Appendix A

Set theory preliminaries

This appendix reviews necessary background on set theory, in particular the notion of countability, which is crucial in probability theory and measure theory.

A.1. Intersections and unions of sets

Let $A, B$ be two sets. The intersection $A \cap B$ is defined as the set of those elements which belong to both $A$ and $B$,

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}. $$

The union $A \cup B$ is defined as the set of those elements which belong to at least one of the sets $A$ and $B$,

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}. $$

More generally, let $(A_i)_{i \in I}$ be a collection of sets $A_i$ indexed by $i \in I$. The union of the collection is defined as

$$\bigcup_{i \in I} A_i = \{ x \mid x \in A_i \text{ for some } i \in I \}. $$

The intersection of the collection is defined provided the collection is non-empty ($I \neq \emptyset$) as

$$\bigcap_{i \in I} A_i = \{ x \mid x \in A_i \text{ for all } i \in I \}. $$

A collection $(A_i)_{i \in I}$ of sets is said to be disjoint if no two different members of the collection have common elements, i.e., for all $i, j \in I, i \neq j$, we have $A_i \cap A_j = \emptyset$. If the collection $(A_i)_{i \in I}$ of sets is disjoint, then we say that the union $\bigcup_{i \in I} A_i$ is a disjoint union. Disjoint unions enjoy additivity properties in probability theory and measure theory, by definition.

A.2. Set differences and complements

Let $A, B$ be two sets. The set difference $A \setminus B$ is defined as the set of those elements which belong to $A$ but do not belong to $B$,

$$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}. $$

When it is clear from the context that we are considering subsets of a particular reference set $S$ (often the sample space $S = \Omega$), then the complement of a subset $A \subset S$, denoted by $A^c$, is the set of those elements (of $S$) which do not belong to $A$

$$A^c = S \setminus A = \{ x \in S \mid x \notin A \}. $$
The following basic results of set theory tell how unions and intersections behave under complements. They are known as De Morgan’s laws.

**Proposition A.1** (De Morgan’s laws).  
Let \((A_i)_{i \in I}\) be a non-empty indexed collection of subsets \(A_i \subset S\) of a fixed set \(S\). Then we have
\[
(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} A_i^c \quad \text{and} \quad (\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c.
\]

**Exercise A.1.** Prove the De Morgan’s laws above.

### A.3. Images and preimages of sets under functions

Let \(S\) and \(S'\) be two sets and \(f : S \to S'\) a function from \(S\) to \(S'\). For \(A \subset S\), the **image** of \(A\) under \(f\) is the subset \(f(A) \subset S'\) consisting of all those elements \(s' \in S'\) such that \(s' = f(s)\) for some \(s \in S\),
\[
f(A) = \{ f(s) \mid s \in S \} \subset S'.
\]

For \(A' \subset S'\), the **preimage** of \(A'\) under \(f\) is the subset \(f^{-1}(A') \subset S\) consisting of all those elements \(s \in S\) whose image \(f(s)\) belongs to the subset \(A'\),
\[
f^{-1}(A') = \{ s \in S \mid f(s) \in A' \} \subset S.
\]

**Exercise A.2.** Show that
(a) \(f^{-1}(A'^c) = (f^{-1}(A'))^c\)
(b) \(f^{-1}(\bigcup_{i} A'_i) = \bigcup_{i} f^{-1}(A'_i)\)
(c) \(f^{-1}(\bigcap_{i} A'_i) = \bigcap_{i} f^{-1}(A'_i)\).

### A.4. Cartesian products

Let \(A, B\) be two sets. The **Cartesian product** \(A \times B\) is defined as the set of ordered pairs \((a, b)\) whose first member \(a\) belongs to the set \(A\) and second member \(b\) belongs to the set \(B\), i.e.,
\[
A \times B = \{ (a, b) \mid a \in A, \ b \in B \}.
\]

More generally, when \(A_1, \ldots, A_n\) are sets, the \(n\)-fold Cartesian product \(A_1 \times \cdots \times A_n\) is the set of ordered \(n\)-tuples \((a_1, \ldots, a_n)\) such that \(a_k \in A_k\) for each \(k = 1, \ldots, n\)
\[
A_1 \times \cdots \times A_n = \{ (a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n \}.
\]

