



Master's Thesis  
Mathematics

**Quantum cohomology of partial flag varieties**

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**Title:** Quantum cohomology of partial flag varieties

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**Month and year:** March 2026

**Page count:** 49

**Abstract:** One natural extension of intersection theory on a variety is intersection theory on various moduli spaces of maps into that variety. This idea can be formalized in terms of quantum cohomology, which deforms cohomology by introducing a  $q$ -parameter keeping track of Gromov–Witten invariants. In general, computing quantum cohomology can be highly non-trivial, but on partial flag varieties  $G/P$  nice descriptions exist in terms of the root datum associated to the semisimple algebraic group  $G$ . In this thesis, we shall explore these themes.

We begin by computing the classical cohomology of  $G/P$ . We define the Schubert basis, identify divisor classes with fundamental weights, and identify curve classes with coroots. Using this, we prove a Chevalley formula expressing multiplication by divisors in terms of the Schubert basis. We then study the moduli spaces of stable curves and stable maps into  $G/P$ . Using these, we define Gromov–Witten invariants on  $G/P$ , and show various formal and enumerative properties of these invariants.

Finally, we define the quantum cohomology ring of  $G/P$  and compute interesting examples. We prove a quantum extension of the Chevalley formula, originally demonstrated by Fulton and Woodward, and discuss Kontsevich’s and Mihalcea’s reconstruction theorems. To conclude, we present a brief exposition of a connection to the affine Grassmannian, first claimed by Peterson and later proven by Shimozono and Lam.

**Keywords:** Quantum cohomology, Gromov–Witten invariants, partial flag varieties

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# Chapter 1

## Introduction

In string theory, one approach to formalizing the notion of a path integral is to consider a space of all holomorphic maps  $\mathcal{C} \rightarrow X$  from a curve  $\mathcal{C}$  to a Kähler manifold  $X$ , and to compute integrals over this space [25]. However, this space often ends up being non-compact and singular, making integration difficult. A working formulation in terms of pseudoholomorphic maps to  $X$  in symplectic geometry was given by Gromov [24], where these integrals came to be known later as *Gromov–Witten (GW) invariants*. An algebro-geometric formulation in terms of stable maps was given by Kontsevich–Manin [38], along with the work of Behrend, Behrend–Fantechi, and Li–Tian [7, 9, 41].

The significance of these ideas in enumerative geometry became quickly understood. Candelas et al. [14] predicted the number of rational curves on generic quintic hypersurfaces in  $\mathbb{P}^4$  using mirror symmetry, and Givental [21] later confirmed these using equivariant GW-invariants. Letting  $N_d$  be the number of degree  $d$  curves in  $\mathbb{P}^2$  passing through  $3d - 1$  generic points, Kontsevich [38] exhibited a recursion

$$N_d = \sum_{d=c+e} N_c N_e \left( c^2 e^2 \binom{3d-4}{3c-2} + c^3 e \binom{3d-4}{3c-1} \right).$$

One way to prove this, and to package GW-invariants in an orderly way, is to  $q$ -deform the cohomology ring  $H^*(\mathbb{P}^2, \mathbb{Q})$  to something where structure constants for multiplication are given by all genus 0 GW-invariants, called *large quantum cohomology* [37]. A more manageable analogue is given by only considering 3-point genus 0 invariants, called *small quantum cohomology* or *quantum cohomology*.

Since then, quantum cohomology has also started living a life of its own in geometric representation theory. Building on Schubert calculus, a natural interest has been in understanding quantum cohomology for the partial flag variety  $G/P$ , where  $G$  is a connected semisimple linear algebraic group over the complex numbers and  $P \subset G$  is a parabolic subgroup. In [20, 31], Givental–Kim proved that the  $G$ -equivariant quantum cohomology of  $G/B$  for  $B \subset G$  a Borel subgroup is determined by the Toda lattice system for the Langlands dual pair  $G^\vee \supset B^\vee$ . In [19], Fulton–Woodward demonstrated a *quantum Chevalley formula*

$$\sigma_{s_\beta} * \sigma_w = \sum_{\gamma} \langle \omega_\beta, \gamma \rangle \cdot \sigma_{ws_\gamma} + \sum_{\gamma} q^\gamma \langle \omega_\beta, \gamma \rangle \cdot \sigma_{ws_\gamma} \quad (1.0.1)$$

expressing quantum multiplication by divisors for  $G/P$ . In [46], Mihalcea proved that an equivariant analogue of this formula is sufficient to determine the whole quantum

cohomology ring.

This quantum Chevalley formula can also be understood as a limit for the same story for  $T^*(G/P)$ , where the quantum cohomology is related to the mirror symmetry for symplectic resolutions [30]. In particular, Braverman–Maulik–Okounkov [12] determined  $G \times \mathbb{C}^*$ -equivariant quantum multiplication by divisors in  $T^*(G/B)$  in terms of a graded affine Hecke algebra integral in the Langlands program [15]. In his thesis, Su [55] extended these ideas to more general parabolics in terms of a *stable basis*. This builds on the earlier work of Maulik–Okounkov [45], where the ideas for  $T^*(G/B)$  are extended to Nakajima quiver varieties with Yangians in the role of the Hecke algebra.

In this thesis, we explore the quantum cohomology of partial flag varieties. In particular, we prove the  $T$ -equivariant analogue of the quantum Chevalley formula (1.0.1), and note some applications and connections. This should be viewed as a generalization of Schubert calculus [35] to both the quantum setting and a more general representation-theoretic setting.

In Chapter 2, we discuss the classical  $T$ -equivariant cohomology  $H_T^*(X, \mathbb{Z})$  for  $X = G/P$ . We begin by recalling all the necessary combinatorial data and the Schubert basis for both the cohomology and the homology. We then interpret the divisor classes in terms of  $T$ -equivariant line bundles on  $X$  corresponding to characters  $Q_L$  of a Levi  $L \subset P$ , and compute pairings with curve classes using the Atiyah–Bott localization formula. Using this, we identify the curve classes in terms of cocharacters  $Q^\vee/Q_P^\vee$ , giving the classes of  $T$ -invariant curves of  $X$  precisely as the positive coroots  $\Phi_+^\vee \setminus \Phi_{P+}^\vee$  of  $L$ . We finally use this to understand multiplication by divisors in  $H_T^*(X, \mathbb{Z})$ .

Furthermore, we introduce stable pointed curves and the moduli stack of these objects in Chapter 3. We then characterize these spaces, namely, compute their dimension, and show that they are smooth. We extend these ideas to the moduli stack of stable maps from pointed nodal curves onto  $G/P$  and prove smoothness in the case of genus 0. We define GW-invariants as integrals over the *virtual fundamental class* of these spaces, and show that these invariants are enumerative in genus 0 and other basic properties. We finish with an appendix giving the construction of a virtual fundamental class using  $P$ -torsors, making our treatment of GW-invariants self-contained.

Finally, in Chapter 4, we define the  $T$ -equivariant quantum cohomology ring  $qH_T^*(G/P, \mathbb{Z})$ , and compute interesting examples. We then prove the  $T$ -equivariant analogue of (1.0.1) using the ideas built in Chapter 2 and Chapter 3. In addition, we state Kontsevich’s theorem of how this compute all  $n$ -point genus 0 GW-invariants of  $G/B$ , and Mihalcea’s theorem of how this computes all 3-point genus 0 GW-invariants of  $G/P$ . To finish, we briefly discuss Shimozono–Lam’s theorem on how this relates to  $T$ -equivariant homology  $H_*^T(\mathcal{G}r_G, \mathbb{Z})$  of the affine Grassmannian  $\mathcal{G}r_G = \mathbf{L}G/\mathbf{L}^+G$  of  $G$ .

# Chapter 2

## Partial flag varieties

### 2.1 Notation

We assume familiarity with the structure theory of reductive groups and Lie algebras [53, 33]. We consider the following data:

- $G$  a semisimple, simply connected algebraic group over  $\mathbb{C}$
- $T$  a maximal torus in  $G$
- $B$  a fixed Borel subgroup in  $G$ , with unipotent radical  $U$
- $T$  the maximal torus contained in  $B$ , so that  $B = T \ltimes U$
- $B^-$  the Borel subgroup opposite to  $B$ , with unipotent radical  $U^-$
- $(Q, \Phi, \Delta, Q^\vee, \Phi^\vee, \Delta^\vee)$  the based root datum of  $G \supset B \supset T$
- $\langle -, - \rangle : Q \times Q^\vee \rightarrow \mathbb{Z}$  the pairing, and  $(-)^\vee : \Phi \rightarrow \Phi^\vee$  the dual
- $\omega_\alpha \in Q$  the fundamental weights for  $\alpha \in \Delta$
- $2\rho \in Q$  the sum of positive roots
- $U_\alpha$  the root subgroup for  $\alpha \in \Phi$
- $W = N_G(T)/T \simeq W(Q, \Phi) \simeq W(Q^\vee, \Phi^\vee)$  with reflections  $s_\alpha$  for  $\alpha \in \Phi$
- $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$  the length function
- $w, v \in W$  are called adjacent, denoted  $w \rightarrow v$ , if  $v = ws_\alpha$  for some  $\alpha \in \Phi$
- $w < v$  if there is a chain  $w = w_0 \rightarrow \dots \rightarrow w_k = v$

In addition, we will consider parabolic subgroups of  $G$  containing  $B$  in order to work in full generality. Classical theory shows that this parabolic is determined by a choice of simple roots in  $\Delta$ , and this ends up generating a parabolic root system of its own.

- $\Delta_P \subseteq \Delta$  a choice of simple roots, with  $\Delta_P^\vee \subset \Delta^\vee$  its dual under  $(-)^\vee$
- $\Phi_P = \Phi \cap \mathbb{Z} \cdot \Delta_P$ ,  $\Phi_P^\vee = \Phi^\vee \cap \mathbb{Z} \cdot \Delta_P^\vee$ , and  $Q_P^\vee = \mathbb{Z} \cdot \Delta_P$ .

- $P$  the parabolic subgroup containing  $B$  associated to  $\Delta_P$ , with unipotent radical  $U_P$
- $L$  the Levi factor of  $P$  containing  $T$ , so that  $P = L \ltimes U_P$
- $Q_L \subset Q$  the characters of  $L$ , with inclusion given by  $T \subset L$ . Equivalently characters of  $Z(L)^\circ \subset T$
- $2\rho_P$  the sum of positive roots in  $\Phi_P$ , and  $2\rho_L = 2\rho - 2\rho_P$  the sum of positive roots in  $\Phi \setminus \Phi_P$ .
- $W_P \subset W$  the subgroup generated by simple reflections  $s_\alpha$  with  $\alpha \in \Delta_P$
- $W^P \subset W$  the set of representatives of  $W/W_P$  of minimal length, with a "product"  $W^P \times W^P \rightarrow W^P$  given by taking the minimal representative of  $[wv] \in W/W_P$  for  $w, v \in W^P$ .
- $w, v \in W^P$  are called adjacent if  $v = ws_\gamma$  for some  $\gamma \in \Phi \setminus \Phi_P$

Since  $G$  is simply connected,  $Q^\vee$  is generated by  $\Delta^\vee$ , and  $Q$  is generated by the fundamental weights  $\omega_\alpha$  for  $\alpha \in \Delta$ . A subindex  $(-)_+$  always denotes being dominant or positive, that is, lying in the  $\mathbb{Z}_{\geq 0}$  span of either of these, and  $(-)_-$  the opposite. In addition, for  $H$  a group, the decapitalized fractura  $\mathfrak{h}$  always denotes its Lie algebra.

*Remark 2.1.1.* Both  $Q$  and  $Q^\vee$  are defined as (co)characters of the groups  $G \supset T$ , but they are equivalently (co)characters of  $\mathfrak{g} \supset \mathfrak{t}$ . This motivates us to use additive notation instead of multiplicative, as is standard with root systems. For example, if  $\alpha, \beta \in Q$  are characters of  $T$ , we will write  $\alpha + \beta \in Q$  to actually denote the character  $\alpha \cdot \beta$ .

*Remark 2.1.2.* Since  $\rho$  can also be understood as the sum of fundamental weights  $\omega_\alpha$  over  $\alpha \in \Delta$ , it follows that  $\rho \in Q$ . However, it is not necessarily true that  $\rho_P \in Q_P$  or  $\rho_L \in Q$  as can be observed already for  $G = \mathrm{SL}_3$ .

## 2.2 Schubert varieties

**Theorem 2.2.1.** *[53, Lemma 6.2.2] The quotient group scheme  $X = G/P$  is a smooth projective variety. We call this the partial flag variety associated to  $G \supset P$ .*

Throughout this thesis,  $X$  will always denote the partial flag variety  $G/P$  except if stated otherwise. We will simultaneously treat  $X$  as an algebraic variety as well as the topological space of its complex points  $X(\mathbb{C})$ .

**Definition 2.2.2.** Let  $w \in W^P$  and  $\dot{w} \in N_G(T)$  an arbitrary representative. Define

(i)  $x_w$  as the point  $\dot{w}P/P \in X$

(ii) The  $B^\pm$ -orbits

$$X_w^\circ = B^- \dot{w}P/P, \quad X_w^{-\circ} = B \dot{w}P/P$$

as the *Schubert cells* in  $X$

(iii) The closures of the Schubert cells

$$X_w = \overline{X_w^\circ} \subseteq X, \quad X_w^- = \overline{X_w^{-\circ}} \subseteq X$$

as the respective *Schubert varieties*.

These points and Schubert cells have a rich description in terms of the  $T$ -action on  $X$ . Namely, the  $x_w$  are the unique  $T$ -fixed points, with an equivalence of  $T$ -modules

$$T_{x_w} X = \mathfrak{g}/\mathrm{Ad}_w \mathfrak{p} = \mathrm{Ad}_w \mathfrak{u}_P = \bigoplus_{\alpha \in w(\Phi_- \setminus \Phi_{P^-})} \mathfrak{g}_\alpha \quad (2.2.1)$$

Furthermore, the Schubert cells  $X_w^\circ$  are the repelling sets to these fixed points, that is,

$$X_w^\circ = \left\{ x \in X \mid \lim_{t \rightarrow \infty} \lambda(t) \cdot x = x_w \right\} \quad (2.2.2)$$

for any antidominant cocharacter  $\lambda \in Q^\vee$ , as shown in [15, Section 3.1]. Thus (i) and (ii) of the following theorem can be viewed as a consequence of a theorem of Białyński-Birula [10].

**Theorem 2.2.3** (Bruhat decomposition). [53, Section 8]

(i) The  $X_w^\circ$  for  $w \in W^P$  are the  $B^-$ -orbits of  $X$ . In particular,

$$X = \bigsqcup_{w \in W^P} X_w^\circ$$

(ii) There is an isomorphism of varieties  $X_w^\circ \simeq \mathbb{A}^{\ell(w)}$

(iii)  $X_w \supset X_v$  if and only if  $w < v$

Since  $X$  is smooth, and has a cell decomposition of even dimensions given by the Schubert varieties, the following follows directly:

**Corollary 2.2.4.**  $H^k(X, \mathbb{Z})$  has  $\mathbb{Z}$ -basis given by the classes  $[X_w]^*$  for  $k = 2\ell(w)$ , where  $[Z]^*$  denotes the Poincaré dual of  $[Z]$  where  $Z \subset X$  is a subvariety.

*Remark 2.2.5.* This also gives that the  $[X_w]^*$  form a basis similarly for the Chow ring  $A_*(X)$  by [56, Theorem 3]. Everything as follows can be done similarly with (operational) Chow classes.

Since this is an even cell decomposition of  $X$ , we find that the subvarieties  $X_w$  generate the cohomology ring of  $X$ . Namely, we define

$$\begin{aligned} \tau_w &= [X_w] \in H_{2 \dim X - 2\ell(w)}(X, \mathbb{Z}) & \tau_w^- &= [X_w^-] \in H_{2\ell(w)}(X, \mathbb{Z}) \\ \sigma_w &= [X_w]^* \in H^{2\ell(w)}(X, \mathbb{Z}) & \sigma_w^- &= [X_w^-]^* \in H^{2 \dim X - 2\ell(w)}(X, \mathbb{Z}) \end{aligned}$$

It is easy to see that  $\sigma_w^- = \sigma_{w_0 w}$  and  $\int \sigma_w \cdot \sigma_v^- = \delta_{vw}$  for  $w, v$  of the same length.

*Remark 2.2.6.* Consider the graded  $\mathbb{Z}$ -algebra  $H_T^*(*, \mathbb{Z})$ , which is equivalently the symmetric algebra  $\mathbb{Z}[Q]$ . Note that since  $X$  and  $BT = (\mathbb{C}\mathbb{P}^\infty)^{\dim T}$  are of even cohomological degrees, the Serre spectral sequence for  $X \rightarrow X \times^T ET \rightarrow BT$  degenerates at page  $E_2$ . Thus the equivariant cohomology  $H_T^*(X, \mathbb{Z})$  decomposes (non-canonically) as  $H^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[Q]$ . It is thus a better idea to study the  $T$ -equivariant cohomology of  $X$ , and we shall use the same notation  $\sigma_w^\pm, \tau_w^\pm$  for the  $T$ -equivariant counterparts.

**Definition 2.2.7.** Let  $H_2(X, \mathbb{Z})_+ \subset H_2(X, \mathbb{Z})$  be the the monoid of positively oriented chains.

*Example 2.1.* Let  $G = \mathrm{SL}_n$  and  $P = B$  consist of upper-triangular matrices. Now  $G$  acts transitively on the set of complete flags

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = \mathbb{C}^n$$

with  $\dim V_i = i$ , and the standard flag  $\mathbf{e}_*$  with  $\mathbf{e}_i = \langle e_1, \dots, e_i \rangle$  has stabilizer  $B$ . Thus  $X = G/B$  parametrizes all complete flags of length  $n$ . In particular,  $W = S_n$  and the Schubert varieties are given by

$$X_w = \{V_* \in X \mid \dim(\mathbf{e}_i^- \cap V_j) \geq r_w(i, j) \forall i, j\}$$

where  $\mathbf{e}_i^- = \langle e_i, \dots, e_n \rangle$  and  $r_w(i, j)$  is the count of numbers  $1 \leq h \leq i$  with  $w(h) \leq j$ .

*Example 2.2.* Let  $G = \mathrm{SL}_n$  and  $P$  consists of matrices with the bottom-left  $k \times (n - k)$  rectangle vanishing and determinant 1. Now  $G$  acts transitively on the set of  $k$ -planes in  $\mathbb{C}^n$  with  $\mathbf{e}_k$  having stabilizer  $P$ , so  $X = G/P$  is the Grassmannian  $\mathrm{Gr}(n, k)$ . Since  $W = S_n$  and  $W_P = S_k \times S_{n-k}$ , we find that  $W^P$  corresponds to  $(k, n - k)$ -shuffles. In particular,  $(k, n - k)$ -shuffles correspond to partitions  $\lambda$  whose Young diagrams fit into a size  $k \times (n - k)$  rectangle by sending  $w$  to  $(\lambda_1, \dots, \lambda_k)$  with  $\lambda_i = w(i) - i$ . With this indexing,

$$X_\lambda = \{V \in \mathrm{Gr}(k, n) \mid \dim(\mathbf{e}_{n-k+i-\lambda_i}^- \cap V) \geq i \forall i\}.$$

*Remark 2.2.8.* Although  $X$  and  $X_w^{\circ\pm}$  are smooth, the Schubert varieties  $X_w$  may be singular. For example, let  $G = \mathrm{SL}_4$  with  $P = B$  and choose  $w = s_2 s_1 s_3 s_2 \in S_4 = W$ . Now by Theorem 2.2.3,  $h^{2i} = \dim H^{2i}(X_w^-, \mathbb{Q})$  is the count of  $v \in W$  with  $v \leq w$  and  $\ell(v) = i$ . The Schubert variety  $X_w^-$  is real 8-dimensional satisfying  $h^2 = 3$  and  $h^6 = 4$ , so it doesn't satisfy Poincaré duality and can not be smooth.

## 2.3 Line bundles

Many bundles on  $X$  naturally carry a  $T$ -equivariant action. We give the following naive definition which suffices for our purposes:

**Definition 2.3.1.** Let  $\mathcal{E} \rightarrow X$  be a vector bundle on  $X$ . A  $T$ -equivariant structure on  $\mathcal{E}$  is a fiberwise linear group action  $\sigma : T \times \mathcal{E} \rightarrow \mathcal{E}$  lifting the action map  $T \times X \rightarrow X$ .

We also define isomorphisms of  $T$ -equivariant bundles as  $T$ -equivariant bundle isomorphisms, and the  $T$ -equivariant Picard group  $\mathrm{Pic}_T(X)$  be the group of  $T$ -equivariant line bundles up to  $T$ -equivariant isomorphism.

**Lemma 2.3.2.** *There is a short exact sequence*

$$0 \longrightarrow Q \longrightarrow \mathrm{Pic}_T(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$$

where the first map sends  $\lambda \in Q$  to the trivial line bundle  $\mathcal{O}_\lambda$  with  $T$ -equivariant structure given by  $\lambda$ , and the second map forgets the  $T$ -equivariant structure.

*Proof.* Clearly the maps are linear and the composition of these maps is zero. It thus suffices to show that a trivial bundle  $\mathcal{L}$  has all its  $T$ -equivariant structures determined by a character of  $T$ . But  $T$  acts by a character on each fiber  $\mathcal{L}_x$  of  $x \in X$ , which must vary continuously along  $X$ . But since  $X$  is connected and  $Q$  discrete, this must be the same everywhere.  $\square$

A  $T$ -equivariant vector bundle  $\mathcal{E}$  also has a  $T$ -equivariant Chern class  $c_n^T(\mathcal{E}) \in H_T^{2n}(X, \mathbb{Z})$  lifting the ordinary Chern class  $c_n(\mathcal{E}) \in H^{2n}(X, \mathbb{Z})$  as in [4]. Interpreting the Chern class as a pullback of  $X \in H_T^{2n}(X, \mathbb{Z})$  along the map  $ET \times^T X \rightarrow BGL_r$  corresponding to  $\mathcal{E}$ , we see that restriction to a fixed point  $x_w \in X$

$$c_n^T(\mathcal{L})|_{x_w} = c_n^T(\mathcal{L}_{x_w}) \in H_T^{2n}(x_w, \mathbb{Z}) \subset \mathbb{Z}[Q]$$

amounts to taking the  $n$ 'th elementary symmetric polynomial of an eigenbasis whenever  $n \leq \text{rank } \mathcal{E}$ . In particular, for a  $T$ -equivariant line bundle  $\mathcal{L}$ , the restriction  $c_1(\mathcal{L}|_{x_w}) \in \mathbb{Z}[Q]$  is simply the corresponding character of  $T$ .

