki polynomin

$$x^4 - 2x^3 + 3x^2 - 2x + 1 = (x^2 - x + 1)^2$$

nollakohtien kertalukuja polynomirenkaassa $K[x]$, kun
    (a) $K = \mathbb{Q}$; V: $m(\alpha) = 0$, kaikilla $\alpha \in \mathbb{Q}$.
    (b) $K = \mathbb{R}$; V: $m(\alpha) = 0$, kaikilla $\alpha \in \mathbb{R}$.
    (c) $K = \mathbb{C}$; V: $m(\frac{1+\sqrt{-3}}{2}) = m(\frac{1-\sqrt{-3}}{2}) = 2$
    (d) $K = \mathbb{Z}_2$; V: $m(\alpha) = 0$, kaikilla $\alpha \in \mathbb{Z}_2$.
    (e) $K = \mathbb{Z}_3$. V: $m(0) = m(1) = 0$, $m(2) = 4$.

12. Jaa/factor polynomit
    (a) $x^2 + 2351x + 125$; Jaoton.
    (b) $6x^2 + 7x - 5 = (2x - 1)(3x + 5)$;
    (c) $6x^4 + 7x^3 + 5x^2 + 7x - 5$; Irreducible. Reduction $\pmod 7$.
    (d) $x^4 + 7$; Jaoton. Eisenstein.
    (e) $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$;

(f) $x^5 + x + 1 = (x^2 + x + 1)(x^3 - x^2 + 1)$;

(g) $x^5 - x + 1$;   V: Jaoton.

jaottomiin primitiivisiin tekijöihin polynomirenkaassa/into irreducible primitive factors $\mathbb{Z}[x]$.

13. Olkoon D kokonaisalue. Osoita, että $D[x]^* = D^*$.

14. Olkoon K kunta, $p(x) \in K[x]$ ja $\deg p(x) \geq 1$. Osoita, että $n_K(p(x)) \leq \deg p(x)$.

15. Näytä, että

    (a) $1 + x + x^2 + ... + x^{10} \in J_{\mathbb{Q}[x]}$;

    (b) $7 + 7x - 14x^3 + 2x^5 \in J_{\mathbb{Q}[x]}$;

    (c) $2 - 14x^2 + 7x^4 + 7x^5 \in J_{\mathbb{Q}[x]}$;

    (d) $2 - 14(x-1)^2 + 7(x-1)^4 + 7(x-1)^5 \in J_{\mathbb{Q}[x]}$.

    Ratkaisu: Katso Esimerkit 22 ja 21.

16. Olkoon
$$x^2 + bx + c = (x - \alpha)(x - \beta) \in \mathbb{Q}[x].$$
    Näytä, että

    (a) $\alpha^2 + \beta^2 \in \mathbb{Q}$;

    (b) $\alpha^3 + 2\alpha\beta + \beta^3 \in \mathbb{Q}$.

    Solution: $-b = \alpha + \beta \in \mathbb{Q}$, $c = \alpha\beta \in \mathbb{Q}$, ...

17. Määrää luvun

    (a) $\sqrt{2} + \sqrt{3}$;

    (b) $2^{1/3} + 3^{1/3}$;

    minimipolynomi.

18. Onko
$$\frac{1 + \sqrt{37}}{3}$$
    kokonainen algebrallinen luku?
    V: EI, koska $M_\alpha(x) = x^2 - \frac{2}{3}x - 4$.

19. Tutki lukujen

    (a) e;

    (b) π;

    (c) $i\pi$;

    (d) $e + i$;

    (e) $i2^{1/3}$;

algebrallisuutta kuntien $\mathbb{Q}$, $\mathbb{R}$ ja $\mathbb{C}$ suhteen. (tiedetään, että e ja π ovat transkendenttisia $\mathbb{Q}$:n yli.)

