THE AFFINE SPRINGER FIBER – SHEAF CORRESPONDENCE

EUGENE GORSKY, OSCAR KIVINEN, AND ALEXEI OBLOMKOV

ABSTRACT. Given a semisimple element in the loop Lie algebra of a reductive group, we construct a quasi-coherent sheaf on a partial resolution of the trigonometric commuting variety of the Langlands dual group. The construction uses affine Springer theory and can be thought of as an incarnation of 3d mirror symmetry. For the group GL_n, the corresponding partial resolution is $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$. We also consider a quantization of this construction for homogeneous elements.

CONTENTS

Date: April 1, 2022.

1. Introduction

Let $\mathbf{k} = \overline{\mathbf{k}}$ be an algebraically closed field of characteristic zero or large enough positive characteristic. We fix a connected reductive group $G/\mathbf{k}$ and a maximal torus $T \subset G$. We let $\mathrm{Lie}(G) =: \mathfrak{g} \supset \mathfrak{t} := \mathrm{Lie}(T)$, and denote by $\check{G}$ the Langlands dual group over $\mathbb{C}$ (or $\overline{\mathbb{Q}}_\ell$).

In this paper we explain how one can naturally associate to an affine Springer fiber for G, or rather to a conjugacy class of the loop Lie algebra $\mathfrak{g}((t))$, a quasi-coherent sheaf on a partial resolution of the commuting variety (henceforth PRCV) associated to $\check{G} \times \check{\mathfrak{g}}$. We prove that the sheaves constructed this way are coherent in a number of cases and conjecture they are coherent in general. The sheaf on the PRCV remembers homological invariants of the affine Springer fiber, and we expect that our construction provides a unified perspective on the affine Springer fibers and their various functorial properties.

Our first main character is a partial resolution of the trigonometric version of the commuting variety for $\check{G}$, which we denote by $\widetilde{\mathfrak{C}}_{\check{G}} = \widetilde{T^*T^\vee/W}$. It is in general a singular Poisson variety which we conjecture to locally agree with those constructed in [53, 1, 66]. There is a natural map $\widetilde{\mathfrak{C}}_{\check{G}} \to \mathrm{Proj} \bigoplus_{d\geq 0} A^d_{\check{G}}$ where $A_{\check{G}} \subset \mathbb{C}[T^*T^\vee]$ is the subspace of antisymmetric polynomials with respect to the diagonal Weyl group action, and $A^d_{\check{G}} = (A_{\check{G}})^d$. For $G = GL_n$ this is an isomorphism, so that we recover the Hilbert scheme of points on $\mathbb{C}^* \times \mathbb{C}$ [64]. A more detailed construction is given in Section 3.

Let Gr_G be the affine Grassmannian of G. On the level of $\mathbf{k}$-points this is $G(\mathcal{K})/G(\mathcal{O})$, where $\mathcal{O} = \mathbf{k}[[t]]$ and $\mathcal{K} = \mathbf{k}((t)) = \mathbf{k}[[t]][t^{-1}]$. Our second main character is the *affine Springer fiber* $\mathrm{Sp}_\gamma \subset \mathrm{Gr}_G$, $\gamma \in \mathfrak{g}(\mathcal{K})$, defined as the fixed locus of the vector field γ. More precisely, let $\gamma \in \mathfrak{g}(\mathcal{K})$ be a compact semisimple element. On the level of R-points

$$\mathrm{Sp}_\gamma(R) := \left\{ gG(R[[t]]) | \,\mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(G)(R[[t]]) \right\} \subset \mathrm{Gr}_G(R).$$

Under these assumptions, Sp_γ is a nonempty ind-scheme over $\mathbf{k}$. If γ is also regular, Sp_γ is finite-dimensional and locally of finite type. We will only be interested in the étale or singular cohomologies of the Sp_γ, so will be writing Sp_γ for the $\mathbf{k}$-points $\mathrm{Sp}_\gamma(\mathbf{k})$. If $\mathbf{k} = \mathbb{C}$ we will use the analytic topology and if $\mathbf{k} = \overline{\mathbb{F}}_q$ we will use the étale topology. Our main result is the following.

Theorem 1.1. *Let $\gamma \in \mathfrak{g}(\mathcal{K})$ be a semisimple element and $K_\gamma = \mathrm{Stab}_{G(\mathcal{K})}$ its centralizer. Then for every compact (in the t−adic sense) subgroup $L_\gamma \subseteq K_\gamma$, there exists a quasi-coherent sheaf $\mathcal{F}_\gamma \in \mathrm{QCoh}_{\mathbb{G}_m}(\widetilde{\mathfrak{C}}_{\check{G}})$ such that:*

1. *$\mathcal{F}_{t\gamma} = \mathcal{L} \otimes \mathcal{F}_\gamma$ where $\mathcal{L} = \mathcal{O}(1)$ is the Serre twisting sheaf coming from the Proj-construction of $\widetilde{\mathfrak{C}}_{\check{G}}$.*
2. *There exists an integer M such that for $m > M$ we have*

$$H^0(\mathcal{F}_{t^m\gamma}, \widetilde{\mathfrak{C}}_{\check{G}}) = H^{L_\gamma}_*(\mathrm{Sp}_{t^m\gamma})$$

Moreover, the homological grading on the affine Springer fiber side can be recovered from the torus action dilating the cotangent fibers on the coherent sheaf side.

We conjecture below that $\mathcal{F}_\gamma$ is actually coherent (see Conjecture 1.7) and prove it in some cases.

Remark 1.2. Note that the quasicoherent sheaf $\mathcal{F}_\gamma$ only depends on the reduced structure of Sp_γ.

Remark 1.3. Here and in the rest of the paper, $H_*(-)$ denotes Borel-Moore homology, defined as $H_*(X) := H^{-*}(X, \omega_X)$. The Borel-Moore homology of an ind(-proper) variety $X = \varinjlim X_i$ will be defined as $H_*(X) = \varinjlim H_*(X_i)$.

It is natural to wonder what kind of sheaves $\mathcal{F}_\gamma$ Theorem 1.1 yields. For $G = GL_n$ and γ *homogeneous*, we have the following.

Theorem 1.4 (Proposition 8.10). *If $G = GL_n$ and γ homogeneous of slope $\frac{kn+1}{n}$ then $\mathcal{F}_\gamma$ agrees with the restriction of $\mathcal{O}(k)$ to the punctual Hilbert scheme at $(1, 0)$.*

Theorem 1.5. *For $G = GL_n$ and γ of integral slope k, the sheaf $\mathcal{F}_\gamma$ agrees with $\mathcal{P} \otimes \mathcal{O}(k)$, where $\mathcal{P}$ is the Procesi bundle restricted to* $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$.

Remark 1.6. For arbitrary G and homogeneous γ of integral slope, we also get an explicit description of $\mathcal{F}_\gamma$, see Theorem 8.12.

We also prove a noncommutative version of the above results. The ring of functions on $\widetilde{\mathfrak{C}}_{\check{G}}$ admits a deformation, or quantization known as the spherical *trigonometric* or *graded* Cherednik algebra (or graded DAHA). By the work of Yun [88, 89], this algebra acts in homology of Sp_γ.

The sheaf $\mathcal{L} = \mathcal{O}(1)$ and its powers are quantized to bimodules between two trigonometric Cherednik algebras with different values of quantization parameters. These algebras and modules are assembled to a large $\mathbb{Z}$-algebra, and one of the main results of the paper (Theorem 7.9) associates a module over this $\mathbb{Z}$-algebra to a collection of affine Springer fibers $\{\mathrm{Sp}_\gamma, \mathrm{Sp}_{t\gamma}, \mathrm{Sp}_{t^2\gamma}, \ldots\}$. Roughly speaking, considering the bimodules between different algebras allows one to move between different affine Springer fibers. For $G = GL_n$, the $\mathbb{Z}$-algebra is similar to the one considered by Gordon and Stafford for rational Cherednik algebras of GL_n in [35, 36].

The main tool we use is a novel construction of $\mathbb{Z}$-algebras related to Coulomb branches of $3d\,\mathcal{N} = 4$ theories. These (but not the $\mathbb{Z}$-algebras we construct) were defined by Braverman, Finkelberg, Nakajima [9, 10] and further explored by Webster [85]. This algebra is defined as the convolution algebra in the equivariant Borel-Moore homology of certain spaced related to the affine Grassmannian of G. This construction admits a natural quantization by considering additional equivariant parameters, and one can study both commutative and quantized versions. We use the machinery of Coulomb branches to achieve the following goals:

- We realize the full graded DAHA as the Coulomb branch algebra associated to the affine flag variety and construct its Dunkl-Cherednik embedding to $\hbar$-difference operators on the Lie algebra of the torus $T^\vee$.
- We give explicit formulas for the Coulomb branch $\mathbb{Z}$-algebra in difference presentation.
- We prove the shift isomorphisms for the spherical/anti-spherical subalgebras of the graded DAHA, in the difference-operator representation. This allows us to define the shift bimodules and $\mathbb{Z}$-algebras associated to graded DAHA in all types.
- In the commutative version, the Coulomb branch $\mathbb{Z}$-algebra is equivalent to a graded algebra which we identify explicitly. This allows us to define $\widetilde{\mathfrak{C}}_{\check{G}}$ using Proj construction.
- Finally, we prove that a collection of affine Springer fibers $\{\mathrm{Sp}_\gamma, \mathrm{Sp}_{t\gamma}, \mathrm{Sp}_{t^2\gamma}, \ldots\}$ yields a module over the Coulomb branch $\mathbb{Z}$-algebra. This is done similarly to "BFN Springer theory" developed by Hilburn, Kamnitzer and Weekes [42], and Garner and the second author [26].

We give a more detailed outline of the results and arguments in Section 1.4. We also comment on various conjectures and connections to physics of "3d Mirror Symmetry" and link homology (for $G = GL_n$).

