Abstract. In this article, we find a $q$-analogue for Fomin’s formulas. The original Fomin’s formulas relate determinants of random walk excursion kernels to loop-erased random walk partition functions, and our formulas analoguously relate conformal block functions of conformal field theories to pure partition functions of multiple SLE random curves. We also provide a construction of the conformal block functions by a method based on a quantum group, the $q$-deformation of $\mathfrak{sl}_2$. The construction both highlights the representation theoretic origin of conformal blocks functions and explains the appearance of $q$-combinatorial formulas.

1. Introduction

Conformal blocks are fundamental building blocks of correlation functions of conformal field theories. In this article, we study the combinatorics of conformal block functions associated to the simplest non-trivial primary fields in conformal field theories.

Following the conventions in the literature about random conformally invariant curves of SLE$_\kappa$ type, we parameterize the central charge of the conformal field theory via a parameter $\kappa > 0$, as

$$c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}. \quad (1.1)$$

We assume $\kappa \in (0, 8) \setminus \mathbb{Q}$. The primary fields whose conformal blocks we study are of conformal weight

$$h = \frac{6 - \kappa}{2\kappa}. \quad (1.2)$$
This is the first non-trivial conformal weight in the Kac table \([Kac78]\), and fields of this type appear in particular as the boundary changing fields that create the tip of an \(SLE_\kappa\) type curve \([BB03a, BB03b, FW03, BBK05, Dub07, Kyt07, KM13, Dub15]\).

We cover some background on conformal blocks in conformal field theories in Section 3. For all other parts of the article, a few key properties of conformal block functions can be taken as their definition. Namely, the partial differential equations, Möbius covariance, and asymptotics given precisely in Section 3.4 serve as their defining properties.

The starting point for the combinatorics is the observation that the conformal block functions are functions
\[
U_\alpha(x_1, \ldots, x_{2N})
\]
of an even number \(n = 2N\) of variables, which are indexed by Dyck paths \(\alpha\) of length \(2N\), i.e., sequences \(\alpha = (\alpha(0), \alpha(1), \ldots, \alpha(2N))\) of non-negative integers with \(|\alpha(j) - \alpha(j - 1)| = 1\) for all \(j\) and \(\alpha(0) = \alpha(2N) = 0\). Figure 1.1 depicts example Dyck paths.

Our first main result, Theorem A in Section 4, relates the conformal block functions via explicit \(q\)-combinatorial formulas to another family of functions: the pure partition functions of multiple \(SLE\)s \([KP16]\), whose precise definition is recalled in Section 4.1. The pure partition functions are a key ingredient in the construction of joint laws of \(N\) curves of \(SLE_\kappa\) type, with deterministic connectivity \([BBK05, KP16, KKP17, PW17]\). They are indexed by the planar connectivities, or equivalently by Dyck paths. In the case \(\kappa = 2\), the relation between the conformal block functions and the pure partition functions arises as a consequence of Fomin’s formulas \([Fom01]\) for loop-erased random walks, as explained in \([KKP17]\), and our result can be seen as a \(q\)-analogue of Fomin’s formulas.

Specifically, we show that for fixed \(N\), the conformal block functions and the multiple \(SLE\) pure partition functions form a basis of the same function space of dimension given by the Catalan number \(C_N = \frac{1}{N+1} \binom{2N}{N}\), and we give an explicit combinatorial formula for the change of basis matrix \(\mathfrak{M}\) from the latter basis to the former, as well as for the inverse \(\mathfrak{M}^{-1}\). The rows and columns of both \(\mathfrak{M}\) and \(\mathfrak{M}^{-1}\) are indexed by Dyck paths, and the entries are rational functions of \(q = e^{i2\pi/\kappa}\). The non-zero entries of \(\mathfrak{M}\) appear where a binary relation introduced in \([SZ12, KW11]\) holds between the two Dyck paths, whereas the non-zero entries of \(\mathfrak{M}^{-1}\) appear where the two Dyck paths are in the natural partial order. Combinatorial formulas for the matrices are given in Section 4.2, but for small values of \(N\) their forms are already illustrated in Figures 1.2 and 1.3.

The second main result of this article, Theorem B in Section 5, is a construction of the conformal block functions via the quantum group based method of \([KP14]\). Our construction expresses the conformal block functions as concrete linear combinations of integrals of Coulomb gas type, similar to \([DF84]\). It also reflects the underlying idea of conformal blocks, according to which the Dyck path serves to label a sequence of intermediate representations.
Figure 1.2. The rows and columns of the matrix $\mathcal{M}$ are indexed by Dyck paths of $2N$ steps. The non-zero entries appear where a certain binary relation — the parenthesis reversal relation — holds between the two Dyck paths. This figure gives the explicit matrix elements of $\mathcal{M}$ in terms of $q = e^{i\pi/\kappa}$ for the cases $N = 2$ and $N = 3$.

Figure 1.3. The rows and columns of the matrix $\mathcal{M}^{-1}$ are indexed by Dyck paths of $2N$ steps. The non-zero entries appear where the natural partial order relation holds between the two Dyck paths; in particular, the matrix is upper triangular. This figure gives the explicit matrix elements of $\mathcal{M}^{-1}$ in terms of $q = e^{i\pi/\kappa}$ for the cases $N = 2$ and $N = 3$.

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2. Combinatorial preliminaries

In this section, we recall some combinatorial definitions and results. A complete account can be found in our previous article [KKP17, Section 2], whose notations and conventions we follow.

2.1. Dyck paths, skew Young diagrams, and Dyck tiles. We denote by $\text{DP}_N$ the set of Dyck paths of $2N$ steps, i.e., sequences $\alpha = (\alpha(0), \alpha(1), \ldots, \alpha(2N))$ such that $\alpha(j) \in \mathbb{Z}_{\geq 0}$ and $|\alpha(j) - \alpha(j-1)| = 1$.
for all \( j \in \{1, \ldots, 2N\} \), and \( \alpha(0) = \alpha(2N) = 0 \). The number of such Dyck paths is a Catalan number,

\[
\#\text{DP}_N = C_N = \frac{1}{N+1} \binom{2N}{N}.
\]

We also denote by \( \text{DP} := \bigsqcup_{N \in \mathbb{Z}_{\geq 0}} \text{DP}_N \) the set of Dyck paths of arbitrary length.

For each \( N \), the set of Dyck paths of \( 2N \) steps has a natural partial ordering: for \( \alpha, \beta \in \text{DP}_N \) we denote \( \alpha \preceq \beta \) if and only if \( \alpha(j) \leq \beta(j) \) for all \( j \in \{0, 1, \ldots, 2N\} \). When \( \alpha \preceq \beta \), the area between the Dyck paths \( \alpha \) and \( \beta \) forms a skew Young diagram, denoted by \( \alpha / \beta \).

The main combinatorial objects for the present article are certain tilings of skew Young diagrams, called Dyck tilings. The tiles \( t \) in these tilings are skew Young diagrams of a particular type: namely \( t = \alpha / \beta \) such that for some \( 0 < x_t \leq x'_t < 2N \) and \( h_t \in \mathbb{Z}_{>0} \) we have

\[
\begin{cases}
\alpha(j) = \beta(j) & \text{for } 0 \leq j < x_t \\
\alpha(j) = \beta(j) - 2 & \text{for } x_t \leq j \leq x'_t \\
\alpha(j) = \beta(j) & \text{for } x'_t < j \leq 2N
\end{cases}
\]

and

\[
\alpha(x_t - 1) = \beta(x_t - 1) = \alpha(x_t + 1) = \beta(x_t + 1) = h_t.
\]

Such tiles \( t = \alpha / \beta \) are called Dyck tiles, the number \( h_t \) is called the height of \( t \), and the intervals \([x_t, x'_t] \) and \((x_t - 1, x'_t + 1)\) are called the horizontal extent and shadow of \( t \), respectively. Figure 2.1 illustrates these notions. We say that a Dyck tile \( t_2 = \alpha_2 / \beta_2 \) covers another Dyck tile \( t_1 = \alpha_1 / \beta_1 \) if there exists a \( j \) such that \( j \in [x_{t_1}, x'_{t_1}] \cap [x_{t_2}, x'_{t_2}] \) and \( \alpha_1(j) < \alpha_2(j) \).

In general, a Dyck tiling \( T \) of a skew Young diagram \( \alpha / \beta \) is a collection of Dyck tiles \( t \) which cover the area of the skew Young diagram, \( \bigcup_{t \in T} t = \alpha / \beta \), and which have no overlap. Specifically, we consider so called nested Dyck tilings and cover-inclusive Dyck tilings illustrated in Figures 2.2 and 2.3 and defined below in sections 2.2 and 2.3, respectively.

2.2. Nested Dyck tilings and the parenthesis reversal relation. A Dyck tiling \( T \) of a skew Young diagram \( \alpha / \beta \) is said to be a nested Dyck tiling if the shadows of any two distinct tiles of \( T \) are either disjoint or one contained in the other, and in the latter case the tile with the larger shadow covers the other. Figure 2.2 exemplifies nested Dyck tilings. It is not difficult to see that if a skew Young diagram \( \alpha / \beta \) admits a nested Dyck tiling, such a tiling is necessarily unique. In this case, we write

\[
\alpha \xrightarrow{\leftarrow} \beta,
\]
and we denote the nested Dyck tiling of $\alpha/\beta$ by $T_0(\alpha/\beta)$. This binary relation $\prec$ on $\text{DP}_N$ was first introduced in [SZ12, KW11], and we call it the *parenthesis reversal relation*, because of a convenient characterization it has in terms of balanced parenthesis expressions, see [KKP17, Lemma 2.7].

2.3. Cover-inclusive Dyck tilings. A Dyck tiling $T$ of a skew Young diagram $\alpha/\beta$ is said to be a *cover-inclusive Dyck tiling* if for any two distinct tiles of $T$, either the horizontal extents are disjoint, or the tile that covers the other has horizontal extent contained in the horizontal extent of the other. Figure 2.3 exemplifies cover-inclusive Dyck tilings. In contrast with nested Dyck tilings, any skew Young diagram has cover-inclusive Dyck tilings. For $\alpha \preceq \beta$, the set of cover-inclusive Dyck tilings of $\alpha/\beta$ is denoted by $C(\alpha/\beta)$.

2.4. Weighted incidence matrices and their inversion. The incidence matrix of the binary relation $\leftarrow$ on the set $\text{DP}_N$ of Dyck paths plays a role in the combinatorics of dimers and groves [KW11], and of uniform spanning tree boundary branches [KKP17]. The rows and columns of this incidence matrix are indexed by Dyck paths, and its entries are 1 or 0 according to whether the relation $\leftarrow$ holds between the two paths. It turns out that an appropriately weighted incidence matrix is relevant for the combinatorics of conformal blocks.

Suppose that a weight $w(t) \in \mathbb{C}$ has been assigned to each Dyck tile $t$. We define the weighted incidence matrix by setting for all $\alpha, \beta \in \text{DP}_N$

$$M_{\alpha,\beta} := \begin{cases} \prod_{t \in T_{0}(\alpha/\beta)} (-w(t)) & \text{if } \alpha \leftarrow \beta \\ 0 & \text{otherwise,} \end{cases}$$

where $T_{0}(\alpha/\beta)$ denotes the unique nested tiling of the skew Young diagram $\alpha/\beta$ when $\alpha \prec \beta$.

We rely on the following combinatorial result, which gives a formula for the inverse of the weighted incidence matrix (2.1) in terms of cover-inclusive Dyck tilings. Such formulas for the inverses appear in [KW11, SZ12, KKP17].
Proposition 1. The weighted incidence matrix $M \in \mathbb{C}^{\text{DP}_N \times \text{DP}_N}$ with entries (2.1) is invertible, and the entries of the inverse matrix $M^{-1}$ are given by the weighted sums

$$M^{-1}_{\alpha,\beta} = \begin{cases} \sum_{T \in \mathcal{C}(\alpha/\beta)} \prod_{t \in T} w(t) & \text{if } \alpha \preceq \beta \\ 0 & \text{otherwise} \end{cases}$$

over the sets $\mathcal{C}(\alpha/\beta)$ of cover-inclusive Dyck tilings of the skew Young diagrams $\alpha/\beta$.