The following Corollary of the Atiyah–Bott localization formula will be useful, and the main reason for considering  $T$ -equivariance:

**Lemma 2.3.3.** *Let  $T$  act on  $\mathbb{P}^1$  with exactly two fixed points  $x, y \in \mathbb{P}^1$  with nonzero weight. Consider  $\mathcal{L} \in \text{Pic}_T(\mathbb{P}^1)$  with  $T$  acting on fibers  $\mathcal{L}_x, \mathcal{L}_y$  by  $\psi_x, \psi_y \in Q$  respectively. Let  $T$  act on  $T_x\mathbb{P}^1, T_y\mathbb{P}^1$  on  $\varphi_x, \varphi_y \in Q$ , respectively. Now, the following holds in a localization of  $H_T^2(\mathbb{P}^1, \mathbb{Z})$ :*

$$\int_{\mathbb{P}^1} c_1(\mathcal{L}) = \frac{\psi_x - \psi_y}{\varphi_x}.$$

*Proof.* According to the Atiyah–Bott localization formula [5] and the earlier discussion,

$$\int_{\mathbb{P}^1}^T c_1^T(\mathcal{L}) = \int_x \frac{c_1^T(\mathcal{L})|_x}{c_1^T(T_x\mathbb{P}^1)} + \int_y \frac{c_1^T(\mathcal{L})|_y}{c_1^T(T_y\mathbb{P}^1)} = \frac{\psi_x}{\varphi_x} + \frac{\psi_y}{\varphi_y}. \quad (2.3.1)$$

Now, an action  $T \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  with two fixed points is a projective representation  $T \rightarrow \text{PGL}_2$  fixing two lines, and thus a conjugate of  $t \mapsto \begin{pmatrix} \lambda(t) & \\ & 1 \end{pmatrix}$  for a character  $\lambda \in Q$ . In this presentation, the fixed points are  $[1 : 0]$  and  $[0 : 1]$ . Now,  $T$  acts on  $T_{[1:0]}\mathbb{P}^1$  by  $\lambda(t)$ , and on  $T_{[0:1]}\mathbb{P}^1$  by  $\lambda(t)^{-1}$ , so it follows that  $\varphi_x = -\varphi_y$ . Finally, since  $H_T^0(\mathbb{P}^1, \mathbb{Z}) = H^0(\mathbb{P}^1, \mathbb{Z})$ , the  $T$ -equivariant integral can be reduced to a regular one, and (2.3.1) reduces to

$$\int_{\mathbb{P}^1} c_1(\mathcal{L}) = \frac{\psi_x - \psi_y}{\varphi_x}. \quad \square$$

**Definition 2.3.4.** Let  $H \subset G$  be a subgroup, and  $Y$  a  $H$ -variety. Define  $G \times^H Y$  as the quotient of  $G \times Y$  by  $H$  where  $H$  acts by  $h \cdot (g, y) = (gh^{-1}, hg)$ .

This is always a variety, as outlined in [1, Definition 6.1.14].

Now consider a character  $\lambda \in Q_L$  of the Levi subgroup  $L$ , which is equivalently a character of  $Z(L)^0 \subset T$ . The inclusion  $T \subset L$  identifies  $Q_L \simeq \bigoplus_{\alpha \in \Delta \setminus \Delta_P} \omega_\alpha$  with  $Q_L^\perp \simeq Q_P^\vee$  under the pairing  $\langle -, - \rangle$ . Consider the composition

$$P \longrightarrow L \xrightarrow{\lambda} \mathbb{C}$$

to get a representation  $\mathbb{C}_\lambda$  of  $P$ . Finally, we define the line bundle

$$\mathcal{L}_\lambda = G \times^P \mathbb{C}_{-\lambda} \longrightarrow G/P = X.$$

$\mathcal{L}_\lambda$  can be equipped with a  $T$ -equivariant structure via  $t \cdot [g, z] = [tgt^{-1}, z]$ .

**Theorem 2.3.5.** *For  $\alpha \in \Delta \setminus \Delta_P$ , we have  $c_1^T(\mathcal{L}_{\omega_\alpha}) = \sigma_{s_\alpha} \in H_T^2(X, \mathbb{Z})$ .*

*Proof.* Let us first show that this holds in the non-equivariant context. In this case, it suffices to show that  $\int c_1(\mathcal{L}_{\omega_\alpha}) \cdot \sigma_{s_\beta}^- = \delta_{\alpha\beta}$  for all  $\beta \in \Delta \setminus \Delta_P$ . But this is simply

$$\int_{X_{s_\beta}^-} c_1(\mathcal{L}_{\omega_\alpha}).$$

Now  $X_{s_\beta}^- \simeq \mathbb{P}^1$  with poles  $x_1$  and  $x_{s_\beta}$ , and  $T$  acts on the fiber of  $\mathcal{L}_{\omega_\beta}$  at  $x_1$  and  $x_{s_\beta}$  via characters 0 and  $\omega_\alpha - s_\beta(\omega_\alpha) = \langle \omega_\alpha, \beta^\vee \rangle \beta$ , respectively, and on  $T_{x_1} X_{s_\beta}^-$  by  $\beta$ . Thus Lemma 2.3.3 guarantees that

$$\int_{X_{s_\beta}^-} c_1(\mathcal{L}_{\omega_\alpha}) = \frac{0 - (\omega_\beta - s_\alpha(\omega_\beta))}{\beta} = \langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha\beta}.$$

It follows that  $c_1(\mathcal{L}_{\omega_\alpha}) = \sigma_{s_\alpha} + \chi$  for a character  $\chi \in H_T^2(*, \mathbb{Z})$ . But since  $T$  acts trivially on the fiber over  $x_1$ , restricting to  $x_1$  shows that  $\chi = 0$  and thus the result follows.  $\square$

**Theorem 2.3.6.** *There are canonical isomorphisms*

$$Q_L \longrightarrow \text{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}) \quad (2.3.2)$$

given by

$$\lambda \longmapsto \mathcal{L}_\lambda \longmapsto c_1(\mathcal{L}_\lambda).$$

*Proof.* A basis for  $Q_L$  is given by  $\omega_\alpha$  for  $\alpha \in \Delta \setminus \Delta_P$ , and for  $H^2(X, \mathbb{Z})$  by  $\sigma_{s_\alpha}$  for  $\alpha \in \Delta \setminus \Delta_P$ . Thus by Theorem 2.2.3, and the maps being homomorphisms, the composition of these maps is an isomorphism. Furthermore, by [28, Proposition 3.3.2]  $c_1 : \text{Pic}(X) \rightarrow H^{1,1}(X, \mathbb{Z}) = H^2(X, \mathbb{Z})$  is an isomorphism, and hence so is the map  $Q_L \rightarrow \text{Pic}(X)$   $\square$

The following will use identifications given in the next section. Nonetheless, we state it here for the sake of organization.

**Theorem 2.3.7.**

(i) *The tangent bundle  $\mathcal{T}_X$  of  $X$  canonically identifies as*

$$\mathcal{T}_X \simeq G \times^P \mathfrak{g}/\mathfrak{p}$$

(ii) *For  $\gamma \in Q^\vee/Q_P^\vee \simeq H_2(X, \mathbb{Z})$  as identified in Corollary 2.4.3, we have*

$$\int_\gamma c_1(\mathcal{T}_X) = \langle 2\rho_L, \gamma \rangle.$$

*In particular, the anticanonical bundle  $\omega_X^{-1} = \det(\mathcal{T}_X)$  is  $T$ -equivariantly isomorphic to  $\mathcal{L}_{2\rho_L} \otimes \mathcal{O}_{-2\rho_L}$ , where  $\mathcal{O}_{-2\rho_L}$  is the trivial line bundle with  $T$ -equivariant structure given by  $2\rho_L \in Q$ .*

(iii) For each  $\gamma \in \Delta^\vee \setminus \Delta_P^\vee$  projected onto  $Q^\vee/Q_P^\vee$ , we have  $\langle 2\rho_L, \gamma \rangle \geq 2$ . In particular,  $\omega_X^{-1}$  is ample, that is,  $X$  is a Fano variety.

*Proof.* (i) Let  $\mathfrak{g}$  be the trivial vector bundle  $X \times \mathfrak{g} \rightarrow X$ . Consider the infinitesimal action bundle map

$$\mathfrak{g} \longrightarrow \mathcal{T}_X$$

given by differentiating the action map

$$G \times X \longrightarrow X$$

on the first coordinate. Now, a point  $x \in X$  is fixed exactly by  $\text{Ad}_x P \subset G$ , so on the fiber over  $x$ , the infinitesimal action  $\mathfrak{g} \rightarrow T_x X$  has kernel  $\text{Ad}_x \mathfrak{p}$ . Thus it follows that there is an exact sequence

$$0 \longrightarrow G \times^P \mathfrak{p} \longrightarrow \mathfrak{g} \longrightarrow \mathcal{T}_X \longrightarrow 0 \quad (2.3.3)$$

which gives an identification  $G \times^P \mathfrak{g}/\mathfrak{p} \simeq \mathcal{T}_X$ .

(ii) Let  $\beta \in \Delta \setminus \Delta_P$ . Using (2.2.1), we know that  $T$  acts on the fiber of  $\omega_X^{-1}$  at  $x_1$  and  $x_{s_\beta}$  by  $2\rho_L$  and  $s_\beta(2\rho_L) = 2\rho_L - \langle 2\rho_L, \beta^\vee \rangle \beta$ . In addition,  $T$  acts on  $T_{x_1} X_{s_\beta}^-$  by  $-\beta$ . Thus again by Lemma 2.3.3

$$\int_{X_{s_\beta}^-} c_1(\mathcal{T}_X) = \int_{X_{s_\beta}^-} c_1(\omega_X^{-1}) = \frac{2\rho_L - (2\rho_L - \langle 2\rho_L, \beta^\vee \rangle \beta)}{-\beta} = \langle 2\rho_L, \beta^\vee \rangle.$$

Since the  $[X_{s_\beta}^-] = \tau_{s_\beta}$  form a basis for  $H^2(X, \mathbb{Z})$  for  $\beta \in \Delta \setminus \Delta_P$ , this gives also the integral over  $\gamma$ . Finally, we get that  $\omega_X^{-1}$  and  $\mathcal{L}_{2\rho_L}$  are isomorphic as non-equivariant bundles, and comparing the  $T$ -actions on the fibers over  $x_1$  shows that twisting  $\mathcal{L}_{2\rho_L}$  by the  $\mathcal{O}_{-2\rho_L}$  upgrades this to an isomorphism of  $T$ -equivariant bundles.

(iii) Lift  $Q_L \times Q^\vee/Q_P^\vee$  to  $Q \times Q^\vee$  via identifying  $Q_L \simeq \bigoplus_{\alpha \in \Delta \setminus \Delta_P} \mathbb{Z} \cdot \omega_\alpha$  and  $Q_P/Q_P^\vee \simeq \bigoplus_{\beta \in \Delta \setminus \Delta_P} \mathbb{Z} \cdot \beta^\vee$ . Now, it is well-known that  $\langle 2\rho, \gamma \rangle = \langle \gamma^\vee, \gamma \rangle = 2$ . In addition,  $\langle 2\rho_P, \gamma \rangle \leq 0$  since  $\gamma$  does not occur as a summand of  $2\rho_P$ . Thus  $\langle 2\rho_L, \gamma \rangle \geq 0$ . Now, since each Hodge class is algebraic by the Bruhat decomposition, numerically trivial curve classes are also homologically trivial. Hence the closure of the Kleiman-Mori cone  $\overline{NE}(X) \simeq \bigoplus_{\gamma \in \Delta^\vee \setminus \Delta_P^\vee} \mathbb{R} \cdot \gamma$  and by Kleiman's criterion [16, Theorem 1.27] and (ii) the line bundle  $\omega_X^{-1}$  is ample.  $\square$

Although the following theorem will not be used in this thesis, we state it for completeness.

**Theorem 2.3.8** (Borel–Weil–Bott). *Equip the sheaf cohomology space  $H^i(X, \mathcal{L}_\lambda)$  with the structure of a  $G$ -module via  $G$  acting on  $X$ . Let  $\lambda \in Q_L \subset Q$ .*

- (i) *If  $\lambda + \rho$  is singular in  $Q$ , then  $H^i(X, \mathcal{L}_\lambda) = 0$ .*
- (ii) *If  $\lambda + \rho$  is regular in  $Q$ , so that there exists a unique  $w \in W$  with  $w(\lambda + \rho) - \rho$  dominant, then  $H^i(X, \mathcal{L}_\lambda) = 0$  for  $i \neq \ell(w)$  and  $H^{\ell(w)}(X, \mathcal{L}_\lambda)^\vee$  is the irreducible  $G$ -module with highest weight  $\lambda$ .*

*Proof.* In the case  $P = B$ , a proof can be found in [43]. In the general case, consider the projection  $\pi : G/B \rightarrow G/P = X$ , and note that Theorem 2.3.6 is functorial in the sense that the diagram

$$\begin{array}{ccc} Q_L & \longrightarrow & \text{Pic}(X) \\ \downarrow & & \downarrow \pi^* \\ Q & \longrightarrow & \text{Pic}(G/B) \end{array}$$

commutes. For  $\lambda \in Q_L$ , the line bundle  $\pi^* \mathcal{L}_\lambda$  is the corresponding line bundle with character  $\lambda \in Q$  for  $G/B$ . By the projection formula

$$\mathbf{R}^i \pi_* \pi^* \mathcal{L}_\lambda = \mathbf{R}^i \pi_* \mathcal{O}_{G/B} \otimes \mathcal{L}_\lambda$$

and we claim that this is  $\mathcal{L}_\lambda$  for  $i = 0$  and vanishes for  $i > 0$ . When  $i = 0$ , we have  $\mathbf{R}^0 \pi_* \mathcal{O}_{G/B} = \mathcal{O}_X$  since  $\pi$  is projective. When  $i > 0$ , we note that  $\pi$  is a locally trivial fibration with fiber  $xP/B \simeq L/(L \cap B)$  at  $x \in X$ , and thus as earlier the cohomology of this is of type  $(p, p)$ . Hence, using flat base change, the fiber of  $\mathbf{R}^i \pi_* \mathcal{O}_{G/B}$  at  $x \in X$  is  $H^i(xP/B, \mathcal{O}_{xP/B}) = H^{0,i}(xP/B) = 0$ .

Now, the Leray spectral sequence

$$E_2^{pq} = H^p(X, \mathbf{R}^q \pi_* \pi^* \mathcal{L}_\lambda) \implies H^{p+q}(G/B, \pi^* \mathcal{L}_\lambda)$$

thus degenerates at page  $E_2$  and gives an isomorphism  $H^i(X, \mathcal{L}_\lambda) \simeq H^i(G/B, \pi^* \mathcal{L}_\lambda)$  for all  $i > 0$ . Since everything happens on the level of  $G$ -modules, and the latter satisfies the theorem, the result follows.  $\square$

*Example 2.3.* Consider  $G = \text{SL}_2$  with  $B$  the upper triangular matrices. Now  $G/B = \mathbb{P}^1$  with  $\mathcal{O}(n) \simeq \mathcal{L}_{\lambda_n}$  for the weight  $\lambda_n : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t^n$ . Now,  $-\rho$  corresponds to  $-1$ , and all sheaf cohomology vanish for  $\mathcal{O}(-1)$ . For  $n > 0$ ,  $H^i(X, \mathcal{O}(n))$  vanishes for  $i > 0$ , and for  $i = 0$  has  $\mathbb{C}$ -basis given by the monomials  $\langle X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n \rangle$ , which is the irreducible representation with lowest weight  $-n$ . For  $n < -1$ , Serre duality similarly gives an isomorphism  $H^1(X, \mathcal{O}(n)) \simeq H^0(X, \mathcal{O}(-n-2))^\vee$  of  $G$ -modules.

*Remark 2.3.9.* Theorem 2.3.8 allows one to compute  $|\omega_X^{-1}| = \dim \mathbb{P}(H^0(X, \omega_X^{-1}))$  using the Weyl dimension formula for irreducible representations of  $\mathfrak{g}$  (equivalently  $G$ ) [33, Corollary 8.40]. When  $P = B$ , we get  $|\omega_X^{-1}| = 3^{\#\Phi_+} - 1$ .

## 2.4 $T$ -invariant curves

Consider a rational smooth connected curve  $\mathcal{C} \subset X$  invariant under the  $T$ -action. Now (2.2.2) tells that  $\mathcal{C}$  must contain a  $T$ -fixed point  $x_w$ , and (2.2.1) implies that locally  $\mathcal{C}$  must be the exponential map of one of these tangent directions  $\mathfrak{g}_\alpha$  at  $x_w$ . Hence  $\mathcal{C}$  is given by

$$\overline{U_\alpha w P} / P = \overline{\dot{w} \cdot U_{w^{-1}(\alpha)} P} / P = \dot{w} \cdot G_{w^{-1}(\alpha)} P / P$$

where  $G_\beta \subset G$  is the group generated by the  $\mathfrak{sl}_2$ -triple  $\mathfrak{g}_\beta \oplus \mathfrak{t}_\beta \oplus \mathfrak{g}_{-\beta}$  for  $\beta \in \Phi$ . This set contains exactly one other  $T$ -stable point, which is  $x_v$  for  $v = w s_{w^{-1}(\alpha)}$ . We get the following:

**Corollary 2.4.1.** For every  $w, v \in W^P$  adjacent in the sense that  $v = ws_\alpha$  for  $\alpha \in \Phi_+ \setminus \Phi_{P+}$ , there exists a unique smooth rational  $T$ -invariant curve  $C_{w,v}$  connecting  $x_w, x_v \in X$ . Furthermore, these characterize all  $T$ -invariant smooth rational curves on  $X$ .

When discussing curves, it is more convenient to talk about coroots since  $H_2(X, \mathbb{Z})$  is dual to  $H^2(X, \mathbb{Z}) \simeq Q_L$ .

**Theorem 2.4.2.** Let  $w, v \in W^P$  with  $v = ws_\gamma$  for  $\gamma \in \Phi_+^\vee \setminus \Phi_{P+}^\vee$ . Then

$$\int_{C_{w,v}} \sigma_{s_\beta} = \langle \omega_\beta, \gamma \rangle$$

for  $\beta \in \Delta \setminus \Delta_P$ . In particular,

$$[C_{w,v}] = \sum_{\beta \in \Delta \setminus \Delta_P} \langle \omega_\beta, \gamma \rangle \tau_{s_\beta} \in H_2(X, \mathbb{Z})_+.$$

*Proof.* Write  $C_\gamma$  for the curve  $\dot{w}^{-1}C_{w,v} = G_{\gamma^\vee}P/P$  containing  $x_1$  and  $x_{s_\gamma^\vee}$  as the  $T$ -fixed points. Since  $[C_\gamma] = [C_{w,v}]$  in  $H_2(X, \mathbb{Z})$ , it suffices to prove the claim for  $\dot{w} = 1$ . Consider the  $T$ -equivariant line bundle  $\mathcal{L}_{\omega_\beta}$  on  $X$  with Chern class  $c_1(\mathcal{L}_{\omega_\beta}) = \sigma_{s_\beta}$  by Theorem 2.3.5. As in the proof of Theorem 2.3.5,  $C_\gamma \simeq \mathbb{P}^1$  with poles  $x_1$  and  $x_{s_\gamma}$ , and  $T$  acts on the fiber of  $\mathcal{L}_{\omega_\beta}$  at  $x_1$  and  $x_{s_\beta}$  via characters 0 and  $\omega_\beta - s_{\gamma^\vee}(\omega_\beta) = \langle \omega_\beta, \gamma \rangle \gamma^\vee$ , respectively, and on  $T_{x_1}C_\gamma$  by  $-\gamma^\vee$ . Thus Lemma 2.3.3 guarantees that

$$\int_{C_\gamma} c_1(\mathcal{L}_{\omega_\beta}) = \frac{0 - (\omega_\beta - s_{\gamma^\vee}(\omega_\beta))}{-\gamma^\vee} = \langle \omega_\beta, \gamma \rangle.$$

In particular, considering the formula for Schubert varieties  $[C_\gamma] = \tau_{s_{\gamma^\vee}}$  for all  $\gamma \in \Delta^\vee \setminus \Delta_P^\vee$  we get

$$[C_\gamma] = \sum_{\beta \in \Delta \setminus \Delta_P} \left( \int_{C_\gamma} \sigma_{s_\beta} \right) \tau_{s_\beta} = \sum_{\beta \in \Delta \setminus \Delta_P} \langle \omega_\beta, \gamma \rangle \cdot \tau_{s_\beta},$$

and the result follows.  $\square$

**Corollary 2.4.3.** There is a canonical identification

$$Q^\vee / Q_P^\vee \longrightarrow H_2(X, \mathbb{Z})$$

satisfying  $[\gamma] \mapsto [C_\gamma]$  for all  $\gamma \in \Phi_+^\vee \setminus \Phi_{P+}^\vee$ . In particular, the effective curve classes  $H_2(X, \mathbb{Z})_+$  can be identified with  $\bigoplus_{\gamma \in \Delta \setminus \Delta_P} \mathbb{Z}_{\geq 0} \cdot \gamma$  modulo  $Q_P^\vee$ .

*Remark 2.4.4.* For a more "canonical" identification, consider morphisms  $X \rightarrow G \setminus X = BP \leftarrow BL$  of topological spaces, with the last map a  $U_P$ -gerbe. This gives a morphism  $H_2(X, \mathbb{Z}) \rightarrow H_2(G \setminus X, \mathbb{Z}) = H_2(BL, \mathbb{Z})$ . Furthermore, the Hurewicz map  $\pi_2(BL) \rightarrow H_2(BL, \mathbb{Z})$  identifies this with  $\pi_2(BL)$ , delooping identifies it with  $\pi_1(L)$ , and this finally identifies with  $Q^\vee / Q_P^\vee$ .

What should be learned from Theorem 2.3.6, Theorem 2.4.2, and Corollary 2.4.3, is that  $H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z})$ , along with its integration pairing  $\int$ , identifies naturally with  $Q_L \times Q^\vee / Q_P^\vee$  with the pairing  $\langle -, - \rangle$ . Effective divisor classes correspond to sums of fundamental weights, and effective curve classes to sums of simple coroots.

In particular, the homology classes of  $T$ -invariant curves correspond to positive coroots.

Furthermore, these identifications work functorially in the sense that if  $P' \supset P$  is another parabolic with Levi  $L' \supset T$ , the pullback  $H^2(X, \mathbb{Z}) \leftarrow H_2(G/P', \mathbb{Z})$  of  $X \rightarrow G/P'$  identifies with  $Q_L \leftarrow Q_{L'}$  induced by  $L' \supset L$ . Similarly,  $H_2(X, \mathbb{Z}) \rightarrow H_2(G/P', \mathbb{Z})$  identifies with  $Q^\vee/Q_P^\vee \rightarrow Q^\vee/Q_{P'}^\vee$  given by  $\Phi_{P'}^\vee \supset \Phi_P^\vee$ . With all this in mind, the following is natural:

**Theorem 2.4.5.** *Let  $w \in W^P$  and  $\beta \in \Delta \setminus \Delta_P$  a simple root. Now*

$$\sigma_{s_\beta} \cdot \sigma_w = \sum_{\gamma} \langle \omega_\beta, \gamma \rangle \sigma_{ws_\gamma} + [\omega_\beta - w(\omega_\beta)] \cdot \sigma_w.$$

where the sum is taken over  $\gamma \in \Phi_+^\vee \setminus \Phi_{P_+}^\vee$  satisfying  $\ell(ws_\gamma) = \ell(w) + 1$  as cosets in  $W^P$ .

