Stochastic relations of random variables and processes

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Definition

Given a closed relation $R \subset S_1 \times S_2$ between Polish spaces S_1 and S_2, denote

- $x \sim y$ for nonrandom elements, if $(x, y) \in R$
- $X \sim_{\rm st} Y$ for random elements, if there exists a coupling $(\hat{X}, \hat{Y})$ of X and Y such that $\hat{X} \sim \hat{Y}$ almost surely
- $\mu \sim_{\rm st} \nu$ for probability measures, if there exists a coupling λ of μ and ν such that $\lambda(R) = 1$

The relation $R_{\rm st} = \{(\mu, \nu) : \mu \sim_{\rm st} \nu\}$ is called the *stochastic relation* generated by R.

Remark. For Dirac measures, $\delta_x \sim_{\rm st} \delta_y$ if and only if $x \sim y$.

Functional characterization

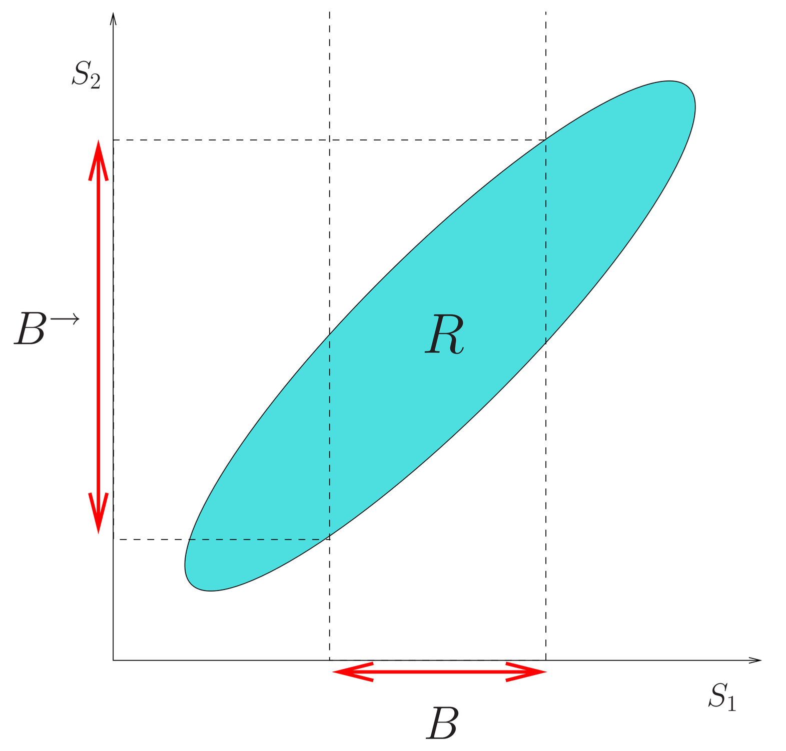
Theorem. The following are equivalent:

(i) $\mu \sim_{\rm st} \nu$.
(ii) $\mu(B) \le \nu(B^{\to})$ for all compact $B \subset S_1$.
(iii) $\int_{S_1} f\, d\mu \le \int_{S_2} f^{\to}\, d\nu$ for all positive upper semicontinuous f on S_1 with compact support.

The *relational conjugates* of sets and functions are defined by

$$B^{\to} = \cup_{x\in B}\{y \in S_2 : x \sim y\},$$

$$f^{\to}(y) = \sup_{x\in S_1 : x\sim y} f(x).$$



Examples

- *Stochastic equality*. For the stochastic relation generated by the equality, we have $X =_{\rm st} Y$ if and only if X and Y have the same distribution.
- *Stochastic ϵ-distance*. Define $x \approx y$ on the real line, if $|x - y| \le \epsilon$. Then $X \approx_{\rm st} Y$ if and only if the corresponding c.d.f.'s satisfy $F_Y(x - \epsilon) \le F_X(x) \le F_Y(x + \epsilon)$ for all x.
- *Stochastic order*. For a stochastic relation generated by an order (reflexive and transitive) relation, $X \le_{\rm st} Y$ if and only if $\mathrm{E}\, f(X) \le \mathrm{E}\, f(Y)$ for all positive increasing f on S.
- *Stochastic induced order*. [1] Given real functions f on S_1 and g on S_2, define $x \le^{f,g} y$ by $f(x) \le g(y)$. Then $\mu \le^{f,g}_{\rm st} \nu$ if and only if $\mu(f^{-1}((\alpha, \infty))) \le \nu(g^{-1}((\alpha, \infty)))$ for all real numbers α.

Preservation of stochastic relations

Two Markov processes X_1 and X_2 are said to *stochastically preserve* a relation R, if for all initial states x and y:

$$x \sim y \implies X_1(x, t) \sim_{\rm st} X_2(y, t) \quad \text{for all } t.$$

Theorem. Two nonexplosive Markov jumps processes X_1 and X_2 stochastically preserve a relation R if and only if the corresponding rate kernels Q_1 and Q_2 satisfy

$$Q_1(x, B) - q_1(x)\delta(x, B) \le Q_2(y, B^{\to}) - q_2(y)\delta(y, B^{\to})$$

for all $x \sim y$ and all compact $B \subset S_1$ such that $\delta(x, B) = \delta(y, B^{\to})$.