We note that the technology of Coulomb branches works in far greater generality than the "adjoint matter" case studied in this paper. The associated $\mathbb{Z}$-algebras and their modules can be defined and studied as well [50]. It is likely that their representation theory would be very interesting as well.

1.1. **3d Mirror Symmetry.** The construction can be thought of as a part of the $3d$ mirror symmetry of $3d\,\mathcal{N} = 4$ gauge theories [45]. In particular, it is known that the Higgs branch of the mirror of the (G, Ad)-theory is a partial resolution of $T^*T^\vee/W$. In physics terms, our construction starts with a "boundary condition" for the category of line operators in the A-twist of the (G, Ad)-theory (skyscraper sheaf on the stack of conjugacy classes in the loop Lie algebra) and produces another line operator in the B-twist of the dual theory (a (quasi-)coherent sheaf on the Higgs branch) [21]. While the present construction is far from giving any sort of categorical equivalence (even definitions of the categories involved is delicate), it gives an explicit "mirror map" for some of the boundary conditions and exchanges the algebras of local operators on a resolved Higgs branch and on a resolved Coulomb branch. We hope this construction gives a starting point for rigorous constructions of 3d mirror symmetry for line operators. The fact that these categories have putative definitions in terms of vertex operator algebras [20] is an interesting topic for further investigations.

1.2. **Conjectures.** The main construction of the paper produces a $\mathbb{C}^*$-equivariant quasi-coherent sheaf

$$\mathcal{F}_\gamma \in \mathrm{QCoh}_{\mathbb{C}^*}(\mathfrak{C}_{\check{G}})$$

for $\gamma \in \mathfrak{g}(\mathcal{K})$. Quasi-coherence of the sheaf follows directly from our construction, but we suspect that a stronger statement is in fact true:

Conjecture 1.7. *For any regular semisimple $\gamma \in \mathfrak{g}(\mathcal{K})$ the sheaf $\mathcal{F}_\gamma$ is coherent:*

$$\mathcal{F}_\gamma \in \mathrm{Coh}_{\mathbb{C}^*}(\mathfrak{C}_{\check{G}}).$$

The coherence conjecture already has interesting numerical corollaries. It is known that $H^*(\mathrm{Sp}_\gamma)$ is finite-dimensional if γ is elliptic and G is simply connected. Thus the conjecture above implies an estimate on the growth of the dimensions of these cohomology spaces, as we multiply γ by t^m.

Conjecture 1.8. *For any elliptic regular semisimple $\gamma \in \mathfrak{g}(\mathcal{K})$ with G being simple and simply connected there are $c_i \in \mathbb{Q}$ and $M \in \mathbb{Z}$ such that*

$$\dim(H^*(\mathrm{Sp}_{t^m\gamma})) = \sum_{i=0}^{r} c_i m^i, \quad m > M, \quad r = \mathrm{rank}(\mathfrak{g}).$$

In the case of homogeneous elliptic γ it was shown in [83] that the conjecture is true, the left-hand side being given by variants of rational Coxeter-Catalan numbers, in particular the c_i can be explicitly computed.[1] Many low-rank examples are treated in a lot of detail in [72]. More complicated non-homogeneous elliptic cases for $G = SL_n$ were studied in [75].

1.3. **Relation to conjectures stemming from knot theory.** The case of $G = GL_n, SL_n$ is of special interest because of the applications to knot theory [33, 31, 32, 71, 70, 68]. In particular, the characteristic polynomial of $\gamma \in \mathfrak{gl}_n(\mathcal{K})$ defines an germ of planar curve singularity and the link of this singularity is the closure of the braid (conjugacy class) $\beta(\gamma) \in \mathfrak{Br}_n$. When $G = SL_n$, the conjecture [70, Conjecture 2 and Proposition 4] predicts an isomorphism between the (reduced) triply graded homology of $\beta(\gamma)$ and $H^*(\mathrm{Sp}_\gamma)$ enhanced with the perverse filtration [61, 60, 62]. Notice that these papers use *cohomology* whereas the present work uses *BM homology*, but this distinction is immaterial for these numerical comparisons of multiply graded finite dimensional vector spaces.

In this paper, we enrich the algebro-geometric side of the above conjectures by considering an infinite family of affine Springer fibers $\{\mathrm{Sp}_\gamma, \mathrm{Sp}_{t\gamma}, \mathrm{Sp}_{t^2\gamma}, \ldots\}$. It is easy to see that multiplication of γ by t^m corresponds to the multiplication of the braid $\beta(\gamma)$ by FT^m, the m-th power of the full twist braid FT. Since FT is central in the braid group, the conjugacy class of $\beta(\gamma)\cdot\mathrm{FT}^m$ is determined by the conjugacy class of $\beta(\gamma)$.

By construction of triply graded homology $\mathrm{HHH}(-)$, there are natural multiplication maps

$$\mathrm{HHH}(\beta(\gamma)) \otimes \mathrm{HHH}(\mathrm{FT}^m) \to \mathrm{HHH}(\beta(\gamma)\cdot\mathrm{FT}^m), \quad \mathrm{HHH}(\mathrm{FT}^m)\otimes\mathrm{HHH}(\mathrm{FT}^{m'}) \to \mathrm{HHH}(\mathrm{FT}^{m+m'}), \tag{1}$$

and hence $\bigoplus_m \mathrm{HHH}(\beta(\gamma)\cdot\mathrm{FT}^m)$ has a structure of a graded module over the graded algebra $\bigoplus_m \mathrm{HHH}(\mathrm{FT}^m)$. The latter graded algebra, as conjectured in [31] and proved in [32], is closely related to the homogeneous coordinate ring of $\mathrm{Hilb}^n(\mathbb{C}^2)$, and to the $\mathbb{Z}$-algebras appearing in this paper. In other words, in this paper we establish a precise analogue of multiplication maps (1) on the affine Springer side by means of geometric representation theory.

In a series of papers [68, 69] the third named author and Rozansky took a different approach to knot invariants and defined a $\mathbb{C}^*\times\mathbb{C}^*$-equivariant complex of coherent sheaves

$$\mathcal{G}_\beta \in D^b_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Coh}(\mathrm{Hilb}_n(\mathbb{C}^2)))$$

such that the hypercohomology $H^*(\mathcal{G}_\beta)$ is isomorphic as a bigraded vector space to the "lowest row" of the triply-graded homology of β.

To connect these constructions with the present one, note that the natural inclusion map $i_{\mathbb{C}^*}: \mathbb{C}^*\times\mathbb{C}\to\mathbb{C}^2$, $i_{\mathbb{C}^*}(x,y) = (x-1,y)$ induces an inclusion $i_{\mathbb{C}^*}: \mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C}) \to \mathrm{Hilb}^n(\mathbb{C}^2)$ which identifies the punctual Hilbert schemes at $(1,0)$ and $(0,0)$.

Conjecture 1.9. *For any regular semisimple $\gamma \in \mathfrak{gl}_n[\![t]\!]$ there is an isomorphism of $\mathbb{C}^*$-equivariant sheaves*

$$\mathcal{F}_\gamma \simeq i^*_{\mathbb{C}^*}(\mathcal{G}_\beta), \quad \beta = \beta(\gamma).$$

Let us point out that the results of Maulik [58], the aforementioned [60, 61, 62], and the results of the third author with Rozansky in [69] can be combined to show that the conjecture is true on the level of Euler characteristics:

$$\chi(\mathrm{Sp}_\gamma) = \chi_{\mathbb{C}^*}(i^*_{\mathbb{C}^*}(\mathcal{F}_\beta)), \quad \beta = \beta(\gamma),$$

for $\gamma \in \mathfrak{sl}_n[\![t]\!]$ elliptic regular semisimple.

In particular, we derive an Euler characteristics version of the weak coherence conjecture:

Corollary 1.10. *For any elliptic regular semisimple $\gamma \in \mathfrak{sl}_n[\![t]\!]$ there are $c_i \in \mathbb{Q}$ and $M \in \mathbb{Z}$ such that*

$$\chi(H^*(\mathrm{Sp}_{t^m\gamma})) = \sum_{i=0}^{n-1} c_i m^i, \quad m > M.$$

[1] For $\widetilde{\mathrm{Sp}}_{t^m\gamma}$ i.e. the version in affine flags, the result is easier to state and simply says $c_i = 0$, $i < r$

Finally, note that Maulik's result actually keeps track of the Euler characteristics of the Hilbert schemes of points on the germ defined by γ and hence the perverse filtration [60, 61]. In particular, for elliptic γ we may also conjecture that there exists a Springer-theoretic construction of a sheaf $\mathrm{gr}^P \mathcal{F}_\gamma \in \mathrm{Coh}_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{Hilb}^n(\mathbb{C}^2))$ which agrees with $\mathcal{F}_\beta$ $\mathbb{C}^\times \times \mathbb{C}^\times$-equivariantly.

It is interesting to consider the extension of the conjecture to other than elliptic elements. While the homology in this case is always infinite-dimensional, one has promising results for for instance the lattice quotients Sp_γ. In the unramified case, the paper [16] proves that the multivariate generating function (over root valuations) of the weight polynomials is rational. On the other hand, the second author has computed in [51] the equivariant cohomology of the affine Springer fibers in the unramified case and related them to the Procesi bundle on the Hilbert scheme of points.