20. Tutki lukujen

    (a) $\sqrt{\pi}$;

    (b) $\sqrt{-\pi}$;

    (c) π^2;

    algebrallisuutta kuntien $\mathbb{Q}$, $\mathbb{Q}(\sqrt{\pi})$ ja $\mathbb{Q}(\sqrt{-\pi})$ suhteen.

21. Olkoon $\mathbb{K}$ lukukunta ja $\sigma: \mathbb{K} \to \mathbb{C}$ monomorfia. Osoita, että

    (a) $\sigma(a) = a \quad \forall a \in \mathbb{Q}$.

    (b) $\sigma(a\alpha + b\beta) = a\sigma(\alpha) + b\sigma(\beta), \quad \forall a, b \in \mathbb{Q}, \alpha, \beta \in \mathbb{K}$.

    (c) $\sigma(p(\beta)) = p(\sigma(\beta)) \quad \forall \beta \in \mathbb{K}, \quad p(x) \in \mathbb{Q}[x]$.

22. Määrää algebrallisten lukujen (kunnan $\mathbb{Q}$ suhteen)

    (a) $\alpha = \sqrt{2} + \sqrt{3}$;
    V: Write $\alpha_1 = \sqrt{2} + \sqrt{3}, \alpha_2 = \sqrt{2} - \sqrt{3}, \alpha_3 = -\sqrt{2} - \sqrt{3}, \alpha_4 = -\sqrt{2} + \sqrt{3}$, then
    $M_{\alpha_i}(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4) = x^4 - 10x^2 + 1$ is irreducible giving $\deg_{\mathbb{Q}} \alpha = 4$ and on the other hand
    $M_\alpha(x) = x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 = x^4 - 10x^2 + 1$ implying
    $s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$,
    $s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = -10$,
    $s_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4 = 0$,
    $s_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = 1$.

    (b) $\alpha = 2^{1/3} + 3^{1/3}$;
    V: $M_\alpha(x) = x^9 - 15x^6 - 87x^3 - 125$, $\deg_{\mathbb{Q}} \alpha = 9$. Optional/EI vaadita.

    (c) $\alpha = \sqrt{2 + \sqrt{3}}$;
    V: $M_\alpha(x) = x^4 - 4x^2 + 1$, $\deg_{\mathbb{Q}} \alpha = 4$.

    asteet ja liittoluvut sekä määrää vastaavien peruspolynomien

    $$s_k(\sigma_1(\alpha), \sigma_2(\alpha), ...), \quad k = 1, 2, ...,$$

    arvot/degrees, conjugates and determine the values of the corresponding elementary symmetric polynomials.

23. Määrää algebrallisten kuntalaajennusten $\langle \mathbb{Q}, \alpha, \beta \rangle$ dimensiot ja kannat kunnan $\mathbb{Q}$ suhteen, kun/ Determine the dimension and a basis of the algebraic field extension $\langle \mathbb{Q}, \alpha, \beta \rangle$ over the field $\mathbb{Q}$, when

    (a) $\alpha = \sqrt{2}, \beta = \sqrt{3}$;
    V: It is proved in Example 33 (lecture notes: small font version) that
    $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. Next

$\langle \mathbb{Q}, \sqrt{2}, \sqrt{3}\rangle = \langle \mathbb{Q}(\sqrt{2}), \sqrt{3}\rangle = \mathbb{Q}(\sqrt{2})(\sqrt{3})$
meaning that first we extend $\mathbb{Q}$ by $\sqrt{2}$ to $\mathbb{Q}(\sqrt{2})$.
Then we extend $\mathbb{Q}(\sqrt{2})$ by $\sqrt{3}$ resulting $\mathbb{Q}(\sqrt{2})(\sqrt{3})$.
By Theorem 42:
$\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}] = \mathbb{Q}\cdot 1 + \mathbb{Q}\cdot\sqrt{2} := M$.
$\mathbb{Q}(\sqrt{2})(\sqrt{3}) = M\cdot 1 + M\cdot\sqrt{3} = \mathbb{Q}\cdot 1 + \mathbb{Q}\cdot\sqrt{2} + \mathbb{Q}\cdot\sqrt{3} + \mathbb{Q}\cdot\sqrt{6} =$
$\langle 1, \sqrt{2}, \sqrt{3}, \sqrt{6}\rangle_{\mathbb{Q}}$ the linear hull/verho over $\mathbb{Q}$ generated by the basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.
Thus $\dim_{\mathbb{Q}}\langle \mathbb{Q}, \sqrt{2}, \sqrt{3}\rangle = 4 = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ rank of the extension/laajennuksen aste.