Proof. In this form, the assertion is proved in [KKP17, Theorem 2.9].

2.5. Slopes and wedges in Dyck paths and a recursion for incidence matrices. Any two consecutive steps of a Dyck path $\alpha$ are said to form either a slope or a wedge, according to the cases illustrated in Figure 2.4: we say that $\alpha$ has a wedge at $j$ if $\alpha(j-1) = \alpha(j+1)$, and that $\alpha$ has a slope at $j$ otherwise.

A slope at $j$ is called an up-slope if $\alpha(j+1) = \alpha(j-1) + 2$ and a down-slope if $\alpha(j+1) = \alpha(j-1) - 2$. Without specifying the type of the slope, we denote the presence of a slope at $j$ by $\times_j \in \alpha$.

A wedge at $j$ is called an up-wedge if $\alpha(j) = \alpha(j \pm 1) + 1$, and a down-wedge if $\alpha(j) = \alpha(j \pm 1) - 1$, and in these two cases we respectively write $\lor_j \in \alpha$ and $\land_j \in \alpha$. Without specifying the type of the wedge, we denote the presence of a wedge at $j$ by $\lor_j \in \alpha$. By removing a wedge at $j$ from a Dyck path $\alpha \in \text{DP}_N$ we obtain a shorter Dyck path $\hat{\alpha} \in \text{DP}_{N-1}$, namely $\hat{\alpha} = (\alpha(0), \alpha(1), \ldots, \alpha(j-1), \alpha(j+2), \ldots, \alpha(2N))$. According to whether the removed wedge is an up-wedge or a down-wedge, we write $\hat{\alpha} = \alpha \setminus \lor_j$ or $\hat{\alpha} = \alpha \setminus \land_j$, or without specifying the type of the removed wedge, we may write $\hat{\alpha} = \alpha \setminus \lor_j$.

Suppose that the weights $w(t)$ of Dyck tiles $t$ are chosen to only depend on the height $h_t$ of the tile. Then, wedge removals allow for a characterization of weighted incidence matrices of the parenthesis reversal relation by the following recursion.

Proposition 2. Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be a given function. Then the collection $(M^{(N)})_{N \in \mathbb{N}}$ of weighted incidence matrices (2.1) with weights of tiles determined by tile heights via $w(t) = f(h_t)$ is the unique collection of matrices $M^{(N)} \in \mathbb{C}^{\text{DP}_N \times \text{DP}_N}$ satisfying the following recursion: we have $M^{(0)} = 1$, and for any $N \in \mathbb{Z}_{>0}$, and $\alpha, \beta \in \text{DP}_N$, and $j \in \{1, \ldots, 2N - 1\}$ such that $\land_j \in \beta$, we have

$$M^{(N)}_{\alpha,\beta} = \begin{cases} 0 & \text{if } \times_j \in \alpha \\ M^{(N-1)}_{\hat{\alpha},\beta} & \text{if } \land_j \in \alpha \\ -f(\alpha(j) + 1) \times M^{(N-1)}_{\hat{\alpha},\beta} & \text{if } \lor_j \in \alpha, \end{cases}$$
where we denote by \( \hat{\alpha} = \alpha \setminus \emptyset_j \in \text{DP}_{N-1} \) and \( \hat{\beta} = \beta \setminus \emptyset^j \in \text{DP}_{N-1} \).

**Proof.** See [KKP17, Lemmas 2.13 and 2.14]. □

3. Conformal block functions

In the operator formalism of quantum field theories, fields correspond to linear operators on the state space of the theory, and correlation functions are written as “vacuum expected values”. Somewhat more concretely, \(n\)-point correlation functions are particular matrix elements of a composition of \(n\) linear operators on the state space. Since the state space carries representations of the symmetries of the quantum field theory and can be split to a direct sum of subrepresentations, it is natural to split these linear operators into corresponding blocks. In conformal field theory (CFT), the state space is a representation of the Virasoro algebra by virtue of conformal symmetry. The term conformal block refers to the idea of splitting the field operators into pieces that go from one Virasoro subrepresentation of the state space to another, and compositions of field operators to pieces that pass through a given sequence of subrepresentations, see [BPZ84a, Fel89, DMS97, Rib14].

The main purpose of this section is to provide background for the definition of conformal block functions that we use in the rest of the article. We give this definition in Section 3.4. The background is included to provide sufficient context and main ideas, but the presentation here is not intended to be fully rigorous — a complete mathematical treatment would require extensive formalism and results on vertex operator algebras.

3.1. Highest weight representations of Virasoro algebra. In usual conformal field theories, the state space splits into a direct sum of highest weight representations of the Virasoro algebra [Kac78, FF90, IK11], with a common central charge \( c \in \mathbb{R} \) and various highest weights \( h \in \mathbb{R} \). We parametrize the central charges \( c \leq 1 \) by \( \kappa > 0 \) via (1.1), as is relevant to the theory of SLE\(_\kappa\) type random curves. A special role is played by a primary field of conformal weight \( h = \frac{6 - \kappa^2}{2 \kappa} \), which through fusion generates the so called first row of the Kac table with conformal weights

\[
h(\lambda) = h_{1, \lambda+1} = \frac{\lambda^2 + 2 \lambda}{\kappa} - \frac{\lambda}{2} \quad \text{for} \quad \lambda \in \mathbb{Z}_{\geq 0}.
\]

In this article, we consider the generic case \( \kappa \not\in \mathbb{Q} \). Then, the irreducible highest weight representation with highest weight \( h(\lambda) \) and central charge \( c \) is a quotient of the corresponding Verma module by a submodule that itself is a Verma module\(^1\). We denote this irreducible quotient by \( Q_\lambda \) and a highest weight vector in it by \( w_\lambda \). The contragredient (graded dual) representation \( Q_\lambda^* \) (see e.g. [IK11]) is isomorphic to \( Q_\lambda \), and we choose a highest weight vector \( w_\lambda^* \) for it so that the normalization \( \langle w_\lambda^*, w_\lambda \rangle = 1 \) holds.

3.2. Conformal blocks.

3.2.1. Intertwining relation for primary field operators. A primary field \( \psi \) of conformal weight \( h \) is characterized by its transformation property

\[
\psi(x) \rightarrow \phi'(x)^h \psi(\phi(x))
\]

under conformal transformations \( \phi \). According to the seminal work [BPZ84a], more general fields can be understood in terms of these primary fields.

\(^1\)At rational values of \( \kappa \), the structure of highest weight representations can be more involved, see, e.g., [IK11].
In the operator formalism, a primary field $\psi(x)$ is then realized by a primary field operator $\Psi(x)$, which we wish to split to conformal blocks between various highest weight representations $Q_\lambda$ and $Q_\mu$, with $\lambda, \mu \in \mathbb{Z}_{\geq 0}$. The intertwining relation

\begin{equation}
L_n \Psi(x) - \Psi(x)L_n = x^{1+n} \frac{\partial}{\partial x} \Psi(x) + (1+n)x^n h \Psi(x)
\end{equation}

with the Virasoro generators $L_n, \ n \in \mathbb{Z},$ is the infinitesimal form of the primary field transformation property under a conformal transformation $\phi$ obtained by varying the identity transformation to the direction of the holomorphic vector field $\ell_x = -x^{1+n} \frac{\partial}{\partial x}$.

3.2.2. **Matrix elements characterizing conformal blocks.** The single matrix element between highest weight vectors,

\begin{equation}
U_{\lambda}^\mu(x) = \langle w_\mu^*, \Psi_{\lambda}^\mu(x) w_\lambda \rangle,
\end{equation}

contains sufficient information to completely determine $\Psi_{\lambda}^\mu(x)$. More generally, a composition of primary field operators splits into conformal blocks indexed by a sequence $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ with $\sigma_j \in \mathbb{Z}_{\geq 0}$ labeling the intermediate representations $Q_{\sigma_0}, Q_{\sigma_1}, \ldots, Q_{\sigma_n}$. Now, the matrix element

\begin{equation}
U_\sigma(x_1, x_2, \ldots, x_n) = \langle w_{\sigma_0}^*, \Psi_{\sigma_{n-1}}^\sigma(x_n) \cdots \Psi_{\sigma_1}^\sigma(x_2) \Psi_{\sigma_0}^\sigma(x_1) w_{\sigma_0} \rangle
\end{equation}

contains sufficient information to uniquely determine the block of the composition. Note that $U_{\lambda}^\mu(x)$ is a special case of $U_\sigma(x_1, x_2, \ldots, x_n)$ with $n = 1, \sigma_0 = \lambda, \text{ and } \sigma_1 = \mu$. We furthermore point out that $U_\sigma(x_1, x_2, \ldots, x_n)$ appears in an actual vacuum expected value of $n$ fields if the highest weight states on the right and left are the absolute vacua, i.e., if $\sigma_0 = 0$ and $\sigma_n = 0$.

In the vertex operator algebra axiomatization of conformal field theory, the block $\Psi_{\lambda}^\mu(x)$ is a formal power series in $x$ with coefficients that are linear operators, and the matrix elements $U_{\lambda}^\mu(x)$ and $U_\sigma(x_1, x_2, \ldots, x_n)$ are formal power series with complex coefficients. For radially ordered variables

$$0 < |x_1| < |x_2| < \cdots < |x_n|,$$

the series are in fact convergent, so we may view (3.3) and (3.4) as actual functions. The fact that they determine the operators and their compositions justifies calling them conformal block functions.

3.3. **Properties of conformal block functions.** We now review properties of conformal block functions $U_{\sigma}(x_1, \ldots, x_n)$, which in particular completely fix the form of the matrix elements $U_{\lambda}^\mu(x)$, and characterize for which $\lambda$ and $\mu$ the block $\Psi_{\lambda}^\mu(x)$ can be non-vanishing in the first place.

3.3.1. **Covariance properties.** The intertwining relation (3.2) for $L_0$ combined with the eigenvalues $L_0 w_{\sigma_0} = h(\sigma_0) w_{\sigma_0}$ and $L_0^* w_{\sigma_n}^* = h(\sigma_n) w_{\sigma_n}^*$ of the highest weight vectors gives

$$\langle h(\sigma_0) - h(\sigma_0) \rangle \ U_{\sigma}(x_1, \ldots, x_n) = \sum_{j=1}^{n} \left( x_j \frac{\partial}{\partial x_j} + \hat{h} \right) \ U_{\sigma}(x_1, \ldots, x_n),$$

with $h = h(1) = \frac{\alpha-\kappa}{2\kappa}$. This infinitesimal relation can be integrated to obtain the homogeneity property

\begin{equation}
U_{\sigma}(rx_1, \ldots, rx_n) = r^{h(\sigma_n)-h(\sigma_0)-nh} U_{\sigma}(x_1, \ldots, x_n), \quad \text{for } r > 0,
\end{equation}

of the conformal block functions. For the simplest conformal block function $U_{\lambda}^\mu(x)$, this homogeneity fixes its form up to a multiplicative constant $C_{\lambda}^\mu$,

\begin{equation}
U_{\lambda}^\mu(x) = \langle w_\mu^*, \Psi_{\lambda}^\mu(x) w_\lambda \rangle = C_{\lambda}^\mu x^{h(\mu)-h(\lambda)-h}.
\end{equation}

Next, if we have $\sigma_0 = 0$, then $L_{-1} w_{\sigma_0} = 0$ is a null vector in the quotient representation $Q_0$. Together with the intertwining relation (3.2) for $L_{-1}$, this gives the infinitesimal form of the translation invariance

\begin{equation}
U_{\sigma}(x_1 + t, \ldots, x_n + t) = U_{\sigma}(x_1, \ldots, x_n), \quad \text{for } t \in \mathbb{R}.
\end{equation}
Likewise, if \( \sigma_n = 0 \), then \( L_1^+ w_\sigma^* = 0 \) is a null vector in the contragredient representation \( Q_\lambda^* \), and together with the intertwining relation (3.2) for \( L_1 \), this gives the infinitesimal form of the following covariance under special conformal transformations:

\[
U_\sigma \left( \frac{x_1}{1-sx_1}, \ldots, \frac{x_n}{1-sx_n} \right) = \prod_{j=1}^{n} (1-sx_j)^{2h} \times U_\sigma(x_1, \ldots, x_n), \quad \text{for } s \in \mathbb{R}.
\]

For the conformal blocks that contribute to the vacuum expected value, we have \( \sigma_0 = 0 \) and \( \sigma_n = 0 \).