1.4. **Outline.**

1.4.1. *Outline of the argument.* The key ingredients of the construction are 1) the technology developed by Braverman, Finkelberg and Nakajima [9] on the affine Springer side (Topology) and 2) noncommutative geometry methods akin to the work by [35] on the Hilbert scheme side (Algebraic Geometry). The theory of the 3) Double affine Hecke algebras (Algebra) links these two theories together. Our work provides a dictionary between objects in the three theories. A part of this dictionary is as follows[2]:

Topology	Algebra	Algebraic Geometry
${}_i\mathcal{A}_i := H_*^{\widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times}({}_i\mathcal{R}_i)$	$\mathbf{e}\mathbb{H}_{c+i\hbar,\hbar}\mathbf{e}$	$\mathbb{C}[T^*T^\vee/W]$
${}_{i-1}\mathcal{A}_i := H_*^{\widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times}({}_{i-1}\mathcal{R}_i)$	${}_{i-1}\mathcal{B}_i = \mathbf{e}_-\mathbb{H}_{c+i\hbar,\hbar}\mathbf{e}$	$\mathcal{O}(1)$
${}_i\widetilde{\mathcal{A}}_i := H_*^{\widetilde{\mathbf{I}}_{\mathcal{O}} \rtimes \mathbb{C}^\times}({}_i\widetilde{\mathcal{R}}_i)$	$\mathbb{H}_{c+i\hbar,\hbar}$	$H^0\left(\widetilde{T^*T^\vee/W}, \mathcal{P}\right)$
${}_{i-1}\widetilde{\mathcal{A}}_i := H_*^{\widetilde{\mathbf{I}}_{\mathcal{O}} \rtimes \mathbb{C}^\times}({}_{i-1}\widetilde{\mathcal{R}}_i)$	${}_{i-1}\widetilde{\mathcal{B}}_i$	$H^0\left(\widetilde{T^*T^\vee/W}, \mathcal{P} \otimes \mathcal{O}(1)\right)$
$H_*^{\mathbb{C}^\times}(\widetilde{\mathrm{Sp}}_{\frac{nk+1}{n}})$	$\mathbb{H}_{c+k\hbar,\hbar} \curvearrowright L_{\frac{nk+1}{n}}(\mathbb{C})$	$\mathcal{O}_{\pi^{-1}(1,0)}(k)$

In the Algebra column of the table we have the algebraic objects related to the representation theory of the graded double affine Hecke algebra $\mathbb{H}_{c,\hbar}$ defined in Definition 4.1. This algebra is also known under the names *trigonometric* or *degenerate* DAHA. The algebra $\mathbb{H}_{c,\hbar}$ contains the finite Weyl group W and $\mathbf{e}, \mathbf{e}_- \in W$ are the projectors for the trivial and sign representations. We define an explicit representation of these algebras using difference operators and use it to prove the following:

Theorem 1.11. *(Theorem 4.12) The spherical subalgebra* $\mathbf{e}\mathbb{H}_{c,\hbar}\mathbf{e}$ *is isomorphic to the anti-spherical subalgebra* $\mathbf{e}_-\mathbb{H}_{c-\hbar,\hbar}\mathbf{e}_-$ *with shifted parameter.*

Similar shift isomorphisms are well known in the theory of rational Cherednik algebras and for the Dunkl differential-difference representation [4, 43, 74], but it appears to be new for the difference representation of trigonometric DAHA.

Thus $\mathbf{e}_-\mathbb{H}_{c+i\hbar,\hbar}\mathbf{e}$ naturally has left $\mathbf{e}\mathbb{H}_{c+(i-1)\hbar,\hbar}\mathbf{e}$ and right $\mathbf{e}\mathbb{H}_{c+i\hbar,\hbar}\mathbf{e}$ action and we set:

$$ {}_i\mathcal{B}_i = \mathbf{e}\mathbb{H}_{c+i\hbar,\hbar}\mathbf{e}, \quad {}_i\mathcal{B}_{i+1} = \mathbf{e}_-\mathbb{H}_{c+i\hbar,\hbar}\mathbf{e}, \quad {}_i\mathcal{B}_{j+1} = {}_i\mathcal{B}_j \otimes_{{}_j\mathcal{B}_j} {}_j\mathcal{B}_{j+1}. $$

The direct sum ${}_\bullet\mathcal{B}_\bullet = \bigoplus_{i \le j} {}_i\mathcal{B}_j$ is an example of a $\mathbb{Z}$-algebra, introduced by Polischuk and Bondal [8] and studied in a setting relevant to us by Gordon and Stafford [35, 36, 76].

We now explain how the above mentioned structures exist in the affine Springer theory, which corresponds to the Topology column. The key geometric object is an ind-scheme ${}_i\widetilde{\mathcal{R}}_j$, a variant of "the space of triples $\mathcal{R}$" central to the work of Braverman, Finkelberg and Nakajima on Coulomb branches:

$$ {}_j\widetilde{\mathcal{R}}_i = \left\{(g, v) \in G(\mathcal{K}) \times t^i \,\mathrm{Lie}(\mathbf{I}) | g \cdot v \in t^j \,\mathrm{Lie}(\mathbf{I})\right\} / \mathbf{I}, $$

where $\mathbf{I}$ is the Iwahori subalgebra. On the level of sets, the quotient space $\mathbf{I} \backslash {}_0\widetilde{\mathcal{R}}_0$ is the quotient $\widetilde{St}/G(\mathcal{K})$ of the affine Steinberg space $\widetilde{St}$ as also explained in the introduction to [10].

It was explained by Varagnolo and Vasserot [82] that the equivariant homology group of the affine Steinberg variety $H_*^{\widetilde{\mathbf{I}} \rtimes \mathbb{C}^*_{\mathrm{rot}}}({}_0\widetilde{\mathcal{R}}_0)$ is isomorphic to $\mathbb{H}_{c,\hbar}$, here the parameter c depends on our choice of the equivariant structure with respect to the loop rotation group $\mathbb{C}^*_{\mathrm{rot}}$. Their work however uses localization techniques which we are able to avoid, thereby providing an isomorphism over the full parameter space, see Theorem 6.1.

[2]For simplicity of introduction we discuss the type A case, for other types see section 8

Similarly, one can define the affine Grassmannian version ${}_i\mathcal{R}_j$ of the above spaces. Since the fibers of the projection ${}_i\widetilde{\mathcal{R}}_j \to {}_i\mathcal{R}_j$ are classical Springer fibers, we have a geometric model for the spherical algebra $\mathbf{e}\mathbb{H}(c)\mathbf{e} = H_*^{G(\mathcal{O})\ltimes\mathbb{C}^*_{\mathrm{rot}}}({}_0\mathcal{R}_0)$. Thus, it is natural to define

$$ {}_i\mathcal{A}_j = H_*^{G(\mathcal{O})\ltimes\mathbb{C}^*_{\mathrm{rot}}}({}_i\mathcal{R}_j), \quad {}_\bullet\mathcal{A}_\bullet = \bigoplus_{i\le j} {}_i\mathcal{A}_j. $$

As explained in [11, 21, 85], there is a natural convolution product

$$ H_*^{G(\mathcal{O})\ltimes\mathbb{C}^*_{\mathrm{rot}}}({}_i\mathcal{R}_j) \otimes H_*^{G(\mathcal{O})\ltimes\mathbb{C}^*_{\mathrm{rot}}}({}_j\mathcal{R}_k) \to H_*^{G(\mathcal{O})\ltimes\mathbb{C}^*_{\mathrm{rot}}}({}_i\mathcal{R}_k). $$

By associativity the convolution descends to give bilinear product maps

$$ {}_i\mathcal{A}_j \bigotimes_{{}_j\mathcal{A}_j} {}_j\mathcal{A}_k \to {}_i\mathcal{A}_k. $$

One of our main results partially identifies the Coulomb branch $\mathbb{Z}$-algebra ${}_\bullet\mathcal{A}_\bullet$ in terms of the algebraic $\mathbb{Z}$-algebra ${}_\bullet\mathcal{B}_\bullet$.

Theorem 1.12. *The Coulomb branch $\mathbb{Z}$-algebra ${}_\bullet\mathcal{A}_\bullet$ satisfies the following properties:*

(a) *For all i the algebras ${}_i\mathcal{A}_i$ and ${}_i\mathcal{B}_i$ are isomorphic.*
(b) *For all i the bimodules ${}_i\mathcal{A}_{i+1}$ and ${}_i\mathcal{B}_{i+1}$ are isomorphic.*
(c) *For $G = GL_n$, the bimodules ${}_i\mathcal{A}_j$ and ${}_i\mathcal{B}_j$ are isomorphic for all $i \le j$. Moreover, the $\mathbb{Z}$-algebras ${}_\bullet\mathcal{A}_\bullet$ and ${}_\bullet\mathcal{B}_\bullet$ are isomorphic.*

We prove part (a) as Theorem 6.1, part (b) as Theorem 6.5 and part (c) as Theorem 6.21. In Proposition 6.10 we also provide an explicit basis for the associated graded of ${}_\bullet\mathcal{A}_\bullet$ with respect to Bruhat filtration in all types, see in particular Theorem 6.8 for $G = GL_n$.

The main difficulty in proving part (c) is that ${}_\bullet\mathcal{B}_\bullet$ is generated by the "degree one bimodules" ${}_i\mathcal{B}_{i+1}$ by definition, while this is not clear at all for ${}_\bullet\mathcal{A}_\bullet$. For $G = GL_n$, we resolve this difficulty by a careful combinatorial analysis of the basis in Theorem 6.8, and using deep results of Gordon and Stafford on $\mathbb{Z}$-algebras for rational Cherednik algebras [35, 36].

Next, we turn to the Algebraic Geometry column of the table. In the commutative limit $c = \hbar = 0$ the $\mathbb{Z}$-algebra ${}_\bullet\mathcal{A}_\bullet^{\hbar=0}$ becomes a graded commutative algebra, as ${}_i\mathcal{A}_j^{\hbar=0}$ only depends on the difference $d = j - i$. For $d = 0$ the algebra ${}_0\mathcal{A}_0^{\hbar=0}$ can be identified with the algebra of symmetric polynomials on $T^*T^\vee$, or, equivalently, the algebra of functions on $T^*T^\vee/W$. For $d = 1$ the module ${}_0\mathcal{A}_1^{\hbar=0}$ can be identified with the space of antisymmetric polynomials on $T^*T^\vee$. We can then define an algebraic variety

$$ \widetilde{\mathfrak{C}}_{\check{G}} := \operatorname{Proj} \bigoplus_{d=0}^{\infty} {}_0\mathcal{A}_d^{\hbar=0} \tag{2} $$

which is a partial resolution of $\operatorname{Spec} {}_0\mathcal{A}_0^{\hbar=0} = T^*T^\vee/W$. In other words, we identify the above graded algebra with the homogeneous graded ring of $\widetilde{\mathfrak{C}}_{\check{G}}$. Our next main result identifies this graded algebra explicitly.