(b) $\alpha = 2^{1/5}, \beta = 0$;

(c) $\alpha = \sqrt{2+\sqrt{3}}, \beta = 0$;

24. Määrää algebrallisten kuntalaajennusten $\langle \mathbb{K}, \alpha, \beta, \gamma\rangle$ dimensiot ja kannat kunnan $\mathbb{K}$ suhteen, kun

(a) $\alpha = \sqrt{3}, \beta = \sqrt{5}, \gamma = \sqrt{15},\ \mathbb{K} = \mathbb{Q}$;
V: $\langle \mathbb{K}, \alpha, \beta, \gamma\rangle = \langle \mathbb{Q}, \sqrt{3}, \sqrt{5}, \sqrt{15}\rangle = \langle \mathbb{Q}, \sqrt{3}, \sqrt{5}\rangle$ ($= \langle \mathbb{Q}, \sqrt{3}, \sqrt{15}\rangle$), because $\sqrt{15} \in \langle \mathbb{Q}, \sqrt{3}, \sqrt{5}\rangle$ (and $\sqrt{5} \in \langle \mathbb{Q}, \sqrt{3}, \sqrt{15}\rangle$).
We denote $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}\cdot 1 + \mathbb{Q}\cdot\sqrt{3} := M$.
Then
$\langle \mathbb{Q}, \sqrt{3}, \sqrt{5}\rangle = \mathbb{Q}(\sqrt{3})(\sqrt{5}) = M\cdot 1 + M\cdot\sqrt{5} =$
$\mathbb{Q}\cdot 1 + \mathbb{Q}\cdot\sqrt{3} + \mathbb{Q}\cdot\sqrt{5} + \mathbb{Q}\cdot\sqrt{15} =$
$\langle 1, \sqrt{3}, \sqrt{5}, \sqrt{15}\rangle_{\mathbb{Q}}$ the linear hull over $\mathbb{Q}$ generated by the basis $\{1, \sqrt{3}, \sqrt{5}, \sqrt{15}\}$.
Thus $\dim_{\mathbb{Q}}\langle \mathbb{Q}, \sqrt{3}, \sqrt{5}, \sqrt{15}\rangle = 4$.

(b) $\alpha = \sqrt{3}, \beta = \sqrt{5}, \gamma = 0,\ \mathbb{K} = \mathbb{Q}(\sqrt{15})$;

(c) $\alpha = \sqrt{3}, \beta = \sqrt{5}, \gamma = \sqrt{7},\ \mathbb{K} = \mathbb{Q}$;

(d) $\alpha = \sqrt{3}, \beta = \sqrt{5}, \gamma = 0,\ \mathbb{K} = \mathbb{Q}(\sqrt{7})$;

25. Esitä kunnat $\langle \mathbb{K}, \alpha, \beta, \gamma\rangle$

(a) $\alpha = \sqrt{3}, \beta = \sqrt{5}, \gamma = \sqrt{15},\ \mathbb{K} = \mathbb{Q}$;

(b) $\alpha = \sqrt{3}, \beta = \sqrt{5}, \gamma = 0,\ \mathbb{K} = \mathbb{Q}(\sqrt{15})$;

(c) $\alpha = \sqrt{3}, \beta = \sqrt{5}, \gamma = \sqrt{7},\ \mathbb{K} = \mathbb{Q}$;

(d) $\alpha = \sqrt{3}, \beta = \sqrt{5}, \gamma = 0,\ \mathbb{K} = \mathbb{Q}(\sqrt{7})$;

muodossa $\mathbb{Q}(\tau)$.