These conformal block functions satisfy the covariance

\[
U_\sigma(x_1, \ldots, x_n) = \prod_{j=1}^{n} \mu'(x_j)^{2h} \times U_\sigma(\mu(x_1), \ldots, \mu(x_n))
\]

under general Möbius transformations \( \mu(x) = \frac{ax+b}{cx+d} \) with \( a, b, c, d \in \mathbb{R} \) and \( ad - bc > 0 \).

### 3.3.2. Partial differential equations.

Suppose now that the primary field \( \Psi(x) \) of conformal weight \( h = h(1) = \frac{\Delta - \lambda}{2\kappa} \) has the same degeneracy at grade two as the quotient representation \( Q_1 \) of highest weight \( h(1) \). Then the conformal block functions satisfy partial differential equations of second order. These PDEs obtain a more symmetric expression in terms of the shifted versions of the conformal block functions defined by

\[
\tilde{U}_\sigma(x_0, x_1, \ldots, x_n) := U_\sigma(x_1 - x_0, \ldots, x_n - x_0).
\]

The partial differential equation arising from the degeneracy of \( \Psi(x_j) \) at grade two takes the form given in [BPZ84a],

\[
\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \frac{2}{x_0 - x_j} \frac{\partial}{\partial x_0} - \frac{2h(\sigma_0)}{(x_0 - x_j)^2} \right. \\
+ \left. \sum_{i=1}^{n} \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{2h(x_i - x_j)^2}{(x_i - x_j)^2} \right\} \tilde{U}_\sigma(x_0, x_1, \ldots, x_n) = 0.
\]

### 3.3.3. Selection rules.

The PDEs above in particular imply selection rules for when non-vanishing conformal blocks can exist. Namely, for \( U^\mu_\Delta(x) = C^\mu_\Delta x^\Delta \), with \( \Delta = h(\mu) - h(\lambda) - h \) as in Equation (3.6), the requirement of the PDE (3.9) amounts to

\[
C^\mu_\Delta \left( \frac{\kappa}{2} \Delta(\Delta - 1) + 2\Delta - 2h(\lambda) \right) x^{\Delta-2} = 0,
\]

which implies either the vanishing of \( C^\mu_\Delta \) and therefore of the entire conformal block, or a quadratic equation relating the conformal weights \( h(\mu) \) and \( h(\lambda) \). For fixed \( \lambda \), the two solutions of this quadratic equation are obtained at \( \mu = \lambda \pm 1 \). One can therefore conclude that the conformal blocks take the form

\[
U^\mu_\lambda(x) = \begin{cases} 
C^\pm_\lambda x^{h(\mu)-h(\lambda)-h} & \text{if } \mu = \lambda \pm 1 \\
0 & \text{if } |\mu - \lambda| \neq 1.
\end{cases}
\]

The normalizations \( C^\pm_\lambda \) are not canonically fixed; in fact, the space of intertwining operators forms a vector space (in the present cases always of dimension one or zero depending on whether the selection rules are fulfilled). One can make any convenient choice, and we will fix our choice later.

In the case \( \lambda = 0 \), there is one further selection rule: by translation invariance, \( U^0_\mu(x) \) is constant. This further restricts the possibilities in (3.6) to \( h(\mu) = h = h(1) \), i.e., \( \mu = 1 \).

In conclusion, a non-vanishing intertwining operator from \( Q_\lambda \) to \( Q_\mu \) can only exist if \( |\mu - \lambda| = 1 \) and \( \mu \geq 0 \). Consequently, the composition of intertwining operators as in the conformal block function (3.4) can only be non-trivial if the sequence \( \tilde{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_n) \) satisfies \( \sigma_j \in \mathbb{Z}_{\geq 0} \) and \( |\sigma_j - \sigma_{j-1}| = 1 \) for all \( j \). This means that non-trivial conformal block functions are indexed by nearest neighbor walks on...
The homogeneity (3.5) can be used to cast the problem into a form where we resort to direct solutions of the PDEs (3.9) in the following. Using this expansion, the conformal block function becomes

\begin{equation}
\Psi^\sigma_0(x_1, x_2) = \sum_{k \in \mathbb{N}} \sum_{n_1, \ldots, n_k > 0} w_{\sigma_0} n_1 \cdots n_k (x_1) L_{-n_k} \cdots L_{-n_1} w_{\sigma_1},
\end{equation}

where in particular the coefficient of the highest weight vector \( w_{\sigma_1} \) is picked by the projection to \( w_{\sigma_1}^* \),

\begin{equation}
a_0(x_1) = \langle w_{\sigma_1}, \Psi^\sigma_0(x_1) w_{\sigma_0} \rangle = U^\sigma_{\sigma_0}(x_1) = C^\sigma_{\sigma_0} x_1^{h(\sigma_1) - h(\sigma_0) - h}.
\end{equation}

Using this expansion, the conformal block function becomes

\begin{equation}
U^\sigma_{\sigma_0}(x_1, x_2) = \sum_{k \in \mathbb{N}} \sum_{n_1, \ldots, n_k > 0} a_{n_1, \ldots, n_k}(x_1) \langle w_{\sigma_2}^*, \Psi^\sigma_{\sigma_1}(x_2) L_{-n_k} \cdots L_{-n_1} w_{\sigma_0} \rangle.
\end{equation}

With the generic form (3.6) of the conformal blocks \( U^\sigma_{\sigma_0}(x_2) \), the intertwining relation (3.2) implies

\begin{equation}
\langle w_{\sigma_2}^*, \Psi^\sigma_{\sigma_1}(x_2) L_{-n_k} \cdots L_{-n_1} w_{\sigma_0} \rangle \propto x_2^{h(\sigma_2) - h(\sigma_1) - h - n_2 - \cdots - n_k}.
\end{equation}

The leading contribution in the limit \( x_2 \to \infty \) comes from the highest weight vector \( w_{\sigma_1} \), since \( n_j > 0 \). Thus, for fixed \( x_1 \) and as \( x_2 \to \infty \), the leading asymptotics of the conformal block function is

\begin{equation}
U^\sigma_{\sigma_0}(x_1, x_2) \approx a_0(x_1) \langle w_{\sigma_2}^*, \Psi^\sigma_{\sigma_1}(x_2) w_{\sigma_0} \rangle
= a_0(x_1) \times U^\sigma_{\sigma_0}(x_2)
= C^\sigma_{\sigma_0} x_1^{h(\sigma_1) - h(\sigma_0) - h} \times C^\sigma_{\sigma_1} x_2^{h(\sigma_2) - h(\sigma_1) - h},
\end{equation}

By a similar argument, for fixed \( x_2 \) and as \( x_1 \to 0 \), the leading asymptotics of the conformal block function is

\begin{equation}
U^\sigma_{\sigma_0}(x_1, x_2) \approx C^\sigma_{\sigma_2} C^\sigma_{\sigma_0} x_2^{h(\sigma_2) - h(\sigma_1) - h} x_1^{h(\sigma_1) - h(\sigma_0) - h}.
\end{equation}

The remaining interesting asymptotics of \( U^\sigma_{\sigma_0}(x_1, x_2) \) concern the limit \( |x_1 - x_2| \to 0 \). To analyze these, we resort to direct solutions of the PDEs (3.9) in the following.

The homogeneity (3.5) can be used to cast the \( n = 2 \) conformal block function into the form

\begin{equation}
U^\sigma_{\sigma_0}(x_1, x_2) = x_2^{h(\sigma_1) - h(\sigma_0) - 2h} g^\sigma \left( \frac{x_1}{x_2} \right),
\end{equation}

and the PDEs (3.9) for \( \tilde{U}^\sigma_{\sigma_0}(x_0, x_1, x_2) = (x_2 - x_0)^{h(\sigma_2) - h(\sigma_0) - 2h} g^\sigma \left( \frac{x_1 - x_0}{x_2 - x_0} \right) \) then translate to the following second order ODE for \( g^\sigma(z) \):

\begin{equation}
0 = \kappa \left( z^2(z - 1)^2 g^\sigma (z) + 8 z(z - 1)(z - \frac{1}{2}) g^\sigma (z) \right.
+ 4 \left( z(z - 2) h - z(z - 1) h(\sigma_0) + (z - 1) h(\sigma_2) \right) g^\sigma(z).
\end{equation}

Such an ODE in principle has a two-dimensional solution space, but the asymptotics

\begin{equation}
g^\sigma(z) \sim C^\sigma_{\sigma_0} C^\sigma_{\sigma_1} z^{h(\sigma_1) - h(\sigma_0) - h} \quad \text{as} \quad z \to 0
\end{equation}

obtained from (3.11) - (3.12) turn out to be enough to fix the solutions. We analyze different cases allowed by the selection rules separately.
Denote $\sigma_0 = \lambda$. By the selection rules, there are four possibilities when $n = 2$, which we label as follows:

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>up-wedge</th>
<th>down-wedge</th>
<th>up-slope</th>
<th>down-slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\wedge)</td>
<td>((\lambda, \lambda + 1, \lambda))</td>
<td>((\lambda, \lambda - 1, \lambda))</td>
<td>((\lambda, \lambda + 1, \lambda + 2))</td>
<td>((\lambda, \lambda - 1, \lambda - 2))</td>
</tr>
<tr>
<td>Figure</td>
<td>4(a)</td>
<td>4(b)</td>
<td>4(c)</td>
<td>4(d)</td>
</tr>
</tbody>
</table>

In the case of an up-slope, the only solution of the ODE (3.13) with the correct asymptotics (3.14) is

$$g_\lambda(z) = C_\lambda^+ C_{\lambda+1}^+ \times (1 - z)^{\frac{2\lambda}{\kappa}} \times z^{h(\lambda+1) - h(\lambda) - h}.$$  

A similar conclusion holds in the case of a down-slope:

$$g_\lambda(z) = C_\lambda^- C_{\lambda-1}^- \times (1 - z)^{\frac{2\lambda}{\kappa}} \times z^{h(\lambda-1) - h(\lambda) - h}.$$  

The solution in the case of an up-wedge is slightly more complicated, non-degenerate hypergeometric function

$$g_\lambda(z) = C_\lambda^+ C_{\lambda+1}^- z^{\frac{2\lambda}{\kappa}} (1 - z)^{\frac{\kappa}{\kappa}} 2F_1\left(\begin{array}{c} \frac{\kappa}{\kappa} \end{array} \left| \begin{array}{c} \frac{\kappa}{\kappa} \end{array} \right| ; \frac{4\lambda + 4}{\kappa}; z \right).$$

Similarly, in the case of a down-wedge, the solution is

$$g_\lambda(z) = C_\lambda^- C_{\lambda-1}^+ z^{\frac{2\lambda}{\kappa}} (1 - z)^{\frac{\kappa}{\kappa}} 2F_1\left(\begin{array}{c} \frac{\kappa}{\kappa} \end{array} \left| \begin{array}{c} \frac{\kappa}{\kappa} \end{array} \right| ; \frac{2\kappa - 8 - 4\lambda}{\kappa}; z \right).$$