Theorem 1.13. *(Theorem 3.8) Let ϵ be the sign representation of $\mathbb{C}[W]$ and $\mathbf{e}_d$ be the idempotent corresponding to $\epsilon^{\otimes d}$. We have*

$$ {}_0\mathcal{A}_d^{\hbar} \simeq \mathbf{e}_d \bigcap_{\alpha\in\Phi^+} \langle 1 - \alpha^\vee, y_\alpha \rangle^d $$

where in the right hand side we have an intersection of ideals in $\mathbb{C}[T^*T^\vee]$. The isomorphism agrees with the convolution structure on the left hand side and the multiplication on the right hand side.

By the work of Haiman [40], for $G = GL_n$ this implies the following:

Theorem 1.14. *For $G = GL_n$, we get*

$$ \operatorname{Proj} \bigoplus_{d=0}^{\infty} {}_0\mathcal{A}_d^{\hbar=0} = \operatorname{Hilb}^n(\mathbb{C}^* \times \mathbb{C}), $$

in particular, it is a smooth resolution of $T^*T^\vee/W \simeq (\mathbb{C}^* \times \mathbb{C})^n/S_n$.

Remark 1.15. A different proof of Theorem 1.14 essentially follows from [11, Theorem 3.10], which identifies a resolution of the Coulomb branch for $(GL_n, \mathfrak{gl}_n \oplus \mathbb{C}^n)$, constructed using flavor symmetry, with $\operatorname{Hilb}^n(\mathbb{C}^2)$. The same proof in *loc. cit.* applied to $(GL_n, \mathfrak{gl}_n)$ yields Theorem 1.14.

Another proof can be extracted from more general results of Nakajima and Takayama [65, 80] which identify the resolved Coulomb branches of quiver gauge theories of affine type A with certain Cherkis bow varieties [19], and certain moduli spaces of parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. See, in particular, [11, Theorem 4.9] for more context and details.

Next, we study graded modules over all of the above $\mathbb{Z}$-algebras and graded algebras. The convolution structure of spaces ${}_i\widetilde{\mathcal{R}}_j$ allows us to define a correspondence between the affine Springer fibers in the affine flag variety $\widetilde{\mathrm{Sp}}_{t^i\gamma}, \widetilde{\mathrm{Sp}}_{t^j\gamma} \subset G(\mathcal{K})/\widetilde{I}$. Similarly, we prove in Theorem 7.9 that $\bigoplus_{j\geq 0} H_*(\mathrm{Sp}_{t^j\gamma})$ is a module over $\mathbb{Z}$-algebra ${}_\bullet\mathcal{A}_\bullet$. This is very similar to the BFN Springer theory developed in [42, 26]. In the commutative variant, we obtain a graded module over the graded algebra, and hence a quasi-coherent sheaf over its Proj construction defined by (2), which proves Theorem 1.1.

Finally, we outline a simple construction of the above action. Recall that the homology of the affine Grassmanian version of the Springer fibers $\mathrm{Sp}_{t^j\gamma} \subset G(\mathcal{K})/G(\mathcal{O})$ can be described in terms of the action of the finite Weyl group:

$$H_*(\mathrm{Sp}_{t^j\gamma}) = H_*(\widetilde{\mathrm{Sp}}_{t^j\gamma})^W, \quad H_*(\mathrm{Sp}_{t^{j-1}\gamma}) = H_*(\widetilde{\mathrm{Sp}}_{t^j\gamma})^\epsilon[-2\dim G/B]. \tag{3}$$

where ϵ is the sign representation of W and $[-2\dim G/B]$ denotes shift in homological degree. We explain the details of the isomorphisms in the Lemma 2.2. Given a class in $H_*(\mathrm{Sp}_{t^{j-1}\gamma})$, we can identify it with a class in $H_*(\mathrm{Sp}_{t^j\gamma})^\epsilon$, then act by an antisymmetric polynomial (using the commutative version of the double affine action of [88]) and get a class in $H_*(\mathrm{Sp}_{t^j\gamma})^W = H_*(\mathrm{Sp}_{t^j\gamma})$. This construction gives a map

$$A_G \otimes H_*(\mathrm{Sp}_{t^{j-1}\gamma}) \to H_*(\mathrm{Sp}_{t^j\gamma}) \tag{4}$$

where $A_G = {}_0\mathcal{A}_1^{\hbar=0}$ is the space of antisymmetric polynomials.

It is unclear if this approach can be used to define the action of the full graded algebra $\bigoplus_{d=0}^{\infty} {}_0\mathcal{A}_d^{\hbar=0}$, the main obstacles are:

- It is unclear how to verify the relations between the products of elements of ${}_0\mathcal{A}_1^{\hbar=0}$ inside ${}_0\mathcal{A}_d^{\hbar=0}$
- For $G \neq GL_n$, it is unclear if the algebra is generated in degree 1.

To avoid these obstacles, we have abandoned this approach altogether and instead used the machinery of Coulomb branches throughout the paper. Nevertheless, *a posteriori* we conclude that the action of the degree 1 part of the algebra agrees with (4), and hence all necessary relations are satisfied.

1.4.2. *Outline of the paper.* In Section 2 we define the affine Springer fibers and some background material. The sheaves we construct live on a partial resolution of $T^*T^\vee/W$, which is introduced in Section 3. In Section 4, we study the trigonometric Cherednik algebra and a natural $\mathbb{Z}$-algebra built out of it, which is the algebraic main part of the construction. In Sections 5,6 and 7 we study the affine Springer fibers using Coulomb algebra machinery, in particular constructing a geometric $\mathbb{Z}$-algebra action and comparing it to the one in Section 4. In Section 8 we prove that the sheaves we construct are coherent whenever γ is homogeneous. In Section 8.2, we study some homogeneous examples in detail.

Acknowledgments. The authors would like to thank Roman Bezrukavnikov, Tudor Dimofte, Pavel Etingof, Pavel Galashin, Niklas Garner, Ivan Losev, Ngô Bao Châu, Hiraku Nakajima, Wenjun Niu, José Simental Rodriguez, Eric Vasserot, Monica Vazirani, Ben Webster, Alex Weekes and Zhiwei Yun for useful discussions. We thank the organizers of the MRSI program "Enumerative Geometry Beyond Numbers" in 2018 for providing an ideal working environment which motivated us to start this project.

The work of E. G. was partially supported by the NSF grant DMS-1760329. The work of A. O. was partially supported by the NSF grants DMS-1352398 and DMS-1760373. The work of O. K. was partially supported by the Finnish Academy of Science and Letters grant for postdoctoral research abroad.

2. Affine Springer fibers

2.1. **Definitions.** Let $G/\mathbf{k}$ be a reductive group over a field $\mathbf{k}$. We assume $\mathbf{k} = \overline{\mathbf{k}}$ and that the characteristic is zero or large enough (no attempt will be made to give bounds). Fix some pinning $T \subset B \subset G$ of the root system of G. Define $N = \dim(G/B)$. Let $\mathcal{K} = \mathbf{k}((t))$ and $\mathcal{O} = \mathbf{k}[\![t]\!]$. Write $G((t)) = G(\mathcal{K})$ and $G[\![t]\!] = G(\mathcal{O})$. Denote also $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}((t)) = \mathfrak{g}(\mathcal{K}), \mathfrak{g}[\![t]\!] = \mathfrak{g}(\mathcal{O})$.

Let $\mathbf{P}$ be a standard parahoric subgroup of $G(\mathcal{K})$. Let $\mathrm{Fl}_G^{\mathbf{P}} = G(\mathcal{K})/\mathbf{P}$ be the corresponding partial affine flag variety. When $\mathbf{P} = \mathbf{I}$ is the Iwahori subgroup of $G(\mathcal{K})$ corresponding to B, we simply write $\mathrm{Fl}_G := \mathrm{Fl}^{\mathbf{I}}$ for the affine flag variety and when $\mathbf{P} = G(\mathcal{O})$ we write $\mathrm{Gr}_G := \mathrm{Fl}^{G(\mathcal{O})}$ for the affine Grassmannian. Since it will usually be clear from the context, we will also omit the subscript G.

For any $\mathbf{P}$ and $\gamma \in \mathfrak{g}(\mathcal{K})$, define the affine Springer fiber

$$\mathrm{Sp}_\gamma^{\mathbf{P}} := \{g\mathbf{P} | \operatorname{Ad}(g^{-1})\gamma \in \operatorname{Lie}(\mathbf{P})\} \subset \mathrm{Fl}^{\mathbf{P}} .$$

When $\mathbf{P} = G(\mathcal{O})$ we omit the superscript and when $\mathbf{P} = \mathbf{I}$ we write $\widetilde{\mathrm{Sp}}_\gamma = \mathrm{Sp}_\gamma^{\mathbf{I}}$.

The space $\mathrm{Sp}_\gamma^{\mathbf{P}}$ is a sub-ind-scheme of $\mathrm{Fl}^{\mathbf{P}}$. It is always nonreduced, but since it makes no difference to us, we will only work with the reduced structure of $\mathrm{Sp}_\gamma^{\mathbf{P}}$. $\mathrm{Sp}_\gamma^{\mathbf{P}}$ is finite-dimensional if and only if γ is regular semisimple [47]. We shall mostly focus on the case γ is regular semisimple from now on. We also assume γ is compact, i.e. contained in some Iwahori subgroup, or equivalently that the affine Springer fiber is nonempty. The ind-scheme Sp_γ is locally of finite type, and by results of [47] there exists a free abelian group L acting on $\mathrm{Sp}_\gamma^{\mathbf{P}}$ freely and a projective scheme $S \subset \mathrm{Sp}_\gamma^{\mathbf{P}}$ such that $L \cdot S = \mathrm{Sp}_\gamma^{\mathbf{P}}$. The free abelian group L can be identified with the cocharacter lattice $X_*(C_{G(\mathcal{K})}(\gamma))$. In particular, L is trivial if and only if γ is elliptic. In this case, $\mathrm{Sp}_\gamma^{\mathbf{P}}$ is a projective variety.