Before proving the theorem, we state a transversality theorem by Kleiman [34], and then state a corollary shown in [19]. We state a slight strengthening, noting that  $Y$  and  $Z$  can be replaced by smooth Deligne-Mumford stacks. Both will be essential in all of the sections.

**Theorem 2.4.6.** [34] *Let  $Y$  and  $Z$  be smooth Deligne-Mumford stack with maps*

$$Y \longrightarrow X \longleftarrow Z.$$

For  $g \in G$ , denote  $gY$  the translate of the image of  $Y$  in  $X$ .

(i) *There exists a Zariski-open subset  $U \subset G$  such that for all  $g \in U$ , the stack  $gY \times_X Z$  is either empty or of pure dimension*

$$\dim Y + \dim Z - \dim X.$$

(ii) *If  $Y$  and  $Z$  are smooth, then  $U$  can be chosen such that so is  $gY \times_X Z$  for all  $g \in U$ .*

Considering this for the group  $G \times G$  and the maps

$$Y \longrightarrow X \times X \longleftarrow X_w \times X_v^-$$

and using [19, Lemma 7.1], we immediately get the following:

**Corollary 2.4.7.** Let  $Y$  be an irreducible  $G$ -variety, let  $G$  act diagonally on  $X \times X$ , and let  $F : Y \rightarrow X \times X$  be a  $G$ -equivariant morphism. Then for every  $w, v \in W^P$ , the scheme  $F^{-1}(X_w \times X_v^-)$  is either empty or of pure codimension  $\ell(w) + \dim X - \ell(v)$  and reducible, locally irreducible, and smooth at points mapping to a smooth point of  $X_w \times X_v^-$ .

Furthermore, letting  $Y = X$  and  $F = \Delta$  the diagonal map, we get:

**Corollary 2.4.8.** If nonempty, the scheme  $X_w \cap X_v^- \subset X$  is irreducible of pure codimension  $\ell(w) + \dim X - \ell(v)$ .

*Proof of Theorem 2.4.5.* Suppose the coefficient of  $\sigma_v$  in  $\sigma_{s_\beta} \cdot \sigma_w$  is nonvanishing. This is given by

$$\int_X^T \sigma_{s_\beta} \cdot \sigma_w \cdot \sigma_v^- \in H_T^{\ell(w)-\ell(v)+1}(*)$$

so  $\ell(w) + 1 \geq \ell(v)$ . In addition,  $v \geq w$  since  $\sigma_{s_\beta} \cdot \sigma_w$  is the Poincaré dual of a set within  $X_w$ . It thus follows that either  $v = ws_\gamma$  for  $\gamma \in \Phi_+^\vee \setminus \Phi_{P_+}^\vee$  or  $v = w$

In the former case, we define  $E = X_w \cap X_v^-$ , which is  $T$ -invariant and according to Corollary 2.4.8, irreducible and of pure dimension 1. Since  $x_w$  and  $x_v$  are the only  $T$ -fixed points in  $E$ ,  $E$  is precisely the  $T$ -invariant curve  $C_{w,v} = \dot{w}C_\gamma$  connecting  $x_w$  and  $x_v$ . Hence, using the projection formula and Theorem 2.4.2,

$$\int_X \sigma_{s_\beta} \cdot \sigma_w \cdot \sigma_v^- = \int_X \sigma_{s_\beta} \cdot [E]^* = \int_{C_\gamma} \sigma_{s_\beta} = \langle \omega_\beta, \gamma \rangle.$$

In the latter case, we can pull back along  $x_w \rightarrow X$  to find that the coefficient of  $\sigma_w$  is given by  $\sigma_{s_\beta}|_{x_w} \in H_T^2(*, \mathbb{Z}) = \mathbb{Q}$ . Since  $\sigma_{s_\beta} = c_1(\mathcal{L}_{\omega_\beta})$ , this is equivalently the weight of  $T$  acting on  $\mathcal{L}_{\omega_\beta}|_{x_w}$ , which can be computed explicitly to be  $\omega_\beta - w(\omega_\beta)$ . The result follows directly.  $\square$

*Remark 2.4.9.* Note that all the earlier discussion guarantees that  $X$  as a  $T$ -space is a *GKM space* in the sense of [22]. Thus the restriction

$$H_T^*(X, \mathbb{Q}) \longrightarrow \bigoplus_{w \in W^P} H_T^*(x_w, \mathbb{Q}) = \bigoplus_{w \in W^P} \mathbb{Q}[Q] = \mathbb{Q}[Q]^{W^P}$$

is injective with image

$$\left\{ (f_w) \in \mathbb{Q}[Q]^{W^P} \mid \frac{f_w - f_{ws_\gamma}}{\alpha} \in \mathbb{Q}[Q] \ \forall w \in W^P, \gamma \in \Phi_+^\vee \setminus \Phi_{P_+}^\vee \right\}.$$

# Chapter 3

## Moduli of curves and maps

### 3.1 Moduli of stable curves

Denote  $*$  =  $\text{Spec } \mathbb{C}$ , which we consider as a scheme or a topological point coming from its unique  $\mathbb{C}$ -point depending on the context. Now, let  $\mathcal{C}$  be a complex curve, that is, a reduced scheme of finite type over  $*$  and of dimension 1.

**Definition 3.1.1.** Let the *arithmetic genus*  $g$  of  $\mathcal{C}$  be

$$g = 1 - \chi(\mathcal{O}_{\mathcal{C}}) = h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) - h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) + 1.$$

We use the arithmetic genus instead of the geometric genus  $g' = h^0(\mathcal{C}, \Omega_{\mathcal{C}}^1)$  due to the arithmetic genus being invariant under deformations. Although  $g$  is not preserved under normalization, in the following case the genus of the curve and its normalization are easy to compare:

**Definition 3.1.2.** Call  $\mathcal{C}$  nodal if, around a singular point,  $\mathcal{C}$  is étale-locally of the form  $\text{Spec } \mathbb{C}[x, y]/(xy)$

*Remark 3.1.3.* A computationally much easier definition is that the singularity is formally of the form  $\text{Spec } \mathbb{C}[[x, y]]/(xy)$ . However, by Artin approximation [2, Theorem B.5.18] these are equivalent.

In addition, consider  $n$  distinct *marked points*  $(p_1, \dots, p_n)$  on the smooth locus of  $\mathcal{C}$ . We call a point  $p \in \mathcal{C}$  a *special point* if  $p$  is either marked or a node. Define a morphism of  $n$ -pointed nodal curves

$$(\mathcal{C}, p_1, \dots, p_n) \longrightarrow (\mathcal{C}', p'_1, \dots, p'_n)$$

to be a morphism  $f : \mathcal{C} \rightarrow \mathcal{C}'$  such that  $f(p_i) = p'_i$  for all  $i = 1, \dots, n$ .

**Definition 3.1.4.** Call the pointed curve  $(\mathcal{C}, p_1, \dots, p_n)$  stable if it is nodal and has finitely many automorphisms.

Consider the normalization morphism  $\nu : \mathcal{C}^\nu \rightarrow \mathcal{C}$  of a nodal curve  $\mathcal{C}$ . This comes with the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \nu_* \mathcal{O}_{\mathcal{C}^\nu} \longrightarrow \bigoplus_{q \text{ node}} \mathbb{C}_q \longrightarrow 0. \quad (3.1.1)$$

Since  $\nu$  is finite,  $\mathbf{R}\nu_* = \nu_*$  and thus  $\chi(\nu_*\mathcal{O}_{\mathcal{C}^\nu}) = \chi(\mathcal{O}_{\mathcal{C}^\nu})$ . Taking the sheaf cohomology long exact sequence of (3.1.1), and Euler characteristics of that, we see that

$$g_{\mathcal{C}} = \left( \sum_{\mathcal{D}} g_{\mathcal{D}} \right) - c + \delta + 1 \quad (3.1.2)$$

where the sum is taken irreducible components  $\mathcal{D}$  of  $\mathcal{C}$ ,  $c$  is the number of irreducible components, and  $\delta$  the number of nodes.

In a stable  $n$ -pointed curve  $(\mathcal{C}, p_1, \dots, p_n)$ , a component of genus 0 must have at least 3 special points, of genus 1 at least 1 special point, and of genus  $g > 1$  need not have special points. This combinatorial definition leads to the following:

**Lemma 3.1.5.** [2, Proposition 6.3.5] *The following are equivalent:*

- (i)  $(\mathcal{C}, p_1, \dots, p_n)$  is stable
- (ii) The line bundle  $\omega_{\mathcal{C}}(p_1 + \dots + p_n)^{\otimes 3}$  is very ample, where  $\omega_{\mathcal{C}}$  is the dualizing sheaf

We wish to consider moduli of these objects. For this, we will use the notion of algebraic stacks over the fpqc topology. In particular, we consider a stack  $\mathcal{X}$  as a fibered category over the category of  $\mathbb{C}$ -schemes, and for a scheme  $S$  we denote by  $\mathcal{X}(S)$  the groupoid over  $S$ . All stacks we encounter will be of Deligne-Mumford type, except for a few Artin stacks. We refer the reader to [8, 40] for the basics of these objects, and to [59, 6] for their Chow groups, cohomology, and homology.

**Definition 3.1.6.** Let  $S$  be a  $\mathbb{C}$ -scheme. A nodal  $n$ -pointed curve of genus  $g$  over  $S$  is an algebraic space  $\mathcal{C}$  over  $S$  with  $n$  non-intersecting sections  $\sigma_1, \dots, \sigma_n : S \rightarrow \mathcal{C}$  such that

- (i)  $\mathcal{C} \rightarrow S$  is flat, proper, and of finite presentation
- (ii) For every geometric point  $s \in S$ ,  $\mathcal{C}_s$  is a nodal curve of genus  $g$ , and  $\sigma_1(s), \dots, \sigma_n(s)$  lie in the smooth locus of  $\mathcal{C}_s$ .

Moreover,  $\mathcal{C}/S$  is stable if  $(\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s))$  is for all geometric points  $s \in S$ .

We define isomorphisms as earlier. Clearly a stable family as defined above has a finite number of automorphisms. Moreover, an argument similar to Lemma 3.1.5 guarantees that a stable family is projective. In general, a family of curves is projective only up to étale base change [2, Proposition 6.4.1]

*Remark 3.1.7.* A test scheme  $S$  in this case need not be locally of finite type, meaning that a geometric point  $\text{Spec } \bar{k} \rightarrow S$  for  $\bar{k}/\mathbb{C}$  a nontrivial extension need not factor through some  $*$   $\rightarrow S$ . This can be fixed by noting that our theory of curves also holds for  $\text{Spec } \bar{k}$ . Alternatively, the spaces we will consider will be a priori of finite presentation, so we can use affine schemes of finite type as our test schemes [54, Proposition 32.6.1]. Nonetheless, a geometric point of  $S$  essentially means a complex point  $s \in S(\mathbb{C})$  to us.

**Definition 3.1.8.** Let  $\overline{\mathcal{M}}_{g,n}$  and  $\mathcal{M}_{g,n}$  be the stacks defined by

$$\overline{\mathcal{M}}_{g,n}(S) = \left\{ \begin{array}{l} \text{stable } n\text{-pointed curves} \\ \text{of genus } g \text{ over } S \end{array} \right\}, \quad \mathcal{M}_{g,n}(S) = \left\{ \begin{array}{l} \text{stable smooth } n\text{-pointed curves} \\ \text{of genus } g \text{ over } S \end{array} \right\}$$

for a scheme  $S$ .

**Theorem 3.1.9.** [2, Theorem 3.19]

- (i)  $\overline{\mathcal{M}}_{g,n}$  is a smooth proper Deligne-Mumford stack
- (ii)  $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$  is open with its boundary a simple normal crossing divisor.
- (iii)  $\overline{\mathcal{M}}_{g,n}$  has coarse moduli space  $\overline{M}_{g,n}$ , which is a projective variety with quotient singularities

*Example 3.1.* Clearly  $\overline{\mathcal{M}}_{g,n} = \emptyset$  for  $2g - 2 + n \leq 0$ . In addition,  $\overline{\mathcal{M}}_{0,3}(\mathbb{C})$  is a single point since an automorphism of  $\mathbb{P}^1$  is determined by three points, and a nodal curve with 3 points can not be stable. By a technical result [52, Proposition 4.2] a genus 0 smooth curve  $\mathcal{C} \rightarrow S$  with three sections  $S \rightarrow \mathcal{C}$  trivializes to  $S \times \mathbb{P}^1 \rightarrow S$ , so we see that  $\mathcal{M}_{0,3} = *$  as a stack, and by properness  $\overline{\mathcal{M}}_{0,3} = *$ .

*Example 3.2 (Bubbling).* Let  $\mathcal{C} = \mathbb{A}^1 \times \mathcal{C}_0 \rightarrow \mathbb{A}^1$  a trivial family of nodal genus  $g$  curves with possibly nontrivial sections  $\sigma_1, \dots, \sigma_n$ , distinct over  $\mathbb{A}^1 \setminus \{0\}$ . To understand the compactification  $\overline{\mathcal{M}}_{g,n}$  one should check what happens when special points coincide in  $\mathcal{C}_0$ . There are two special cases to consider, with the rest generalizing simply:

- $\sigma_i(0) = \sigma_j(0)$  are smooth points for some  $i \neq j$ . In this case, we can replace the family  $\mathcal{C} \rightarrow \mathbb{A}^1$  by  $\tilde{\mathcal{C}} = \text{Bl}_{\sigma_i(0)}\mathcal{C} \rightarrow \mathbb{A}^1$ . This family remains flat since  $\mathbb{A}^1$  is Dedekind, and adds a rational "bubble" in place of two coinciding points. This introduces new  $n$  sections  $\tilde{\sigma}_0, \dots, \tilde{\sigma}_n$ , with  $n - 2$  of them coinciding with  $\sigma_1, \dots, \sigma_n$ . If  $\tilde{\sigma}_i(0) = \tilde{\sigma}_j(0)$  the process can be repeated. We can check étale-locally that the  $k$ 'th blowup always records the direction of the  $k$ 'th derivative, and one of these must differ at some point due to  $\sigma_i$  and  $\sigma_j$  being generically different, so the process terminates.
- $\sigma_i(0) = q$  for a node  $q$ . In this case, we blow up at the node instead, and get a bubble in place of the node. We similarly repeat the process until necessary.

If we impose  $\mathcal{C}_0$  to be stable and wish to remain in the stable context, we must also contract all the added bubbles with less than 3 special points. When  $\mathcal{C}_0$  is smooth, these bubbles are  $(-1)$ -curves and can thus be contracted using Castelnuovo's criterion. This works in the singular case by normalizing, doing the construction, and then gluing back to nodal.

**Definition 3.1.10.**

- (i) For a  $(n+1)$ -pointed stable curve  $(\mathcal{C}, p_1, \dots, p_{n+1})$ , define the  $n$ -pointed *contracting stabilization*  $(\mathcal{C}, p_1, \dots, p_n)^{\text{stab}}$  as  $(\mathcal{C}, p_1, \dots, p_n)$  if it is stable, and  $(\mathcal{C}', p_1, \dots, p'_n)$  where  $\mathcal{C}'$  is  $\mathcal{C}$  with the component including  $p_n$  contracted to a point  $p'_n$ .
- (ii) For a  $n$ -pointed stable curve  $(\mathcal{C}, p_1, \dots, p_n)$  and  $q \in \mathcal{C}$ , define the *bubbling stabilization*  $(\mathcal{C}, p_1, \dots, p_n, q)^{\text{stab}}$  as  $(\mathcal{C}, p_1, \dots, p_n, q)$  if the marked points are distinct, and if  $q = p_i$  for some  $i$ , as  $(\mathcal{C}', p_1, \dots, \tilde{p}_i, \dots, p_n, \tilde{q})$  for  $\mathcal{C}' = \mathcal{C} \cup \mathbb{P}^1$  intersecting at  $q = p_i \in \mathcal{C}$  and  $\tilde{p}_i, \tilde{q} \in \mathbb{P}^1$  are any two distinct points away from the node.

*Remark 3.1.11.* Contraction can actually be defined in families: If  $(\mathcal{C}/S, \sigma_1, \dots, \sigma_n)$  is as in Definition 3.1.6, consider the relative dualizing bundle  $\mathcal{L} = \omega_{\mathcal{C}/S}(\sigma_1 + \dots + \sigma_n)^{\otimes 3}$ , and consider the map  $f$  from  $\mathcal{C}/S$  onto the projective bundle  $|\mathcal{L}|/S$ . For every geometric point  $s \in S$ , the restriction  $\mathcal{L}_s$  is simply  $\omega_{\mathcal{C}_s}(p_1 + \dots + p_n)^{\otimes 3}$ , and hence it

is of degree 0 on unstable components and very ample on stable ones. Thus the image of  $f$  gives  $(\mathcal{C}/S, \sigma_1, \dots, \sigma_n)^{\text{stab}}$ . Similarly stabilization can be defined in families as gluing  $\mathcal{M}_{g,n}$  to  $\mathcal{M}_{0,3}$  along  $p_i = q$  [2, Corollary 6.6.16].

**Definition 3.1.12.**

- (i) Define the forgetful map  $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  given by sending a pointed family  $(\mathcal{C}/S, \sigma_1, \dots, \sigma_n, \sigma_{n+1})$  to  $(\mathcal{C}/S, \sigma_1, \dots, \sigma_n)^{\text{stab}}$  over a scheme  $S$  as in Remark 3.1.11.
- (ii) Define the section  $\sigma_i : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n+1}$  for  $i = 1, \dots, n$  given by sending a pointed family  $(\mathcal{C}/S, \sigma_1, \dots, \sigma_n)$  to  $(\mathcal{C}/S, \sigma_1, \dots, \sigma_n, \sigma_i)^{\text{stab}}$  over a scheme  $S$

**Lemma 3.1.13.** [2, Proposition 6.6.12] *The forgetful map  $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  defines a universal family for  $\overline{\mathcal{M}}_{g,n}$ : For every scheme  $S$ , a stable  $n$ -pointed family of genus  $g$  stable curves  $(\mathcal{C}/S, \sigma_1, \dots, \sigma_n)$  is a base change of  $(\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}, \sigma_1, \dots, \sigma_n)$  along a unique morphism  $S \rightarrow \overline{\mathcal{M}}_{g,n}$ .*

Following this, we often denote the family  $\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}$  by  $\mathfrak{C}/\overline{\mathcal{M}}_{g,n}$  and call it the *universal curve*.

**Definition 3.1.14.** Define the Hodge bundle

$$\mathbb{E} = \mathbf{R}^0 \pi_* \omega_{\mathfrak{C}/\overline{\mathcal{M}}_{g,n}},$$

where  $\omega_{\mathfrak{C}/\overline{\mathcal{M}}_{g,n}}$  is the relative dualizing sheaf of the universal curve.

*Remark 3.1.15.* The fiber of  $\mathbb{E}$  over a point  $[\mathcal{C}] \in \overline{\mathcal{M}}_{g,n}(\mathbb{C})$  is given by  $H^0(\mathcal{C}, \omega_{\mathcal{C}})$ . Recall that by Grothendieck–Serre duality, the dual bundle  $\mathbb{E}^\vee$  is canonically identified with  $\mathbf{R}^1 \pi_* \mathcal{O}_{\mathfrak{C}}$ .

## 3.2 Deformation theory

In order to study the local geometry of  $\overline{\mathcal{M}}_{g,n}$ , we need to study the deformation theory of curves. For a complex point  $[\mathcal{C}] \in \overline{\mathcal{M}}_{g,n}(\mathbb{C})$  corresponding a curve  $\mathcal{C}$ , a tangent vector is given by a lift  $\text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2) \rightarrow \overline{\mathcal{M}}_{g,n}$ , so a first-order deformation  $\mathcal{C}' \rightarrow \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$ . By the infinitesimal lifting condition [2, Theorem 4.7.1], the smoothness of this point  $[\mathcal{C}]$  is recorded by the obstructions to this deformation.

We use the calculus of cotangent complexes [29] (see also [2, Appendix C.3]) for schemes (or sometimes Artin stacks [49]) for our deformation theory. For  $f : X \rightarrow Y$  a map of stacks, we consider the *cotangent complex*  $\mathbb{L}_{X/Y} \in \mathbf{D}_{\text{qcoh}}^-(X)$  in the bounded-below derived category of sheaves on  $X$  with quasi-coherent cohomology sheaves. For  $Y = *$ , we simply write  $\mathbb{L}_{X/*} = \mathbb{L}_X$ . This complex possesses the following properties, which we shall take for granted:

**Theorem 3.2.1.**

- (i) [29, Corollaire III.3.1.3] *If  $f : X \rightarrow Z$  factors as a closed immersion  $i : X \rightarrow Y$  with ideal sheaf  $\mathcal{I}$ , composed with a smooth morphism  $Y \rightarrow Z$ , then  $\mathbb{L}_{X/Y}$  is quasi-isomorphic to the complex*

$$\mathcal{I}/\mathcal{I}^2 \rightarrow i^* \Omega_{Y/Z}^1$$

in degrees  $[-1, 0]$ . In particular, for  $f$  a closed immersion, this reduces to  $\mathcal{N}_{X/Y}^\vee[1] = \mathcal{I}/\mathcal{I}^2[1]$ . For  $f$  smooth, this reduces to  $\Omega_{X/Y}^1[0]$ .

(ii) [29, (II.1.5.6)] If  $X \xrightarrow{f} Y \rightarrow Z$  induces an exact triangle

$$\mathbf{L}f^*\mathbb{L}_{Y/Z} \longrightarrow \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y} \longrightarrow \quad (3.2.1)$$

in  $\mathbf{D}_{\text{qcoh}}^-(X)$ .

(iii) [29, Proposition II.2.2.1, Corollaire II.2.2.3] Consider the Cartesian square of schemes

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{y} & Y' \end{array}$$

with either  $f'$  or  $y$  flat. Now, the canonical morphisms

$$\mathbf{L}x^*\mathbb{L}_{X'/Y'} \longrightarrow \mathbb{L}_{X/Y}, \quad \mathbf{L}x^*\mathbb{L}_{X'/Y'} \oplus \mathbf{L}f^*\mathbb{L}_{Y/Y'} \longrightarrow \mathbb{L}_{X/Y'}$$

are quasi-isomorphisms in  $\mathbf{D}_{\text{qcoh}}^-(X)$ .