The asymptotics as $z \to 1$ of such hypergeometric functions can be obtained from the identities $2F_1(a, b; c; 0) = 1$ and [AS64, Equation (15.3.6)]:

$$2F_1(a, b; c; z) = (1 - z)^{c-a-b} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} 2F_1(c - a, c - b; c - a - b + 1; 1 - z)$$

$$+ \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} 2F_1(a, b; a + b - c + 1; 1 - z).$$

Because we assume $0 < \kappa < 8$, the parameters $a, b, c$ of the hypergeometric functions in both $g_\lambda(z)$ and $g_\lambda(z)$ satisfy $c - a - b = \frac{8 - \kappa}{\kappa} > 0$. Thus, in the limit $z \to 1$ of the hypergeometric function, the first term above vanishes and the second term tends to $\frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}$. This shows that we have

$$(1 - z)^{\frac{8-\kappa}{\kappa}} \times g_\lambda(z) \rightarrow C_\lambda^+ C_{\lambda+1}^- \frac{\Gamma(\frac{4\lambda + 4\kappa}{\kappa}) \Gamma(\frac{8 - \kappa}{\kappa})}{\Gamma(\frac{8 - \kappa + 4\kappa}{\kappa}) \Gamma(\frac{4\kappa}{\kappa})},$$

$$(1 - z)^{\frac{8-\kappa}{\kappa}} \times g_\lambda(z) \rightarrow C_\lambda^- C_{\lambda-1}^+ \frac{\Gamma(\frac{2\kappa - 4 - 4\kappa}{\kappa}) \Gamma(\frac{8 - \kappa}{\kappa})}{\Gamma(\frac{\kappa}{\kappa}) \Gamma(\frac{8 - 4\kappa}{\kappa})},$$

and we can write down explicit asymptotics of the conformal block functions $U_\lambda(x_1, x_2)$ as $x_1, x_2 \to \xi$:

$$(3.15) \quad \frac{U_\lambda(x_1, x_2)}{(x_2 - x_1)^{2h}} \rightarrow C_\lambda^+ C_{\lambda+1}^- \frac{\Gamma(\frac{4\kappa + 4\lambda}{\kappa}) \Gamma(\frac{8 - \kappa}{\kappa})}{\Gamma(\frac{8 - \kappa + 4\kappa}{\kappa}) \Gamma(\frac{4\kappa}{\kappa})}.$$  

$$(3.16) \quad \frac{U_\lambda(x_1, x_2)}{(x_2 - x_1)^{2h}} \rightarrow C_\lambda^- C_{\lambda-1}^+ \frac{\Gamma(\frac{2\kappa - 4 - 4\lambda}{\kappa}) \Gamma(\frac{8 - \kappa}{\kappa})}{\Gamma(\frac{\kappa}{\kappa}) \Gamma(\frac{8 - 4\kappa}{\kappa})}.$$  

$$(3.17) \quad \frac{U_\lambda(x_1, x_2)}{(x_2 - x_1)^{h(2) - 2h}} \rightarrow C_\lambda^+ C_{\lambda+1}^+ \xi^{h(\lambda+2) - h(\lambda) - h(2)}.$$  

$$(3.18) \quad \frac{U_\lambda(x_1, x_2)}{(x_2 - x_1)^{h(2) - 2h}} \rightarrow C_\lambda^- C_{\lambda-1}^- \xi^{h(\lambda-2) - h(\lambda) - h(2)}.$$  

The last two expressions above, (3.17) and (3.18), are proportional to one-point conformal block functions from $Q_{\lambda}$ to $Q_{\lambda+2}$ of a primary field operator of conformal weight $h(2) = \frac{8 - \kappa}{\kappa}$, whereas the first two, (3.15) and (3.16), are proportional to an one-point conformal block function from $Q_{\lambda}$ to itself of a
primary field operator of conformal weight \( h(0) = 0 \). The latter is the identity operator, whose one-point conformal block function is just the constant 1. We now choose the normalization constants \( C_\lambda^\pm \) so that the coefficient of the identity operator in (3.15) equals one, i.e., we set

\[
C_\lambda^+ = \frac{1}{C_{\lambda-1}^+} \times \frac{\Gamma\left(\frac{4-\kappa+4\lambda}{2\kappa}\right) \Gamma\left(\frac{\kappa}{2}\right)}{\Gamma\left(\frac{4-\kappa}{2\kappa}\right) \Gamma\left(\frac{\kappa-4\lambda}{2}\right)} \quad \text{for all } \lambda > 0.
\]

Then, the coefficients \( C_\lambda^+ \) are the remaining free parameters. The coefficient of the identity operator in (3.16) then becomes a ratio of gamma-functions, which can be further simplified to

\[
C_\lambda^{-1} C_{\lambda-1}^+ \frac{\Gamma\left(\frac{2\kappa-4-4\lambda}{\kappa}\right) \Gamma\left(\frac{\kappa-4\lambda}{\kappa}\right)}{\Gamma\left(\frac{\kappa}{\kappa}\right) \Gamma\left(\frac{\kappa-4\lambda}{\kappa}\right)} = \frac{\Gamma\left(\frac{2\kappa-4-4\lambda}{\kappa}\right) \Gamma\left(\frac{\kappa+4\lambda}{\kappa}\right)}{\Gamma\left(\frac{\kappa}{\kappa}\right) \Gamma\left(\frac{\kappa-4\lambda}{\kappa}\right)}
\]

using the identity \( \Gamma(w)\Gamma(1-w) = \pi/\sin(\pi w) \) twice. Introducing the parameter \( q = e^{i\pi 4/\kappa} \) and \( q \)-integers \([n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sin(4\pi n/\kappa)}{\sin(\pi n/\kappa)} \), this takes the simple form

\[
C_\lambda^{-1} C_{\lambda-1}^+ \frac{\Gamma\left(\frac{2\kappa-4-4\lambda}{\kappa}\right) \Gamma\left(\frac{\kappa-4\lambda}{\kappa}\right)}{\Gamma\left(\frac{\kappa}{\kappa}\right) \Gamma\left(\frac{\kappa-4\lambda}{\kappa}\right)} = - \frac{[\lambda]}{[\lambda + 1]}.
\]

With the chosen normalization convention and when \( 0 < \kappa < 8 \), the leading asymptotics (3.15) – (3.18) as \( x_1, x_2 \to \xi \in (0, \infty) \) of the conformal block functions can thus be summarized as

\[
(3.19) \quad \frac{U_\hat{\sigma}(x_1, x_2)}{(x_2 - x_1)^{-2h}} \longrightarrow \begin{cases} 
1 & \text{if } \hat{\sigma} = \wedge \\
-\frac{[\lambda]}{[\lambda + 1]} & \text{if } \hat{\sigma} = \vee \\
0 & \text{if } \hat{\sigma} = \hat{\sigma}^\wedge \text{ or } \hat{\sigma} = \hat{\sigma}^\vee.
\end{cases}
\]

In the case of general \( n \) and \( \hat{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_n) \), the leading asymptotics on pairwise diagonals can be inferred recursively from the above calculation. Specifically, we get

\[
\frac{U_\hat{\sigma}(x_1, \ldots, x_n)}{(x_{j+1} - x_j)^{-2h}} \longrightarrow \begin{cases} 
U_\hat{\sigma}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) & \text{if } \sigma_{j-1} = \sigma_{j+1} = \sigma_j - 1 \\
\frac{[\sigma_j + 1]}{[\sigma_j + 2]} \times U_\hat{\sigma}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_n) & \text{if } \sigma_{j-1} = \sigma_{j+1} = \sigma_j + 1 \\
0 & \text{if } \sigma_{j-1} \neq \sigma_{j+1},
\end{cases}
\]

as \( x_j, x_{j+1} \to \xi \in (x_{j-1}, x_{j+2}) \), where we denote \( \hat{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{j-1}, \sigma_{j+2}, \ldots, \sigma_{2N}) \).

### 3.4. Defining properties of conformal block functions.

So far in this section we have provided background on conformal block functions, so as to have a self-contained justification of their properties that we use as their definition in the rest of this article. We only consider the conformal block functions that contribute to vacuum expected values, in which case \( \sigma_0 = 0 \) and \( \sigma_n = 0 \), and \( n \) is necessarily even: \( n = 2N \). The sequence \((\sigma_0, \sigma_1, \ldots, \sigma_n)\) then forms a Dyck path of \( n = 2N \) steps, see Figure 1.1. Instead of the notation \( \hat{\sigma} \), we use the notation \( \alpha \in \text{DP}_N \) for this Dyck path, and

\[
U_\alpha(x_1, \ldots, x_{2N})
\]

for the corresponding conformal block function.

We next list the defining properties of \((U_\alpha)_{\alpha \in \text{DP}_N}\) in the form that they will be used. Denote \( h = h(1) = \frac{6-\kappa}{2\kappa} \) as before. The required properties of \( U_\alpha \) for \( \alpha \in \text{DP}_N \) are the partial differential equations

\[
(\text{PDE}) \quad \left[ \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i=1}^{2N} \left( \frac{2}{x_1 - x_j} \frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2} \right) \right] U_\alpha(x_1, \ldots, x_{2N}) = 0
\]

for all \( j \in \{1, \ldots, 2N\} \).
the Möbius covariance
\[
(U_{\text{COV}}) \quad U_{\alpha}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \mu'(x_i)^h \times U_{\alpha}(\mu(x_1), \ldots, \mu(x_{2N}))
\]
for all \( \mu(z) = \frac{az + b}{cz + d} \), with \( a, b, c, d \in \mathbb{R} \), \( ad - bc > 0 \), such that \( \mu(x_1) < \cdots < \mu(x_{2N}) \),
and the recursive asymptotics properties
\[
(U_{\text{ASY}}) \quad \lim_{x_j, x_{j+1} \to \xi} \frac{U_{\alpha}(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = \begin{cases} 
0 & \text{if } x_j \in \alpha \\
\frac{[\alpha(j)+1]}{[\alpha(j)+2]} \times U_{\alpha \setminus \{j\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \bigwedge j \in \alpha, \\
\text{for any } j \in \{1, \ldots, 2N - 1 \} \text{ and } \xi \in (x_{j-1}, x_{j+2}),
\end{cases}
\]
where the square bracket expressions are the \( q \)-integers \( [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \) with the parameter \( q = e^{i \pi 4/\kappa} \) depending on \( \kappa \). Finally, the case \( N = 0 \) fixes an overall normalization when we require that \( U_{\alpha}(0) = 1 \) for the Dyck path \( (0) \in D_{2N} \).

In one of the main results of this article, Theorem A in Section 4.2, we in particular prove that the conformal block functions \( U_{\alpha} \) are uniquely determined by the properties above.

4. CHANGE OF BASIS BETWEEN CONFORMAL BLOCK FUNCTIONS AND PURE PARTITION FUNCTIONS

This section contains our first main result. It states first of all that the conformal block functions and the multiple SLE pure partition functions, whose definition will be recalled below in Section 4.1, both form a basis of the same solution space of the system (PDE) of partial differential equations. Moreover, it gives an explicit \( q \)-combinatorial formula for the change of basis matrix.