Even more generally, if \((A_j)_{j \in J}\) is a collection of sets indexed by \(j \in J\), the Cartesian product \(\prod_{j \in J} A_j\) is the set of indexed collections \((a_j)_{j \in J}\) such that \(a_j \in A_j\) for each \(j \in J\),
\[
\prod_{j \in J} A_j = \{ (a_j)_{j \in J} \mid \forall j \in J : a_j \in A_j \}.
\]
A.6. Sequences of sets

As a particular case, if each member is the same set, \( A_j = A \) for all \( j \in J \), then the Cartesian product is alternatively denoted by \( A^J := \prod_{j \in J} A \). In that case an element of \( A^J \) is an indexed collection \((a_j)_{j \in J}\) of elements of \( A \), which can be naturally identified with the function \( j \mapsto a_j \) from \( J \) to \( A \). Therefore \( A^J \) is the set of functions from \( J \) to \( A \),

\[
A^J = \{ f : J \to A \text{ function} \}.
\]

A.5. Power set

Given a set \( S \), the set \( P(S) \) of all subsets \( A \subset S \) of it is called the power set of \( S \),

\[
P(S) = \{ A \mid A \subset S \}.
\]

A subset \( A \subset S \) can be specified by indicating for each element \( s \in S \) whether it belongs to \( A \) or not, so it is natural to identify the the power set \( P(S) \) of \( S \) with the set \( \{0, 1\}^S \) of functions \( S \to \{0, 1\} \). In particular if \( S \) is a finite set with \( \#S = n \) elements, then its power set is a finite set with \( \#P(S) = 2^n \) elements.

It is good to keep in mind that the power set readily provides an easy first example of many notions introduced in probability theory. For instance, in view of Definitions I.1, II.5, and C.2, the collection \( P(S) \) of all subsets of \( S \) is obviously a sigma algebra on \( S \), a \( \pi \)-system on \( S \), a d-system on \( S \), etc.

A.6. Sequences of sets

A sequence \( A_1, A_2, \ldots \) of sets is said to be increasing if

\[
A_1 \subset A_2 \subset A_3 \subset \ldots.
\]

In this case we denote

\[
A_n \uparrow A,
\]

where the limit \( A \) of the increasing sequence of sets is defined as the union

\[
A = \bigcup_{n=1}^{\infty} A_n.
\]

Likewise, a sequence \( A_1, A_2, \ldots \) of sets is said to be decreasing if

\[
A_1 \supset A_2 \supset A_3 \supset \ldots.
\]

In this case we denote

\[
A_n \downarrow A,
\]

where the limit \( A \) of the decreasing sequence of sets is defined as the intersection

\[
A = \bigcap_{n=1}^{\infty} A_n.
\]

Sequences of sets which are either increasing or decreasing are said to be monotone.

Exercise A.3. Show that the limits of monotone sequences of sets can be characterized as follows.
(a) Suppose that $A_n \uparrow A$. Show that

$x \in A \iff \exists m \in \mathbb{N} \text{ such that } \forall n \geq m : x \in A_n$

$x \notin A \iff \forall n \in \mathbb{N} : x \notin A_n$.

(b) Suppose that $A_n \downarrow A$. Show that

$x \in A \iff \forall n \in \mathbb{N} : x \in A_n$

$x \notin A \iff \exists m \in \mathbb{N} \text{ such that } \forall n \geq m : x \notin A_n$.

For a sequence $A_1, A_2, \ldots$ of sets, we define

$$
\limsup_{n} A_n := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} A_n.
$$

Note that if we define $C_m = \bigcup_{n \geq m} A_n$, then the sequence $C_1, C_2, \ldots$ of sets is decreasing, and $\limsup_n A_n$ is its limit.

We also define

$$
\liminf_{n} A_n := \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} A_n.
$$

Note that if we define $C_m = \bigcap_{n \geq m} A_n$, then the sequence $C_1, C_2, \ldots$ of sets is increasing, and $\liminf_n A_n$ is its limit.

**Exercise A.4.** Show that the limsup and liminf of a sequence $A_1, A_2, \ldots$ of sets can be characterized as follows:

(a): $\limsup_{n} A_n = \{ s \mid \forall m \in \mathbb{N} : \exists n \geq m : s \in A_n \}$

(b): $\liminf_{n} A_n = \{ s \mid \exists m \in \mathbb{N} : \forall n \geq m : s \in A_n \}$.