Remark 2.1. By the Jordan decomposition, we can write any γ as the sum of commuting semisimple and nilpotent elements: $\gamma = \gamma_s + \gamma_n$. Therefore, we can reduce the study of general γ to nilpotent and semisimple γ, as $\mathrm{Sp}_\gamma = \mathrm{Sp}_{\gamma_s} \cap \mathrm{Sp}_{\gamma_n}$. While it is also interesting to study non-regular semisimple elements, much about this case can in principle be extracted from the regular semisimple case by the fact that the centralizer of a semisimple element is a reductive group over $\mathcal{K}$. In the present work, we are mostly concerned with $H_*^{L_\gamma}(\mathrm{Sp}_\gamma)$ for L_γ a compact subgroup of the centralizer of γ. For example, the natural action of the dDAHA on the coordinate ring of the universal centralizer in Theorem 3.21 can be interpreted as taking $\gamma = 0$ and $L_\gamma = G_\mathcal{O} \subseteq G_\mathcal{K}$.

However, the nilpotent elements seem more mysterious from our point of view. It is clear e.g. by the convergence of the corresponding orbital integrals that the centralizers of nilpotent elements are quite large. In the case where γ is nilpotent, it is not even known if there is a Levi factor of the centralizer. It does make sense to ignore the centralizer (or at most use compact subgroups thereof) and use finite-dimensional approximation to study the Borel-Moore homology of Sp_γ even in the nilpotent cases.

2.2. **The Springer action.** Assume for now that $\mathbf{k} = \mathbb{C}$ or that we are using étale cohomology over $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ for $\ell \neq \mathrm{char}(p)$. One of the remarkable things about $\widetilde{\mathrm{Sp}}_\gamma$ is that $H_*(\widetilde{\mathrm{Sp}}_\gamma) = H_*(\widetilde{\mathrm{Sp}}_\gamma, \mathbb{C})$ has an action of the extended affine Weyl group $\widetilde{W} = W \ltimes X_*(T)$ as shown by Lusztig [57] (in the adjoint case) and Yun [88] (in general), analogously to the Weyl group action in the cohomology of classical Springer fibers.

Lemma 2.2. *Let $\gamma \in \mathfrak{g}(\mathcal{K})$ be an element such that $\gamma = t\gamma_0$ for a regular semisimple compact element $\gamma_0 \in \mathfrak{g}(\mathcal{K})$. Then under the Springer action of $W \subset \widetilde{W}$ on $H_*(\widetilde{\mathrm{Sp}}_\gamma)$, we have a natural identification*

$$H_*(\widetilde{\mathrm{Sp}}_\gamma)^W = H_*(\mathrm{Sp}_\gamma) \tag{5}$$

and an isomorphism

$$H_*(\widetilde{\mathrm{Sp}}_\gamma)^\epsilon \cong H_*(\mathrm{Sp}_{\gamma_0})[-2N]. \tag{6}$$

Here $[-2N]$ means a shift in homological degree and $N = \dim G/B$.

Proof. The first part is due to [89, Section 2.6], and the second part is well-known but not found in the literature, so we give a proof here.

First recall the construction of the Springer action for the subgroup $W \subset \widetilde{W}$. Since $\mathbf{I} \subset G[\![t]\!]$, we have natural projections $\mathrm{Fl} \to \mathrm{Gr}$ and $\widetilde{\mathrm{Sp}}_\gamma \to \mathrm{Sp}_\gamma$. For any $\mathbf{P}$, write $P = \mathbf{P}/t\mathbf{P}$ and $\mathfrak{p}$ for its Lie algebra. There are natural maps of fpqc sheaves $\mathrm{Sp}_\gamma^{\mathbf{P}} \to [\mathfrak{p}/P]$, which send the cosets $g\mathbf{P}$ to the respective images of $g^{-1}\gamma g$ under the projections $\mathrm{Lie}(\mathbf{P}) \to \mathfrak{p}$. In particular there is a cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Sp}}_\gamma & \xrightarrow{\varphi'} & [\widetilde{\mathfrak{g}}/G] = [\mathfrak{b}/B] \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathrm{Sp}_\gamma & \xrightarrow{\varphi} & [\mathfrak{g}/G] \end{array} \tag{7}$$

where the right-hand side is naturally identified with the Grothendieck-Springer resolution for G. Since $\gamma = t\gamma_0$, it is clear that the image of φ will be contained in $[\mathcal{N}/G] \subset [\mathfrak{g}/G]$ and the image of φ' will be contained in $[\widetilde{\mathcal{N}}/G]$. The restriction $S := \pi'_*\mathbb{C}|_\mathcal{N}$ is perverse, and is called the Springer sheaf. It is in fact isomorphic to a direct sum of IC complexes on nilpotent orbits, and it is known by classical Springer theory that

$$\mathrm{End}_{\mathrm{Perv}^G(\mathcal{N})}(S) \cong \mathbb{C}[W].$$

In particular, there is a map

$$\mathbb{C}[W] \to \mathrm{End}_{\mathrm{Perv}(\mathrm{Sp}_\gamma)}(\varphi^* S)$$

and hence an action of W on $H_*(\widetilde{\mathrm{Sp}}_\gamma)$. Decomposing the regular representation of W, we see that there is some IC complex $\mathcal{F}$ on $[\mathcal{N}/G]$ corresponding to the sign representation ϵ. From classical Springer theory it follows that this complex is isomorphic as a perverse sheaf to the shifted skyscraper sheaf $\mathbb{C}_{\{0\}}[-2N]$.

By proper base change,

$$H_*(\widetilde{\mathrm{Sp}}_\gamma)^\epsilon = \mathrm{Hom}(\varphi^*\mathcal{F}, \pi_*\mathbb{C}) \cong \mathrm{Hom}(\varphi^*\mathcal{F}, \varphi^* S).$$

Now note that

$$\varphi^{-1}(0) = \{gG(\mathcal{O}) \in \mathrm{Sp}_\gamma | g^{-1}\gamma g \equiv 0 \ (\mathrm{mod}\ t)\} = \mathrm{Sp}_{\gamma_0}.$$

This is a closed subspace, so $\varphi^*\mathcal{F}$ is isomorphic to the (shifted) extension by zero of the constant sheaf on Sp_{γ_0}. In particular, $R\Gamma(\varphi^*\mathcal{F}) \cong H_*(\mathrm{Sp}_{\gamma_0})$. On the other hand, by the adjunction

$$\mathrm{Hom}(\varphi^*\mathcal{F}, \varphi^* S) = \mathrm{Hom}(\mathcal{F}, \varphi_*\varphi^* S)$$

it is clear that $\mathrm{Hom}(\varphi^*\mathcal{F}, \varphi^* S) \cong R\Gamma(\varphi^*\mathbb{C}_{\{0\}})[-2N]$. Thus $H_*(\widetilde{\mathrm{Sp}}_\gamma)^\epsilon \cong H_*(\mathrm{Sp}_{\gamma_0})[-2N]$.

□

We can rephrase Lemma 2.2 as follows. We have the Leray filtration on the Borel-Moore homology of $\widetilde{\mathrm{Sp}}_\gamma$ such that

$$\mathrm{gr}\, H_k(\widetilde{\mathrm{Sp}}_\gamma) = \bigoplus_{i+j=k} H_i(\mathrm{Sp}_\gamma, R^j\pi_*\mathbb{C}),\ 0 \le j \le 2N. \tag{8}$$

Corollary 2.3. *a) The W-invariant part of the homology of $\widetilde{\mathrm{Sp}}_\gamma$ is canonically isomorphic to the $j = 0$ part of* (8)*:*

$$H_k(\widetilde{\mathrm{Sp}}_\gamma)^W = H_k(\mathrm{Sp}_\gamma, R^0\pi_*\mathbb{C}) = H_k(\mathrm{Sp}_\gamma, \mathbb{C}).$$

b) The associated graded of the W-antiinvariant part is isomorphic to the $j = 2N$ part of (8)*. In other words, the restriction of the obvious map*

$$H_k(\widetilde{\mathrm{Sp}}_\gamma)^\epsilon \to H_{k-2N}(\mathrm{Sp}_\gamma, R^{2N}\pi_*\mathbb{C}) \cong H_{k-2N}(\mathrm{Sp}_{\gamma_0})$$

is an isomorphism.

Remark 2.4. The result is also true in cohomology, cohomology with compact supports and BM homology (where we replace the constant sheaf by $\omega_{\widetilde{\mathrm{Sp}}_\gamma}$) by identical reasoning.

Remark 2.5. The stabilizer $K_\gamma := \mathrm{Stab}_{G(\mathcal{K})}(\gamma)$ acts naturally on $\widetilde{\mathrm{Sp}}_\gamma$, inducing an action of the component group in cohomology. In particular, we have the Springer action for equivariant versions of any of the above theories, as well as a commuting action of the component group of the centralizer.

2.3. **Extended symmetry.** In addition to the action of $\widetilde{W}$ on $H_*(\widetilde{\mathrm{Sp}}_\gamma)$ there is a degenerate action of the character lattice $X^*(T)$ on $H_*(\widetilde{\mathrm{Sp}}_\gamma)$ defined as follows. There is a natural map $\mathbf{I} \to T$ realizing T as the reductive quotient of $\mathbf{I}$. In particular, each character $\chi : T \to \mathbb{G}_m$ gives a map $\mathbf{I} \to \mathbb{G}_m$ and in particular a $\mathbb{G}_m$-torsor $\mathcal{L}(\chi)$ on Fl. As a line bundle, we can write this as

$$G \times^{\mathbf{I},\chi} \mathbb{A}^1 \to \mathrm{Fl}.$$

Cap product with $c_1(\mathcal{L}(\chi))$ defines an action of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ on $H_*(\widetilde{\mathrm{Sp}}_\gamma)$.