We begin by discussing the space of functions. Fix \( N \in \mathbb{N} \), and consider the system of \( 2N \) second order partial differential equations and Möbius covariance conditions as in Section 3.4,
\[
(PDE) \quad \left[ \frac{\kappa}{2} \frac{\partial^2}{\partial x^2_j} + \sum_{i=1, i \neq j}^{2N} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2} \right) \right] F(x_1, \ldots, x_{2N}) = 0
\]
for all \( j \in \{1, \ldots, 2N\} \),
\[
(COV) \quad F(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \mu'(x_i)^h \times F(\mu(x_1), \ldots, \mu(x_{2N}))
\]
for all \( \mu(z) = \frac{az + b}{cz + d} \), with \( a, b, c, d \in \mathbb{R} \), \( ad - bc > 0 \), such that \( \mu(x_1) < \cdots < \mu(x_{2N}) \),
for complex valued functions \( F \) defined on the set
\[
\mathcal{X}_{2N} := \left\{ (x_1, x_2, \ldots, x_{2N}) \mid x_1 < x_2 < \cdots < x_{2N} \right\}
\]
of \( 2N \)-tuples of real variables in increasing order. Require moreover that \( F \) has at most polynomial growth on pairwise diagonals and at infinity in the sense that
there exist positive constants \( C, p > 0 \) such that we have
\[
(GROW) \quad |F(x_1, \ldots, x_{2N})| \leq C \times \prod_{i < j} \max \left( (x_j - x_i)^p, (x_j - x_i)^{-p} \right) \quad \text{for all } (x_1, \ldots, x_{2N}) \in \mathcal{X}_{2N}.
\]
We consider the following space of solutions:
\[
(4.1) \quad \mathcal{S}_N := \left\{ F : \mathcal{X}_{2N} \to \mathbb{C} \mid F \text{ satisfies } (PDE), (COV), \text{ and } (GROW) \right\}.
\]
The dimension of this space is known to be the \( N \)-th Catalan number, \( \dim(S_N) = C_N \), and the multiple SLE pure partition functions form a basis for \( S_N \), as we recall precisely in Section 4.1.

### 4.1. Multiple SLE pure partition functions

In many situations in planar statistical physics, boundary conditions force the existence of multiple macroscopic interfaces, and in the scaling limit at criticality, such interfaces are described by multiple SLE\(_{\kappa} \) curves with \( \kappa \) depending on the model. Contrary to, e.g., a single chordal SLE\(_{\kappa} \) curve, the law of a multiple SLE\(_{\kappa} \) with fixed number \( N \) of curves is not unique, instead the possible laws form a non-trivial convex set. It is thus natural to express any multiple SLE\(_{\kappa} \) as a convex combination of the extremal points of this convex set — the pure geometries, in which the curves connect the starting points of the interfaces pairwise in a deterministic planar pair partition. The pure geometries are thus indexed by planar pair partitions, which in turn are in bijection with Dyck paths. For background on multiple SLEs, we refer to [BBK05, Dub07, KL07, KP16], and for results on their role as scaling limits, to [CS12, Izy17, Wu16, KKP17, PW17, KS17].

For the purposes of this article, the important aspect of multiple SLEs is their partition functions \( Z \), which essentially define the multiple SLE\(_{\kappa} \) by giving the Radon-Nikodym density of its law with respect to independent chordal SLE\(_{\kappa} \) laws, see [Dub07, KP16]. In particular, each pure geometry with connectivity encoded by a Dyck path \( \alpha \) has a partition function denoted by \( Z_\alpha \). These functions satisfy the partial differential equations (PDE) and Möbius covariance (COV) as before, and the following recursive asymptotics:

\[
\text{(Z-ASY)} \quad \lim_{x_j, x_{j+1} \to \xi} \frac{Z_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{-2\kappa}} = \begin{cases} 
Z_\alpha \land \land (x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \land \land \in \alpha \\
0 & \text{if } \land \land \notin \alpha 
\end{cases}
\]

for any \( j \in \{1, \ldots, 2N - 1\} \), and \( \xi \in (x_{j-1}, x_{j+2}) \).

As stated in the following proposition, these requirements together with the normalization condition \( Z_{(0)} = 1 \) uniquely determine the functions \( Z_\alpha \), called the multiple SLE pure partition functions.

**Proposition 3.** Let \( \kappa \in (0, 8) \setminus \mathbb{Q} \). There exists a unique collection of functions \( (Z_\alpha)_{\alpha \in DP} \), such that \( Z_\alpha \in S_N \) when \( \alpha \in DP_N \), \( Z_{(0)} = 1 \), and (Z-ASY) holds for all \( \alpha \). Moreover, for any \( N \in \mathbb{Z}_{\geq 0} \), the functions \( (Z_\alpha)_{\alpha \in DP_N} \) form a basis of the solution space \( S_N \).

**Proof.** By [FK15c, Theorem 8], we have \( \dim(S_N) = C_N \). On the other hand, [KP16, Theorem 4.1] shows that the pure partition functions \( (Z_\alpha)_{\alpha \in DP_N} \) form a linearly independent set in this space (the power-law bound (GROW) can be verified from the explicit form of the functions as Coulomb gas integrals, see [KP14, KP16]). The assertion follows, since \( \#DP_N = C_N \). \( \square \)

### 4.2. The change of basis result

We now show how to express the conformal block functions \( U_\alpha \) in the basis of the multiple SLE pure partition functions \( Z_\alpha \) using weighted incidence matrices of the parenthesis reversal relation. It will follow that the conformal block functions are well-defined and also form a basis.

We take the weight of a Dyck tile \( t \) at height \( h_t \) to be

\[
(4.2) \quad w(t) := \frac{[h_t]}{[h_t + 1]},
\]

where \([n] = \frac{q^n - q^{-n}}{q - q^{-1}}\) and \( q = e^{i\pi \kappa / 4}\) as before. Denote by \( \mathfrak{M} = (\mathfrak{M}_{\alpha, \beta}) \) the correspondingly weighted incidence matrix (2.1): its non-zero elements are

\[
\mathfrak{M}_{\alpha, \beta} = \prod_{t \in T_0(\alpha/\beta)} (-w(t)), \quad \text{for } \alpha \prec \beta,
\]
where $T_0(\alpha/\beta)$ is the nested Dyck tiling of the skew Young diagram $\alpha/\beta$. Proposition 1 shows that the matrix $\mathcal{M}$ is invertible and the non-zero matrix elements of its inverse are

$$
\mathcal{M}_{\alpha,\beta}^{-1} = \sum_{T \in \mathcal{C}(\alpha/\beta)} \prod_{t \in T} w(t), \quad \text{for } \alpha \preceq \beta,
$$

with $\mathcal{C}(\alpha/\beta)$ the family of cover-inclusive Dyck tilings of the skew Young diagram $\alpha/\beta$. Examples of these matrices are shown in Figures 1.2 and 1.3.

**Theorem A.** There exists a unique collection $(\mathcal{U}_\alpha)_{\alpha \in \text{DP}_N}$ such that $\mathcal{U}_\alpha \in \mathcal{S}_N$ when $\alpha \in \text{DP}_N$, $\mathcal{U}_{(0)} = 1$, and the asymptotics ($U$-ASY) hold. For any $\alpha \in \text{DP}_N$, the function $\mathcal{U}_\alpha$ of this collection can be written in the basis $(Z_\beta)_{\beta \in \text{DP}_N}$ of Proposition 3 as

$$
\mathcal{U}_\alpha = \sum_{\beta \in \text{DP}_N} \mathcal{M}_{\alpha,\beta} Z_\beta,
$$

where $\mathcal{M}$ is the weighted incidence matrix of the parenthesis reversal relation with weights (4.2). Moreover, for any $N \in \mathbb{Z}_{\geq 0}^+$, the functions $(\mathcal{U}_\alpha)_{\alpha \in \text{DP}_N}$ form a basis of the solution space $\mathcal{S}_N$ and

$$
Z_\alpha = \sum_{\beta \in \text{DP}_N} \mathcal{M}_{\alpha,\beta}^{-1} \mathcal{U}_\beta.
$$

**Proof.** Let $(\mathcal{U}_\alpha)_{\alpha \in \text{DP}_N}$ be any collection of functions such that $\mathcal{U}_\alpha \in \mathcal{S}_N$ for $\alpha \in \text{DP}_N$ and the asymptotics ($U$-ASY) hold with $\mathcal{U}_{(0)} = 1$. Write $\mathcal{U}_\alpha$ in the basis $(Z_\beta)_{\beta \in \text{DP}_N}$ of $\mathcal{S}_N$ as

$$
\mathcal{U}_\alpha = \sum_{\beta \in \text{DP}_N} M_{\alpha,\beta}^{(N)} Z_\beta,
$$

which, for each $N \in \mathbb{N}$, defines a matrix $M_{\alpha,\beta}^{(N)} \in \mathbb{C}^{\text{DP}_N \times \text{DP}_N}$. The recursive asymptotics ($U$-ASY) then equivalently require that the matrices $M_{\alpha,\beta}^{(N)}$ satisfy the initial condition $M^{(0)} = 1$ and the recursion

$$
\lim_{x_j,x_{j+1} \rightarrow 0} \frac{1}{(x_{j+1} - x_j)^{-2h}} \sum_{\beta \in \text{DP}_N} M_{\alpha,\beta}^{(N-1)} Z_\beta(x_1, \ldots, x_{2N})
$$

(4.3)

equals

$$
\begin{align*}
&= \left\{ \begin{array}{ll}
0 & \text{if } x_j \in \alpha \\
\sum_{\hat{\beta} \in \text{DP}_{N-1}} M_{\hat{\alpha},\hat{\beta}}^{(N-1)} Z_\hat{\beta}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \wedge^j \in \beta \\
-\frac{[\alpha(j)+1]}{[\alpha(j)+2]} \times \sum_{\hat{\beta} \in \text{DP}_{N-1}} M_{\hat{\alpha},\hat{\beta}}^{(N-1)} Z_\hat{\beta}(x_1, \ldots, x_{j-2}, x_{j+1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \forall j \in \alpha
\end{array} \right.
\end{align*}
$$

where we denote $\alpha \setminus \hat{\beta} = \hat{\alpha}$. Now recall the asymptotics properties (Z-ASY) of pure partition functions, and note that each $\hat{\beta} \in \text{DP}_{N-1}$ determines a unique $\beta \in \text{DP}_N$ with $\wedge^j \in \beta$ such that $\beta \setminus \wedge^j = \hat{\beta}$. The left-hand side of (4.3) then becomes

$$
\lim_{x_j,x_{j+1} \rightarrow 0} \frac{1}{(x_{j+1} - x_j)^{-2h}} \sum_{\hat{\beta} \in \text{DP}_{N-1}} M_{\alpha,\hat{\beta}}^{(N)} Z_{\hat{\beta}}(x_1, \ldots, x_{2N})
$$

(4.4)

$$
= \sum_{\hat{\beta} \in \text{DP}_{N-1}} M_{\alpha,\hat{\beta}}^{(N)} Z_{\hat{\beta}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}).
$$

Since $(Z_\beta)_{\beta \in \text{DP}_{N-1}}$ is a basis, the recursion (4.3) is equivalent to the following: for any $j \in \{1, \ldots, 2N - 1\}$ and any $\hat{\beta} \in \text{DP}_N$ such that $\wedge^j \in \beta$, we have

$$
M_{\alpha,\beta}^{(N)} = \left\{ \begin{array}{ll}
0 & \text{if } x_j \in \alpha \\
M_{\hat{\alpha},\hat{\beta}}^{(N-1)} & \text{if } \wedge^j \in \alpha \\
-\frac{[\alpha(j)+1]}{[\alpha(j)+2]} \times M_{\hat{\alpha},\hat{\beta}}^{(N-1)} & \text{if } \forall j \in \alpha,
\end{array} \right.
$$

where we denote by $\hat{\alpha} = \alpha \setminus \hat{\beta} \in \text{DP}_{N-1}$ and $\hat{\beta} = \beta \setminus \wedge^j \in \text{DP}_{N-1}$. Finally, Proposition 2 states that the recursion (4.4) holds if and only if the matrices $M^{(N)}$ are, for any $N$, the weighted incidence
matrix of the parenthesis reversal relation, $M = \mathcal{M}$. The rest follows since the matrix $\mathcal{M}$ is, for any $N$, invertible by Proposition 1.