### A.7. Countable and uncountable sets

In probability theory we need to distinguish between sets of different sizes: finite sets, countably infinite sets, and uncountably infinite sets. In fact, if one had to summarize probability theory (and measure theory) in a single phrase, it might be:

All countable operations in probability theory are defined to behave just as intuition dictates.

#### Comparison of cardinalities

Cardinality is the set-theoretic notion of the size or “number of elements” of a set. The idea is that when comparing the sizes of two sets $A$ and $B$, we attempt to match the elements of $A$ to the elements of $B$ by functions $f : A \rightarrow B$. Under a surjective function, each element of $B$ has at least some element(s) of $A$ matched to it, and we then would interpret that $A$ has at least as many elements as $B$. Under an injective function $f$, all elements of $A$ are matched to some elements of $B$ without any two different elements ever being matched to the same element, and then we would interpret that $A$ has at most as many elements as $B$. Comparison of cardinalities is done by asking about the existence of such functions.
Definition A.1. Let $A$ and $B$ be sets. We say that the cardinality of $A$ is less than or equal to the cardinality of $B$ if there exists an injective function $f: A \rightarrow B$.

Remark A.2. Suppose that $A$ and $B$ are two finite sets. Let $n = \#A$ be the number of elements in $A$ and $m = \#B$ be the number of elements in $B$. Then it is easy to see that there exists an injective function $f: A \rightarrow B$ if and only if we have $n \leq m$. Thus for finite sets, the comparison of cardinalities amounts to just the comparison of the number of elements.

Example A.3 (Subsets can not have larger cardinality). If $A$ is a subset of $B$, $A \subseteq B$, then the cardinality of $A$ is less than or equal to the cardinality of $B$, because the inclusion mapping $\iota: A \rightarrow B$ defined by $\iota(x) = x$ for all $x \in A$ is injective.

Example A.4 (Transitivity of comparison of cardinalities). Let $A, B, C$ be sets. Suppose that the cardinality of $A$ is less than or equal to the cardinality of $B$ and the cardinality of $B$ is less than or equal to the cardinality of $C$. In that case there exists injective functions $f: A \rightarrow B$ and $\tilde{f}: B \rightarrow C$. The composition $\tilde{f} \circ f: A \rightarrow C$ is also injective, so we get that the cardinality of $A$ is less than or equal to the cardinality of $C$. In other words, the comparison of cardinalities is transitive.

As suggested before Definition A.1, instead of requiring the existence of injective functions in the comparison of cardinalities, one can alternatively require the existence of surjective functions in the opposite direction. The following two exercises establish this alternative characterization. To solve these exercises, you are allowed to use the axiom of choice.

Exercise A.5. Show that if the cardinality of a non-empty set $A \neq \emptyset$ is less than or equal to the cardinality of a set $B$, then there exists a surjective function $g: B \rightarrow A$.

Exercise A.6. Show that if there exists a surjective function $g: B \rightarrow A$, then the cardinality of the set $A \neq \emptyset$ is less than or equal to the cardinality of the set $B$.

Equal cardinalities

Definition A.2. We say that $A$ and $B$ have equal cardinalities if there exists an injective function $f: A \rightarrow B$ and an injective function $g: B \rightarrow A$.

Clearly if there exists a bijective function $f: A \rightarrow B$, then $A$ and $B$ have equal cardinalities (we may then take $g = f^{-1}$). The converse is also true, but it is not as obvious. The Schröder - Bernstein theorem states that if $A$ and $B$ have equal cardinalities, then there exists a bijective function $f: A \rightarrow B$ (you could try to prove this as an exercise).

Countable sets

For the purposes of probability theory and measure theory, countable cardinalities are the most crucial. We begin with the definition.

Definition A.3. A set $A$ is said to be countable if the cardinality of $A$ is less than or equal to the cardinality of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers.
Example A.5 (Subsets of natural numbers are countable).
From Example A.3 it follows that any subset $S \subset \mathbb{N}$, including the set $\mathbb{N}$ of natural numbers itself, is countable.

Since countable sets are so important, we unravel the definition once more, and provide an alternative characterization and two useful sufficient conditions.