(iv) [29, (III.2.1.7)] Let  $f : X \rightarrow Y$  be flat and consider a closed inclusion  $Y \rightarrow Y'$  with ideal sheaf  $\mathcal{I}$  satisfying  $\mathcal{I}^2 = 0$ . Consider the set (or groupoid) of deformations, that is, flat families  $X' \rightarrow Y'$  with a closed inclusion  $X \rightarrow X'$  making the following diagram Cartesian:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ f \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

Now,  $\mathcal{I}$  has the structure of a  $\mathcal{O}_Y$ -module, and the following hold:

- For a fixed deformation, its space of automorphisms is given by  $\text{Ext}_{\mathcal{O}_X}^0(\mathbb{L}_{X/Y}, f^*\mathcal{I})$
- If a deformation exists, their set is a torsor under  $\text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}, f^*\mathcal{I})$
- There is a canonical element  $\text{ob}_{X/Y} \in \text{Ext}_{\mathcal{O}_X}^2(\mathbb{L}_{X/Y}, f^*\mathcal{I})$  whose vanishing is necessary and sufficient for the existence of a deformation

We care, in particular, about first-order deformations  $Y' = Y \times_{\mathbb{C}} \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$  so that  $f^*\mathcal{I}$  is a trivial  $\mathcal{O}_X$ -module. In this case,  $X' = X \times_{\mathbb{C}} \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$  is a canonical "trivial" deformation, so  $\text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}, \mathcal{O}_X)$  can be considered as the space of deformations with the trivial deformation corresponding to 0.

**Lemma 3.2.2.** *For  $\mathcal{C}$  a family of nodal curves over a scheme  $S$ , we have  $\mathbb{L}_{\mathcal{C}/S} = \Omega_{\mathcal{C}/S}^1[0]$ .*

*Proof.* In the smooth locus this is clear. Around a node  $p \in \mathcal{C}$ , we can choose an affine open set  $U = \text{Spec } R \rightarrow S$  and an étale neighborhood  $V = \text{Spec } R[x, y]/(xy - r)$  so that by flat base change  $\mathbb{L}_{\mathcal{C}/S|_V} = \mathbb{L}_{V/U}$ . Now,  $V \rightarrow U$  factors through  $V \rightarrow \mathbb{A}_U^2 \rightarrow U$ , so we  $\mathbb{L}_{V/U}$  as the complex  $[\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{\mathbb{A}_U^2/U}^1]$ . Since this is injective over the generic point, the conormal exact sequence implies that this is quasi-isomorphic to  $\Omega_{V/U}^1[0]$ , and the result follows.  $\square$

A first-order deformation of a smooth curve  $\mathcal{C}$  can be easily seen to be an element of  $\check{H}^1(\mathcal{C}, \mathcal{T}_{\mathcal{C}})$  since the tangent bundle  $\mathcal{T}_{\mathcal{C}}$  is the sheaf of automorphisms for first-order deformations. When considering deformations of a  $n$ -pointed curve  $(\mathcal{C}, p_1, \dots, p_n)$ , we instead expect the points  $p_i$  to remain fixed, so we are led to consider the bundle  $\mathcal{T}_{\mathcal{C}}(-D) = \mathcal{H}om(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}})$  instead for  $D = p_1 + \dots + p_n$ . Since the marked points are smooth, and deformation theory happens locally, it follows that the deformation theory of a nodal curve  $\mathcal{C}$  is controlled by  $\mathbb{L}_{\mathcal{C}}(D) = \Omega_{\mathcal{C}}^1(D)[0]$  and we are led to consider the following spaces:

**Theorem 3.2.3.** *Let  $(\mathcal{C}, p_1, \dots, p_n)$  be a complex nodal curve of genus  $g$  with  $D = p_1 + \dots + p_n$ . Then the following hold:*

(i)

$$\mathrm{Ext}^2(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) = 0$$

(ii)

$$\dim \mathrm{Ext}^1(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) - \dim \mathrm{Ext}^0(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) = 3g - 3 + n$$

(iii) For  $(\mathcal{C}, p_1, \dots, p_n)$  stable,

$$\mathrm{Ext}^0(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) = 0$$

*Proof.* The Grothendieck spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{E}xt^q(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}})) \implies \mathrm{Ext}^{p+q}(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}})$$

degenerates on the second page by Grothendieck's vanishing theorem. Writing  $\mathcal{F}^i = \mathcal{E}xt^i(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}})$ <sup>1</sup> for  $i = 0, 1, 2$ , we thus get non-canonical splittings

$$\mathrm{Ext}^2(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) = H^1(\mathcal{C}, \mathcal{F}^1) \oplus H^0(\mathcal{C}, \mathcal{F}^2) \quad (3.2.2)$$

$$\mathrm{Ext}^1(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) = H^1(\mathcal{C}, \mathcal{F}^0) \oplus H^0(\mathcal{C}, \mathcal{F}^1) \quad (3.2.3)$$

while  $\mathrm{Ext}^0(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) = H^0(\mathcal{C}, \mathcal{F}^0)$  by definition. Hence it suffices to study the sheaves  $\mathcal{F}^i$ . Let us first study behavior around a smooth point. Around a smooth trivialization  $U$ , we get  $\mathcal{F}_U^i = \mathcal{E}xt^i(\mathcal{O}_U, \mathcal{O}_U)$ , which is trivial for  $i = 0$  and vanishes for  $i = 1, 2$ . Hence  $\mathcal{F}^1$  and  $\mathcal{F}^2$  are supported at most at nodes.

Around a node, choose an étale neighborhood  $U = \mathrm{Spec} R$  with  $R = S^{-1}\mathbb{C}[x, y]/(xy)$  where  $S$  is a sufficient localization removing marked points from  $U$ .  $\Omega_{\mathcal{C}}^1(D)|_U$  as an  $R$ -module is canonically isomorphic to  $\Omega = \Omega_R^1 = R\langle dx, dy \rangle / (xdy + ydx)$ , with a free resolution

$$R \cdot (xdy + ydx) \xrightarrow{\phi} R \cdot dx \oplus R \cdot dy \longrightarrow \Omega \quad (3.2.4)$$

Dualizing (3.2.4), we get  $\mathrm{Ext}_R^i(\Omega, R)$  as the cohomology of the complex

$$R \oplus R \xrightarrow{(r,s) \mapsto rx+ys} R$$

giving  $\Omega^\vee = \mathrm{Hom}_R(\Omega, R) \simeq R \cdot x \oplus R \cdot y$ ,  $\mathrm{Ext}_R^1(\Omega, R) \simeq R/(x, y)$ , and  $\mathrm{Ext}_R^2(\Omega, R) = 0$ . To be more precise, the  $R$ -module  $\Omega^\vee$  is generated by  $\xi$  sending  $dx \mapsto x, dy \mapsto 0$

<sup>1</sup>Also known as the pointed  $T^i$  functors [26]

and  $\zeta$  sending  $dx \mapsto 0, dy \mapsto y$ . In particular,  $\Omega \rightarrow \Omega/y\Omega \oplus \Omega/x\Omega$  induces a canonical isomorphism

$$(\Omega/y\Omega)^\vee \oplus (\Omega/x\Omega)^\vee \longrightarrow \Omega^\vee. \quad (3.2.5)$$

That is, a tangent vector on  $U$  corresponds to a tangent vector on  $x = 0$  vanishing at  $y = 0$ , along with a tangent vector  $y = 0$  vanishing at  $x = 0$ . If we consider the normalization  $R = R/(x) \times R/(y) = R^\nu$ , we learn that if  $\Omega^\vee$  is canonically  $(\Omega_{R/(x)}^1)^\vee \cdot y \oplus (\Omega_{R/(y)}^1)^\vee \cdot x$  as a  $R$ -module.

Globalizing, we learn that  $\mathcal{T}^2 = 0$ ,  $\mathcal{T}^1 = \bigoplus_{q \text{ node}} \mathbb{C}$ , and  $\mathcal{T}^0 = \nu_* \mathcal{V}$  for  $\nu : \mathcal{C}^\nu \rightarrow \mathcal{C}$  the normalization and  $\mathcal{V} = \mathcal{T}_{\mathcal{C}^\nu}(-D - \sum_{q \text{ node}} \nu^* q)$ . It thus follows that  $H^0(\mathcal{C}, \mathcal{T}^2) = H^1(\mathcal{C}, \mathcal{T}^1) = 0$ , and  $\dim H^0(\mathcal{C}, \mathcal{T}^1)$  is the number of nodes  $\delta$ . In particular, this proves (i). Since  $\mathbf{R}\nu_* = \nu_*$  and  $\deg \mathcal{T}_{\mathcal{D}} = 2 - 2g_{\mathcal{D}}$  for  $\mathcal{D}$  a component of  $\mathcal{C}^\nu$ , the Riemann–Roch theorem on  $\mathcal{T}^0 = \nu_* \mathcal{V}$  and gives

$$\dim H^1(\mathcal{C}, \mathcal{T}^0) - \dim H^0(\mathcal{C}, \mathcal{T}^0) = - \left( \sum_{\mathcal{D}} (2 - 2g_{\mathcal{D}}) - n - 2\delta \right) + \sum_{\mathcal{D}} g_{\mathcal{D}} - c$$

where the sum is taken over components  $\mathcal{D}$  of  $\mathcal{C}^\nu$  and  $c$  is the number of components of  $\mathcal{C}^\nu$ . Simplifying and using (3.1.2), this evaluates to

$$3 \left( \sum_{\mathcal{D}} g_{\mathcal{D}} + 3c + 3\delta \right) + n - \delta = 3g - 3 + n - \delta.$$

Now, combining (3.2.2) with  $H^0(\mathcal{C}, \mathcal{T}^1) = \delta$ , we get

$$\dim \text{Ext}^1(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) - \dim \text{Ext}^0(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}) = 3g - 3 + n,$$

which proves (ii). Finally, since  $\text{Ext}^0(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}})$  parametrizes the infinitesimal automorphisms of  $(\mathcal{C}, p_1, \dots, p_n)$ , this must vanish by the definition of stability, proving (iii). Alternatively, one can use the combinatorial description of stability and count that  $\deg \mathcal{V}_{\mathcal{D}} < 0$  and thus  $H^0(\mathcal{C}, \mathcal{T}^0) = 0$ . □

The space  $\text{Ext}^2(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}})$  controls the higher-order deformations of  $(\mathcal{C}, p_1, \dots, p_n)$  as in [2, Proposition C.2.4]. Thus by the infinitesimal lifting condition [2, Theorem 4.7.1] we get the following:

**Corollary 3.2.4.** Let  $[\mathcal{C}] \in \overline{\mathcal{M}}_{g,n}(\mathbb{C})$  represent a stable  $n$ -pointed curve  $(\mathcal{C}, p_1, \dots, p_n)$ . Now,  $[\mathcal{C}]$  is a smooth point of  $\overline{\mathcal{M}}_{g,n}$  with tangent space

$$T_{[\mathcal{C}]} \overline{\mathcal{M}}_{g,n} = \text{Ext}^1(\Omega_{\mathcal{C}}^1(D), \mathcal{O}_{\mathcal{C}}).$$

for  $D = p_1 + \dots + p_n$ . In particular,  $\overline{\mathcal{M}}_{g,n}$  is smooth of dimension

$$\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n.$$

### 3.3 Moduli of stable maps

Consider morphisms  $f : \mathcal{C} \rightarrow X$  from the  $n$ -pointed nodal curve  $(\mathcal{C}, p_1, \dots, p_n)$  to  $X$ . In general, we write  $f : (\mathcal{C}, p_1, \dots, p_n) \rightarrow X$  to keep track of the marked points. An isomorphism of  $n$ -pointed maps  $f : (\mathcal{C}, p_1, \dots, p_n) \rightarrow X$  and  $f' : (\mathcal{C}', p'_1, \dots, p'_n) \rightarrow X$  is a morphism  $(\mathcal{C}, p_1, \dots, p_n) \rightarrow (\mathcal{C}', p'_1, \dots, p'_n)$  of pointed curves making the natural diagram commute. We call an  $n$ -pointed map *stable* if its automorphism group is finite.

**Definition 3.3.1.** Let  $(\mathcal{C}/S, \sigma_1, \dots, \sigma_n)$  be a genus  $g$   $n$ -pointed nodal curve over  $S$  and  $\gamma \in H_2(X, \mathbb{Z})_+$ . A stable map  $f : (\mathcal{C}/S, \sigma_1, \dots, \sigma_n) \rightarrow X$  of class  $\gamma$  is a morphism  $f : \mathcal{C} \rightarrow X$  such that for each geometric point  $s \in S$ , the map  $f_s : (\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s)) \rightarrow X$  is stable and  $f_{s*}[\mathcal{C}_s] = \gamma$  in  $H_2(X, \mathbb{Z})$ .

**Definition 3.3.2.** We define the moduli stack of stable maps  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  as

$$\overline{\mathcal{M}}_{g,n}(X, \gamma)(S) = \left\{ \begin{array}{l} \text{nodal } n\text{-pointed curves } (\mathcal{C}/S, \sigma_1, \dots, \sigma_n) \text{ along with} \\ \text{stable maps } f : (\mathcal{C}/S, \sigma_1, \dots, \sigma_n) \rightarrow X \text{ of class } \gamma \end{array} \right\},$$

and  $\mathcal{M}_{g,n}(X, \gamma)$  as the substack with  $\mathcal{C}/S$  smooth.

**Theorem 3.3.3.** [18, 3]

- (i)  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  is a proper Deligne-Mumford stack
- (ii)  $\mathcal{M}_{g,n}(X, \gamma) \subset \overline{\mathcal{M}}_{g,n}(X, \gamma)$  is an open dense substack
- (iii)  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  has a coarse moduli space  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  which is a complex projective variety with quotient singularities.

**Definition 3.3.4.** Define the *evaluation morphism*  $e_i : \overline{\mathcal{M}}_{g,n}(X, \gamma) \rightarrow X$  for  $i = 1, \dots, n$  as the morphism

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}(X, \gamma)(S) & \longrightarrow & X(S) \\ (\mathcal{C}/S, \sigma_1, \dots, \sigma_n, f : \mathcal{C} \rightarrow X) & \longmapsto & f \circ \sigma_i \end{array}$$

over a scheme  $S$ .

If  $[f] \in \overline{\mathcal{M}}_{g,n+1}(X, \gamma)(\mathbb{C})$  given by  $f : (\mathcal{C}, p_1, \dots, p_n, p_{n+1}) \rightarrow X$  is a stable map, it descends to a stable map on  $(\mathcal{C}, p_1, \dots, p_n)^{\text{stab}}$ : If  $f$  is unstable without  $p_{n+1}$ , then  $f$  sends the component containing  $p_{n+1}$  to a point, and  $f$  descends to a map on this contracted curve. Similarly, for  $[f] \in \overline{\mathcal{M}}_{g,n}(X, \gamma)(\mathbb{C})$  given by  $f : (\mathcal{C}, p_1, \dots, p_n) \rightarrow X$ ,  $f$  also lifts to  $(\mathcal{C}, p_1, \dots, p_n, p_i)^{\text{stab}}$  by mapping the resulting bubble to a point. As with Definition 3.1.12, this construction also works in families, and we have an analogue of Theorem 3.1.13 using the same proof.

**Lemma 3.3.5.**

- (i) There exists a forgetful morphism  $\Phi : \overline{\mathcal{M}}_{g,n+1}(X, \gamma) \rightarrow \overline{\mathcal{M}}_{g,n}(X, \gamma)$  along with sections  $\sigma_1, \dots, \sigma_n : \overline{\mathcal{M}}_{g,n}(X, \gamma) \rightarrow \overline{\mathcal{M}}_{g,n+1}(X, \gamma)$  as defined above for  $\mathbb{C}$ -points

(ii)  $\Phi : \overline{\mathcal{M}}_{g,n+1}(X, \gamma) \rightarrow \overline{\mathcal{M}}_{g,n}(X, \gamma)$  defines a universal family for  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$ : For every scheme  $S$ , a family of stable  $n$ -pointed family of genus  $g$  stable maps onto  $X$  ( $\mathcal{C}/S, f, \sigma_1, \dots, \sigma_n$ ), is given by a base change of

$$(\overline{\mathcal{M}}_{g,n+1}(X, \gamma)/\overline{\mathcal{M}}_{g,n}(X, \gamma), e_{n+1}, \sigma_1, \dots, \sigma_n)$$

along a unique morphism  $S \rightarrow \overline{\mathcal{M}}_{g,n}(X, \gamma)$ .

Following this, we often denote  $\overline{\mathcal{M}}_{g,n+1}(X, \gamma)/\overline{\mathcal{M}}_{g,n}(X, \gamma)$  by  $\mathfrak{C}/\overline{\mathcal{M}}_{g,n}(X, \gamma)$  along with the universal map  $e_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \gamma) \rightarrow X$  being written by  $\mathfrak{f} : \mathfrak{C} \rightarrow X$ . This should not be confused with the universal curve  $\mathfrak{C}/\overline{\mathcal{M}}_{g,n}$  although closely related (3.4.1).

*Example 3.3.* A stable morphism  $f : \mathcal{C} \rightarrow X$  of class 0 is always a constant morphism. Thus  $\overline{\mathcal{M}}_{g,n}(X, 0) = \overline{\mathcal{M}}_{g,n} \times X$ .

*Example 3.4.* For  $X = \mathbb{P}^N$  with  $g = 0$ ,  $n = 0$  and  $\gamma = 1$  the generator of  $H_2(X, \mathbb{Z})_+$ , each stable map is from a smooth curve, and such a map is equivalently a line  $\ell \subset \mathbb{P}^N$ . Thus,  $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^N, 1)$  is equivalently the Grassmannian  $\text{Gr}(2, N + 1)$ .

*Example 3.5.* For  $X = \mathbb{P}^2$ , with  $g = 1$ ,  $n = 3$  and  $\gamma = 3$ , things are already more difficult.  $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3)$  has three components, with  $\mathbb{C}$ -points corresponding to:

- (i)  $\mathcal{C} = E$  smooth genus 1 curve, with  $f : \mathcal{C} \rightarrow \mathbb{P}^2$  a parametrization of a smooth cubic
- (ii)  $\mathcal{C} = E \cup \mathbb{P}^1$  a smooth genus 1 curve with a rational tail, and  $f : \mathcal{C} \rightarrow \mathbb{P}^2$  a parametrization of a rational (singular) cubic with  $E$  contracted
- (iii)  $\mathcal{C} = \mathbb{P}^1 \cup E \cup \mathbb{P}^1$  smooth genus 1 curve with two rational tails, and  $f : \mathcal{C} \rightarrow \mathbb{P}^2$  a parametrization of a cubic and a line tangent to it, with  $E$  contracted to their point of tangency.

A naive dimension count shows that locus (i) has dimension 9, locus (ii) dimension 10 and locus (iii) dimension 8. See also [58, 57].

It follows that  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  is not pure-dimensional in general. We still have the following approximation for the dimension:

**Definition 3.3.6.** Let the *virtual dimension* of  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  be

$$\text{vdim } \overline{\mathcal{M}}_{g,n}(X, \gamma) = \int_{\beta} c_1(\mathcal{T}_X) + (\dim X - 3)(1 - g) + n.$$

The virtual dimension can fail to approximate the dimensions of irreducible components very badly, as can be seen from Example 3.3 where the space of obstructions is of size  $g \cdot \dim X$ . It still provides a lower bound for the dimension, and is equal to the dimension in cases like Theorem 3.3.8.

For an unpointed stable map  $f : \mathcal{C} \rightarrow X$ , over  $\mathcal{C}$  the deformation theory of  $f$  is controlled by the cotangent complex  $\mathbb{L}_f$ . Since this fits in the exact triangle

$$f^*\mathbb{L}_X \longrightarrow \mathbb{L}_{\mathcal{C}} \longrightarrow \mathbb{L}_f \longrightarrow,$$

$f^*\mathbb{L}_X \simeq f^*\Omega_X^1[0]$  and  $\mathbb{L}_{\mathcal{C}} \simeq \Omega_{\mathcal{C}/S}^1[0]$  by Lemma 3.2.2, we get that  $\mathbb{L}_f$  is quasi-isomorphically the 2-term complex  $[f^*\Omega_X^1 \longrightarrow \Omega_{\mathcal{C}}^1]$  concentrated in amplitude  $[-1, 0]$ . In the pointed

case, we consider the complex  $\mathcal{E} = [f^*\Omega_X^1 \rightarrow \Omega_{\mathbb{C}}^1(D)]$  on  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  for  $f: \mathcal{C} \rightarrow X$  the universal map and  $D = \sigma_1 + \dots + \sigma_n$ . Over a point  $[f] \in \overline{\mathcal{M}}_{g,n}(X, \gamma)(\mathbb{C})$  represented by  $f: (\mathcal{C}, p_1, \dots, p_n) \rightarrow X$ , this restricts to  $\mathcal{E}_f = [f^*\Omega_X^1 \rightarrow \Omega_{\mathbb{C}}^1(D)]$  with  $D = p_1 + \dots + p_n$  and we naturally get the following:

**Theorem 3.3.7.** *For a  $\mathbb{C}$ -point  $[f]$  of  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$*

$$(i) \ T_{[f]}\overline{\mathcal{M}}_{g,n}(X, \gamma) = \text{Ext}^1(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}) \text{ and } \text{Ext}^0(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}) = 0$$

$$(ii) \ [f] \text{ is a smooth point if } \text{Ext}^2(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}) = 0$$

$$(iii) \ \text{vdim } \overline{\mathcal{M}}_{g,n}(X, \gamma) = \dim \text{Ext}^1(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}) - \dim \text{Ext}^2(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}})$$

*Proof.*  $E = (\mathbf{R}\pi_*\mathcal{E}^\vee)^\vee[-1]$  along with a map  $E \rightarrow \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X, \gamma)}$  as constructed in [48, Section 5.3.5], gives a perfect obstruction theory for  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  in the sense of [9]. Considering a point  $[f] \in \overline{\mathcal{M}}_{g,n}(X, \gamma)(\mathbb{C})$  given by  $f: (\mathcal{C}, p_1, \dots, p_n) \rightarrow X$ , restriction gives  $E_{[f]} = \mathbf{R}\Gamma(\mathcal{C}, \mathcal{E}_f^\vee)^\vee[-1]$  and we have

$$\text{Ext}^i(E_{[f]}, \mathbb{C}) = \mathbf{R}^{i+1}\Gamma(\mathcal{C}, \mathcal{E}_f^\vee) = \text{Ext}^{i+1}(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}).$$

Since by [9, Theorem 4.5] the spaces  $\text{Ext}^i(E_{[f]}, \mathbb{C})$  give the automorphisms, deformations and obstructions of  $[f]$  for  $i = -1, 0, 1$ , (i) and (ii) follow using the infinitesimal lifting condition [2, Theorem 4.7.1]. Alternatively, see [48, Lemma 5.1] For (iii), note that  $\mathcal{E}_f$  fits into the exact triangle

$$f^*\Omega_X^1 \longrightarrow \Omega_{\mathbb{C}}^1(D) \longrightarrow \mathcal{E}_f \longrightarrow,$$

so taking Ext-groups gives a long exact sequence

$$\begin{array}{ccccccc} 0 & \longleftarrow & \text{Ext}^0(\Omega_{\mathbb{C}}^1(D), \mathcal{O}_{\mathcal{C}}) & \longleftarrow & H^0(\mathcal{C}, f^*\mathcal{T}_X) & & \\ & & & \searrow & \nearrow & & \\ \text{Ext}^1(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}) & \longleftarrow & \text{Ext}^1(\Omega_{\mathbb{C}}^1(D), \mathcal{O}_{\mathcal{C}}) & \longleftarrow & H^1(\mathcal{C}, f^*\mathcal{T}_X) & & (3.3.1) \\ & & & \searrow & \nearrow & & \\ \text{Ext}^2(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}) & \longleftarrow & 0 & & & & \end{array}$$