5. DIRECT CONSTRUCTION OF CONFORMAL BLOCK FUNCTIONS BY A QUANTUM GROUP METHOD

In the preceding section, we expressed the conformal block functions $U_\alpha$ as linear combinations of multiple SLE pure partition functions $Z_\alpha$ and vice versa, generalizing Fomin’s formula [Fom01, KKP17]. These expressions can also be viewed as a construction of the conformal block functions, which however relies on an earlier construction of the multiple SLE pure partition functions and detailed information about the solution space [FK15a, FK15b, FK15c, KP16]. In this section, we provide an alternative, more direct construction of the conformal block functions $U_\alpha$ based on a quantum group method developed in [KP14]. In analogy with the core underlying idea of conformal blocks as discussed in Section 3, the present construction employs the Dyck path $\alpha$ as labeling a sequence of representations of the quantum group $U_q(\mathfrak{sl}_2)$. This quantum group construction furthermore sheds some light on why $q$-combinatorial formulas for conformal blocks appear in the first place.

Generalizations of this construction for conformal blocks in representations of the quantum group $U_q(\mathfrak{sl}_2)$ and in relation to CFT correlation functions are used and studied in [FP17a, FP17b, FP17c].

5.1. The quantum group and its representations. We begin by introducing the needed definitions and notation about the quantum group $U_q(\mathfrak{sl}_2)$ and its representations. Let $q = e^{i\pi/4}/\kappa$ as before. As a $\mathbb{C}$-algebra, $U_q(\mathfrak{sl}_2)$ is generated by the elements $K, K^{-1}, E, F$ subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2EK, \quadKF = q^{-2}FK,$$

(5.1)
$$EF - FE = \frac{1}{q - q^{-1}}(K - K^{-1}).$$

It has a Hopf algebra structure, with coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ given on its generators by

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(K) = K \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$ (5.2)

The coproduct is used to define the action of the Hopf algebra $U_q(\mathfrak{sl}_2)$ on tensor products of representations as follows. If the coproduct of an element $X \in U_q(\mathfrak{sl}_2)$ reads

$$\Delta(X) = \sum_i X'_i \otimes X''_i,$$

and if $V'$ and $V''$ are two representations, then $X$ acts on a tensor $v' \otimes v'' \in V' \otimes V''$ by the formula

$$X.(v' \otimes v'') = \sum_i X'_i.v' \otimes X''_i.v''.$$ (5.3)

Tensor product representations with $n$ tensor components are defined using the $(n - 1)$-fold coproduct

$$\Delta^{(n)} = (\Delta \otimes \text{id})^{\otimes(n-2)} \circ (\Delta \otimes \text{id})^{\otimes(n-3)} \circ \cdots \circ (\Delta \otimes \text{id}) \circ \Delta, \quad \Delta^{(n)} : U_q(\mathfrak{sl}_2) \to \left(U_q(\mathfrak{sl}_2)\right)^{\otimes n}.$$ (5.4)

By the coassociativity property $(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$ the tensor products of representations thus defined are associative, i.e., there is no need to specify the order in which the tensor products are formed. For each $d \in \mathbb{N}$, the quantum group $U_q(\mathfrak{sl}_2)$ has an irreducible representation $M_d$ of dimension $d$, obtained by suitably $q$-deforming the $d$-dimensional irreducible representation of the simple Lie algebra $\mathfrak{sl}_2$. Of primary importance to us is the two-dimensional irreducible representation $M_2$: it has a basis $\{e_0, e_1\}$ on which the generators act by

$$K.e_0 = q.e_0, \quad K.e_1 = q^{-1}.e_1, \quad E.e_0 = 0, \quad E.e_1 = e_0, \quad F.e_0 = e_1, \quad F.e_1 = 0.$$ (5.5)

A similar explicit definition of the $d$-dimensional irreducible $M_d$ can be found in, e.g., [KP14]. The tensor product of two two-dimensional irreducibles decomposes as a direct sum of subrepresentations,

$$M_2 \otimes M_2 \cong M_1 \oplus M_3,$$ (5.6)
where $M_1$ is a one-dimensional subrepresentation spanned by the vector
\begin{equation}
(5.3) \quad s = \frac{1}{q-q^{-1}} (e_1 \otimes e_0 - q e_0 \otimes e_1),
\end{equation}
and $M_3$ is a three-dimensional irreducible subrepresentation with basis
\[ t_+ = e_0 \otimes e_0, \quad t_0 = q^{-1} e_0 \otimes e_1 + e_1 \otimes e_0, \quad t_- = [2] e_1 \otimes e_1. \]
We denote the projection onto the one-dimensional subrepresentation $M_1 \subset M_2 \otimes M_2$
by
\[ \pi : M_2 \otimes M_2 \rightarrow M_2 \otimes M_2, \quad \pi(s) = s \quad \text{and} \quad \pi(t_+) = \pi(t_0) = \pi(t_-) = 0. \]
The one-dimensional representation $M_1$ is trivial in the sense that it is the neutral element for tensor
products of representations: for any representation $V$, we have $M_1 \otimes V \cong V \otimes M_1$, and $M_1$ can thus
simply be identified with the scalars $\mathbb{C}$. Using the identification $s \mapsto 1 \in \mathbb{C}$, we denote the projection
from $M_2 \otimes M_2$ to $M_1 \cong \mathbb{C}$ by
\begin{equation}
(5.4) \quad \hat{\pi} : M_2 \otimes M_2 \rightarrow \mathbb{C}, \quad \hat{\pi}(s) = 1 \quad \text{and} \quad \hat{\pi}(t_+) = \hat{\pi}(t_0) = \hat{\pi}(t_-) = 0.
\end{equation}
More generally, we have the $q$-Clebsch-Gordan formula
\begin{equation}
(5.5) \quad M_{d_2} \otimes M_{d_1} \cong M_{d_1+d_2-1} \oplus M_{d_1+d_2-3} \oplus \cdots \oplus M_{d_1-d_2+3} \oplus M_{d_1-d_2+1}
\end{equation}
for the direct sum decomposition of the tensor product of the irreducible representations of dimensions
$d_1$ and $d_2$, see e.g. [KP14, Lemma 2.4]. Repeated application of the decomposition (5.5) gives
\begin{equation}
(5.6) \quad M_2^{\otimes n} \cong \bigoplus_d m_d^{(n)} M_d,
\end{equation}
where the irreducible $M_d$ of dimension $d$ appears with multiplicity $m_d^{(n)}$, see e.g. [KP16, Lemma 2.2].
When $n = 2N$, the trivial subrepresentation
\begin{equation}
H_{2N}^{(0)} := \left\{ v \in M_2^{\otimes 2N} \left| E.v = 0, \ K.v = v \right. \right\}
\end{equation}
coincides with the sum of all copies of $M_1$, and has dimension equal to a Catalan number
\[ \dim(H_{2N}^{(0)}) = m_1^{(2N)} = C_N. \]
Finally, in the tensor product $M_2^{\otimes n}$, we denote by $\pi_j$ and $\hat{\pi}_j$ the projections $\pi$ and $\hat{\pi}$ acting on the $j$:th
and $(j+1)$:st tensor components counting from the right, i.e.,
\[ \pi_j := \text{id}^{\otimes(n-1-j)} \otimes \pi \otimes \text{id}^{\otimes(j-1)} : M_2^{\otimes n} \rightarrow M_2^{\otimes n}, \]
\[ \hat{\pi}_j := \text{id}^{\otimes(n-1-j)} \otimes \hat{\pi} \otimes \text{id}^{\otimes(j-1)} : M_2^{\otimes n} \rightarrow M_2^{\otimes(n-2)}. \]

5.2. Constructing conformal blocks. The purpose of this section is to give a construction of the
conformal block functions. Our construction relies on the method introduced in [KP14], called "spin
chain - Coulomb gas correspondence", which is allows to solve conformal field theory PDEs with given
boundary conditions by quantum group calculations. We use the correspondence in the following form,
which combines a special case of a more general theorem in [KP14] with additional information available
in that special case [KP16].

Proposition 4. Let $\kappa \in (0, 8) \setminus \mathbb{Q}$ and $q = e^{i \pi 4/\kappa}$. There exist explicit linear isomorphisms
\[ \mathcal{F} : H_{2N}^{(0)} \rightarrow S_N, \]
for all $N \in \mathbb{Z}_{\geq 0}$, with the following property. Let $v \in H_{2N}^{(0)}$, and $j \in \{1, 2, \ldots, 2N-1\}$, and denote
\[ \hat{\nu} = \hat{\pi}_j(v) \in H_{2N}^{(0)}(2N-1). \] Then, for any $\xi \in (x_{j-1}, x_{j+2})$, the function $\mathcal{F}[v] : \mathbb{R}_{2N} \rightarrow \mathbb{C}$ has the asymptotics
\[ \lim_{x_j, x_{j+1} \rightarrow \xi} \mathcal{F}[v](x_1, \ldots, x_{2N}) = B \times \mathcal{F}[\hat{\nu}](x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), \]
where $B = \frac{\Gamma(1-4/\kappa)^2}{\Gamma(2-8/\kappa)}$. 

Proof. Such a map $F$ was constructed in [KP14] and it follows from the explicit expressions of the functions $F[v]$ as Coulomb gas integrals that $F[v] \in \mathcal{S}_N$ for all $v \in H_2^{(0)}$. That $F$ is injective is proven in [KP16], and comparison of dimensions then shows that $F$ is a linear isomorphism. Finally, the asymptotics property follows immediately from [KP14, Theorem 4.17(ASY)].

With the help of the correspondence $F$ of Proposition 4, the task of constructing the conformal block functions is reduced to the task of constructing suitable vectors in the trivial subrepresentation $H_2^{(0)}$ of a tensor product $M_2^{\otimes 2N}$ of two-dimensional irreducible representations of the quantum group. This is achieved in the following proposition, which we prove in the end of this section.

**Proposition 5.** There exists a unique collection of vectors $\{u_\alpha\}_{\alpha \in \text{DP}}$, with $u_\alpha \in H_2^{(0)}$ when $\alpha \in \text{DP}_N$, such that $u_{\alpha(0)} = 1$ and the following projection properties hold:

$$
\hat{\pi}_j(u_\alpha) = \begin{cases} 
0 & \text{if } \alpha \not\in \alpha_j \\
\frac{1}{[\alpha(j+1)]} u_{\alpha \backslash \alpha_j} & \text{if } \alpha \supset \alpha_j \\
-\frac{[\alpha(j)]}{[\alpha(j)]+2} u_{\alpha \backslash \alpha_j} & \text{if } \alpha \not\supset \alpha_j \end{cases} \quad \text{for all } j \in \{1, \ldots, 2N-1\}.
$$

Moreover, for any $N \in \mathbb{Z}_{\geq 0}$, the collection $\{u_\alpha\}_{\alpha \in \text{DP}_N}$ is a basis of $H_2^{(0)}$.

Once Proposition 5 is established, the construction is immediate:

**Theorem B.** Let $\{u_\alpha\}_{\alpha \in \text{DP}}$ be the collection of vectors in Proposition 5, and let $F: H_2^{(0)} \to \mathcal{S}_N$ be the linear isomorphisms of Proposition 4. Then the functions

$$
U_\alpha := \frac{1}{\mathcal{B}_N} \times F[u_\alpha], \quad \text{for } \alpha \in \text{DP}_N,
$$

satisfy the defining properties (PDE), (COV), and (U-ASY) of conformal block functions.

Proof. This follows by combining Propositions 5 and 4. □

5.3. **Proof of Proposition 5.** The rest of this section constitutes the proof of Proposition 5, divided in four parts: uniqueness, construction, linear independence, and verification of projection properties. The uniqueness is routine by considering the corresponding homogeneous problem. The explicit construction of the vectors $u_\alpha$ is the essence of the proof. In the construction, each Dyck path $\alpha \in \text{DP}_N$ specifies a sequence of representations of the quantum group, and we recursively assemble the vector $u_\alpha$ proceeding along this sequence. Linear independence is transparent in the construction. Finally, for the projection properties we just have to inspect a number of cases explicitly.