Lemma A.6 (Criteria for countability).
(a) A set $A$ is countable if and only if there exists an injective function $f : A \to \mathbb{N}$.
(b) A non-empty set $A \neq \emptyset$ is countable if and only if there exists a surjective function $g : \mathbb{N} \to A$.
(c) If $B$ is a countable set and there exists an injective function $f : A \to B$, then also the set $A$ is countable.
(d) If $B$ is a countable set and there exists a surjective function $g : B \to A$, then also the set $A$ is countable.

Proof. Assertion (a) follows directly by combining Definitions A.1 and A.3.

Assertion (b) follows by combining Definition A.3 with the characterization of Exercises A.5 and A.6.

Assertions (c) and (d) are similarly obtained using the transitivity in Example A.4. □

Enumerations of countable sets

If $A$ is countable and non-empty, then from Exercise A.5 it follows that there exists a surjective function $g : \mathbb{N} \to A$. We see that all elements of $A$ are obtained in the following “list”

$$A = \{g(1), g(2), g(3), \ldots\}.$$

Note, however, that repetitions are allowed in the above “list”, as $g$ does not have to be injective. It is possible to remove repetitions and obtain an enumeration of the elements of $A$. To do this, one defines $a_k \in A$ as the $k$:th value in the list above omitting repetitions. If the set $A$ is finite, however, then there are only finitely many different values and the enumeration terminates at some point. Thus, for a finite set $A$ with $n$ elements, we have an enumeration

$$A = \{a_1, a_2, \ldots, a_n\}.$$

An infinite set which is countable is said to be countably infinite, and for such a set $A$, we have an enumeration

$$A = \{a_1, a_2, a_3, \ldots\}.$$

Note also that if the elements of $A$ can be enumerated as above without repetition, then the mapping $a_k \mapsto k$ is well defined and injective $A \to \mathbb{N}$. Therefore any set whose elements can be enumerated is countable.

Operations that preserve countability

In probability theory and measure theory, countable operations work well. It is therefore crucial to understand clearly which set theoretic operations preserve countability.
Suppose that $A_1$ and $A_2$ are countable sets. By definition, there exists injective functions $f_1: A_1 \to \mathbb{N}$ and $f_2: A_2 \to \mathbb{N}$. Consider the union $A_1 \cup A_2$, and note that it can be expressed as $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$, where the latter is a disjoint union. The function $f: A_1 \cup A_2 \to \mathbb{N}$ defined “piecewise” by
\[
f(x) = \begin{cases} 
2f_1(x) & \text{if } x \in A_1 \\
2f_2(x) + 1 & \text{if } x \in A_2 \setminus A_1
\end{cases}
\]
is clearly injective: it maps elements of $A_1$ injectively to even natural numbers and the remaining elements injectively to odd natural numbers. From the existence of such an injective function we conclude that the union $A_1 \cup A_2$ is countable. Using this argument inductively, we get that finite unions of countable sets remain countable.

Lemma A.7. Let $A_1, \ldots, A_n$ be countable sets. Then the union
\[
A_1 \cup \cdots \cup A_n = \bigcup_{j=1}^n A_j
\]
is also countable.

Example A.8 (The set of integers is countable).

Consider the three sets:
\[
A_1 = \{1, 2, 3, \ldots\} \\
A_2 = \{0\} \\
A_3 = \{-1, -2, -3, \ldots\}.
\]
Each of them is countable: the set $A_1 = \mathbb{N}$ is countable by Example A.5, the set $A_2$ is countable because it is finite, and the set $A_3$ is countable because it is in bijection with $\mathbb{N}$ via $x \mapsto -x$. The set of all integers is the union of these three
\[
\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = A_1 \cup A_2 \cup A_3
\]
and as such $\mathbb{Z}$ is itself countable by Lemma A.7.