Taking the alternating sum of the dimensions of the complex, it suffices to compute the sum of  $\dim \text{Ext}^1(\Omega_{\mathbb{C}}^1(D), \mathcal{O}_{\mathcal{C}}) - \dim \text{Ext}^0(\Omega_{\mathbb{C}}^1(D), \mathcal{O}_{\mathcal{C}})$  and  $\dim H^0(\mathcal{C}, f^*\mathcal{T}_X) - \dim H^1(\mathcal{C}, f^*\mathcal{T}_X)$ . From Theorem 3.2.3, we know the former evaluates to

$$\dim \text{Ext}^1(\Omega_{\mathbb{C}}^1(D), \mathcal{O}_{\mathcal{C}}) - \dim \text{Ext}^0(\Omega_{\mathbb{C}}^1(D), \mathcal{O}_{\mathcal{C}}) = 3g - 3 + n. \quad (3.3.2)$$

For the latter, consider the normalization  $\mathcal{C}^\nu \rightarrow \mathcal{C}$  and twist the short exact sequence (3.1.1) by  $f^*\mathcal{T}_X$  to get the short exact sequence.

$$0 \longrightarrow f^*\mathcal{T}_X \longrightarrow \nu_*(f\nu)^*\mathcal{T}_X \longrightarrow \bigoplus_{q \text{ node}} \mathbb{C}_q \otimes f^*\mathcal{T}_X \longrightarrow 0. \quad (3.3.3)$$

Taking the long exact sequence in sheaf cohomology, we get

$$\begin{array}{ccccccc}
& & & & & & 0 \\
& & & & & & \swarrow \\
H^0(\mathcal{C}, f^*\mathcal{T}_X) & \longleftarrow & H^0(\mathcal{C}^\nu, (f\nu)^*\mathcal{T}_X) & \longrightarrow & \bigoplus_{q \text{ node}} H^0(q, (f^*\mathcal{T}_X)|_q) & & (3.3.4) \\
& & & & & & \swarrow \\
H^1(\mathcal{C}, f^*\mathcal{T}_X) & \longleftarrow & H^1(\mathcal{C}^\nu, (f\nu)^*\mathcal{T}_X) & \longrightarrow & & & 0
\end{array}$$

Taking the alternating sum of the dimensions of this complex, it suffices to compute the difference of  $\dim H^0(\mathcal{C}^\nu, (f\nu)^*\mathcal{T}_X) - \dim H^1(\mathcal{C}^\nu, (f\nu)^*\mathcal{T}_X)$  and  $\dim \bigoplus_{q \text{ node}} H^0(q, (f^*\mathcal{T}_X)|_q)$ . Using the Hirzebruch–Riemann–Roch theorem on each component  $\mathcal{D}$  of  $\mathcal{C}^\nu$ , the former evaluates to

$$\dim H^0(\mathcal{C}, (\nu f)^*\mathcal{T}_X) - \dim H^1(\mathcal{C}, (\nu f)^*\mathcal{T}_X) = \int_{\gamma} c_1(\mathcal{T}_X) + \dim X \left( \sum_{\mathcal{D}} (1 - g_{\mathcal{D}}) \right),$$

and the latter to  $-\dim X$  times the number of nodes. Taking the difference of these values, and using (3.1.2), we end up with

$$\dim H^0(\mathcal{C}, f^*\mathcal{T}_X) - \dim H^1(\mathcal{C}, f^*\mathcal{T}_X) = \int_{\gamma} c_1(\mathcal{T}_X) + \dim X(1 - g). \quad (3.3.5)$$

Finally, summing (3.3.2) and (3.3.5), we end up with

$$\dim \text{Ext}^1(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}) - \dim \text{Ext}^2(\mathcal{E}_f, \mathcal{O}_{\mathcal{C}}) = \int_{\gamma} c_1(\mathcal{T}_X) + (\dim X - 3)(1 - g) + n,$$

which is by definition the virtual dimension  $\text{vdim } \overline{\mathcal{M}}_{g,n}(X, \gamma)$ .  $\square$

**Theorem 3.3.8.**

- (i) For any map  $f : \mathcal{C} \rightarrow X$  with  $\mathcal{C}$  of genus 0,  $H^1(\mathcal{C}, f^*\mathcal{T}_X) = 0$
- (ii)  $\overline{\mathcal{M}}_{0,n}(X, \gamma)$  is smooth and of pure dimension. Furthermore, identifying  $H_2(X, \mathbb{Z}) \simeq Q^\vee / Q_P^\vee$  as in Corollary 2.4.3, we have

$$\dim \overline{\mathcal{M}}_{0,n}(X, \gamma) = \langle 2\rho_L, \gamma \rangle + \dim X + (n - 3)$$

*Proof.* (i) First, note that  $\mathcal{T}_X$  is globally generated by sections as can be seen in the proof of Lemma 2.3.7, and hence so is  $f^*\mathcal{T}_X$  on  $\mathcal{C}$ . Furthermore, if  $\nu : \mathcal{C}^\nu \rightarrow \mathcal{C}$  is the normalization, then  $(f\nu)^*\mathcal{T}_X$  is also globally generated by sections. Since  $\mathcal{C}^\nu$  is a disjoint union of curves  $\mathbb{P}^1$ , it follows that on each component,  $(f\nu)^*\mathcal{T}_X$  splits as a sum of line bundles of non-negative degree. In particular,  $H^1(\mathcal{C}^\nu, (f\nu)^*\mathcal{T}_X) = 0$ .

Now consider the normalization  $\mathcal{C}^\nu \rightarrow \mathcal{C}$  and twist (3.1.1) by  $f^*\mathcal{T}_X$ . Taking the long exact sequence in sheaf cohomology, we get

$$0 \rightarrow H^0(\mathcal{C}, f^*\mathcal{T}_X) \rightarrow H^0(\mathcal{C}^\nu, (f\nu)^*\mathcal{T}_X) \xrightarrow{r} \bigoplus_{q \text{ node}} H^0(q, (f^*\mathcal{T}_X)|_q) \rightarrow H^1(\mathcal{C}, f^*\mathcal{T}_X) \rightarrow 0.$$

For the vanishing of  $H^1(\mathcal{C}, f^*\mathcal{T}_X)$ , it thus suffices to prove that the restriction map  $r$  is surjective. This can be done inductively as follows:

Since  $\mathcal{C}$  is of arithmetic genus 0, its dual graph is a tree. Now, choose a component  $\mathcal{D}$  of  $\mathcal{C}^\nu$  which corresponds to a leaf in this dual graph and let  $p$  be the unique node on it. Since  $(f\nu)^*\mathcal{T}_X$  is globally generated, we know that the restriction

$$H^0(\mathcal{D}, ((f\nu)^*\mathcal{T}_X)|_{\mathcal{D}}) \longrightarrow H^0(p, (f^*\mathcal{T}_X)|_p) \quad (3.3.6)$$

is surjective. Since  $H^0(\mathcal{C}^\nu, (f\nu)^*\mathcal{T}_X)$  splits as a sum of these components, quotienting  $r$  by 3.3.6 reduces the surjectivity of  $r$  to the same problem for the restriction  $f' : \mathcal{C}' \rightarrow X$  with  $\mathcal{C}'$  being the curve  $\mathcal{C}$  with  $\mathcal{D}$  contracted. Inducting this process, we eventually reach a point where  $\mathcal{C}'$  is smooth and no node remains. Thus  $r$  is surjective, and  $H^1(\mathcal{C}, f^*\mathcal{T}_X) = 0$ .

- (ii) For a point  $[f] \in \overline{\mathcal{M}}_{0,n}(X, \gamma)(\mathbb{C})$ , the long exact sequence (3.3.1) combined with (i), implies that  $\dim \text{Ext}^2(\mathcal{E}_f, \mathcal{O}_C) = 0$ . Thus by Theorem 3.3.7,  $[f]$  is a smooth point with local dimension given by  $\dim \text{Ext}^1(\mathcal{E}_f, \mathcal{O}_C)$ , which evaluates to  $\text{vdim } \overline{\mathcal{M}}_{0,n}(X, \gamma)$ . Finally, by Theorem 2.3.7, we get

$$\dim \overline{\mathcal{M}}_{0,n}(X, \gamma) = \langle 2\rho_L, \gamma \rangle + \dim X + (n - 3).$$

□

**Theorem 3.3.9.** *For  $\gamma \in H_2(X, \mathbb{Z})_+$ ,  $\overline{\mathcal{M}}_{0,n}(X, \gamma)$  is nonempty.*

*Proof.* It suffices to consider a single unpointed stable map  $f : \mathcal{C} \rightarrow X$ . Consider path graphs  $\Gamma_\alpha$  of length  $\langle \omega_\alpha, \gamma \rangle$  for each  $\alpha \in \Delta \setminus \Delta_P$  with a chosen endpoint. Now form a graph  $\Gamma$  as follows

- If there is a single nonempty  $\Gamma_\alpha$ , add a special half-edge  $h$  to the chosen endpoint, and call the resulting graph  $\Gamma$ .
- If there are two nonempty  $\Gamma_\alpha$ , connect the chosen endpoints by a special edge  $e$ , and call the resulting graph  $\Gamma$ .
- If there are at least three nonempty  $\Gamma_\alpha$ , connect the chosen endpoints to a special vertex  $v$ , and call the resulting graph  $\Gamma$ .

Consider a nodal curve  $\mathcal{C}$  of genus 0 with dual graph  $\Gamma$ , with a half-edge  $h$ , corresponding to a marked point, an edge  $e$  to a node, and  $v$  to an irreducible component. Whether  $\Gamma$  has a special  $h$ ,  $v$ , or  $e$ , send it to the point  $x_1 \in X$ . Also, send each component  $\Gamma_\alpha$  bijectively to  $C_\alpha = X_{s_\alpha}^-$  in a well-defined manner. This gives

$$f_*[\mathcal{C}] = \sum_{\alpha \in \Delta \setminus \Delta_P} \langle \omega_\alpha, \gamma \rangle \cdot [X_{s_\alpha}^-] = \sum_{\alpha \in \Delta \setminus \Delta_P} \langle \omega_\alpha, \gamma \rangle \cdot \tau_{s_\alpha} = \gamma.$$

Since the only irreducible component sent to a point already has three nodes,  $f$  is stable. □

*Remark 3.3.10.* In [50], smooth rational curves of all classes  $\gamma$  for  $X = G/P$  are constructed, except for the case of

$$\mathbb{P}^1 = \mathrm{SL}_2/B, \quad \mathbb{P}^2 = \mathrm{SL}_3/B, \quad \mathbb{P}^1 \times \mathbb{P}^1 = \mathrm{SO}_4/P.$$

In the two latter cases, their (bi)degree-genus formulae bound the degree of a smooth rational curve. However, nodal representatives can be exhibited easily.

*Remark 3.3.11.* An existence of a rational curve is not guaranteed in general. For example, an abelian variety  $A$  of dimension 2 has  $h^{3,3}(A) = 2$  yet these classes are not represented by rational curves: If there were a non-constant map  $\mathbb{P}^1 \rightarrow A$ , pulling back a nonzero holomorphic 1-form along it would produce a nonzero holomorphic 1-form on  $\mathbb{P}^1$ , which is absurd.

**Theorem 3.3.12.** [32] *For  $X = G/P$ , the space  $\overline{\mathcal{M}}_{g,n}(X, \beta)$  is connected.*

### 3.4 Gromov–Witten invariants

As a proper Deligne–Mumford stack, the moduli space  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  has a rational Chow group  $A^k(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q})$  of codimension  $k$  closed integral substacks by [59]. Similarly,  $\overline{\mathcal{M}}_{g,n}(X, \gamma)(\mathbb{C})$ , also written  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  as abuse of notation, acquires the structure of a topological stack. As in [6], we can thus talk about its (singular) homology  $H_*(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q})$  and cohomology  $H^*(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q})$ . We also have an operator

$$A_k(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q}) \longrightarrow H_{2k}(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q})$$

sending a closed integral dimension  $k$  substack to the corresponding cycle.

**Definition 3.4.1.** Let  $\gamma \in H_2(X, \mathbb{Z})_+$ , and  $\phi_1, \dots, \phi_n \in H^*(X, \mathbb{Q})$ . We define the genus  $g$   $n$ -point Gromov–Witten invariant of these classes to be the rational number

$$\langle \phi_1, \dots, \phi_n \rangle_{g,n,\gamma} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\gamma)]^{\mathrm{vir}}} e_1^* \phi_1 \cdots e_n^* \phi_n.$$

where the *virtual fundamental class*  $[\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q})$  is as defined in Section 3.5.

Consider the projection  $\eta : \overline{\mathcal{M}}_{g,n}(X, \gamma) \rightarrow \overline{\mathcal{M}}_{g,n}$  defined by  $(\mathcal{C}/S, f, \sigma_1, \dots, \sigma_n) \mapsto (\mathcal{C}/S, \sigma_1, \dots, \sigma_n)^{\mathrm{stab}}$  over a scheme  $S$ , and call a pair  $(g, n)$  *admissible* if it is not one of  $(0, 0)$ ,  $(0, 1)$ ,  $(0, 2)$ ,  $(1, 0)$ . The *tree-level* strenghtenings

$$\mathfrak{J}_{g,n,\gamma}(\phi_1, \dots, \phi_n) = (\eta_*(e_1^* \phi_1 \cdots e_n^* \phi_n \cap [\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\mathrm{vir}}))^* \in H^{\mathrm{vdim}-3g+3-n}(\overline{\mathcal{M}}_{g,n})$$

of GW-invariants for admissible  $(g, n)$  satisfy the *Gromov–Witten axioms* [38]. These give a basis for a cohomological field theory, and their integrals over  $\overline{\mathcal{M}}_{g,n}$  recover the numerical GW-invariants as defined earlier. We will not state the axioms except for Lemmas 3.4.3, 3.4.4, 3.4.5 and 3.4.6, which are integral to our computations.

*Remark 3.4.2.* Although the tree-level GW-invariants  $\mathfrak{J}_{g,n,\gamma}$  are not defined for non-admissible  $(g, n)$ , their numerical counterparts are still extremely valuable. The following results will be stated in terms of the tree-level invariants, but we will also state the numerical results for these non-admissible  $(g, n)$ .

**Lemma 3.4.3** ( $S_n$ -invariance). *The tree-level GW-invariant operator*

$$\mathfrak{I}_{g,n,\gamma} : H^*(X, \mathbb{Q})^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

is  $S_n$ -invariant, with the symmetric group  $S_n$  acting on  $H^*(X, \mathbb{Q})^{\otimes n}$  via the natural action. In particular,  $\langle -, \dots, - \rangle_{g,n,\gamma}$  is independent of the order of elements.

We are especially interested in 3-point genus 0 GW-invariants. These classes are exceptional in the sense that no well-defined  $\mathfrak{I}_{0,n,\gamma}$  for  $n < 3$  exists and thus the operators  $\mathfrak{I}_{0,3,\gamma}$  are "new" in some sense.

**Lemma 3.4.4** (Mapping to point in genus 0). *For  $\gamma = 0$ ,  $g = 0$ , and  $n \geq 3$  we have*

$$\mathfrak{I}_{0,n,0}(\phi_1, \dots, \phi_n) = \left( \int_X \phi_1 \cdots \phi_n \right) \cdot [\overline{\mathcal{M}}_{0,n}]^*.$$

In particular, the GW-invariants  $\langle \phi_1, \dots, \phi_n \rangle_{0,n,0}$  vanish in all cases except for

$$\langle \phi_1, \phi_2, \phi_3 \rangle_{0,3,0} = \int_X \phi_1 \cdot \phi_2 \cdot \phi_3.$$

*Proof.* Since  $\overline{\mathcal{M}}_{0,n}(X, 0) = \overline{\mathcal{M}}_{0,n} \times X$  with virtual fundamental class  $[\overline{\mathcal{M}}_{0,n} \times X]$ , the evaluation morphisms  $e_i : \overline{\mathcal{M}}_{0,n} \rightarrow X$  identify with projection  $q : \overline{\mathcal{M}}_{0,n} \times X \rightarrow X$ . It thus follows that  $e_1^* \phi_1 \cdots e_n^* \phi_n = q^*(\phi_1 \cdots \phi_n)$  and

$$\begin{aligned} \mathfrak{I}_{0,n,0}(\phi_1, \dots, \phi_n)^* &= \eta_*([\overline{\mathcal{M}}_{0,n} \times X] \cap q^*(\phi_1 \cdots \phi_n)) \\ &= \eta_*([\overline{\mathcal{M}}_{0,n}] \boxtimes [X] \cap [\overline{\mathcal{M}}_{0,n}] \boxtimes \phi_1 \cdots \phi_n) \\ &= \eta_*([\overline{\mathcal{M}}_{0,n}] \boxtimes ([X] \cap \phi_1 \cdots \phi_n)). \end{aligned}$$

Now if  $[X] \cap \phi_1 \cdots \phi_n$  is of positive degree, the GW-invariant vanishes for dimensional reasons. If it is of degree 0, the GW-invariant evaluates to  $\int_X \phi_1 \cdots \phi_n$  and we get

$$\mathfrak{I}_{0,n,0}(\phi_1, \dots, \phi_n)^* = \left( \int_X \phi_1 \cdots \phi_n \right) \cdot [\overline{\mathcal{M}}_{0,n}].$$

Taking Poincaré duals again, we get the desired result for tree-level invariants. Finally, integrating over  $\overline{\mathcal{M}}_{0,n}$  we get the result for numerical invariants for  $n \geq 3$  by noting that  $\overline{\mathcal{M}}_{0,3} = *$ . For non-admissible  $(0, n)$ , the proof is exactly the same, but mapping to a point instead of mapping via  $\eta$ .  $\square$

**Lemma 3.4.5** (Fundamental class insertion). *Let  $\mathbf{1} = [X]^* \in H^0(X, \mathbb{Q})$  be the identity. We have*

$$\mathfrak{I}_{g,n+1,\gamma}(\phi_1, \dots, \phi_n, \mathbf{1}) = \pi^* \mathfrak{I}_{g,n,\gamma}(\phi_1, \dots, \phi_n).$$

for  $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  the forgetful map. In particular, the GW-invariants  $\langle \phi_1, \dots, \phi_n, \mathbf{1} \rangle_{0,n+1,\gamma}$  vanish in all cases except for

$$\langle \phi_1, \phi_2, \mathbf{1} \rangle_{0,3,0} = \int_X \phi_1 \cdot \phi_2.$$

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{e_i} \overline{\mathcal{M}}_{g,n+1}(X, \gamma) & \xrightarrow{\eta} \overline{\mathcal{M}}_{g,n+1} \\ & \searrow e_i & \downarrow \Phi \\ & & \overline{\mathcal{M}}_{g,n}(X, \gamma) & \xrightarrow{\eta} \overline{\mathcal{M}}_{g,n} \end{array} \quad (3.4.1)$$

with the right square 2-Cartesian. By [7, Axiom IV.],  $\Phi$  admits a Gysin pullback  $\Phi^!$  on homology satisfying  $\Phi^![\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} = [\overline{\mathcal{M}}_{g,n+1}(X, \gamma)]^{\text{vir}}$ . Also, the evaluation maps  $e_i$  on  $\overline{\mathcal{M}}_{g,n+1}(X, \gamma)$  factor as  $e_i \circ \Phi$ , and  $e_{n+1}^* \mathbf{1} \in H^0(\overline{\mathcal{M}}_{g,n+1}(X, \gamma), \mathbb{Z})$  is the identity, and thus also the identity for the cap product. It follows that

$$\begin{aligned} \mathfrak{J}_{g,n+1,\gamma}(\phi_1, \dots, \phi_n, \mathbf{1})^* &= \eta_*([\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} \cap e_1^* \phi_1 \cdots e_n^* \phi_n) \cap e_{n+1} \mathbf{1} \\ &= \eta_*(\Phi^![\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} \cap \Phi^*(e_1^* \phi_1 \cdots e_n^* \phi_n)) \\ &= \eta_* \Phi^!([\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} \cap e_1^* \phi_1 \cdots e_n^* \phi_n) \\ &= \pi^! \eta_*([\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} \cap e_1^* \phi_1 \cdots e_n^* \phi_n) \\ &= \pi^! \mathfrak{J}_{g,n,\gamma}(\phi_1, \dots, \phi_n)^*, \end{aligned}$$

where the second last equality follows from the Cartesianity of (3.4.1). Taking Poincaré duals, we get the desired result for the tree-level invariants. Taking integrals, dimensional reasons give the vanishing of numerical GW-invariants for admissible pairs. For non-admissible pairs  $(g, n)$  with  $\gamma \neq 0$ , the same argument follows by mapping to a point instead of mapping by  $\eta$ , and for  $(g, n) = (0, 0), (0, 1), (1, 0)$  and  $\gamma = 0$  the result is trivial. For  $(g, n, \gamma) = (0, 2, 0)$ , direct computation as in the proof Lemma 3.4.4 gives the result.  $\square$

**Lemma 3.4.6** (Divisor insertion). *Let  $D \in H^2(X, \mathbb{Q})$  be a divisor. Now*

$$\pi_! \mathfrak{J}_{g,n+1,\gamma}(\phi_1, \dots, \phi_n, D) = \left( \int_{\gamma} D \right) \cdot \mathfrak{J}_{g,n,\gamma}(\phi_1, \dots, \phi_n).$$

In particular,

$$\langle \phi_1, \dots, \phi_n, D \rangle_{g,n+1,\gamma} = \left( \int_{\gamma} D \right) \cdot \langle \phi_1, \dots, \phi_n \rangle.$$

*Proof.* Noting that  $\pi \circ \eta$  on  $\overline{\mathcal{M}}_{g,n+1}(X, \gamma)$  is equivalent to as  $\eta \circ \Phi$ , and following along the proof of Lemma 3.4.5, we get

$$\pi_* \mathfrak{J}_{g,n+1,\gamma}(\phi_1, \dots, \phi_n, D)^* = \eta_* \Phi_*(\Phi^!([\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} \cap e_1^* \phi_1 \cdots e_n^* \phi_n) \cap e_{n+1}^* D).$$

Using the projection formula, this evaluates to

$$\eta_*([\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} \cap e_1^* \phi_1 \cdots e_n^* \phi_n) \cap \Phi_! e_{n+1}^* D \quad (3.4.2)$$

Now  $\Phi_! e_{n+1}^* D \in H_0(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q})$  is a multiple of the identity  $\mathbf{1} \in H_0(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q})$ . Since  $\Phi_!$  is integration over the fibers of the universal curve  $\Phi : \overline{\mathcal{M}}_{g,n+1}(X, \gamma) \rightarrow \overline{\mathcal{M}}_{g,n}(X, \gamma)$  (see Lemma 3.3.5), this multiple is given by

$$\int_{\mathcal{C}} e_{n+1}^* D = \int_{e_{n+1}^*[\mathcal{C}]} D = \int_{\gamma} D$$

for any  $\mathbb{C}$ -point of  $f : \mathcal{C} \rightarrow X$  of  $\overline{\mathcal{M}}_{g,n+1}(X, \gamma)$ . Thus (3.4.2) evaluates to

$$\left( \int_{\gamma} D \right) \cdot \eta_*([\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} \cap e_1^* \phi_1 \cdots e_n^* \phi_n) = \left( \int_{\gamma} D \right) \cdot \mathfrak{J}_{g,n,\gamma}(\phi_1, \dots, \phi_n)^*.$$