26. Olkoon $\deg_{\mathbb{Q}} \alpha = n$. Määrää lukukunta $\mathbb{K} = \mathbb{Q}(\alpha)$ muodossa

$$\mathbb{Q} + \mathbb{Q}\alpha + ... + \mathbb{Q}\alpha^{n-1},$$

kun

(a) $\alpha^2 + 1 = 0$;
V: Because $\deg_{\mathbb{Q}} \alpha = 2$, therefore $\mathbb{Q}(\alpha) = \mathbb{Q}[\alpha] = \mathbb{Q} + \mathbb{Q}\alpha$.

(b) $\alpha^2 - 3 = 0$;

(c) $\alpha^2 + \alpha + 1 = 0$;

(d) $\alpha^2 + 2\alpha + 1 = 0$;
V: Because $\deg_{\mathbb{Q}} \alpha = 1$, therefore $\mathbb{Q}(\alpha) = \mathbb{Q}[\alpha] = \mathbb{Q}$.

(e) $\alpha^4 - 10\alpha^2 + 1 = 0$.
V: Because $\deg_{\mathbb{Q}} \alpha = 4$, therefore $\mathbb{Q}(\alpha) = \mathbb{Q}[\alpha] = \mathbb{Q} + \mathbb{Q}\alpha + \mathbb{Q}\alpha^2 + \mathbb{Q}\alpha^3$.

Näytä vielä laskemalla/by computing, että

$$\frac{\alpha^3 - 7}{\alpha^5 + \alpha + 2} \in \mathbb{Q} + \mathbb{Q}\alpha + ... + \mathbb{Q}\alpha^{n-1}$$

mikäli lauseke on määritelty/whenever the expression is determined.
a) $\alpha^2 = -1, \alpha^3 = -\alpha, \ldots$, joten $\frac{\alpha^3-7}{\alpha^5+\alpha+2} = \frac{-\alpha-7}{2\alpha+2} = \frac{-1}{2}\frac{\alpha+7}{\alpha+1} = \frac{-1}{2}\frac{(\alpha+7)(\alpha-1)}{\alpha^2-1} = -\frac{\alpha^2+6\alpha-7}{4} = -\frac{3}{2}\alpha + 2$.

27. Olkoon $\mathbb{K} = \mathbb{Q}(\tau)$ lukukunta ja $[\mathbb{K} : \mathbb{Q}] = m$. Osoita, että

(a) $N(\alpha\beta) = N(\alpha)N(\beta)$;

(b) $T(r\alpha + s\beta) = rT(\alpha) + sT(\beta)$;

(c) $N(r) = r^m, \quad T(r) = mr$;

kaikilla $\alpha, \beta \in \mathbb{K}$, $r, s \in \mathbb{Q}$.

28. Osoita, että $\alpha \notin \mathbb{K} = \mathbb{Q}(\tau)$, kun

(a) $\alpha = 3^{1/2}$ ja $\tau = 2^{1/2}$;
V: Example 33 (small font version).

(b) $\alpha = 3^{1/2}$ ja $\tau = 2^{1/3}$; V: $\deg_{\mathbb{Q}} \alpha = 2$ and $[\mathbb{Q}(\tau) : \mathbb{Q}] = 3$. Use then Theorem 42 C(small font version).

(c) $\alpha = 3^{1/2}$ ja $\tau = 2^{1/3}$;

(d) $\alpha = 3^{1/2}$ ja $\tau = 2^{1/4}$;

(e) $\alpha = 3^{1/3}$ ja $\tau = 2^{1/3}$;
See the method used in case 28a.

(f) $\alpha = 3^{1/3}$ ja $\tau = 2^{1/4}$;
See the method used in case 28b.