5.3.1. **Uniqueness.** Uniqueness of the collection $\{u_\alpha\}_{\alpha \in \text{DP}}$ of vectors satisfying the projection properties (5.8) follows from arguments exploited in similar contexts in the articles [KP16, Pel16]. The crucial observation is the following lemma about the homogeneous problem.

**Lemma 6.** If a vector $v \in H_2^{(0)}$ satisfies the property $\hat{\pi}_j^{(1)}(v) = 0$ for all $j \in \{1, \ldots, 2N-1\}$, then $v = 0$.

Proof. See, e.g., [KP16, Corollary 2.5]. □

As a corollary, the solution space of the recursive projection properties (5.8) is one dimensional, with initial condition $u_{\alpha(0)} \in M_2^{\otimes 0} \cong \mathbb{C}$ determining the solution.

**Corollary 7.** Let $\{u_\alpha\}_{\alpha \in \text{DP}}$ and $\{u'_\alpha\}_{\alpha \in \text{DP}}$ be two collections of vectors $u_\alpha, u'_\alpha \in H_2^{(0)}$ satisfying the projection properties (5.8), and having same initial condition $u'_{\alpha(0)} = u_{\alpha(0)}$. Then we have

$$
u'_\alpha = u_\alpha \quad \text{for all } \alpha \in \text{DP}.$$
Proof. Let \( N \geq 1 \) and suppose the condition \( u'_\beta = u_\beta \) holds for all \( \beta \in \text{DP}_N - 1 \). Then, for any \( \alpha \in \text{DP}_N \), the difference \( v = u'_\alpha - u_\alpha \) satisfies \( \hat{\sigma}_j(v) = 0 \) for all \( j \in \{1, \ldots, 2N - 1\} \), so \( v = 0 \) by Lemma 6. The assertion follows by induction on \( N \).

5.3.2. Construction. We now construct the vectors \( u_\alpha \) of Proposition 5 and show that they lie in the correct subspaces \( \mathbb{H}^{(s)}_N \). In the intermediate steps of the construction, we encounter vectors in the highest weight vector spaces

\[
H^{(s)}_n = \left\{ v \in M_2^\otimes_n \mid E.v = 0, \ K.v = q^s v \right\}.
\]

These spaces consist of generators of the \( M_d \)-isotypic components in the tensor product (5.6) with \( d = s + 1 \): for any non-zero \( v \in H^{(s)}_n \), the collection \( (F^k_0)^s_{k=0} \) obtained from \( v \) by the action of the generator \( F \) spans a subrepresentation isomorphic to \( M_d \) in \( M_2^\otimes_n \). For each \( d \), the dimension of the linear space (5.9) equals \( m_d^{(u)} \). When \( s \neq 0 \), the spaces (5.9) themselves are not representations of \( U_q(\mathfrak{sl}_2) \).

For \( k = 1, 2, \ldots, 2N \), we will first construct vectors \( u^{(k)}_\alpha \in \mathbb{H}^{(s)}_k \), which can be thought of as being indexed by the first \( k \) steps of the walk \( \alpha \). From these vectors we will then construct the vectors \( u_\alpha \) — see Equation (5.12) below.

Let \( u^{(0)}_\alpha := 1 \in \mathbb{C} \cong M_2^\otimes^0 \). Define recursively \( u^{(k+1)}_\alpha \in M_2^\otimes(k+1) \) in terms of \( u^{(k)}_\alpha \in M_2^\otimes k \) by

\[
u^{(k+1)}_\alpha := \begin{cases} u^{(k)}_\alpha & \text{if } \alpha(k + 1) = \alpha(k) + 1 \\ \left( \frac{1}{q-q^{-1}} \left( e_0 \otimes u^{(k)}_\alpha - \frac{q^{\alpha(k)}}{[\alpha(k)]} e_0 \otimes F u^{(k)}_\alpha \right) \right) & \text{if } \alpha(k + 1) = \alpha(k) - 1. \end{cases}
\]

Lemma 8. For \( k = 0, 1, \ldots, 2N \), the vectors \( u^{(k)}_\alpha \in M_2^\otimes k \) satisfy \( u^{(k)}_\alpha \in \mathbb{H}^{(s)}_k \), that is, we have

\[
E.u^{(k)}_\alpha = 0 \quad \text{and} \quad K.u^{(k)}_\alpha = q^{\alpha(k)} u^{(k)}_\alpha.
\]

Proof. We prove the assertion by induction on \( k \) relying on a direct calculation. The base case \( k = 0 \) is clear. Assuming that the claim holds for \( u^{(k)}_\alpha \), we verify it for \( u^{(k+1)}_\alpha \). Recall that the actions of \( E \) and \( K \) on \( M_2 \otimes M_2^\otimes k \) are given by the coproduct (5.2). We use the identities

\[
E.e_0 = 0, \quad E.e_1 = e_0, \quad K.e_0 = q e_0, \quad K.e_1 = q^{-1} e_1, \quad E.u^{(k)}_\alpha = 0, \quad K.u^{(k)}_\alpha = q^{\alpha(k)} u^{(k)}_\alpha.
\]

If \( \alpha(k + 1) = \alpha(k) + 1 \), we have \( u^{(k+1)}_\alpha = e_0 \otimes u^{(k)}_\alpha \) and we easily calculate

\[
E.u^{(k+1)}_\alpha = E.e_0 \otimes K.u^{(k)}_\alpha + e_0 \otimes E.u^{(k)}_\alpha = 0, \\
K.u^{(k+1)}_\alpha = K.e_0 \otimes K.u^{(k)}_\alpha = q^{1+\alpha(k)} u^{(k+1)}_\alpha = q^{\alpha(k+1)} u^{(k+1)}_\alpha.
\]

If \( \alpha(k + 1) = \alpha(k) - 1 \), then we have \( u^{(k+1)}_\alpha = \frac{1}{q-q^{-1}} \left( e_0 \otimes u^{(k)}_\alpha - \frac{q^{\alpha(k)}}{[\alpha(k)]} e_0 \otimes F u^{(k)}_\alpha \right) \), and we similarly get

\[
E.u^{(k+1)}_\alpha = \frac{1}{q-q^{-1}} \left( E.e_1 \otimes K.u^{(k)}_\alpha - q^{\alpha(k)} E.e_0 \otimes K F u^{(k)}_\alpha \right) + e_0 \otimes E.u^{(k)}_\alpha - q^{\alpha(k)} e_0 \otimes E F u^{(k)}_\alpha = 0,
\]

where we also used the commutation relation \( EF - FE = \frac{1}{q-q^{-1}} \left( K - K^{-1} \right) \) from (5.1).
Finally, using the commutation relation $KF = q^{-2}FK$ from (5.1), we get (still with $\alpha(k+1) = \alpha(k)-1$)

$$K.u_{\alpha}^{(k+1)} = \frac{1}{q-q^{-1}} \left( K.e_1 \otimes K.u_{\alpha}^{(k)} - q^{\alpha(k)} \frac{K.e_0 \otimes K.F.u_{\alpha}^{(k)}}{[\alpha(k)]]} \right)$$

$$= \frac{1}{q-q^{-1}} \left( q^{-1+\alpha(k)} e_1 \otimes u_{\alpha}^{(k)} - q^{1-2+2\alpha(k)} \frac{e_0 \otimes F.u_{\alpha}^{(k)}}{[\alpha(k)]} \right)$$

$$= q^{\alpha(k)-1} u_{\alpha}^{(k+1)} = q^{\alpha(k+1)} u_{\alpha}^{(k+1)}.$$

This concludes the proof. □

The vectors $u_{\alpha}$ corresponding to the conformal block functions $U_{\alpha}$ are obtained by taking the last of the recursively defined vectors above, $u_{\alpha}^{(2N)}$, and normalizing it appropriately. Specifically, for $\alpha \in DP_N$, we set

$$u_{\alpha} := [2]^N c_{\alpha} \times u_{\alpha}^{(2N)}, \quad \text{where} \quad c_{\alpha} := \left( \prod_{\wedge \in \alpha} \left( \frac{1}{[\alpha(i)+1]} \right) \prod_{\vee \in \alpha} [\alpha(i)+1] \right).$$

We finish this subsection by noting that these vectors indeed belong to the trivial subrepresentation (5.7).

**Corollary 9.** We have $u_{\alpha} \in \mathcal{H}_{2N}^{(0)}$ for all $\alpha \in DP_N$.

**Proof.** This follows immediately from the properties (5.11) of $u_{\alpha}^{(k)}$ with $k = 2N$. □

5.3.3. **Linear independence.** We now quickly verify the linear independence of the vectors $u_{\alpha}$ constructed in (5.10) and (5.12). Since we have $\dim(\mathcal{H}_{2N}^{(0)}) = C_N = \#DP_N$, linear independence also implies that the collection $(u_{\alpha})_{\alpha \in DP_N}$ is a basis of $\mathcal{H}_{2N}^{(0)}$.

By the recursive construction (5.10), the first $k$ steps of $\alpha$ determine a vector $u_{\alpha}^{(k)} \in M_2^{\otimes k}$. Inductively on $k$, it is clear that all different initial segments of $k$ steps define linearly independent vectors. The linear independence of $(u_{\alpha})_{\alpha \in DP_N}$ follows from the case $k = 2N$.

5.3.4. **Projection properties.** To prove the projection properties (5.8) for the vectors $u_{\alpha}$ constructed in (5.10) and (5.12), we use a recursion property of the normalization coefficients $c_{\alpha}$.

**Lemma 10.** The coefficients $c_{\alpha}$ satisfy the following recursion: for any $j$, we have

$$c_{\alpha} = \begin{cases} \frac{[\alpha(j)]}{[\alpha(j)+1]} \times c_{\alpha \setminus \wedge} & \text{if } \wedge \in \alpha \\ \frac{[\alpha(j)+1]}{[\alpha(j)+2]} \times c_{\alpha \setminus \vee} & \text{if } \vee \in \alpha. \end{cases}$$

**Proof.** Observe that the coefficients in (5.12) can be written in the form

$$\prod_{i=1}^{2N} \sqrt{\min\{\alpha(i-1),\alpha(i)\} + 1} = \prod_{\wedge \in \alpha} \frac{1}{[\alpha(i)+1]} \prod_{\vee \in \alpha} [\alpha(i)+1] = c_{\alpha}. \tag{5.13}$$

The expression on the left clearly satisfies the recursion (5.13). □

We also make use of the following explicit formulas for the projection $\hat{\pi}$ defined in Equation (5.4).

**Lemma 11.** With $s \in M_1$ defined in (5.3), we have $\pi(v) = \hat{\pi}(v)s$ for any $v \in M_2 \otimes M_2$, and

$$\hat{\pi}(e_0 \otimes e_0) = 0, \quad \hat{\pi}(e_1 \otimes e_0) = 0,$$

$$\hat{\pi}(e_0 \otimes e_1) = \frac{q^{-1} - q}{[2]}, \quad \hat{\pi}(e_1 \otimes e_1) = \frac{1 - q^{-2}}{[2]}.$$

**Proof.** See, e.g., [KP16, Lemma 2.3]. □
We are now ready to prove the projection properties (5.8) of the vectors $u_\alpha$.

**Proposition 12.** The vectors $(u_\alpha)_{\alpha \in \mathrm{DP}}$, defined in (5.12), satisfy the projection properties (5.8).

**Proof.** Fix $j \in \{1, \ldots, 2N-1\}$. As the projection $\pi_j$ acts locally on the $j$:th and $(j+1)$:st tensor components, the value of $\hat{\pi}_j(u_\alpha)$ can be calculated using the explicit construction (5.10) and the recursion (5.13) of Lemma 10 for the normalization constants appearing in the definition (5.12) of $u_\alpha$. We treat separately each possible local shape of a Dyck path $\alpha$ at $j$, i.e., the cases depicted in Figure 2.4.