Taking Poincaré duals, the result follows for tree-level invariants. Integrating, the result also follows for numerical invariants for admissible pairs. Case-checking as earlier gives the numerical result also for non-admissible pairs  $(g, n)$ .  $\square$

**Lemma 3.4.7.** *For fixed  $\phi_1, \dots, \phi_n \in H^*(X, \mathbb{Q})$ , there are only finitely many  $\gamma \in H_2(X, \mathbb{Z})_+$  such that*

$$\langle \phi_1, \dots, \phi_n \rangle_{g,n,\gamma} \neq 0$$

*Proof.* This only relies on  $X$  being a Fano variety as shown in Theorem 2.3.7. Namely, the operator  $\int_{(-)} c_1(\mathcal{T}_X)$  on  $H_2(X, \mathbb{Z})_+$ , and thus also  $\text{vdim } \overline{\mathcal{M}}_{g,n}(X, (-))$ , has finite sublevel sets. Now, choose  $L > 0$  such that  $\phi_1 \cdots \phi_n$  is of degree less than  $L$ . Then  $e_1^* \phi_1 \cdots e_n^* \phi_n$  is also of degree less than  $L$ , and integrating over  $[\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}}$  gives a class of degree less than  $L - \text{vdim } \overline{\mathcal{M}}_{g,n}(X, \gamma)$ . But now the set

$$S_L = \{\gamma \in H_2(X, \mathbb{Z})_+ \mid \text{vdim } \overline{\mathcal{M}}_{g,n}(X, \gamma) \leq L\} \subset H_2(X, \mathbb{Z})_+$$

is a convex finite set, and  $I_\gamma$  vanishes outside of it, and thus the result follows.  $\square$

**Theorem 3.4.8.** *Let  $n \geq 3$  and  $\gamma \neq 0$ , and  $Y_i \subset X$  for  $i = 1, \dots, n$  be pure-dimensional subvarieties. For  $(g_1, \dots, g_n) \in G^n$  a general element and  $\gamma$ , the GW-invariant  $\langle [Y_1]^*, \dots, [Y_n]^* \rangle_{0,n,\gamma}$  is the count of rational curves in  $X$  of class  $\gamma$  intersecting all  $g_i Y_i$  if  $\sum_i \text{codim } Y_i = \dim \overline{\mathcal{M}}_{0,n}(X, \gamma)$ , and vanishes otherwise.*

*Proof.* Let  $\mathcal{M}_{0,n}^*(X, \gamma) \subset \mathcal{M}_{0,n}(X, \gamma)$  be the automorphism-free locus, which is dense by [18, Lemma 13] as well as a scheme, and let  $\partial$  be the complement of  $\mathcal{M}_{0,3}^*(X, \gamma) \subset \overline{\mathcal{M}}_{0,3}(X, \gamma)$ . Let  $Y = Y_1 \times \dots \times Y_n$ , and let  $Y^{\text{sm}}$  be the smooth locus of  $Y$  and  $Y^{\text{sing}}$  the singular locus. Recall Lemma 2.4.6 with  $X = (G/P)^n$  and choose  $g = (g_1, \dots, g_n) \in G^n$  general. Applying it on

$$Y \longrightarrow X^n \longleftarrow \overline{\mathcal{M}}_{0,n}(X, \gamma)$$

gives  $gY \times_{X^n} \overline{\mathcal{M}}_{0,n}(X, \gamma)$  either empty or dimension 0. Applying on

$$Y^{\text{sing}} \longrightarrow X^n \longleftarrow \overline{\mathcal{M}}_{0,n}(X, \gamma),$$

we learn that  $gY^{\text{sing}} \times_{X^n} \overline{\mathcal{M}}_{0,n}(X, \gamma)$  must be empty since  $\dim Y^{\text{sing}} < \dim Y$ . Applying on

$$Y \longrightarrow X^n \longleftarrow \partial$$

we learn that  $gY \times_{X^n} \partial$  must also be empty. Finally, applying on

$$Y^{\text{sm}} \longrightarrow X^n \longleftarrow \mathcal{M}_{0,n}^*(X, \gamma)$$

we learn that  $gY^{\text{sm}} \times_{X^n} \mathcal{M}_{0,n}^*(X, \gamma)$  is either empty or smooth of dimension 0. It thus follows that

$$gY \times_{X^n} \overline{\mathcal{M}}_{0,n}(X, \gamma) = e_1^{-1}(g_1 Y_1) \cap \dots \cap e_n^{-1}(g_n Y_n) \quad (3.4.3)$$

consists of reduced points of  $\overline{\mathcal{M}}_{0,n}(X, \gamma)$  lying in the automorphism-free locus  $\mathcal{M}_{0,n}^*(X, \gamma)$ . But this parametrizes precisely maps  $f : \mathbb{P}^1 \rightarrow X$  with  $f_*[\mathbb{P}^1] = \gamma$  with  $p_i \in g_i Y_i$  for all  $i$ . Since  $f$  has no pointed automorphisms, this data is equivalent to  $f(\mathbb{P}^1)$  along with  $f(p_1), \dots, f(p_n)$ . Arguing as in [37, Proposition 4.1.5], we see that the marked points are irrelevant, and thus (3.4.3) parametrizes rational curves intersecting all  $g_i Y_i$ .

Now, considering the 2-Cartesian square

$$\begin{array}{ccc} gY \times_{X^n} \overline{\mathcal{M}}_{0,n}(X, \gamma) & \longrightarrow & \overline{\mathcal{M}}_{0,n}(X, \gamma) \times gY \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{0,n}(X, \gamma) & \xrightarrow{i} & \overline{\mathcal{M}}_{0,n}(X, \gamma) \times X^n \end{array}$$

with  $i$  the graph and thus a regular embedding, we see that

$$[gY \times_{X^n} \overline{\mathcal{M}}_{0,n}(X, \gamma)] = i^*[\overline{\mathcal{M}}_{0,n}(X, \gamma) \times gY]$$

in the Chow ring  $A_0(\overline{\mathcal{M}}_{0,n}(X, \gamma))$ , and finally this can be computed to be  $\prod_i e_1^*[g_i Y_i] \cap [\overline{\mathcal{M}}_{0,n}(X, \gamma)]$ , which finally integrates to  $\langle [g_1 Y_1], \dots, [g_n Y_n] \rangle_{0,n,\gamma} = \langle [Y_1], \dots, [Y_n] \rangle_{0,n,\gamma}$ .  $\square$

**Corollary 3.4.9.** For  $g = 0$ , GW-invariants of classes in  $H^*(X, \mathbb{Z})$  on  $X$  are elements of  $\mathbb{Z}$ .

Consider the  $T$ -action on  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  given by acting on  $X$ , so that the evaluation morphisms  $e_i$  are also  $T$ -equivariant. The  $T$ -equivariant virtual fundamental class can similarly be defined as the pushforward of  $[\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}}$  along  $t : X \rightarrow [T \backslash X]$ .

**Definition 3.4.10.** For  $\phi_1, \dots, \phi_n \in H_T^*(X, \mathbb{Q})$ , we define the  $T$ -equivariant GW-invariant to be the element of  $H_T^*(*, \mathbb{Q}) = \mathbb{Q}[Q]$  given by

$$\langle \phi_1, \dots, \phi_n \rangle_{g,n,\gamma}^T = \int_{[\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}}} e_1^* \phi_1 \cdots e_n^* \phi_n.$$

These  $T$ -equivariant invariants are no more enumerative but their theory is much richer and all the earlier non-enumerative results have  $T$ -equivariant analogues. For example, in the foundational paper [23] Graber–Pandharipande gave an analogue of the Atiyah–Bott localization formula for virtual fundamental classes. Using this, they computed a recursion for all GW-invariants of  $\mathbb{P}^n$  in terms of decorated graphs. These same combinatorics can be used to give one for  $X = G/P$  [42, 27].

**Lemma 3.4.11.** For  $\phi_1, \dots, \phi_n \in H_T^*(X, \mathbb{Q}) = H^*([T \backslash X], \mathbb{Q})$  and  $\phi'_1, \dots, \phi'_n$  their pullbacks along  $X \rightarrow [T \backslash X]$ , the reduction of  $\langle \phi_1, \dots, \phi_n \rangle_{g,n,\gamma}^T \in \mathbb{Q}[Q]$  along  $\mathbb{Q}[Q] \rightarrow \mathbb{Q}$  is  $\langle \phi'_1, \dots, \phi'_n \rangle_{n,g,\gamma}$ .

*Proof.* This follows directly by observing the commutativity of the square

$$\begin{array}{ccc} X & \xleftarrow{e_i} & \overline{\mathcal{M}}_{g,n}(X, \gamma) \\ \downarrow & & \downarrow t \\ [T \backslash X] & \xleftarrow{e_i} & T \backslash \overline{\mathcal{M}}_{g,n}(X, \gamma) \end{array}$$

and using the projection formula  $\phi \cap t_*[\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} = t_*(t^* \phi \cap [\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}})$ .  $\square$

**Lemma 3.4.12.** For  $g = 0$ , the  $T$ -equivariant GW-invariants of classes in  $H_T^*(X, \mathbb{Z})$  are elements of  $\mathbb{Z}[Q]$ .

*Proof.* Proceeding exactly as in the proof of Theorem 3.4.8, we see that intersections occur generally on the automorphism-free smooth  $\mathcal{M}_{0,n}^*(X, \gamma)$  part of  $\overline{\mathcal{M}}_{0,n}(X, \gamma)$ . Thus all computations remain with  $\mathbb{Z}$ -coefficients.  $\square$

### 3.5 Virtual fundamental class

When no good fundamental class is available for a Deligne-Mumford stack  $\mathcal{X}$ , the common strategy is to find a cone  $\mathcal{C} \rightarrow \mathcal{X}$  of pure dimension  $c$  embedding in a vector bundle  $\mathcal{E} \rightarrow \mathcal{X}$  of rank  $e$ , and considering the sequence

$$A_*(\mathcal{C}, \mathbb{Q}) \longrightarrow A_*(\mathcal{E}, \mathbb{Q}) \xrightarrow{\simeq} A_{*-e}(\mathcal{X}, \mathbb{Q}) \quad (3.5.1)$$

where the first map is given by pushforward and the second by flat pullback. Using this, we define the *virtual dimension*  $\text{vdim } \mathcal{X} = c - e$  and the *virtual fundamental class*  $[\mathcal{X}]^{\text{vir}} \in A_{\text{vdim } \mathcal{X}}(\mathcal{X}, \mathbb{Q})$  to be the image of the fundamental class  $[\mathcal{C}] \in A_c(\mathcal{C}, \mathbb{Q})$  along these morphisms.

In [7, 48], the required data is constructed from the perfect obstruction theory  $\phi : E \rightarrow \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X, \gamma)}$ . The complex  $E^\vee$  can be written as a complex of vector bundles  $[E^0 \rightarrow E^1]$  concentrated in  $[0, 1]$ , and thus we can define the vector bundle stack  $[E^1/E^0]$  as in [9]. The perfect obstruction theory gives a closed embedding  $\phi^\vee : \mathfrak{C} \rightarrow [E^1/E^0]$  of the intrinsic normal cone of  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$ , giving a 2-Cartesian square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & E^1 \\ \downarrow & & \downarrow \\ \mathfrak{C} & \longrightarrow & [E^1/E^0] \end{array}$$

By consider (3.5.1) on  $\mathcal{C} \rightarrow E^1 \rightarrow \overline{\mathcal{M}}_{g,n}(X, \gamma)$  and noting that  $\dim \mathcal{C} - \text{rk } E^1 = \text{vdim}$ , we get a virtual fundamental class for  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$ .

We sketch a more concrete construction for partial flag varieties  $X = G/P$  given in [7], abusing the geometry of torsors as in Lemma 3.5.2. Recall that by Theorem 2.3.2, a basis for  $\text{Pic}(X)$  is given by the bundles  $\mathcal{L}_{\omega_\alpha}$  for  $\alpha \in \Delta \setminus \Delta_P$  with their first Chern classes giving a basis for  $H^2(X, \mathbb{Z})$ .

**Definition 3.5.1.** Let  $\mathcal{C}$  be a nodal connected curve of genus 0. For  $\mathcal{B} \rightarrow \mathcal{C}$  a  $P$ -torsor, define the multidegree of  $\mathcal{B}$  as a tuple

$$\left[ \int_{\mathcal{C}} c_1(\mathcal{B} \times^P \mathcal{L}_{\omega_\alpha}) \right]_{\alpha \in \Delta \setminus \Delta_P} \in \mathbb{Z}^{\Delta \setminus \Delta_P}$$

For a map  $f : \mathcal{C} \rightarrow X$  of class  $\gamma \in H_2(X, \mathbb{Z})_+$ , and  $f^*G \rightarrow \mathcal{C}$  the associated  $P$ -torsor, we see that  $f^*G \times^P \mathcal{L}_{\omega_\alpha}$  is the line bundle  $f^*\mathcal{L}_\lambda$ . Hence  $f^*G \rightarrow \mathcal{C}$  has multidegree  $\left[ \int_{\mathcal{C}} \sigma_{s_\alpha} \right]_{\alpha \in \Delta \setminus \Delta_P} = [\langle \omega_\alpha, \gamma \rangle]_{\alpha \in \Delta \setminus \Delta_P}$  using Theorem 2.4.2. We may thus consider the multidegree of a torsor to be an effective curve class via Corollary 2.4.3.

**Lemma 3.5.2.**  $\mathcal{C}$  be a connected nodal curve. The following data up to the natural isomorphisms are equivalent:

- (i) Morphism  $f : \mathcal{C} \rightarrow X$  of class  $\gamma$ , up to the  $G$ -action on  $X$
- (ii)  $P$ -torsor  $\mathcal{B} \rightarrow \mathcal{C}$  of multidegree  $\gamma$ , such that there exists a trivialization of  $G$ -bundles  $\mathcal{B} \times^P (G \times \mathcal{C}) \simeq G \times \mathcal{C}$ .

*Proof.* A morphism  $f : \mathcal{C} \rightarrow X$  corresponds to a  $P$ -torsor  $\mathcal{B} \rightarrow \mathcal{C}$  with a  $P$ -equivariant map  $\mathcal{B} \rightarrow G \times \mathcal{C}$  where  $P$  acts on  $G \times \mathcal{C}$  via translations from the right on  $G$ . This induces a trivialization  $\mathcal{B} \times^P (G \times \mathcal{C}) \simeq G \times \mathcal{C}$  of  $G$ -bundles. Similarly, consider a  $P$ -torsor  $\mathcal{B} \rightarrow \mathcal{C}$  with a trivialization  $\mathcal{B} \times^P (G \times \mathcal{C}) \simeq G \times \mathcal{C}$ . For a choice of  $g \in G$ , We get a  $P$ -equivariant map

$$\mathcal{B} \xrightarrow{\text{id} \times g} \mathcal{B} \times^P (G \times \mathcal{C}) \simeq G \times \mathcal{C} \rightarrow G,$$

giving a map  $\mathcal{C} \rightarrow G/P$ . Since automorphisms of the trivial  $G$ -bundle  $G \times \mathcal{C}$  are translations of  $G$ , the trivialization is irrelevant up to the  $G$ -action on morphisms. Finally, the preservation of the class and multidegree follows from the earlier discussion.  $\square$

The same argument even works over families, which is the content of Theorem 3.5.6. Before stating this theorem, we need to define various stacks.

**Definition 3.5.3.** Let  $\mathfrak{M}_{g,n}$  be the stack of all genus  $g$  nodal  $n$ -pointed curves. Namely, over a scheme  $T$  the stack  $\mathfrak{M}_{g,n}$  is the groupoid of algebraic spaces  $f : \mathcal{C} \rightarrow T$  with  $n$  non-intersecting smooth sections  $\sigma_1, \dots, \sigma_n : T \rightarrow \mathcal{C}$  as in Definition 3.1.6.

This is indeed an Artin stack by [54, Theorem 99.15.11] and by noting that the nodal locus is always open in a family of curves. Although  $\mathfrak{M}_{g,n}$  is not a Deligne-Mumford stack nor is it separated, it is still smooth by the same deformation theory arguments. In addition,  $\overline{\mathfrak{M}}_{g,n}$  is an open substack of  $\mathfrak{M}_{g,n}$ : Suppose  $f : \mathcal{C} \rightarrow T$  is given by  $T \rightarrow \mathfrak{M}_{g,n}$ , with  $f$  projective and  $T$  locally Noetherian<sup>2</sup>. Then the stable locus  $\overline{\mathfrak{M}}_{g,n} \times_{\mathfrak{M}_{g,n}} T$  is the locus where  $\varpi = \omega_{\mathcal{C}/T}(\sigma_1 + \dots + \sigma_n)^{\otimes 3}$  is relatively very ample, equivalently  $H^1(\mathcal{C}_t, \varpi_{\mathcal{C}_t}) = 0$  vanishes. But by upper semicontinuity of sheaf cohomology, this is a closed set.

**Definition 3.5.4.** For  $\gamma \in H_2(X, \mathbb{Z})_+$ , let  $\mathfrak{M}_{g,n}(X, \gamma)$  be the stack of maps to  $X$  from genus  $g$  nodal  $n$ -pointed curves of class  $\gamma$ . Namely, over a scheme  $T$  the stack  $\mathfrak{M}_{g,n}(X, \gamma)$  is the groupoid of algebraic spaces data  $\mathcal{C} \rightarrow T$  and  $\sigma_1, \dots, \sigma_n : T \rightarrow \mathcal{C}$  as in Definition 3.5.3 and a map  $f : \mathcal{C} \rightarrow X$  with  $f_{t*}[\mathcal{C}_t] = \gamma$  for all geometric points  $t$ .

Demonstrating that this satisfies Artin's axioms as in [54, Theorem 99.15.11] shows that this is indeed an Artin stack. Again,  $\overline{\mathfrak{M}}_{g,n}(X, \gamma)$  is an open substack by noting that the stable locus of  $f : \mathcal{C} \rightarrow X$  is the locus of relative very ampleness of the line bundle  $\omega_{\mathcal{C}/T}(\sigma_1 + \dots + \sigma_n)^{\otimes 3} \otimes f^* \mathcal{K}$  for some very ample line bundle  $\mathcal{K}$  on  $X$ . In addition, there is a canonical bundle for  $\mathfrak{M}_{g,n}(X, \gamma) \rightarrow \mathfrak{M}_{g,n}$  forgetting the map without stabilizing.

**Definition 3.5.5.** Let  $H$  be a linear algebraic group, and  $\gamma \in H_2(X, \mathbb{Z})_+$

- (i) Let  $\mathfrak{h}(H)$  be the stack of  $n$ -pointed nodal genus  $g$  curves and  $H$ -torsors over them. Namely, over a scheme  $T$ ,  $\mathfrak{h}(H)$  is the groupoid of  $\mathcal{C} \rightarrow T, \sigma_1, \dots, \sigma_n : T \rightarrow \mathcal{C}$  as in Definition 3.5.3 and  $\mathcal{H} \rightarrow \mathcal{C}$  a  $H$ -torsor.
- (ii) Let  $\mathfrak{h}(P, \gamma)$  be the stack of  $n$ -pointed nodal genus  $g$  curves and  $P$ -torsors of multidegree  $\gamma$  over them. Namely, over a scheme  $T$ ,  $\mathfrak{h}(P, \gamma)$  is the groupoid of  $\mathcal{C} \rightarrow T, \sigma_1, \dots, \sigma_n : T \rightarrow \mathcal{C}$  as in Definition 3.5.3, and  $\mathcal{B} \rightarrow \mathcal{C}$  a  $P$ -torsor of multidegree  $\gamma$ .

<sup>2</sup>This suffices since  $f$  can be projectivized over an étale cover, [2, Proposition 5.4.1] and  $\mathfrak{M}_{g,n}$  is locally Noetherian [2, Theorem 5.4.6]

- (iii) Let  $\mathfrak{h}(P, \gamma) \rightarrow \mathfrak{h}(G)$  be the map sending the  $P$ -bundle to its induced  $G$ -bundle. Namely, it sends  $(\mathcal{C} \rightarrow T, \sigma_i, \mathcal{B} \rightarrow \mathcal{C})$  to  $(\mathcal{C} \rightarrow T, \sigma_i, \mathcal{B} \times^P (G \times \mathcal{C}) \rightarrow \mathcal{C})$ .

Both are again Artin stacks: the algebraicity of  $\mathfrak{h}(H)$  can be shown following the proof for  $\mathfrak{M}_{g,n}$  and  $\text{Bun}_H(\mathcal{C})$ . In addition, one can observe that the locus of fixed multidegree of a  $P$ -bundle is clopen, and thus  $\mathfrak{h}(P, \gamma)$  is clopen in  $\mathfrak{h}(P)$  and thus algebraic.

Let  $G$  act on  $\mathfrak{M}_{g,n}(X, \gamma)$  by acting on  $X$  via left translations. Over a scheme  $T$ , this is the groupoid of  $\mathcal{G} \rightarrow T$  a  $G$ -torsor and  $\mathcal{G} \leftarrow \mathcal{C} \rightarrow X$  with the first map a  $G$ -equivariant curve, second of class  $\gamma$ , and everything  $G$ -equivariant, with sections  $\sigma_i$ . We get a map  $G \backslash \mathfrak{M}_{g,n}(X, \gamma) \rightarrow \mathfrak{h}(P, \gamma)$  by sending  $(\mathcal{G} \rightarrow T, \mathcal{G} \leftarrow \mathcal{C} \xrightarrow{f} X, \sigma_i)$  to  $(\mathcal{C} \rightarrow \mathcal{G}, f^*G \rightarrow \mathcal{C}, \sigma_i)$  and quotienting this data by  $G$ .

Finally, let  $G$  act trivially on  $\mathfrak{M}_{g,n}$ , so  $G \backslash \mathfrak{M}_{g,n} \simeq \mathfrak{M}_{g,n} \times BG$ . Let  $F : G \backslash \mathfrak{M}_{g,n}(X, \gamma) \rightarrow G \backslash \mathfrak{M}_{g,n}$  be the quotient of the forgetful morphism by  $G$ . In addition, let  $G \backslash \mathfrak{M}_{g,n} \xrightarrow{i} \mathfrak{h}(G)$  be the morphism sending  $(\mathcal{C} \rightarrow T, \mathcal{G} \rightarrow T)$  to  $(\mathcal{C} \rightarrow T, \mathcal{G} \times \mathcal{C} \rightarrow \mathcal{C})$  over a scheme  $T$ . This is a closed immersion with normal bundle  $\mathcal{N}_i = \mathbf{R}^1 \pi_* \underline{\mathfrak{g}} \simeq \mathbf{R}^1 \pi_* \mathcal{O} \otimes \underline{\mathfrak{g}}$ , where  $\pi : G \backslash \mathcal{C} \rightarrow G \backslash \mathfrak{M}_{g,n}$  is the quotient of the universal bundle, and  $\underline{\mathfrak{g}}$  the trivial bundle with fiber  $\mathfrak{g}$  and  $G$ -equivariant structure given by the adjoint action.