Theorem 2.6. *Let $X_*(T) \subset \widetilde{W}$ be the translation part of the extended affine Weyl group. Then the Springer action of $X_*(T)$ and the action of $X^*(T)$ defined above commute.*

Proof. This is proved in [88, Corollary 6.1.7]. □

Remark 2.7. Note that this is not true in *equivariant* Borel-Moore homology, as one gets essentially relations in the degenerate DAHA.

Note that we can identify the action of $X_*(T)$ with the action of $\mathbb{C}[T^\vee]$ and the action of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ with the action of $\mathbb{C}[\mathfrak{t}]$, and summarize the above results in the following

Proposition 2.8. *The non-equivariant Borel-Moore homology $H_*(\widetilde{\mathrm{Sp}}_\gamma)$ is a (right) module over $\mathbb{C}[T^*T^\vee] \rtimes W$.*

Proposition 2.9. *The non-equivariant Borel-Moore homology $H_*(\mathrm{Sp}_\gamma)$ is a module over $\mathbb{C}[T^*T^\vee]^W$.*

Proof. By Lemma 2.2 we have $H_*(\mathrm{Sp}_\gamma) = H_*(\widetilde{\mathrm{Sp}}_\gamma)^W$, and by Proposition 2.8 $H_*(\widetilde{\mathrm{Sp}}_\gamma)$ has an action of $\mathbb{C}[T^*T^\vee]$. By symmetrizing, we get the action of $\mathbb{C}[T^*T^\vee]^W$ on $H_*(\mathrm{Sp}_\gamma)$. □

We record the following lemma here.

Lemma 2.10. *The action of $X^*(T)\otimes_{\mathbb{Z}}\mathbb{C}$ on $H_*(\mathrm{Sp}_\gamma)$ from Section 2.3 decreases the Leray filtration* (8) *by two.*

Proof. This follows from the fact that the action comes from cap product with Chern classes of the line bundles $L(\chi)$ constructed in the beginning of this section. □

Let Δ be the top-dimensional class in $H^{2N}(G/B)$. By the isomorphism $H^*(G/B)\cong\mathbb{C}[\mathfrak{t}]_W$, we can identify Δ with an antisymmetric polynomial in $\mathbb{C}[\mathfrak{t}]$, namely $\Delta = \prod_{\alpha\in\Phi^+} y_\alpha$. Moreover, note that we may write antisymmetric polynomials as $\mathbb{C}[\mathfrak{t}]^\epsilon = \Delta\cdot\mathbb{C}[\mathfrak{t}]^W$.

Lemma 2.11. *We have that*

$$i_*(H_{k-2N}(\mathrm{Sp}_{t^{-1}\gamma})) = \Delta\cdot H_k(\widetilde{\mathrm{Sp}}_\gamma)^\epsilon.$$

where $i:\mathrm{Sp}_{t^{-1}\gamma}\to\mathrm{Sp}_\gamma$ is the natural inclusion.

Proof. By Lemma 2.10 the operator Δ preserves the decomposition (8) and decreases the j-grading by $2N$. Since $j\le 2N$, the action of Δ kills all the summands in (8) with $j<2N$, so that

$$\Delta\cdot H_k(\widetilde{\mathrm{Sp}}_\gamma)^\epsilon = \Delta\cdot H_{k-2N}(\mathrm{Sp}_\gamma, R^{2N}\pi_*\mathbb{C}).$$

At $p\in\mathrm{Sp}_\gamma$ where $\pi^{-1}(p)$ is a proper subset in the flag variety, $R^{2N}\pi_*\mathbb{C}|_p = 0$. On the other hand, $\pi^{-1}(p)$ is the full flag variety if and only if $p\in i(\mathrm{Sp}_{t^{-1}\gamma})$ and in this case $\Delta: R^{2N}\pi_*\mathbb{C}\to R^0\pi_*\mathbb{C}$ is the isomorphism. Therefore

$$\Delta\cdot H_{k-2N}(\mathrm{Sp}_\gamma, R^{2N}\pi_*\mathbb{C}) = i_*(H_{k-2N}(\mathrm{Sp}_{t^{-1}\gamma})).$$

□

2.4. **The lattice action.** Let G_γ be the stabilizer of γ in $G(\mathcal{K})$. Obviously, G_γ acts on Sp_γ, giving an action (of its component group) on the homology of Sp_γ. Since the action in this case is proper [47], we also get an action on the Borel-Moore homology $H_*(\mathrm{Sp}_\gamma)$.

The W-invariant translation part of the stabilizer action restricts to an action of the "spherical part" $\mathbb{C}[X_*(T)]^W$ on $H_*(\widetilde{\mathrm{Sp}}_\gamma)$. This action commutes with the Springer action, and the local main theorem of [89] identifies the spherical part of the Springer action with the spherical part of the lattice action. More precisely,

Theorem 2.12. [89] *One can define a canonical homomorphism $\mathbb{C}[X_*(T)]^W \twoheadrightarrow \mathbb{C}[\pi_0(G_\gamma)]$ and the spherical part of the Springer action on $H_*(\widetilde{\mathrm{Sp}}_\gamma)$ factors through this map.*

By Propositions 2.8 and 2.9, the (BM) homology of Sp_γ and the homology of $\widetilde{\mathrm{Sp}}_\gamma$ define quasicoherent sheaves $\mathcal{F}'_\gamma$ and $\widetilde{\mathcal{F}}'_\gamma$ on $(T^*T^\vee)/W$.

Lemma 2.13. *These quasicoherent sheaves are actually coherent, and set-theoretically supported on the Lagrangian subvariety $\{0\}\times T^\vee\subset T^*T^\vee$. Moreover, the dimension of their support equals the rank of the centralizer.*

Proof. It is clear that the action of $\mathfrak{t}$ is nilpotent, and symmetric functions in $\mathfrak{t}$ act by 0, as they do in $H_*(G/B)$.

By Theorem 2.12 the homology of Sp_γ is finitely generated over $\mathbb{C}[T^\vee]$ and the lattice part of the centralizer acts freely on the components by Kazhdan-Lusztig [47]. Therefore the support is exactly $\{0\}\times T^\vee_\gamma/W$, which has dimension the rank of the cocharacter lattice of the centralizer of γ. □

2.5. **Equivariant versions, endoscopy.** We can in fact upgrade this construction with the addition of equivariance to the picture. The centralizer $G_\gamma/\mathcal{K}$ has a smooth integral model J_γ over $\mathcal{O}$, see e.g. [67]. The stabilizer action factors through the local Picard group $P_\gamma = G_\gamma(\mathcal{K})/J_\gamma(\mathcal{O})$ whose underlying reduced scheme is finite-dimensional and locally of finite type. Consider the connected component of the identity P^0_γ of this group scheme. This is a linear algebraic group over $\mathbb{C}$ whose maximal reductive quotient contains a split maximal torus of rank $\mathrm{rank}(X^*(G_\gamma))$. Call this torus T_γ. It also acts on Sp_γ, and we may take the equivariant BM homology as in [51, Section 3]. The construction of the Springer action etc. from this section go through T_γ-equivariantly. For simplicity, assume $T_\gamma\hookrightarrow T$. Then we have:

Proposition 2.14. *The equivariant BM homology $H_*^{T_\gamma}(\mathrm{Sp}_\gamma)$ is a quasi-coherent sheaf on $T^*T^\vee/W$. Its support is contained in the subvariety $T^*T^\vee_\gamma/W\subset T^*T^\vee/W$. In particular, if γ is elliptic, the support is zero-dimensional.*

Proof. The equivariant cohomology of a point $H^*_{T_\gamma}(pt)$ acts on $H^{T_\gamma}_*(Sp_\gamma)$, giving that the support is contained in $\mathfrak{t}_\gamma \times T^\vee/W$. Results of Yun [89] then imply that the spherical part of the Springer action is given by $\mathbb{C}[\Lambda_\gamma]^W \cong \mathbb{C}[T^\vee_\gamma]^W$ just as in Theorem 2.12. This gives that the support lies in $T^*T^\vee_\gamma/W$ (and in fact projects surjectively to $T^\vee_\gamma/W$). □

Remark 2.15. For split equivalued elements, or whenever we have equivariant formality of the $H^*_{T_\gamma}(pt)$-action, the proof shows the support is all of $T^*T^\vee_\gamma/W$.

If $G \neq GL_n$, we note that in the determination of the above support we run into issues of endoscopy. Already for $G = SL_n$, the center μ_n will be contained in T_γ for any γ. For example for elliptic γ, we get support at n points in $T^*T^\vee/W$. Let us now illustrate how this plays out in the case of general G and γ elliptic. Recall that we may decompose

$$H^{K_\gamma}_*(\mathrm{Sp}_\gamma) = \bigoplus_{\kappa \in X_*(T)} H^{K_\gamma}_*(\mathrm{Sp}_\gamma)_\kappa$$

where either $K_\gamma = \{1\}$ or $K_\gamma = T_\gamma$ (or any subgroup of G_γ for which the definitions make sense) as above. The homological statement of the Fundamental Lemma, proved by Ngô, is essentially that

$$H^{K_\gamma}_*(\mathrm{Sp}_\gamma)_\kappa \cong H^{K_\gamma}_*(\mathrm{Sp}^H_{\gamma_H})_{st} \tag{9}$$

for some affine Springer fiber $\mathrm{Sp}^H_{\gamma_H}$ of an endoscopic group of G. Here "st" denotes the stable part, or in other words the part of the BM homology where the lattice acts unipotently (see e.g. [89]). Since $H_*(\mathrm{Sp}_\gamma)$ is finite-dimensional, by using Lemma 2.13, the above κ-decomposition and Eq. (9), we deduce the following version of the fundamental lemma.