Remark. For order relations, the above result reduces to Massey [6] and Whitt [7]. López and Sanz have an alternate characterization in terms of a subtle order construction [4].

Bounds for stationary distributions

Problem. How to show that the stationary distributions of irreducible positive recurrent Markov processes X_1 and X_2 with values on an ordered space satisfy

$$\mu_1 \le_{\rm st} \mu_2, \tag{1}$$

without explicitly knowing μ_1?

A well-known sufficient condition for (1) is that X_1 and X_2 stochastically preserve the order, that is,

$$Q_1(x, B) - q_1(x)\delta(x, B) \le Q_2(y, B) - q_2(y)\delta(y, B)$$

for all $x \le y$ and upper sets B such that $\delta(x, B) = \delta(y, B)$ [2, 6, 7].

Key observation. *A less stringent sufficient condition for (1) is that X_1 and X_2 stochastically preserve some (not necessarily symmetric or transitive) subrelation of the order.*

Subrelation algorithm

Theorem. A pair (P_1, P_2) of continuous probability kernels stochastically preserves a subrelation of R if and only if

$$R^* = \bigcap_{n=0}^{\infty} R^{(n)} \ne \emptyset,$$

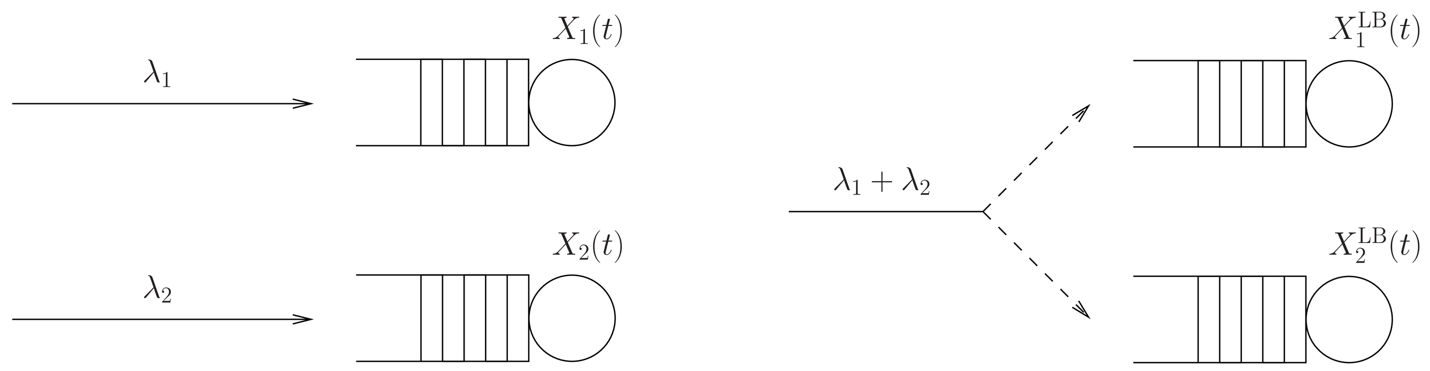
where the sequence $R^{(n)}$ is defined by $R^{(0)} = R$, and

$$R^{(n+1)} = \left\{(x, y) \in R^{(n)} :\ (P_1(x, \cdot), P_2(y, \cdot)) \in R^{(n)}_{\rm st}\right\}.$$

In this case R^* is the maximal subrelation of R that is stochastically preserved by (P_1, P_2).

Remark. An analogous result holds for rate kernels of Markov jump processes.

Application: Load balancing



Common sense suggests that load balancing reduces the net queue length:

$$\mathrm{E}(X_1^{\rm LB}(t) + X_2^{\rm LB}(t)) \le \mathrm{E}(X_1(t) + X_2(t)),$$

but $X^{\rm LB}$ and X do *not* stochastically preserve the coordinatewise order *nor* the order $R^{\rm sum} = \{(x, y) : |x| \le |y|\}$ on $\mathbb{Z}_+^2$, where $|x| = x_1 + x_2$.

Theorem. Starting from $R^{(0)} = R^{\rm sum}$, the subrelation algorithm applied to $(Q^{\rm LB}, Q)$ produces the relations

$$R^{(n)} = \{(x, y) : |x| \le |y| \text{ and } x_1 \vee x_2 \le y_1 \vee y_2 + (y_1 \wedge y_2 - n)^+\},$$
$$\downarrow$$
$$R^* = \{(x, y) : |x| \le |y| \text{ and } x_1 \vee x_2 \le y_1 \vee y_2\}.$$

The relation R^* is known as the *weak majorization order* on $\mathbb{Z}_+^2$, usually denoted by $x \preceq^{\rm wm} y$ [5]. As a consequence,

$$X^{\rm LB}(0) \preceq^{\rm wm} X(0) \implies X^{\rm LB}(t) \preceq^{\rm wm}_{\rm st} X(t) \quad \text{for all } t.$$

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