**Theorem 3.5.6.**

(i) *The diagram*

$$\begin{array}{ccc} G \backslash \mathfrak{M}_{g,n}(X, \gamma) & \xrightarrow{h} & \mathfrak{h}(P, \gamma) \\ \text{for} \downarrow & & \downarrow \kappa \\ G \backslash \mathfrak{M}_{g,n} & \xrightarrow{i} & \mathfrak{h}(G) \end{array} \quad (3.5.2)$$

*of stacks is 2-Cartesian. In particular,  $h$  is an immersion.*

(ii)  $\mathfrak{H}(P, \gamma) \rightarrow \mathfrak{M}_{g,n}$  *is smooth of relative dimension*

$$\int_{\gamma} c_1(\mathcal{T}_X) + \dim P \cdot (g - 1).$$

(iii)  $\mathfrak{H}(G) \rightarrow \mathfrak{M}_{g,n}$  *is smooth of relative dimension*

$$\dim G \cdot (g - 1).$$

*Remark 3.5.7.* The morphisms in (ii) and (iii) are smooth in the sense of [54, Definition 101.33.1] but not representable. For example, base changing  $\mathfrak{h}(G)$  along geometric point  $* \rightarrow \mathfrak{M}_{g,n}$  corresponding to a curve  $\mathcal{C}$  returns the Artin stack  $\text{Bun}_G(\mathcal{C})$  of  $G$ -bundles on  $\mathcal{C}$ .

*Proof.* For  $\mathcal{C}$  a nodal curve, we can just work in the normalization in the following arguments. Thus we assume that all curves are smooth.

(i) This follows from the arguments of Lemma 3.5.2.

- (ii) For  $\mathcal{C}$  a smooth curve and  $\mathcal{B} \rightarrow \mathcal{C}$  a  $P$ -torsor over it, corresponding to a map  $k : \mathcal{C} \rightarrow BP$ , the first-order deformations of  $k$  with fixed domain and codomain is controlled by  $k^*\mathbb{L}_{BP} = k^*\mathfrak{p}^\vee = \mathcal{B} \times^P \mathfrak{p}^\vee$ . Thus the automorphisms, deformations and obstructions of  $\mathcal{B}$  are recorded by  $H^i(\mathcal{C}, \mathcal{B} \times^P \mathfrak{p})$  for  $i = 0, 1, 2$ . Since  $H^2(\mathcal{C}, \mathcal{B} \times^P \mathfrak{p}) = 0$ , there are no obstructions to extending the torsor, and the infinitesimal lifting condition guarantees smoothness. It follows that the dimension at  $\mathcal{B}$  is given by

$$h^1(\mathcal{C}, \mathcal{B} \times^P \mathfrak{p}) - h^0(\mathcal{C}, \mathcal{B} \times^P \mathfrak{p}) = - \int_{\mathcal{C}} c_1(\mathcal{B} \times^P \mathfrak{p}) - \text{rank}(\mathcal{B} \times^P \mathfrak{p}) \cdot (1 - g) \quad (3.5.3)$$

by the Hirzebruch–Riemann–Roch theorem.

Since  $\pi_0(\text{Bun}_P(\mathcal{C}))$  is characterized by multidegrees, we see that  $\mathcal{B}$  can be deformed to  $f^*G \times^P \mathfrak{p} = f^*(G \times^P \mathfrak{p})$  for a map  $f : \mathcal{C} \rightarrow X$  of class  $\gamma$ . Since the integral of the first Chern class is deformation invariant, it suffices to consider  $\mathcal{B} \times^P \mathfrak{p} = f^*(G \times^P \mathfrak{p})$ . Finally, the short exact sequence 2.3.3 gives

$$c_1(G \times^P \mathfrak{p}) + c_1(\mathcal{T}_X) = c_1(X \times \mathfrak{g}) = 0$$

meaning

$$\int_{\mathcal{C}} c_1(\mathcal{B} \times^P \mathfrak{p}) = \int_{\mathcal{C}} f^* c_1(G \times^P \mathfrak{p}) = - \int_{\mathcal{C}} f^* c_1(\mathcal{T}_X) = - \int_{\gamma} c_1(\mathcal{T}_X).$$

The dimension count now follows from Equation 3.5.3 along with  $\text{rank}(\mathcal{B} \times^P \mathfrak{p}) = \dim P$ .

- (iii) Let  $\mathcal{C}$  be a smooth curve, and  $\mathcal{G}$  a  $G$ -torsor on it. Note that the Killing form gives a  $G$ -equivariant identification  $\mathfrak{g} \simeq \mathfrak{g}^\vee$ . It follows that  $\mathcal{G} \times^G \mathfrak{g}$  is self-dual, hence has a trivial determinant bundle and a vanishing first Chern class. Again, the automorphisms, deformations and obstructions of a  $G$ -torsor  $\mathcal{G}$  are recorded by  $H^i(\mathcal{C}, \mathcal{G} \times^G \mathfrak{g})$  for  $i = 0, 1, 2$ , and  $H^2(\mathcal{C}, \mathcal{G} \times^G \mathfrak{g}) = 0$ . The same infinitesimal lifting and the Hirzebruch–Riemann–Roch computation hence gives the result.  $\square$

Now consider the following full diagram.

$$\begin{array}{ccccccc} \overline{\mathcal{M}}_{g,n}(X, \gamma) & \xrightarrow{j} & \mathfrak{M}_{g,n}(X, \gamma) & \xrightarrow{p} & G \backslash \mathfrak{M}_{g,n}(X, \gamma) & \xrightarrow{h} & \mathfrak{h}(P, \gamma) \\ & & & & \downarrow F & & \downarrow \\ & & & & G \backslash \mathfrak{M}_{g,n} & \xrightarrow{i} & \mathfrak{h}(G) \end{array}$$

Since  $h$  is an immersion, it has normal cone  $\mathcal{C}_h$ . Thus  $p^*\mathcal{C}_h$  is a Deligne–Mumford–stacky cone on  $\mathfrak{M}_{g,n}(X, \gamma)$ , and the same holds for  $\mathcal{C} = j^*p^*\mathcal{C}_h$  on  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$ . In addition,  $\mathcal{C}_h$  is closed immersed in  $F^*\mathcal{N}_i$ , so  $\mathcal{C}$  must also embed in  $\mathcal{E} = j^*p^*F^*\mathcal{N}_i$ .

**Corollary 3.5.8.** (i) The normal cone  $\mathcal{C}_h$  to  $h$  is of pure dimension

$$\int_{\beta} c_1(\mathcal{T}_X) + (\dim P + 3)(g - 1) + n,$$

and  $\mathcal{C}$  over  $\overline{\mathcal{M}}_{g,n}(X, \beta)$  is of pure dimension

$$\int_{\beta} c_1(\mathcal{T}_X) + (\dim P + 3)(g - 1) + n + \dim G \quad (3.5.4)$$

(ii) The normal bundle  $\mathcal{N}_i$  to  $i$  is of rank

$$\dim G \cdot (g - 1) + \dim G, \quad (3.5.5)$$

and the same for the pullback bundle  $\mathcal{E} = j^*p^*F^*\mathcal{N}_i$ .

Note that the difference of (3.5.4) and (3.5.5) is precisely the virtual dimension  $\text{vdim}$ . Now, we have a vector bundle  $\mathcal{E}$  over  $\overline{\mathcal{M}}_{g,n}(X, \gamma)$  containing a Deligne-Mumford-stacky cone  $\mathcal{C}$  of pure dimension. Thus, the sequence of maps

$$A_*(\mathcal{C}, \mathbb{Q}) \longrightarrow A_*(\mathcal{E}, \mathbb{Q}) \xrightarrow{\sim} A_{*-\text{rank}\mathcal{E}}(X, \mathbb{Q}).$$

allows us to transport the fundamental class  $[\mathcal{C}] \in A_{\dim \mathcal{C}}(\mathcal{C}, \mathbb{Q})$  to a virtual fundamental class  $[\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}} \in A_{\text{vdim}\overline{\mathcal{M}}_{g,n}(X, \gamma)}(\overline{\mathcal{M}}_{g,n}(X, \gamma), \mathbb{Q})$ , where the virtual dimension

$$\text{vdim}\overline{\mathcal{M}}_{g,n}(X, \gamma) = \dim \mathcal{C} - \text{rank} \mathcal{E} = \int_{\gamma} c_1(\mathcal{T}_X) + (\dim G/P - 3)(g - 1) + n.$$

*Remark 3.5.9.* In the case of  $g = 0$ , the bundle  $\mathcal{N}_i$  is trivial, and thus  $\mathcal{C} = X$  and  $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} = [\overline{\mathcal{M}}_{g,n}(X, \beta)]$ . This can also be observed by Weyl uniformization: the  $\mathbb{C}$ -points of  $\text{Bun}_G(\mathbb{C})$  in genus 0 correspond to double cosets

$$G(\mathbb{C}[t^{-1}]) \backslash G(\mathbb{C}((t))) / G(\mathbb{C}[[t]]),$$

which by the Birkhoff decomposition is given by the coweight lattice. Thus  $i$  is not only a regular immersion, but an isomorphism onto a component.

*Remark 3.5.10.* By [7, Proposition 13],  $[\overline{\mathcal{M}}_{g,n}(X, \gamma)]^{\text{vir}}$  agrees with the virtual fundamental class constructed for the relative perfect obstruction theory  $(\mathbf{R}\Phi_*\mathbf{f}^*\mathcal{T}_X)^\vee \rightarrow \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X, \gamma)/\mathfrak{M}_{g,n}}$ . Furthermore, by observing the diagram [48, (5.12)] and using [9, Proposition 2.7] on both rows, we learn that this coincides with the virtual fundamental class constructed for the absolute perfect obstruction theory  $E \rightarrow \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X, \gamma)}$  as in the proof of Theorem 3.3.7.

**Lemma 3.5.11.** *For  $\gamma = 0$ , so that  $\overline{\mathcal{M}}_{g,n}(X, 0) = \overline{\mathcal{M}}_{g,n} \times X$  is smooth,*

$$[\overline{\mathcal{M}}_{g,n}(X, 0)]^{\text{vir}} = c_{g-\dim X}(\mathbb{E}^\vee \boxtimes \mathcal{T}_X) \cap [\overline{\mathcal{M}}_{g,n}(X, 0)] \in A_{\text{vdim}}(\overline{\mathcal{M}}_{g,n}(X, 0), \mathbb{Q}).$$

*Proof.* The morphism  $G \backslash \mathfrak{M}_{g,n}(X, 0) \rightarrow \mathfrak{h}(P, 0)$  naturally identifies with

$$\mathfrak{M}_{g,n} \times [G \backslash X] \simeq \mathfrak{M}_{g,n} \times [P \backslash *] \longrightarrow \mathfrak{h}(P, 0)$$

sending a curve and a  $P$ -torsor to the curve along with the  $P$ -torsor pulled onto it. This gives  $\mathcal{C}_h \rightarrow \mathfrak{M}_{g,n} \times [P \backslash *]$  as a normal bundle  $\mathbf{R}^1\pi_*\mathfrak{p}$ , identifying with  $\mathbf{R}^1\pi_*(G \times^P \mathfrak{p})$  on  $\mathfrak{M}_{g,n} \times [G \backslash X]$ . Thus  $\mathcal{C} = j^*p^*\mathcal{C}_h = \mathbb{E}^\vee \boxtimes (G \times^P \mathfrak{p})$ , and  $\mathcal{E} = j^*p^*F^*\mathcal{N}_i = \mathbb{E}^\vee \boxtimes \mathfrak{g}$ . By (2.3.3), the excess bundle  $\mathcal{E}/\mathcal{C}$  is thus given by  $\mathbb{E}^\vee \boxtimes \mathcal{T}_X$ , and  $[\mathcal{C}] \in A_*(\mathcal{E})$  on  $A_{\text{vdim}}(\overline{\mathcal{M}}_{g,n} \times X, \mathbb{Q})$  is thus given by  $c_{g-\dim X}(\mathbb{E}^\vee \boxtimes \mathcal{T}_X) \cap [\overline{\mathcal{M}}_{g,n}(X, 0)]$ .  $\square$

# Chapter 4

## Quantum cohomology

### 4.1 Quantum cohomology

Consider the graded algebra  $\mathbb{Z}[\mathbf{q}] = \mathbb{Z}[q^{\beta^\vee} \mid \beta^\vee \in \Delta^\vee \setminus \Delta_P^\vee]$  with

$$\deg q^{\beta^\vee} = \text{vdim } \overline{\mathcal{M}}_{0,3}(X, \beta^\vee) - \dim X = \langle 2\rho_L, \beta^\vee \rangle.$$

For  $\gamma \in Q^\vee/Q_P^\vee$  not in the  $\mathbb{Z}$ -span of  $\Delta^\vee \setminus \Delta_P^\vee$ , we define  $q^\gamma \in \mathbb{Z}[\mathbf{q}]$  by projecting  $\gamma$  onto  $Q^\vee/Q_P^\vee$ , so  $q^\gamma = q^{\sum_{\beta \in \Delta \setminus \Delta_P} \langle \omega_{\beta, \gamma} \rangle \beta^\vee}$ , giving  $\deg q^\gamma = \langle 2\rho_L, \gamma \rangle$  in general. From now on, we identify  $H_2(X, \mathbb{Z}) \simeq Q^\vee/Q_P^\vee$ , and  $H^2(X, \mathbb{Z}) \simeq Q_L$  as in Chapter 2, and treat them interchangeably.

Now, consider the  $q$ -deformation  $qH_T^*(X, \mathbb{Z}) = H_T^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}]$  of  $H_T^*(X, \mathbb{Z})$ , and extend the integral operator  $\int_X^T : H_T^*(X, \mathbb{Z}) \rightarrow \mathbb{Z}[Q]$  linearly to a map  $qH_T^*(X, \mathbb{Z}) \rightarrow \mathbb{Z}[Q] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}]$ .

**Definition 4.1.1.** For  $\psi, \phi, \xi \in H_T^*(X, \mathbb{Z})$ , define the  $T$ -equivariant quantum product

$$\int_X^T (\psi * \phi) \cdot \xi = \sum_{\gamma \in H_2(X, \mathbb{Z})_+} q^\gamma \cdot \langle \psi, \phi, \xi \rangle_{0,3,\gamma}^T \quad (4.1.1)$$

*Remark 4.1.2.* In general, there is no guarantee for GW-invariants to be integral and to vanish for large enough curve classes (see Example 4.4). Thus a more general definition should be defined over the rational power series  $H_T^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}[[\mathbf{q}]]$ . However, by Lemma 3.4.7 and Lemma 3.4.12 this is fine in the case of  $X = G/P$ .

**Lemma 4.1.3.**

(i) The quantum product  $*$  is commutative, that is,

$$\phi * \psi = \psi * \phi$$

in  $qH_T^*(X, \mathbb{Z})$  for all  $\phi, \psi \in H_T^*(X, \mathbb{Z})$

(ii) The quantum product  $*$  is associative, that is,

$$(\phi * \psi) * \xi = \phi * (\psi * \xi) \quad (4.1.2)$$

in  $qH_T^*(X, \mathbb{Z})$  for all  $\phi, \psi, \xi \in H_T^*(X, \mathbb{Z})$

(iii)  $\mathbf{1} = [X]^* \in H_T^0(X, \mathbb{Z})$  is an identity element for  $*$

*Proof.* (i) This is trivial from Lemma 3.4.3.

(ii) Choosing a basis for  $qH_T^*(X, \mathbb{Z})$ , the associativity equation rewrites to the WDVV equations for 3-point GW-invariants. A proof can be found in [3, Part 1, Section 4] (see also [37, Theorem 5.3.1]).

(iii) Using Lemma 3.4.5, we know that  $\langle \phi, \mathbf{1}, \psi \rangle_{0,3,\gamma} = 0$  if  $\gamma \neq 0$  and  $\int_X^T \phi \cdot \psi$  for  $\gamma = 0$ . Thus we see directly from (4.1.1) that quantum multiplication by  $\mathbf{1}$  is equivalent to classical multiplication by  $\mathbf{1}$ , which is the identity.  $\square$

**Definition 4.1.4.** Define the  $T$ -equivariant quantum cohomology ring  $qH_T^*(X, \mathbb{Z}) = H_T^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}]$  with quantum product  $*$  with gradings coming from  $H_T^*(X, \mathbb{Z})$  and  $\mathbb{Z}[\mathbf{q}]$ .

*Remark 4.1.5.* We can similarly define the regular quantum cohomology ring  $qH^*(X, \mathbb{Z})$  using non-equivariant GW-invariants. By Lemma 3.4.11, this is given by the reduction  $\mathbb{Z}[Q] \rightarrow \mathbb{Z}$ .

*Example 4.1.* Let  $X = \text{Gr}(1, N+1) = \mathbb{P}^N$ , and  $\sigma = c_1(\mathcal{O}(1))$  the generator of  $H^2(X, \mathbb{Z})$ . Now,

$$\dim \overline{\mathcal{M}}_{0,3}(\mathbb{P}^n, d) = (N+1)d + N = (d+1)N + d$$

so in order for  $\langle \sigma, \sigma^a, \sigma^b \rangle_{0,3,d} \neq 0$ , we must have

$$1 + a + b = (d+1)N + d.$$

This forces either  $d = 0$  or  $d = 1$ : In the former case,  $a + b = N - 1$  and

$$\langle \sigma, \sigma^a, \sigma^b \rangle_{0,3,0} = \int_{\mathbb{P}^1} \sigma^{1+a+b} = 1.$$

In the latter case we have  $a = b = N$ , so  $\sigma^a = \sigma^b = \sigma^N$  represent generic points. Since there exists a unique line passing through two generic points (and a hyperplane), the enumerative interpretation Theorem 3.4.8 gives us

$$\langle \sigma, \sigma^N, \sigma^N \rangle_{0,3,1} = 1.$$

Hence by the definition, for  $a \leq N$ ,

$$\sigma * \sigma^a = \begin{cases} \sigma^{a+1}, & \text{if } a < N \\ q, & \text{if } a = N \end{cases}$$

It follows that  $qH^*(\mathbb{P}^N, \mathbb{Z}) = \mathbb{Z}[\sigma, q]/(\sigma^{N+1} - q)$  with  $\deg \sigma = 2$ ,  $\deg q = 2(N+1)$

*Example 4.2.* Consider  $X = \text{Gr}(2, 4)$ . Following the notation of Example 2.2, the basis for  $H^*(X, \mathbb{Z})$  is given by classes  $\sigma_0$  in degree 0,  $\sigma_1$  in degree 2,  $\sigma_{1,1}, \sigma_2$  in degree 4,  $\sigma_{2,1}$  in degree 6 and  $\sigma_{2,2}$  in degree 8. Following Theorem 2.4.5, we know the non-equivariant Chevalley formulae for  $H^*(X, \mathbb{Z})$  are given by

$$\begin{array}{ll} \sigma_1 \cdot \sigma_0 = \sigma_1 & \sigma_1 \cdot \sigma_1 = \sigma_2 + \sigma_{1,1} \\ \sigma_1 \cdot \sigma_{1,1} = \sigma_{2,1} & \sigma_1 \cdot \sigma_2 = \sigma_{2,1} \end{array}$$

$$\sigma_1 \cdot \sigma_{2,1} = \sigma_{2,2}$$

$$\sigma_1 \cdot \sigma_{2,2} = 0$$

Let us compute their quantum analogues. Since

$$\dim \overline{\mathcal{M}}_{0,3}(X, d \cdot \tau_{2,1}) = 5d + 4,$$

we have  $\langle \sigma_1, \sigma_\mu, \sigma_\lambda \rangle_{0,3,d} \neq 0$  only if  $d = 1$ , forcing  $\{\mu, \lambda\} = \{(2, 1), (2, 2)\}$ . We claim that  $\langle \sigma_1, \sigma_{2,1}, \sigma_{2,2} \rangle_{0,3,1} = 1$ , giving that in  $qH^*(X, \mathbb{Z})$ , we instead have the formulae.

$$\sigma_1 * \sigma_{2,1} = \sigma_{2,2} + q,$$

$$\sigma_1 * \sigma_{2,2} = q \cdot \sigma_1$$

with  $\deg q = 8$ . Let us study maps  $f : \mathbb{P}^1 \rightarrow X$  of class  $\tau_{2,1}$ . If  $\mathcal{U} \subset \mathcal{O}_X^4$  is the universal bundle over  $X$ ,  $f$  corresponds to the bundle  $f^*\mathcal{U} \subset \mathcal{O}_{\mathbb{P}^1}^4$ . Recalling that  $c_1(\mathcal{U}) = -\sigma_{2,1}$ , we have

$$-1 = - \int_{\tau_{2,1}} \sigma_{2,1} = \int_{\mathbb{P}^1} c_1(f^*\mathcal{U}).$$

Since  $f^*\mathcal{U}$  splits into line bundles on  $\mathbb{P}^1$ , embeds in  $\mathcal{O}_{\mathbb{P}^1}^4$ , and has degree  $-1$ , it must be isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ . This guarantees, that there exist trivial vector bundles  $\ell$  and  $H$  of rank 1 and 3, respectively, such that  $\ell \subset f^*\mathcal{U} \subset H$ . Thus the image of  $f$  is simply translate of  $gX_{2,1}$ , and  $f$  is unique up to automorphisms on its domain. In particular, all linear pencils of lines in  $\mathbb{P}^3$  are just lines contained in a fixed hyperplane, passing through a fixed point. By Example 2.2, the image is thus a Schubert variety  $gX_{1,1}$  for  $g \in \mathrm{SL}_4$ .

Now, Example 2.2 demonstrates, that  $\sigma_{2,1}$  represents a linear pencil of lines, and  $\sigma_{2,2}$  a fixed line. It thus follows by Lemma 3.4.8 that  $\langle \sigma_1, \sigma_{2,1}, \sigma_{2,2} \rangle_{0,3,1}$  is the answer to "for a fixed line  $\ell$ , and a linear pencil of lines  $p \subset k \subset H$ , all in generic position, how many linear pencils of lines exist containing  $\ell$  and a line  $k$ ". This is classical geometry: this pencil must pass through  $p' = \ell \cap H$ , and be contained in  $H' = \langle \ell, pp' \rangle$ . Thus only one such pencil exists.

*Remark 4.1.6.* The Chevalley formulae for  $H^*(\mathrm{Gr}(2, 4), \mathbb{Z})$ , with no additional geometric information such as  $\sigma_2 = \sigma_2^-$ , are not enough to generate the structure constants for multiplication of Schubert classes. However, the equivariant formula, along with a formula for  $\sigma_{\mu|x_\mu} \in \mathbb{Z}[Q]$  for all  $\mu$  for  $H_T^*(\mathrm{Gr}(2, 4), \mathbb{Z})$  allows for recovery of all structure constants by induction on degree, as shown nicely in [36]. This furthermore works for all  $qH_T^*(X, \mathbb{Z})$  as outlined in Theorem 4.2.7.

