Stochastic ordering of network throughputs using flow couplings

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Stochastic ordering of network flows

Consider two Markov processes X and X' on $\mathbb{Z}_+^n$, both describing populations of particles (customers, jobs, individuals) in a network of n nodes. Classical coupling results on the stochastic ordering of X and X' require strong monotonicity assumptions [3, 7, 8] which are often violated in practice. However, in most real-world applications we care more about what goes through a network than what sits inside it. This poster describes a new approach for ordering *flows instead of populations* by augmenting the network states X and X' with their associated flow-counting processes F and F', and developing order-preserving couplings of the state–flow processes (X, F) and (X', F').

Population processes on networks

Consider a network consisting of nodes $N = \{1, \dots, n\}$ where particles randomly move across directed links $L \subset (N \cup \{0\})^2$, and where node 0 represents the outside world. The network dynamics is presented by a Markov jump process $X = (X_1(t), \dots, X_n(t))_{t \ge 0}$ on $\mathbb{Z}_+^n$ with transitions

$$x \mapsto x - e_i + e_j \quad \text{at rate } \alpha_{i,j}(x), \quad (i,j) \in L,$$

where e_i denotes the i-th unit vector in $\mathbb{Z}^n$, and e_0 stands as a synonym for zero.

- $X_i(t)$ is the number of particles in node i at time t
- $\alpha_{i,j}(x)$ for $i, j \in N$ is the transition rate from node i to node j
- $\alpha_{0,i}(x)$ and $\alpha_{i,0}(x)$ are the arrival and departure rates of node i

Redundant state–flow presentation

The *state–flow process* associated to X is a Markov jump process (X, F) on $\mathbb{Z}_+^n \times \mathbb{Z}_+^L$ with transitions

$$(x, f) \mapsto (x - e_i + e_j,\ f + e_{i,j}) \quad \text{at rate } \alpha_{i,j}(x), \quad (i,j) \in L.$$

- $X_i(t)$ is the number of particles in node i at time t
- $F_{i,j}(t)$ is the number of transitions across link (i, j) during $(0, t]$.

This process is *redundant* because the second component of (X, F) may be recovered from the path of X by the formula

$$F_{i,j}(t) - F_{i,j}(0) \;=\; \#\left\{ s \in (0,t] :\ X(s) - X(s-) = -e_i + e_j \right\},$$

where $X(s-)$ denotes the left limit of X at time s. Adding this redundancy allows to derive useful non-Markov couplings of X in terms of Markov couplings of (X, F).

Flow balance

Any coupling of state–flow processes always preserves the relation

$$x_i - \sum_{j:(j,i)\in L} f_{j,i} + \sum_{j:(i,j)\in L} f_{i,j} \;=\; x'_i - \sum_{j:(j,i)\in L} f'_{j,i} + \sum_{j:(i,j)\in L} f'_{i,j}, \quad i \in N. \tag{1}$$

Ordering flows in closed cyclic networks

Let X and X' be population processes on a closed cyclic network generated by transition rates $\alpha_{i,j}(x)$ and $\alpha'_{i,j}(x')$, respectively.

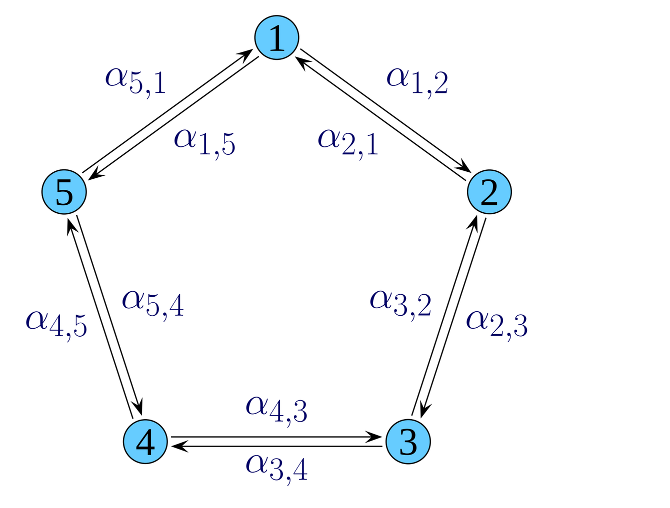
Theorem 1. Assume that for all $i \in N$ and all $x, x' \in \mathbb{Z}_+^n$:

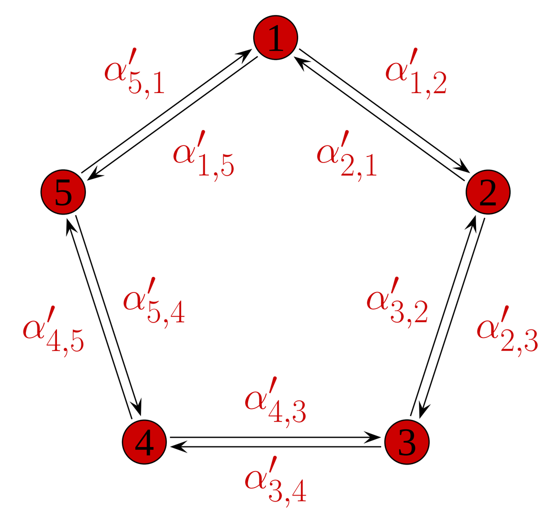
$$x_i \le x'_i \text{ and } x_{i+1} \ge x'_{i+1} \implies \alpha_{i,i+1}(x) \le \alpha'_{i,i+1}(x') \text{ and } \alpha_{i+1,i}(x) \ge \alpha'_{i+1,i}(x').$$

Then the associated flow counting processes are ordered according to

$$(F_{i,i+1}(t) - F_{i+1,i}(t))_{t \ge 0} \;\le_{\mathrm{st}}\; (F'_{i,i+1}(t) - F'_{i+1,i}(t))_{t \ge 0}$$

for all $i \in N$, whenever $X(0) =_{\mathrm{st}} X'(0)$.





Marching soldiers coupling

The *marching soldiers coupling* [1] of state–flow processes (X, F) and (X', F') is a Markov process on $(\mathbb{Z}_+^n \times \mathbb{Z}_+^L)^2$ having the transitions

$$((x,f),(x',f')) \mapsto \begin{cases} (T_{i,j}(x,f), T_{i,j}(x',f')) & \text{at rate } \alpha_{i,j}(x) \wedge \alpha'_{i,j}(x'), \\ ((x,f), T_{i,j}(x',f')) & \text{at rate } (\alpha'_{i,j}(x') - \alpha_{i,j}(x))_+, \\ (T_{i,j}(x,f), (x',f')) & \text{at rate } (\alpha_{i,j}(x) - \alpha'_{i,j}(x'))_+, \end{cases}$$

where $T_{i,j}(x, f) = (x - e_i + e_j, f + e_{i,j})$.

Proof of Theorem 1

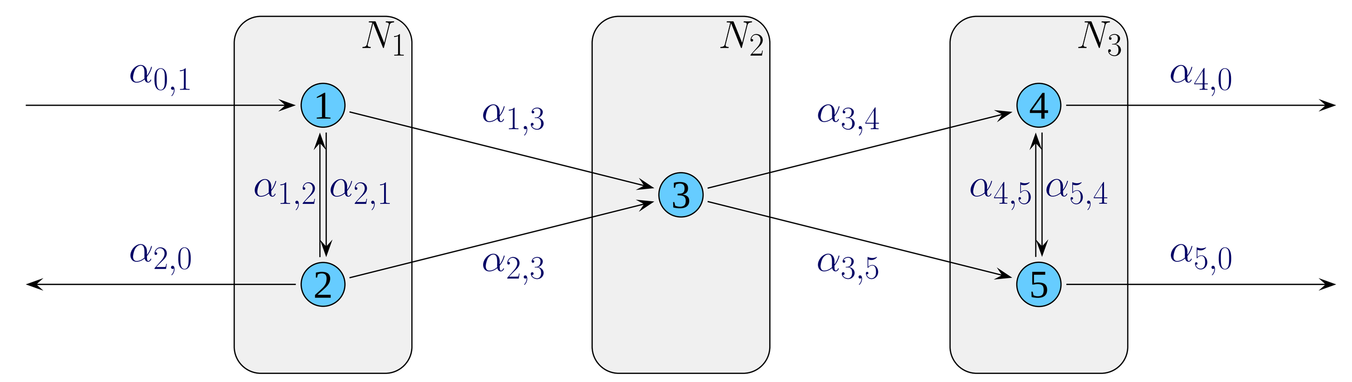
A state–flow pair (x, f) has a *smaller clockwise netflow* than (x', f') if

$$f_{i,i+1} - f_{i+1,i} \;\le\; f'_{i,i+1} - f'_{i+1,i} \tag{2}$$

for all $i \in N$, where $i + 1 := 1$ for $i = n$. The marching soldiers coupling of (X, F) and (X', F') preserves the state–flow relation defined by (1) and (2). An alternative proof can be obtained by applying the theory of monotone generalized semi-Markov processes developed by Glasserman and Yao [4].

Ordering flows through open linear clusters

Consider a network consisting of a linear sequence of clusters $(N_1, \dots, N_m)$ so that only nodes in the boundary clusters N_1 and N_m have links to the exterior of the network, and within the network there are links only between nodes in the same or neighboring clusters.