Theorem 2.16. *The sheaf $\mathcal{F}'_\gamma$ is supported at finitely many points. Each stalk $(\mathcal{F}'_\gamma)_{(0,\kappa)}$ is isomorphic to the stalk at $(0,1)$ of an "endoscopic sheaf" on $T^*T^\vee_H/W_H$ for some endoscopic group H of G.*

3. The commuting variety

3.1. **The commuting scheme.** In this section, we introduce the partial resolution of the commuting variety we will be considering. The construction is algebraic in nature. We show the partial resolution coincides with $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$ in the case $G = GL_n$. In general, we show it is a normal variety and conjecture that locally its singularities are modeled by the $\mathbb{Q}$-factorial terminalizations constructed by Losev in [54, 1]. Finally, we introduce a certain open chart of the partial resolution, which turns out to be isomorphic to the universal centralizer of $\check{G}$.

As above, let $\check{\mathfrak{g}}$ be the Lie algebra of $\check{G}$, $\check{\mathfrak{t}} = \mathfrak{t}^*$ is the Lie algebra of $\check{T}$ and W is the Weyl group. We define two versions of the commuting scheme: $\mathfrak{C}'_{\check{\mathfrak{g}}}$ is a subscheme of $\check{\mathfrak{g}} \times \check{\mathfrak{g}}$ cut out by the equation $[x,y] = 0$, while $\mathfrak{C}'_{\check{G}} = \mathfrak{C}'_{\check{G},\check{\mathfrak{g}}}$ is a subscheme of $\check{G} \times \check{\mathfrak{g}}^*$ cut out by the equation $\mathrm{Ad}_g(x) = x$.

Define $\mathfrak{C}_{\check{\mathfrak{g}}} := \mathfrak{C}'_{\check{\mathfrak{g}}} /\!\!/ \check{G} = \mathrm{Spec}\ \mathbb{C}[\mathfrak{C}'_{\check{\mathfrak{g}}}]^{\check{G}}$ and $\mathfrak{C}_{\check{G}} = \mathfrak{C}'_{\check{G}} /\!\!/ \check{G} = \mathrm{Spec}\ \mathbb{C}[\mathfrak{C}'_{\check{G}}]^{\check{G}}$. It is a long-standing open question if these schemes are reduced. We collect some facts about $\mathfrak{C}_{\mathfrak{g}}$ and $\mathfrak{C}$ here.

There are natural restriction maps $\mathbb{C}[\mathfrak{C}_{\check{\mathfrak{g}}}] \to \mathbb{C}[\mathfrak{t}^* \times \mathfrak{t}^*]^W$ and $\mathbb{C}[\mathfrak{C}_{\check{G}}] \to \mathbb{C}[\check{T} \times \mathfrak{t}]^W$, which induce maps $(\mathfrak{t}^* \times \mathfrak{t}^*)/W \to \mathfrak{C}_{\check{\mathfrak{g}}}$ and $(T^*\check{T})/W \to \mathfrak{C}_{\check{G}}$. The former is surjective by the result of Joseph [46] and defines an isomorphism $(\mathfrak{t}^* \times \mathfrak{t}^*)/W \simeq [\mathfrak{C}_{\check{\mathfrak{g}}}]_{\mathrm{red}}$.

In [12, Proposition 5.24], the following is proved by essentially reducing to Joseph's results in the rational case:

Theorem 3.1. *The restriction of functions induces an isomorphism $(T^*\check{T})/W \simeq [\mathfrak{C}_{\check{G}}]_{\mathrm{red}}$.*

We note that there are alternative proofs of the theorem, such as the one given by Gan-Ginzburg in type A:

Theorem 3.2 ([25])**.** *For $G = GL_n$ the scheme $\mathfrak{C}_{\mathfrak{g}}$ is reduced.*

From this it is easy to deduce

Corollary 3.3. *For $G = GL_n$ the scheme $\mathfrak{C}_G$ is reduced.*

Proof. For $G = GL_n$ we have a natural embedding $G \subset \mathfrak{g}$ which induced embedding $\mathfrak{C}' \subset \mathfrak{C}'_{\mathfrak{g}}$ and $\mathfrak{C}_G \subset \mathfrak{C}_{\mathfrak{g}}$. Since $\mathfrak{C}_{\mathfrak{g}}$ is reduced, $\mathfrak{C}_G$ is reduced too. □

Recently, Chen-Ngô [17] proved that $\mathfrak{C}_{\mathfrak{g}}$ is reduced for $\mathfrak{g} = \mathfrak{sp}_{2n}$ and subsequently, Losev [56] showed that $\mathfrak{C}'_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{sp}_{2n}$ is reduced as well.

3.2. **Partial resolutions.** In this subsection we define several graded commutative algebras closely related to the commuting variety. By applying Proj construction to these graded algebras, we recover partial resolutions of $\mathfrak{C}_{\check{G}}$. We summarize various maps between the algebras in the following commutative diagram:

$$\begin{array}{ccccc} & & \bigoplus J^d & \longrightarrow & \bigoplus I^{(d)} \\ & & \uparrow & & \uparrow \\ \bigoplus {}_0\mathcal{A}_d^{\hbar=0} & \xleftarrow[a]{} & \bigoplus A^d & \xrightarrow{b} & \bigoplus \mathbf{e}_d I^{(d)}. \end{array} \tag{10}$$

All direct sums are over $d \geq 0$. Next, we define all of the entries in this diagram, starting with the middle column.

We denote by ϵ the one-dimensional sign representation of W. We define $A_{\mathfrak{t}}$, respectively A, as the subspace of W-antiinvariant functions (that is, ϵ-isotypic component) in $\mathbb{C}[\mathfrak{t}^* \times \mathfrak{t}]$, respectively $\mathbb{C}[T^\vee \times \mathfrak{t}]$. Also, let $J_{\mathfrak{t}}$, respectively J be the ideal in $\mathbb{C}[\mathfrak{t}^* \times \mathfrak{t}]$ (resp. $\mathbb{C}[T^\vee \times \mathfrak{t}^*]$) generated by $A_{\mathfrak{t}}$ (resp. A). Now A^d and J^d are the powers of A and J inside the respective polynomial rings, with the assumption that $A^0 = \mathbb{C}[T^\vee \times \mathfrak{t}]^W$ and $J^0 = \mathbb{C}[T^\vee \times \mathfrak{t}]$.

In the right column we have a family of ideals $I^{(d)} := \bigcap_{\alpha\in\Phi^+} \langle 1 - \alpha^\vee, y_\alpha \rangle^d$. In particular, for $d = 1$ we get the defining ideal of the union of the codimension 2 hyperplanes corresponding to the roots of $\mathfrak{g}$. It is easy to see that $I^{(d_1)} \cdot I^{(d_2)} \subset I^{(d_1+d_2)}$, so we have a graded algebra structure on the direct sum of all $I^{(d)}$. Furthermore, let $\mathbf{e}_d$ denote the projector to the representation ϵ^d in $\mathbb{C}[W]$, that is, symmetrizer $\mathbf{e}$ for d even and antisymmetrizer $\mathbf{e}_-$ for d odd.

Lemma 3.4. *a) We have $\mathbf{e}_d J^d = A^d$ for all $d \geq 0$.*
b) There are natural inclusions $b_d : J^d \to I^{(d)}, A^d \to \mathbf{e}_d I^{(d)}$.

Proof. a) For $d = 0$ this is clear from the definition, so we focus on $d > 0$. Since J is the ideal generated by A, it is spanned by elements of the form $a \cdot f$ for $a \in A$ and $f \in \mathbb{C}[T^\vee \times \mathfrak{t}]$, and J^d is spanned by elements of the form $a_1 \cdots a_d \cdot f$ for $a_1, \ldots, a_d \in A$ and $f \in \mathbb{C}[T^\vee \times \mathfrak{t}]$. Since $a_1, \ldots, a_d$ are antisymmetric, we get

$$\mathbf{e}_d a_1 \cdots a_d \cdot f = a_1 \cdots a_d \mathbf{e}(f) \in A^d.$$

b) It is easy to see that an antisymmetric polynomial vanishes on all codimension 2 hyperplanes, so we get $A \subset I^{(1)}$ and hence $J \subset I^{(1)}$. Therefore $J^d \subset (I^{(1)})^d \subset I^{(d)}$ and the result follows. □

Finally, in the left column we have *commutative Coulomb branch $\mathbb{Z}$-algebra* $\bigoplus {}_0\mathcal{A}_d^{\hbar=0}$ to be defined in Section 6. It is a generalization of the commutative Coulomb branch appearing in the work of Braverman, Finkelberg and Nakajima [9, 10]. It is defined as the convolution algebra in the equivariant Borel-Moore homology of a certain space related to the affine Grassmannnian of G, and we postpone its definition to Section 6. Here we summarize some of its basic properties with pinpoint references to the proofs later in the paper.

Theorem 3.5. *The algebras ${}_0\mathcal{A}_d^{\hbar=0}$ have the following properties:*
a) For all d the module ${}_0\mathcal{A}_d^{\hbar=0}$ is free over $\mathbb{C}[\mathfrak{t}]^W$ and embeds into $\mathbf{e}_d\mathbb{C}[T^\vee \times \mathfrak{t}]$.
b) For d = 0, we have ${}_0\mathcal{A}_0^{\hbar=0} = \mathbb{C}[T^\vee \times \mathfrak{t}]^W$.
c) For d = 1, we have ${}_0\mathcal{A}_1^{\hbar=0} = A$.
d) In type A, we have ${}_0\mathcal{A}_d^{\hbar=0} = ({}_0\mathcal{A}_1^{\hbar=0})^d = A^d$ for all d.