Consider now the set $\mathbb{N} \times \mathbb{N}$, the Cartesian product of the set of natural numbers with itself. We claim that $\mathbb{N} \times \mathbb{N}$ is countable. To see this, note that the elements can be enumerated
\[
\mathbb{N} \times \mathbb{N} = \{(n, m) \mid n, m \in \mathbb{N}\} = \{(1, 1), \\
(2, 1), (1, 2), \\
(3, 1), (2, 2), (1, 3), \\
(4, 1), (3, 2), (2, 3), (1, 4), \\
\ldots \}
\]
The enumeration shows that $\mathbb{N} \times \mathbb{N}$ is also countable, as it gives rise to an injective function $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Suppose now that $A_1$ and $A_2$ are countable sets. Then there exists injective functions $f_1: A_1 \to \mathbb{N}$ and $f_2: A_2 \to \mathbb{N}$. Now define $f: A_1 \times A_2 \to \mathbb{N}$ by
\[
f(x_1, x_2) = h(f_1(x_1), f_2(x_2)) \quad \text{for } x_1 \in A_1, x_2 \in A_2,
\]
where \( h : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is the injective function given by the above enumeration of \( \mathbb{N} \times \mathbb{N} \). The function \( f : A_1 \times A_2 \to \mathbb{N} \) is injective, and we thus see that the Cartesian product \( A_1 \times A_2 \) of countable sets \( A_1 \) and \( A_2 \) is again countable. Using this observation inductively, we get that Cartesian products of finitely many countable sets remain countable.

**Proposition A.9.** Let \( A_1, \ldots, A_n \) be countable sets. Then the Cartesian product

\[
A_1 \times \cdots \times A_n
\]

is also countable.

**Example A.10** (The \( d \)-dimensional integer lattice \( \mathbb{Z}^d \) is countable).

Let \( d \in \mathbb{N} \). The set

\[
\mathbb{Z}^d = \mathbb{Z} \times \cdots \times \mathbb{Z} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1, \ldots, x_d \in \mathbb{Z} \right\}
\]

of points with integer coordinates in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) is countable, since it is the Cartesian product of \( d \) copies of the countable set \( \mathbb{Z} \).

**Example A.11** (The set of rational numbers is countable).

Consider the set \( \mathbb{Q} \subset \mathbb{R} \) of rational numbers. The mapping \( g : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q} \) defined by

\[
g(n, m) = \frac{n}{m}
\]

is surjective onto \( \mathbb{Q} \). As a Cartesian product of the countable sets \( \mathbb{Z} \) and \( \mathbb{N} \), the set \( \mathbb{Z} \times \mathbb{N} \) is countable by Proposition A.9. From Lemma A.6(d) and the existence of the surjective function \( g \) we get that also the set \( \mathbb{Q} \) of rational numbers is countable.

Using the above proposition about Cartesian products of countable sets we can strengthen our earlier observation about unions of countable sets: it turns out that countable unions of countable sets remain countable.

**Proposition A.12.** Let \( A_1, A_2, A_3, \ldots \) be countable sets. Then the union

\[
\bigcup_{j=1}^{\infty} A_j = A_1 \cup A_2 \cup A_3 \cup \cdots
\]

is also countable.

**Proof.** We may assume that the sets \( A_j \) are non-empty, \( A_j \neq \emptyset \), for all \( j \in \mathbb{N} \) (empty terms can be omitted from the union). Then, for each \( j \in \mathbb{N} \) there exists a surjective function

\[
g_j : \mathbb{N} \to A_j.
\]

We now define a function \( g \) on \( \mathbb{N} \times \mathbb{N} \) by

\[
g(j, k) = g_j(k) \quad \text{for } j, k \in \mathbb{N},
\]

and observe that this function

\[
g : \mathbb{N} \times \mathbb{N} \to \bigcup_{j=1}^{\infty} A_j
\]

is surjective onto the union \( \bigcup_{j=1}^{\infty} A_j \). This shows that the cardinality of \( \bigcup_{j=1}^{\infty} A_j \) is less than or equal to the cardinality of \( \mathbb{N} \times \mathbb{N} \). Since \( \mathbb{N} \times \mathbb{N} \) is countable, this shows that the union \( \bigcup_{j=1}^{\infty} A_j \) is also countable. \( \square \)
Uncountable sets

A set which is not countable is said to be uncountable. Since all finite sets are countable, an uncountable set is necessarily infinite.

In the previous section we saw that some rather nontrivial set theoretic operations preserve countability. We now provide examples of uncountable sets by a useful standard argument known as Cantor’s diagonal extraction. The argument shows that countable Cartesian products of countable sets (or even of finite sets) are generally not countable.

Example A.13 (The set of binary sequences is uncountable).