A state–flow pair (x, f) has a *smaller netflow through* $(N_1, \dots, N_m)$ than (x', f') if

$$\sum_{i \in N_r,\, j \in N_{r+1}} (f_{i,j} - f_{j,i}) \;\le\; \sum_{i \in N_r,\, j \in N_{r+1}} (f'_{i,j} - f'_{j,i}) \tag{3}$$

for all $r = 0, 1, \dots, m$, where $N_0 := \{0\}$, $N_{m+1} := \{0\}$.

Theorem 2. There exists a Markov coupling of state–flow processes (X, F) and (X', F') which preserves the relation defined by (1) and (3) if and only if for all $x, x' \in \mathbb{Z}_+^n$:

$$|x_{N_1}| \ge |x'_{N_1}| \implies \begin{cases} \alpha_{\{0\},N_1}(x) \le \alpha'_{\{0\},N_1}(x') \\ \alpha_{N_1,\{0\}}(x) \ge \alpha'_{N_1,\{0\}}(x') \end{cases}$$

$$\left.\begin{aligned} |x_{N_k}| &\le |x'_{N_k}| \\ |x_{N_{k+1}}| &\ge |x'_{N_{k+1}}| \end{aligned}\right\} \implies \begin{cases} \alpha_{N_k,N_{k+1}}(x) \le \alpha'_{N_k,N_{k+1}}(x') \\ \alpha_{N_{k+1},N_k}(x) \ge \alpha'_{N_{k+1},N_k}(x') \end{cases}$$

$$|x_{N_m}| \le |x'_{N_m}| \implies \begin{cases} \alpha_{N_m,\{0\}}(x) \le \alpha'_{N_m,\{0\}}(x') \\ \alpha_{\{0\},N_m}(x) \ge \alpha'_{\{0\},N_m}(x'), \end{cases}$$

where $|x_I| := \sum_{i \in I} x_i$ and $\alpha_{N_r,N_s} := \sum_{i \in N_r, j \in N_s} \alpha_{i,j}$.

The marching soldiers coupling does not work for proving Theorem 2. A proof based on a general coupling result [5, Thm. 5.6] will be available in [6].

Application: Product-form throughput estimates

A linear network of two queues with buffer capacities n_1 and n_2 is fed by a Poisson process of rate λ and serviced at nondecreasing service rates $\mu_1(x_1)$ and $\mu_2(x_2)$. Arrivals are lost when buffer 1 is full, and server 1 halts when buffer 2 is full. Van Dijk and van der Wal [2] proved that the stead-state mean throughput rate of the network can be bounded by using the following modifications having a product-form equilibrium distribution:

	Modification 1	Original network	Modification 2
$\alpha_{0,1}(x_1,x_2)$	$\lambda 1(x_1{<}n_1, x_2{<}n_2)$	$\lambda 1(x_1{<}n_1)$	$\lambda 1(x_1 + x_2 < n_1 + n_2)$
$\alpha_{1,2}(x_1,x_2)$	$\mu_1(x_1) 1(x_2{<}n_2)$	$\mu_1(x_1) 1(x_2{<}n_2)$	$\mu_1(x_1)$
$\alpha_{2,0}(x_1,x_2)$	$\mu_2(x_2) 1(x_1{<}n_1)$	$\mu_2(x_2)$	$\mu_2(x_2)$

An application of Theorem 2 now yields a stronger result: *The flow counting processes are ordered according to*

$$(F^{\mathrm{mod1}}_{i,i+1}(t))_{t \ge 0} \le_{\mathrm{st}} (F^{\mathrm{orig}}_{i,i+1}(t))_{t \ge 0} \le_{\mathrm{st}} (F^{\mathrm{mod2}}_{i,i+1}(t))_{t \ge 0}, \quad i = 0, 1, 2.$$

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References

[1] Chen, M.-F. (2005). *Eigenvalues, Inequalities, and Ergodic Theory*. Springer.
[2] van Dijk, N. M. and van der Wal, J. (1989). Simple bounds and monotonicity results for finite multi-server exponential tandem queues. *Queueing Syst.*, 4(1):1–15.
[3] Economou, A. (2003). Necessary and sufficient conditions for the stochastic comparison of Jackson networks. *Probab. Eng. Inform. Sci.*, 17:143–151.
[4] Glasserman, P. and Yao, D. D. (1994). *Monotone Structure in Discrete-Event Systems*. Wiley.
[5] Leskelä, L. (2010). Stochastic relations of random variables and processes. *J. Theor. Probab.*, 23(2):523–546.
[6] Leskelä, L. (2012). Stochastic ordering of network throughputs using flow couplings. `http://www.iki.fi/lsl/paper016.html`.
[7] Massey, W. A. (1987). Stochastic orderings for Markov processes on partially ordered spaces. *Math. Oper. Res.*, 12(2):350–367.
[8] Whitt, W. (1986). Stochastic comparisons for non-Markov processes. *Math. Oper. Res.*, 11(4):608–618.

Acknowledgements

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