*Example 4.3.* Let  $T = \mathbb{C}^*$  act on  $\mathbb{P}^1 = \mathrm{SL}_2/B$  as usual. Now, as in 4.1, all non-trivial GW-invariants have no equivariant contribution. Thus the equivariant contribution is given from Theorem 4.2.1:

$$\sigma * \sigma = q + \alpha \cdot \sigma \tag{4.1.3}$$

where  $\alpha \in Q$  is the unique root of  $T$ . This gives  $qH_{\mathbb{C}^*}^*(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}[\sigma, q, \alpha]/(\sigma^2 - \alpha\sigma - q)$  with  $\deg \sigma = \deg \alpha = 2, \deg q = 4$

*Example 4.4.* (Out of scope of the definitions given in this thesis.) Let us compute the  $T \times \mathbb{C}^*$ -equivariant quantum cohomology of  $T^*\mathbb{P}^1$ , which is the essential step to computing that of  $T^*(G/P)$  as in [12, 55]<sup>1</sup>. In this case, curves are contracted along the affinization  $T^*(G/P) \rightarrow \mathcal{O} \subset \mathfrak{g}$  known as the *partial Springer resolution*, so curves

<sup>1</sup>as well as ADE resolutions [13, 44], or more generally resolutions of various Slodowy slices [12].

are supported on the partial Springer fibers [11]. Furthermore, since the affinization of  $T^*\mathbb{P}^1$  is contraction of the zero section, the curves are also supported the zero section  $\mathbb{P}^1$ . Nonetheless, the quantum cohomology is different from  $\mathbb{P}^1$  due to the GW-invariant being a virtual count.

Let us begin by fitting  $T^*\mathbb{P}^1$  into a nice family. Consider  $G = \mathrm{SL}_2$ , so  $G/B = \mathbb{P}^1$  and  $\mathfrak{g}$  acts on  $\mathbb{C}^2$ . Let

$$\tilde{\mathfrak{g}} = \{(\ell, M) \in \mathbb{P}^1 \times \mathfrak{g} \mid M \cdot \ell \subseteq \ell\}$$

and consider the locally trivial fibration  $\tilde{\chi} : \tilde{\mathfrak{g}} \rightarrow \mathbb{A}^1$  given by  $\tilde{\chi}(\ell, M) = \det(M|_\ell)$ . Observe that  $\tilde{\mathfrak{g}}_0 = T^*\mathbb{P}^1$ , whereas  $\tilde{\mathfrak{g}}_\lambda \simeq G/T$  is affine for  $\lambda \neq 0$ . Now,  $\tilde{\mathfrak{g}}$  is a (noncompact) Calabi-Yau threefold: it has a trivial fundamental group using the long exact sequence of homotopy groups, and taking the determinant on the short exact sequence

$$0 \longrightarrow \tilde{\chi}^* \Omega_{\mathbb{A}^1}^1 \longrightarrow \Omega_{\tilde{\mathfrak{g}}}^1 \longrightarrow \Omega_{\tilde{\mathfrak{g}}/\mathbb{A}^1}^1 \longrightarrow 0$$

gives  $\omega_{\tilde{\mathfrak{g}}} = \mathcal{O}_{\tilde{\mathfrak{g}}}$ . In addition, note that the short exact sequence

$$0 \longrightarrow \mathcal{N}_{\mathbb{P}^1/\tilde{\mathfrak{g}}_0} \longrightarrow \mathcal{N}_{\mathbb{P}^1/\tilde{\mathfrak{g}}} \longrightarrow \mathcal{N}_{\tilde{\mathfrak{g}}_0/\tilde{\mathfrak{g}}|\mathbb{P}^1} \longrightarrow 0$$

along with  $\mathcal{N}_{\tilde{\mathfrak{g}}_0/\tilde{\mathfrak{g}}} = \mathcal{O}_{\mathbb{P}^1}$ ,  $\mathcal{N}_{\mathbb{P}^1/\tilde{\mathfrak{g}}_0} = \mathcal{O}_{\mathbb{P}^1}(-2)$ , and  $\mathbb{P}^1$  being rigid in  $\tilde{\mathfrak{g}}$ , implies that  $\mathcal{N}_{\mathbb{P}^1/\tilde{\mathfrak{g}}_0} = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ . That is,  $\mathbb{P}^1$  is a  $(-1, -1)$ -curve in the (noncompact) Calabi-Yau threefold  $\tilde{\mathfrak{g}}$ , and  $\overline{\mathcal{M}}_{0,n}(\tilde{\mathfrak{g}}, d)$  has virtual dimension  $n$ .

Now we are ready to compute the GW-invariants. As in [12, Section 4.3], the embeddings  $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, d) \xrightarrow{j} \overline{\mathcal{M}}_{0,n}(T^*\mathbb{P}^1, d) \xrightarrow{i} \overline{\mathcal{M}}_{0,n}(\tilde{\mathfrak{g}}, d)$  are isomorphisms of stacks, and

$$i_*[\overline{\mathcal{M}}_{0,n}(T^*\mathbb{P}^1, d)]^{\mathrm{vir}} = \hbar \cdot i_*[\overline{\mathcal{M}}_{0,n}(T^*\mathbb{P}^1, d)]^{\mathrm{red}} = \hbar \cdot [\overline{\mathcal{M}}_{0,n}(\tilde{\mathfrak{g}}, d)]^{\mathrm{vir}}$$

where  $\hbar$  is the  $\mathbb{C}^*$ -equivariant parameter, and  $[\overline{\mathcal{M}}_{0,n}(T^*\mathbb{P}^1, d)]^{\mathrm{red}}$  the  $T$ -equivariant *reduced virtual fundamental class* with no  $\mathbb{C}^*$ -contribution<sup>2</sup>. Thus, for  $d \neq 0$ , the  $T \times \mathbb{C}^*$  GW-invariants of  $T^*\mathbb{P}^1$  are given by integrating against the class  $\hbar \cdot [\overline{\mathcal{M}}_{0,n}(\tilde{\mathfrak{g}}, d)]^{\mathrm{vir}}$  restricted to  $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, d)$ . For  $n = 0$ , the GW-invariant with 0 insertions is thus  $\langle - \rangle_{0,0,d} = \hbar/d^3$  by the Aspinwall-Morrison multiple cover formula [17, Theorem 3].

Now, consider the pullbacks of the Schubert classes  $1, \sigma \in H_T^*(\mathbb{P}^1, \mathbb{Z})[\hbar] = H_{T \times \mathbb{C}^*}^*(\mathbb{P}^1, \mathbb{Z})$  as in Example 4.1 along  $T^*\mathbb{P}^1 \rightarrow \mathbb{P}^1$ , and also denote these as  $1, \sigma$ . These give a  $\mathbb{Z}[Q] \otimes \mathbb{Q}[\mathbf{q}]$ -basis for  $qH_{T \times \mathbb{C}^*}^*(T^*\mathbb{P}^1, \mathbb{Q})$ , and the only non-vanishing 3-point genus 0 GW-invariant of is given by  $\langle \sigma, \sigma, \sigma \rangle_{0,3,d}$ , which evaluates to  $(\int_{d, [\mathbb{P}^1]} \sigma)^3 \cdot \langle - \rangle_{0,0,d} = \hbar$  using the divisor equation three times and the earlier formula. It follows that  $qH_{T \times \mathbb{C}^*}^*(T^*\mathbb{P}^1, \mathbb{Q})$  is determined by the equation

$$\sigma * \sigma = \sigma \cdot \sigma + \sum_{d>0} q^d \langle \sigma, \sigma, \sigma \rangle_{0,3,d}^{T \times \mathbb{C}^*} = \alpha \cdot \sigma + \hbar \cdot \frac{q^d}{1 - q^d} \quad (4.1.4)$$

where  $\alpha$  is the unique root of  $G = \mathrm{SL}_2$ , and  $\deg \alpha = 2$ ,  $\deg \hbar = 2$ , and  $\deg q = 0$ . Under the change of variables  $q \rightarrow \hbar^{-1}q'$  we instead have  $\deg q' = 2$  and taking the limit as  $\hbar \rightarrow \infty$  turns (4.1.4) into (4.1.3), and thus recovering  $qH_T^*(\mathbb{P}^1, \mathbb{Q})$ . This is made rigorous in terms of the quantum connection in [12, Section 7].

<sup>2</sup>If  $\mathbb{C}^*$ -equivariance were forgotten,  $\hbar = 0$  and the virtual fundamental class would vanish. This is expected from the deformation invariance of GW-invariants: The family  $\tilde{\chi}$  deforms  $T^*\mathbb{P}^1$  to something affine with no curves, but considering  $\mathbb{C}^*$ -equivariance restricts this deformation

## 4.2 Quantum Chevalley formula

We now wish to establish a quantum analogue of Chevalley's Theorem 4.2.1.

**Lemma 4.2.1.** *The substack  $E = e_2^{-1}X_w \cap e_3^{-1}X_v^-$  in  $\overline{\mathcal{M}}_{0,3}(X, \gamma)$  is either empty or reducible, locally irreducible, and of pure codimension  $\ell(w) + \dim X - \ell(v)$ . Furthermore, each component of  $E$  intersects  $\mathcal{M}_{0,3}(X, \gamma)$  and intersects the boundary components transversally.*

*Proof.* Apply Corollary 2.4.7 for  $Z = \overline{\mathcal{M}}_{0,3}(X, \gamma)$  and  $F = e_2 \times e_3$ .  $\square$

Recall also from Theorem 3.3.8 that

$$\dim \overline{\mathcal{M}}_{0,3}(X, \gamma) = \int_{\gamma} c_1(\mathcal{T}_X) + \dim X = \langle 2\rho_L, \gamma \rangle + \dim X \geq 2 + \dim X$$

**Theorem 4.2.2.** *Multiplication by divisors  $\sigma_{s_\beta} \in H_T^2(X, \mathbb{Z})$  in the  $T$ -equivariant quantum cohomology ring  $qH_T^*(X, \mathbb{Z})$  is as follows:*

$$\sigma_{s_\beta} * \sigma_w = \sum_{\gamma} \langle \omega_\beta, \gamma \rangle \cdot \sigma_{ws_\gamma} + \sum_{\gamma} q^\gamma \langle \omega_\beta, \gamma \rangle \cdot \sigma_{ws_\gamma} + [\omega_\beta - w(\omega_\beta)] \cdot \sigma_w. \quad (4.2.1)$$

Here the first sum is over coroots  $\gamma \in \Phi_+^\vee \setminus \Phi_{P_+}^\vee$  with  $\ell(ws_\gamma) = \ell(w) + 1$  in  $W^P$ , and the second sum over coroots  $\gamma \in \Phi_+^\vee \setminus \Phi_{P_+}^\vee$  with  $\ell(ws_\gamma) = \ell(w) + 1 - \langle 2\rho_L, \gamma \rangle$  as cosets in  $W^P$ .

Note that each coefficient is that the coefficient  $c_{\gamma,v}$  for the term  $q^\gamma \sigma_v$  is given by the  $T$ -equivariant 3-point GW-invariant  $\langle \sigma_{s_\beta}, \sigma_w, \sigma_v^- \rangle_{0,3,\gamma}^T \in H_T^*(X, \mathbb{Z})$ . Furthermore, nonvanishing  $c_{\gamma,v}$  are polynomials of degree

$$d_{\gamma,v} = \ell(w) - \ell(v) + 1 - \langle 2\rho_L, \gamma \rangle.$$

as lengths in  $W^P$ . The following lemma guarantees that the  $T$ -equivariant structure has no contribution to the coefficients  $c_{v,\gamma}$  with  $\gamma \neq 0$ .

**Lemma 4.2.3.** *Let  $v \in W^P$  and  $\gamma \neq 0$ . If  $d_{\gamma,v} > 0$ , then  $c_{\gamma,v} = 0$ .*

*Proof.* We begin by noting that  $X_w \cap X_v^- = \emptyset$ : if not, by Corollary 2.4.7 and the assumption  $d_{v,\gamma} > 0$  it has pure dimension  $\ell(v) - \ell(w) < 1 - \langle 2\rho_L, \gamma \rangle < 0$ , a contradiction. In addition, by Lemma 4.2.1 the set  $E = e_2^{-1}X_w \cap e_3^{-1}X_v^- \subset \overline{\mathcal{M}}_{0,3}(X, \gamma)$  is either empty, of pure dimension  $\langle 2\rho_L, \gamma \rangle - \ell(w) + \ell(v) = 1 - d_{\gamma,v}$ . The bound  $d_{\gamma,v} > 0$  hence implies that  $E$  of dimension at most 0, and we claim that  $E$  is empty.

First by Lemma 4.2.1, we know that no point of  $E$  lies on the boundary. Thus, a point  $\phi \in E(\mathbb{C})$ , if it exists, is a map from  $(\mathbb{P}^1, p_1, p_2, p_3)$ . Since  $E$  is  $T$ -invariant and  $e_2$   $T$ -equivariant, we know that  $\phi(p_2)$  is a  $T$ -invariant point in  $X_w$ . But now since  $X_w \cap X_v^- = \emptyset$  and  $\phi(p_3) \in X_v^-$ , the rational curve in  $X$  connecting these points must pass through a non- $T$ -invariant point  $r \in X$ . Choosing  $\phi(p_1) = r$  gives that  $\phi$  is not a fixed point under the  $T$ -action on  $\overline{\mathcal{M}}_{0,3}(X, \gamma)$ , forcing  $E$  to be positive-dimensional, a contradiction.  $\square$

**Lemma 4.2.4.** *Let  $v \in W^P$  and  $\gamma \neq 0$ , and assume that  $c_{\gamma,v} \neq 0$ . Then  $v = ws_\gamma$ , and  $E = e_2^{-1}X_w \cap e_3^{-1}X_v^- \subset \overline{\mathcal{M}}_{0,3}(X, \gamma)$  maps isomorphically to the  $T$ -invariant curve  $C_{w,v} \subset X$  via  $e_1$ .*

*Proof.* By Lemma 4.2.1,  $E$  is reduced, locally irreducible, and of pure codimension  $\ell(w) + \dim X - \ell(v)$ . Furthermore, the assumption  $c_{\gamma,v} \neq 0$  implies that  $E$  is nonempty and by Lemma 4.2.3  $d_{\gamma,v} = 0$ , so  $\ell(v) = \ell(w) + 1 - \langle 2\rho_L, \gamma \rangle$ . Thus  $E$  is of pure dimension 1. Also, note that  $X_w \cap X_v^- = \emptyset$  as earlier.

Now, consider a regular point  $\phi \in E(\mathbb{C})$  corresponding to a map  $\phi : \mathbb{P}^1 \rightarrow X$  with  $\phi(p_2) \in X_w$  and  $\phi(p_3) \in X_v^-$ . We claim that  $\phi(p_2) = x_w$  and  $\phi(p_3) = x_v$ . If the former fails to hold, then due to  $T$ -invariance  $e_2(E)$  contains another point  $x_{w'}$  for  $w' > w$ . But now  $E' = e_2^{-1}X_{w'} \cap e_3^{-1}X_v^-$  is empty as shown in the proof of Lemma 4.2.3, a contradiction. The similar results holds for  $\phi(p_3)$ . Now, in order for  $\dim E = 1$ , we must have the image of  $\phi$  has to lie in a  $T$ -invariant curve connecting  $x_w$  and  $x_v$ . This forces  $v = ws_\beta$  by our classification of  $T$ -invariant curves, so  $\phi(p_1) \in C_{w,v} \setminus \{x_w, x_v\}$ .

We furthermore claim, that  $\phi : \mathbb{P}^1 \rightarrow C_{w,v}$  is an isomorphism. If  $\phi$  is a ramified cover of degree  $k \geq 1$ , then  $E'' = e_2^{-1}x_w \cap e_3^{-1}x_v$  in  $\overline{\mathcal{M}}_{0,3}(C_{w,v}, k)$  is nonempty and embeds naturally in  $E$ . However, by Lemma 2.4.7  $E''$  has codimension 2, in a stack of dimension  $2k + 1$ . Thus  $2k - 1 \leq 1$  and  $k = 1$ . Since an automorphism of  $\mathbb{P}^1$  is determined by 3 points, it also follows that  $\phi$  is determined by the choice of  $\phi(p_1) \in C_{w,v} \setminus \{x_w, x_v\}$ , and thus  $e_1 : E \rightarrow C_{w,v} \setminus \{x_w, x_v\}$  is an isomorphism. By taking the closure, we see that  $e_1 : E \rightarrow C_{w,v}$  is an isomorphism.  $\square$

*Proof of Theorem 4.2.2.* Let  $E = e_1^{-1}X_w \cap e_2^{-1}X_v^-$ . Note that

$$e_2^* \sigma_w \cdot e_3^* \sigma_v^- = e_1^* [X_w]^* \cdot e_2^* [X_v^-]^* = [E]^*$$

so

$$c_{\gamma,v} = \int_{\overline{\mathcal{M}}_{0,n}(X,\gamma)} e_1^* \sigma_{s_\beta} \cdot [E]^* = \int_{[E]} e_1^* \sigma_{s_\beta} = \int_{e_1^*[E]} \sigma_{s_\beta}$$

Now, since  $e_1 : E \rightarrow C_{w,v}$  is an isomorphism by Lemma 4.2.4, Theorem 2.4.2 gives

$$\int_{e_1^*[E]} \sigma_{s_\beta} = \int_{[C_\gamma]} \sigma_{s_\beta} = \langle \omega_\beta, \gamma \rangle.$$

$\square$

*Remark 4.2.5.* For  $P = B$  Borel, consider the equivariant homology  $\mathbf{H} = H_*^G(X \times X, \mathbb{Q})$  acting on  $H_G^*(G/B \times X, \mathbb{Q}) \simeq H_T^*(X, \mathbb{Q})$  as in (see also [15, Section 2.7], [47, Section 5.5]) and let  $c \star (-)$  be the operator given by a cycle  $c \in \mathbf{H}$ .  $\mathbf{H}$  has a basis consisting of  $[\mathcal{O}_w]$  for  $w \in W$  with  $\mathcal{O}_w = G \cdot (wB, B) \subset G/B \times G/B$  a  $G$ -orbit, and we call the operator  $\partial_w = [\mathcal{O}_w] \star (-)$  the *Demazure operator* associated to  $w \in W$ .

Now for  $\lambda \in Q \simeq H^2(X, \mathbb{Z})$ , projection formulae and divisor insertion give the quantum multiplication in Theorem 4.2.2 in the basis-free form

$$\lambda * (-) = \lambda \cdot (-) + \sum_{\gamma \in H_2(X, \mathbb{Z})_+, \gamma \neq 0} q^\gamma \langle \lambda, \gamma \rangle \cdot F_*^\gamma [\overline{\mathcal{M}}_{0,2}(X, \gamma)] \star (-)$$

where  $F^\gamma = e_1 \times e_2 : \overline{\mathcal{M}}_{0,2}(X, \gamma) \rightarrow X \times X$ . In [12] it is shown that the non-vanishing terms are precisely  $q^\gamma \langle \lambda, \gamma \rangle \cdot \partial_{s_\gamma}$  for  $\gamma \in \Phi_+^\vee$  satisfying  $\ell(s_\gamma) = \langle 2\rho, \gamma \rangle - 1$ . This can also be observed directly from Theorem 4.2.2 by computing the  $\mathbf{H}$ -action on the Schubert basis.

Using Lemma 3.4.6, the invariants  $\langle \sigma_{s_\beta}, \sigma_w, \sigma_v^- \rangle_{0,3,\gamma}$  also determine all 2-point genus 0 GW-invariants. Since 0- and 1-point genus 0 GW-invariants vanish due to dimensional reasons, the following theorem demonstrates that these classes determine all genus 0 GW-invariants in special cases:

**Theorem 4.2.6** (Kontsevich's reconstruction theorem). *[38, Theorem 3.1] When  $H^*(X, \mathbb{Q})$  is generated by divisors, the tree-level genus 0 GW-invariants  $\mathfrak{I}_{0,n,\gamma}$  can be recovered from numerical GW-invariants  $\langle \sigma_{s_\beta}, \sigma_w, \sigma_v^- \rangle_{0,3,\gamma}$  with  $\beta \in \Delta \setminus \Delta_P$  and  $w, v \in W^P$ .*

Complete flag varieties  $G/B$  and projective spaces  $\mathbb{P}^N$  are generated by divisors, so all their genus 0,  $n$ -point GW-invariants can be algorithmically computed using Kontsevich's reconstruction formula. This is not the case for all partial flag varieties, as can be observed for  $\text{Gr}(2, 4)$ . However, the following theorem still shows that Theorem 4.2.2 determines all 3-point genus 0 GW-invariants for a partial flag variety:

**Theorem 4.2.7.** *[46, Corollary 8.2] Let  $\mathcal{A}$  be a graded commutative  $\mathbb{Z}[Q] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}]$ -algebra such that*

- (i)  $\mathcal{A}$  has a  $\mathbb{Z}[Q] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}]$ -basis  $\{S_w\}_{w \in W^P}$  with grading  $\deg S_w = 2\ell(w)$
- (ii) The basis of  $S_w$  of  $\mathcal{A}$  satisfies the formula (4.2.1).

Then, the morphism

$$\mathcal{A} \longrightarrow qH_T^*(X, \mathbb{Z}), \quad S_w \longmapsto \sigma_w$$

gives an isomorphism of  $\mathbb{Z}[Q] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}]$ -algebras.

One interesting consequence is as follows: Consider the affine Grassmannian  $\mathcal{G}r$  of  $G$ , and let  $\mathcal{A} = H_*^T(\mathcal{G}r, \mathbb{Z})$  be its  $T$ -equivariant homology. By [60, Theorem 1.6.1], the loop group  $\Omega K$  is homotopy equivalent to  $\mathcal{G}r$ , and thus  $\mathcal{A}$  acquires the structure of a graded algebra.

Let  $\widetilde{W} = W \ltimes Q^\vee$  be the affine Weyl group, and  $W^- \subset \widetilde{W}$  the set of elements of minimal length of  $\widetilde{W}/W$ , by [39, Lemma 3.3], elements of  $W^-$  are uniquely of the form  $w\lambda$  for  $\lambda \in Q_-^\vee$  and  $w \in W$  minimal in  $W/W_\lambda$ . In particular,  $\ell(w\lambda) = \ell(\lambda) - \ell(w)$ . Now,  $\mathcal{A}$  is generated by the classes of Iwahori-orbits  $S_{w\lambda} \in H_{\ell(w)-\ell(\lambda)}^T(\mathcal{G}r, \mathbb{Z})$ .

**Theorem 4.2.8.** *[51, 39] There is a well-defined isomorphism of graded  $\mathbb{Z}[Q]$ -algebras*

$$\begin{aligned} H_*^T(\mathcal{G}r, \mathbb{Z})[S_\lambda^{-1}, \lambda \in Q_-^\vee] &\longrightarrow qH_T^*(G/B, \mathbb{Z})[q^{-\gamma}, \gamma \in \Delta^\vee] \\ S_{w\lambda} \cdot S_\eta^{-1} &\longmapsto q^{\lambda-\eta} \sigma_w \end{aligned}$$

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