For each $j \in \mathbb{N}$, let $A_j = \{0, 1\}$. Consider the Cartesian product

$$B = \prod_{j=1}^{\infty} A_j = \{0, 1\}^\mathbb{N} = \left\{(b_1, b_2, \ldots) \mid b_1, b_2, \ldots \in \{0, 1\}\right\}$$

of the sets $A_1, A_2, \ldots$, which is most concretely interpreted as the set of infinite binary sequences $b = (b_1, b_2, \ldots)$ of zeroes and ones. The set $B$ is a countable Cartesian product of finite sets. We claim that $B$ itself is uncountable.

The diagonal argument proceeds by supposing, on the contrary, that $B$ is countable. If this were the case, then we could find an enumeration

$$B = \left\{b^{(1)}, b^{(2)}, b^{(3)}, \ldots\right\},$$

where the $m$:th element $b^{(m)}$ is a binary sequence

$$b^{(m)} = (b_1^{(m)}, b_2^{(m)}, b_3^{(m)}, \ldots).$$

Now define a binary sequence $b' = (b'_1, b'_2, b'_3, \ldots)$ by choosing for each $j \in \mathbb{N}$ the $j$:th “digit” $b'_j \in \{0, 1\}$ to be different from $b_j^{(j)}$, the $j$:th “digit” of the $j$:th element $b^{(j)}$ in the enumeration. This construction of $b' \in B$ is known as diagonal extraction. Now for any $m \in \mathbb{N}$, the binary sequence $b'$ differs from $b^{(m)}$ at least in the $m$:th digit, so $b' \neq b^{(m)}$. But the element $b' \in B$ should appear in the enumeration of $B$, so we have derived a contradiction. We conclude that $B$ cannot be enumerated. Therefore $B$ is in fact uncountable.

Example A.14 (The set of real numbers is uncountable).

Let $B = \{0, 1\}^\mathbb{N}$ be the set of binary sequences as in the previous example. Consider also the subset $B' \subset B$ of those binary sequences which contain infinitely many zeroes,

$$B' = \left\{(b_1, b_2, \ldots) \in \{0, 1\}^\mathbb{N} \mid \forall m \in \mathbb{N} : \exists n \geq m : b_n = 0\right\}.$$

We first claim that $B'$ is also uncountable.

The complement $B \setminus B'$ is the set of binary sequences which end with repeated ones,

$$B \setminus B' = \left\{(b_1, b_2, \ldots) \in \{0, 1\}^\mathbb{N} \mid \exists m \in \mathbb{N} : \forall n \geq m : b_n = 1\right\}.$$

Let

$$R_m = \left\{(b_1, b_2, \ldots) \in \{0, 1\}^\mathbb{N} \mid \forall n \geq m : b_n = 1\right\}.$$ 

denote the set of sequences where a repetition of ones has started by the $m$:th “digit”. Note that $R_m$ is a finite set, $\#R_m = 2^{m-1}$, since we are only free to choose the values of the first $m - 1$ “digits”. As a countable union of these finite sets, the complement

$$B \setminus B' = \bigcup_{m=1}^{\infty} R_m.$$ 

is countable by Proposition A.12. Now if $B'$ would be countable, then the union $B = B' \cup (B \setminus B')$ would also be countable, which is a contradiction with the conclusion of Example A.13. We thus conclude that $B'$ is uncountable.
To prove that the set of real numbers is uncountable, we note that any real number has a binary expansion. More specifically, any number \( x \in [0,1) \) has a binary expansion with its “digit sequence” in \( B' \). Indeed, define a function \( f \) on \([0,1)\) by
\[
f(x) = (b_1, b_2, \ldots) \quad \text{where} \quad b_j = \lfloor 2^j x \rfloor.
\]
The sequence \( f(x) = (b_1, b_2, \ldots) \) is a binary expansion of \( x \),
\[
x = \sum_{j=1}^{\infty} b_j 2^{-j}.
\]
It is easy to see that \( f(x) \in B' \) and that \( f: [0,1) \to B' \) is surjective onto \( B' \) (for \( b \in B' \) and \( x = \sum_{j=1}^{\infty} b_j 2^{-j} \) we indeed have \( f(x) = b \)). Therefore we conclude that the cardinality of \( B' \) is less than or equal to the cardinality of \([0,1)\). But since \( B' \) is uncountable, also the set \([0,1)\) is uncountable.

Since \([0,1) \subset \mathbb{R}\) is a subset and \([0,1)\) is uncountable, also the set \( \mathbb{R} \) of real numbers is uncountable.