Suppose first that $\alpha$ contains a slope at $j$, i.e., $\times_j \in \alpha$. We need to show that in this case, we have $\hat{\pi}_j(u_\alpha) = 0$, or, equivalently, that $\hat{\pi}_j(u_\alpha^{(j+1)}) = 0$. Depending whether the slope is an up-slope or a down-slope, we study the two cases in (5.10).

In the easiest case of an up-slope, that is, when we have $\alpha(j) = \alpha(j-1) + 1$ and $\alpha(j+1) = \alpha(j) + 1$, the tensor components $j$ and $j+1$ in $u_\alpha^{(j+1)}$ (counting from the right) are proportional to $e_0 \otimes e_0$, and $\hat{\pi}_j$ thus annihilates the vector $u_\alpha^{(j+1)}$ by Lemma 11(a). Equations (5.10) and (5.12) then show that we also have $\hat{\pi}_j(u_\alpha) = 0$, as asserted in (5.8).

In the case of a down-slope, that is, when we have $\alpha(j) = \alpha(j-1) - 1$ and $\alpha(j+1) = \alpha(j) - 1$, the tensor components $j$ and $j+1$ in $u_\alpha^{(j+1)}$ have several terms. To perform the calculations, it is convenient to first write down the action of $F$ on $u_\alpha^{(j)}$. The action is given by the coproduct (5.2) as follows:

\[
(q - q^{-1}) F.u_\alpha^{(j)} = F. \left( e_1 \otimes u_\alpha^{(j-1)} - \frac{q^{\alpha(j-1)}}{[\alpha(j-1)]} e_0 \otimes F. u_\alpha^{(j-1)} \right)
\]

\[
= F.e_1 \otimes 1.u_\alpha^{(j-1)} + K^{-1}.e_1 \otimes F. u_\alpha^{(j-1)} - q^{\alpha(j-1)} \frac{F.e_0 \otimes F. u_\alpha^{(j-1)} - K^{-1}.e_0 \otimes F^2. u_\alpha^{(j-1)}}{[\alpha(j-1)]}
\]

\[
= q e_1 \otimes F. u_\alpha^{(j-1)} - \frac{q^{\alpha(j-1)}}{[\alpha(j-1)]} \left( e_1 \otimes F. u_\alpha^{(j-1)} - q^{-1} e_0 \otimes F^2. u_\alpha^{(j-1)} \right)
\]

\[
= \left( q - \frac{q^{\alpha(j-1)}}{[\alpha(j-1)]} \right) e_1 \otimes F. u_\alpha^{(j-1)} - \frac{q^{\alpha(j-1)-1}}{[\alpha(j-1)]} e_0 \otimes F^2. u_\alpha^{(j-1)},
\]

where we used the identities $F.e_1 = 0$, $F.e_0 = e_1$, $K^{-1}.e_1 = q e_1$, and $K^{-1}.e_0 = q^{-1} e_0$. The vector $u_\alpha^{(j+1)}$ now reads

\[
\begin{align*}
&u_\alpha^{(j+1)} \propto e_1 \otimes u_\alpha^{(j)} - \frac{q^{\alpha(j)}}{[\alpha(j)]} e_0 \otimes F. u_\alpha^{(j)} \\
&\quad \propto e_1 \otimes \left( e_1 \otimes u_\alpha^{(j-1)} - \frac{q^{\alpha(j-1)}}{[\alpha(j-1)]} e_0 \otimes F. u_\alpha^{(j-1)} \right) \\
&\quad - \frac{q^{\alpha(j)}}{[\alpha(j)]} e_0 \otimes \left( q - \frac{q^{\alpha(j-1)}}{[\alpha(j-1)]} \right) e_1 \otimes F. u_\alpha^{(j-1)} - \frac{q^{\alpha(j-1)-1}}{[\alpha(j-1)]} e_0 \otimes F^2. u_\alpha^{(j-1)}.
\end{align*}
\]
Using Lemma 11(a), the down-step $\alpha(j) = \alpha(j-1) - 1$, and the geometric sum expansion of the $q$-integers $[n] = q^{n-1} + q^{n-3} + \ldots + q^{3-n} + q^{1-n}$, we verify that

$$
\hat{\pi}_j(u_{\alpha}^{(j+1)}) \propto \left(-q^{\alpha(j-1)} \hat{\pi}(e_1 \otimes e_0) - \frac{q^{\alpha(j)+1} - q^{\alpha(j)+1}}{[\alpha(j)]} \hat{\pi}(e_0 \otimes e_1)\right) \otimes F.u_{\alpha}^{(j-1)}
$$

$$
= \left(-q^{\alpha(j)+1} \frac{1 - q^{-2}}{[2]} - \frac{q^{\alpha(j)+1} - q^{\alpha(j)+1}}{[\alpha(j)]} \frac{q^{-1} - q}{[2]}\right) \otimes F.u_{\alpha}^{(j-1)}
$$

$$
= \frac{q^{\alpha(j)+1}(q - q^{-1})}{[2] \lfloor \alpha(j) \rfloor} \times \left(-q^{-1} \lfloor \alpha(j) \rfloor + \lfloor \alpha(j) \rfloor + 1 - q^{\alpha(j)}\right) \otimes F.u_{\alpha}^{(j-1)}
$$

$$
= 0.
$$

It thus follows by Equations (5.10) and (5.12) that the asserted property $\hat{\pi}_j(u_{\alpha}) = 0$ holds also with $\alpha$ having an down-slope at $j$.

Suppose then that $\alpha$ contains an up-wedge at $j$, i.e., $\land^j \in \alpha$. We need to show that in this case, we have $\hat{\pi}_j(u_{\alpha}) = u_{\alpha \land^j}$. Now $\alpha(j) = \alpha(j-1) + 1$ and $\alpha(j+1) = \alpha(j) - 1$ and the vector $u_{\alpha}^{(j+1)}$ reads

$$
u_{\alpha}^{(j+1)} = \frac{1}{q - q^{-1}} \left(e_1 \otimes (e_0 \otimes u_{\alpha}^{(j-1)}) - \frac{q^{\alpha(j)}}{[\alpha(j)]} e_0 \otimes F.(e_0 \otimes u_{\alpha}^{(j-1)})\right)
$$

$$
= \frac{1}{q - q^{-1}} \left(e_1 \otimes (e_0 \otimes u_{\alpha}^{(j-1)}) - \frac{q^{\alpha(j)}}{[\alpha(j)]} e_0 \otimes (F.e_0 \otimes 1.u_{\alpha}^{(j-1)} + K^{-1} e_0 \otimes F.u_{\alpha}^{(j-1)})\right)
$$

$$
= \frac{1}{q - q^{-1}} \left(e_1 \otimes (e_0 \otimes u_{\alpha}^{(j-1)}) - \frac{q^{\alpha(j)}}{[\alpha(j)]} e_0 \otimes (e_1 \otimes u_{\alpha}^{(j-1)} + q^{-1} e_0 \otimes F.u_{\alpha}^{(j-1)})\right).
$$

Applying the projection $\hat{\pi}_j$ on both sides and using Lemma 11(a), we obtain

$$
\hat{\pi}_j(u_{\alpha}^{(j+1)}) = \frac{1}{q - q^{-1}} \left(\hat{\pi}(e_1 \otimes e_0) - \frac{q^{\alpha(j)}}{[\alpha(j)]} \hat{\pi}(e_0 \otimes e_1)\right) \otimes u_{\alpha}^{(j-1)}
$$

$$
= \frac{1}{q - q^{-1}} \left(1 - q^{-2} \frac{[2]}{[\alpha(j)]} - \frac{q^{\alpha(j)} q^{-1} - q}{[\alpha(j)]} \frac{[2]}{[2]}\right) \times u_{\alpha}^{(j-1)}
$$

$$
= \frac{1}{[2] \lfloor \alpha(j) \rfloor} \times (q^{-1} \lfloor \alpha(j) \rfloor + q^{\alpha(j)}) \times u_{\alpha}^{(j-1)}.
$$

Using again the geometric sum expansion of the $q$-integers, we simplify the multiplicative factor by $q^{-1} \lfloor \alpha(j) \rfloor + q^{\alpha(j)} = \lfloor \alpha(j) \rfloor + 1$, which yields

$$
\hat{\pi}_j(u_{\alpha}^{(j+1)}) = \frac{\lfloor \alpha(j) \rfloor + 1}{[2] \lfloor \alpha(j) \rfloor} \times u_{\alpha}^{(j-1)}.
$$

By Equations (5.10) and (5.12) and the recursion (5.13), the asserted property (5.8) follows:

$$
\hat{\pi}_j(u_{\alpha}) = [2]^N c_{\alpha} \times \hat{\pi}_j(u_{\alpha}^{(2N)})
$$

$$
= [2]^N \frac{\lfloor \alpha(j) \rfloor}{\lfloor \alpha(j) + 1 \rfloor} \times c_{\alpha \land^j} \times \frac{\lfloor \alpha(j) + 1 \rfloor}{[2] \lfloor \alpha(j) \rfloor} \times u_{\alpha}^{(2N-2)}
$$

$$
= [2]^{N-1} c_{\alpha \land^j} \times u_{\alpha}^{(2N-2)}
$$

$$
= u_{\alpha \land^j}.
$$
Finally, suppose that $\alpha$ contains a down-wedge at $j$, i.e., $\forall j \in \alpha$. We need to show that in this case, we have $\hat{\pi}_j(u_\alpha) = \frac{\alpha(j) + 1}{\alpha(j) + 2} \times u_{\alpha \setminus \{j\}}$. Now $\alpha(j) = \alpha(j - 1) - 1$ and $\alpha(j + 1) = \alpha(j) + 1$ and $u_{\alpha}^{(j+1)}$ reads

$$u_{\alpha}^{(j+1)} = \frac{1}{q - q^{-1}} \left( e_0 \otimes (e_1 \otimes u_{\alpha}^{(j-1)}) - \frac{q^{\alpha(j-1)}}{[\alpha(j-1)]_q} e_0 \otimes (e_0 \otimes F u_{\alpha}^{(j-1)}) \right).$$

Applying the projection $\hat{\pi}_j$ on both sides and using Lemma 11(a), we obtain

$$\hat{\pi}_j(u_{\alpha}^{(j+1)}) = \frac{1}{q - q^{-1}} (\hat{\pi}(e_0 \otimes e_1)) \otimes u_{\alpha}^{(j-1)} = \frac{q^{-1} - q}{q - q^{-1}} [2] \times u_{\alpha}^{(j-1)} = -\frac{1}{[2]} \times u_{\alpha}^{(j-1)},$$

and again, by Equations (5.10) and (5.12) and the recursion (5.13), the asserted property (5.8) follows:

$$\hat{\pi}_j(u_\alpha) = [2]^N c_\alpha \times \hat{\pi}_j(u_{\alpha}^{(2N)})$$

$$= [2]^N \frac{\alpha(j) + 1}{\alpha(j) + 2} \times c_{\alpha \setminus \{j\}} - \frac{1}{[2]} \times u_{\alpha \setminus \{j\}}^{(2N-2)}$$

$$= -\frac{\alpha(j) + 1}{\alpha(j) + 2} [2]^{N-1} c_{\alpha \setminus \{j\}} \times u_{\alpha \setminus \{j\}}^{(2N-2)}$$

$$= -\frac{\alpha(j) + 1}{\alpha(j) + 2} \times u_{\alpha \setminus \{j\}}.$$

This concludes the proof. \qed

5.3.5. **Proof of Proposition 5.** The vectors $(u_\alpha)_{\alpha \in \mathcal{DP}}$ constructed in (5.10) and (5.12) are in the space $H_{2N}^{(0)}$ by Corollary 9 and satisfy the projection properties by Proposition 12. Such a collection is unique by Corollary 7. In Section 5.3.3 we verified that $(u_\alpha)_{\alpha \in \mathcal{DP}_N}$ forms a basis of $H_{2N}^{(0)}$. \qed

**References**