(g) $\alpha = 3^{1/4}$ ja $\tau = 2^{1/6}$;

Vihje: Käytä jälkifunktiota ja tai dimensiotuloksia.

29. Olkoon $n \in \mathbb{Z}_{\geq 2}$. Osoita, että

$$2^{1/n} + 3^{1/n} \notin \mathbb{Q}.$$

V: Numerically $3 < 2^{1/2} + 3^{1/2} < 4$ and $2 < 2^{1/n} + 3^{1/n} < 3$ for all $n \geq 3$. Then proceed as in Example 37 (small font version).

30. Olkoon $\mathbb{K} = \mathbb{Q}(\sqrt{d})$. Määrää renkaiden $\mathbb{Z}[\sqrt{d}]$ ja $\mathbb{Z}_{\mathbb{K}}$ kannat $\mathbb{Z}$:n suhteen tapauksissa

(a) $d = -1$;
V: $\mathbb{Z}[\sqrt{-1}] = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \sqrt{-1}$, thus $\{1, \sqrt{-1}\}$ is a basis.
Now $-1 \equiv 3 \pmod 4$. Then Theorem 70, from the lectures, says:
$\mathbb{Z}_{\mathbb{Q}(\sqrt{-1})} = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \sqrt{-1}$. Thus $\{1, \sqrt{-1}\}$ forms a basis.

(b) $d = -2$;

(c) $d = -3$;
V: $\mathbb{Z}[\sqrt{-3}] = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \sqrt{-3}$, thus base is $\{1, \sqrt{-3}\}$.
From the lectures $\mathbb{Z}_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \frac{1+\sqrt{-3}}{2}$, thus base is $\{1, \frac{1+\sqrt{-3}}{2}\}$.

(d) $d = -4$;

(e) $d = -5$.

Käytä luentojen Lausetta 70.

31. Olkoon $\mathbb{K} = \mathbb{Q}(\sqrt{d})$. Määrää yksikköryhmät $\mathbb{Z}_{\mathbb{K}}^*$ tapauksissa

(a) $d = -1$;

(b) $d = -2$;

(c) $d = -3$;

(d) $d = -5$.

32. Olkoon $\mathbb{K} = \mathbb{Q}(\sqrt{d})$. Osoita, että $\mathbb{Z}_{\mathbb{K}}$ on Eukleideen alue, kun

(a) $d = -1$;

(b) $d = -2$;

(c) $d = -3$.

33. Olkoon $\mathbb{K} = \mathbb{Q}(\sqrt{d})$. Osoita, että $\mathbb{Z}_{\mathbb{K}}$ ei ole Eukleideen alue, kun

(a) $d = -5$, esimerkiksi tutkimalla identiteettiä
$3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$.

34. Määrää kaikki Gaussin kokonaislukujen renkaan $\mathbb{Z}[i]$ alkualkiot eli Gaussin alkuluvut $\pi = a + ib \in P_{\mathbb{Z}[i]}$, joille pätee

(a) $N(\pi) \le 13, \quad 0 \le b \le a$.
V: $3, 1+i, 2+i, 3+2i$. Apply Theorems 68 and 73.
E.g. $N(1+i) = (1+i)(1-i) = 2 \in \mathbb{P}$ (is a rational prime), thus $1+i \in P_{\mathbb{Z}[i]}$, see Theorem 68 case (14.6) and (16.27).

(b) $N(\pi) \le 13$.

Piirrä kuva Gaussin tasoon.

35. Ratkaise Diofantoksen yhtälö

$$y^2 + 4 = x^3.$$

36. (a) Näytä, että
$1 + i \sim 1 - i$;

(b) Näytä, että
$2 + i \not\sim 1 + 2i$;

(c) Muodosta alkioiden 6 ja 10 alkutekijäkehitelmät;
V: $6 = (1+i)(1-i)3$, $10 = (1+i)(1-i)(2+i)(2-i)$.

Gaussin kokonaislukujen renkaassa $\mathbb{Z}[i]$.