Proof. a) We regard $\mathbb{C}[\mathfrak{t}]^W$ as the equivariant cohomology of the point, and ${}_0\mathcal{A}_d^{\hbar=0}$ is realized as the equivariant Borel-Moore homology of a certain space which admits an affine paving by Bruhat cells. Therefore it is equivariantly formal and its equivariant cohomology is a free module over $H^*_G(pt)$. The embedding to $\mathbb{C}[T^\vee \times \mathfrak{t}]$ is realized by the inclusion to equivariant Borel-Moore homology of the fixed point set. See Section 6 for more details. That we land in the $\mathbf{e}_d$-isotypic component follows from the fact that the localization is defined using T-equivariant cohomology and to pass to G-equivariant cohomology we take W-invariants. See for instance [9, Remark 5.23].

b) This is a specialization of Corollary 6.3 at $\hbar = 0$.

c) By part (a) we have inclusion ${}_0\mathcal{A}_1^{\hbar=0} \subset \mathbf{e}_-\mathbb{C}[T^\vee \times \mathfrak{t}] = A$. By Theorem 6.5 (specialized at $\hbar = 0$) this is an isomorphism.

d) This is a specialization of Theorem 6.21 at $\hbar = 0$. □

Note that part (c) of the theorem yields the map $A \to {}_0\mathcal{A}_1^{\hbar=0}$ and hence a family of maps $A^d \to {}_0\mathcal{A}_d^{\hbar=0}$. These are denoted by a_d in the commutative diagram (10).

Corollary 3.6. *If G has rank 1 then $A^d \simeq {}_0\mathcal{A}_d^{\hbar=0} \simeq \mathbf{e}_d I^{(d)}$ for all d.*

Proof. By Theorem 3.5(d) we get $A^d \simeq {}_0\mathcal{A}_d^{\hbar=0}$. On the other hand, in rank 1 we have only one codimension 2 hyperplane and it is easy to see that $I^{(d)} = A^d$. □

Lemma 3.7. *There is a natural inclusion ${}_0\mathcal{A}_d^{\hbar=0} \to \mathbf{e}_d I^{(d)}$ corresponding to the dotted arrow in* (10).

Proof. By Theorem 3.5(a) we know that ${}_0\mathcal{A}_d^{\hbar=0}$ is contained in the $\mathbf{e}_d$-isotypic component, so it is sufficient to check that it is contained in $I^{(d)}$ or, equivalently, in $\langle 1-\alpha^\vee, y_\alpha\rangle^d$ for all α. For this it is sufficient to restrict to a rank 1 subgroup of G, similarly to [11] and [59, Section 3].

On the other hand, if G has rank 1 then the result follows from Corollary 3.6. □

Theorem 3.8. *We have ${}_0\mathcal{A}_d^{\hbar=0} \simeq \mathbf{e}_d I^{(d)}$ for all d.*

Proof. By Lemma 3.7 we have an inclusion of ${}_0\mathcal{A}_d^{\hbar=0}$ into $\mathbf{e}_d I^{(d)}$. Since ${}_0\mathcal{A}_d^{\hbar=0}$ is free over $\mathbb{C}[\mathfrak{t}]^W$ and $\mathbf{e}_d I^{(d)}$ is torsion free, it is sufficient to prove that the inclusion is an isomorphism outside of codimension 2 subset.

Both $\mathbb{C}[\mathfrak{t}]^W$-modules are supported on the union of the root hyperplanes in $\mathfrak{t}^*/W$. If we specialize to a generic point in one of the hyperplanes, we can replace G by its rank 1 subgroup, and the isomorphism follows from Corollary 3.6. Therefore the two modules are isomorphic outside of the union of pairwise intersections of hyperplanes, which has codimension 2. □

We can use the above graded algebras to construct projective varieties

$$\widetilde{\mathfrak{C}}_{\check{G}} := \operatorname{Proj}\bigoplus_d {}_0\mathcal{A}_d^{\hbar=0} \simeq \operatorname{Proj}\bigoplus_d \mathbf{e}_d I^{(d)},\ Y_G = \operatorname{Proj}\bigoplus_d I^{(d)}.$$

By the work of Haiman [40] for $G = GL_n$ we have $\widetilde{\mathfrak{C}}_{\check{G}} = \operatorname{Hilb}^n(\mathbb{C}^* \times \mathbb{C})$ and Y_G is isomorphic to the isospectral Hilbert scheme of $\mathbb{C}^* \times \mathbb{C}$:

$$\begin{array}{ccc} Y_G & \longrightarrow & (\mathbb{C}^* \times \mathbb{C})^n \\ \downarrow & & \downarrow \\ \operatorname{Hilb}^n(\mathbb{C}^* \times \mathbb{C}) & \longrightarrow & S^n(\mathbb{C}^* \times \mathbb{C}). \end{array}$$

We claim that the varieties $\widetilde{\mathfrak{C}}_{\check{G}}$ and Y_G can be considered as the partial resolutions of the commuting variety which we identify with $T^*T^\vee/W$.

Remark 3.9. In [27], Ginzburg defines and studies the *isospectral commuting variety* for general G. On the other hand, the variety $Y_G = \operatorname{Proj}\bigoplus_{d\geq 0} I_G^{(d)}$ is another candidate for the isospectral commuting variety. It is natural to wonder how the two constructions are related.

Remark 3.10. In [29], Ginzburg-Kaledin prove that there are no crepant resolutions of $T^*\mathfrak{t}/W$ for W outside types A, B, C. Their definition of symplectic resolution includes the crepant condition, so their statement is non-existence of symplectic resolutions. Indeed, these conditions are equivalent in this case. This non-existence of a symplectic resolution is thus likely the case for T^*T/W as well. We note however that from the results of [7], it follows that $T^*\mathfrak{t}/W$ and T^*T/W admit birational maps from the universal centralizer group schemes appearing Theorem 3.21, which are smooth for simply connected groups in any type. Note that these are not resolutions of singularities in the usual sense, since they are not proper.

Proposition 3.11. *Y_G is normal.*

Proof. We will prove the homogeneous coordinate ring $\bigoplus J^{(d)}$ is integrally closed. Indeed, $1-\alpha^\vee, y_\alpha$ is a regular sequence. The powers of an ideal generated by a regular sequence are integrally closed, and intersection preserves integral closedness. So the symbolic blow-up considered here is integrally closed. □

Corollary 3.12. *$\widetilde{\mathfrak{C}}_{\check{G}}$ is normal.*

Proof. If a normal variety Y is acted upon by a finite group Γ, Y/Γ is normal [77, Chapter II.5., top of page 128] (note that Shafarevich assumes that Y is affine, but the argument works for any variety since this is a local property). □

In addition to normality, we have

Proposition 3.13. *The variety $\widetilde{\mathfrak{C}}_{\check{G}}$ has only terminal singularities.*

Proof. This follows from [87, Theorem 15]. □

Remark 3.14. An alternative proof of normality of $\widetilde{\mathfrak{C}}_{\check{G}}$ follows from [87, Theorem 14].

Remark 3.15. Fix some symplectic $\mathbb{Q}$-factorial terminalization of $T^*T^\vee/W$, as constructed in [2, 66]. Denote it by $\widetilde{X}_{\check{G}}$. The formal Poisson deformations of $\widetilde{X}_{\check{G}}$ are parameterized by $H^2(\widetilde{X}^{\mathrm{reg}}_{\check{G}},\mathbb{C})$. There is a Lie algebra version $\widetilde{X}_{\check{\mathfrak{g}}} \to \mathfrak{t}\oplus\mathfrak{t}^*/W$ where $\widetilde{X}_{\check{\mathfrak{g}}}$ is the $\mathbb{Q}$-factorial terminalization constructed in [54]. This is a conical symplectic partial resolution with $\mathbb{Q}$-factorial singularities such that $\mathrm{codim}(\widetilde{X}_{\check{\mathfrak{g}}} - \widetilde{X}^{\mathrm{reg}}_{\check{\mathfrak{g}}}) \geq 4$. By [66], this implies that $\widetilde{X}_{\check{\mathfrak{g}}}$ is terminal. In this case, it is possible to prove [66, 54] that the filtered quantizations of $\widetilde{X}_{\check{\mathfrak{g}}}$ are also parametrized by $H^2(\widetilde{X}^{\mathrm{reg}}_{\check{\mathfrak{g}}},\mathbb{C})$. We do not know if this is the case for $\widetilde{X}_{\check{G}}$, since the results in the Lie algebra case heavily use the fact that $\mathfrak{t}\oplus\mathfrak{t}^*/W$ has *conical* symplectic singularities.

Finally, we note that similar to [1, Proposition 2.1.] the normal intermediate partial resolutions

$$\widetilde{X}_{\check{G}} \to X_{\check{G}} \to T^*T^\vee/W$$

are classified by faces of the ample cone of $\widetilde{X}$ such that for a given face F and a rational point $f\in F$ a positive rational multiple of f is the first Chern class of an ample line bundle on the corresponding partial resolution.

It seems reasonable thus to expect that $\mathfrak{P} = H^2(\widetilde{X}^{\mathrm{reg}}_{\check{G}},\mathbb{C})$ parametrizes both the filtered quantizations of $\widetilde{X}_{\check{G}}$ as well as the partial resolutions between $\widetilde{X}_{\check{G}} \to T^*T^\vee/W$, just as it does in the Lie algebra case. Further, it seems by Corollary 3.12 and Proposition 3.13 reasonable to expect that $\widetilde{\mathfrak{C}}_{\check{G}}$ equals a partial resolution constructed this way, and that the singularities of $\widetilde{X}_{\check{G}} \to T^*T^\vee/W$ are locally modeled on those of $\widetilde{X}_{\check{\mathfrak{g}}}$.

To support this remark, we note the following about the local structure of our algebras.

Lemma 3.16. *Upon completion at $a\in T^\vee$, the Coulomb branch algebra $({}_i\mathcal{A}^\hbar_d)^{\wedge a} \cong \mathbb{H}^{\wedge a}_{G,c+i\hbar,\hbar}$ is isomorphic to $\mathbb{H}^{\mathrm{rat},\wedge a}_{\mathfrak{g},c+i\hbar,\hbar}$ for some $\mathfrak{g}$ coming from the Borel-de Siebenthal algorithm for G.*

Proof. The first part is [22, Proof of Theorem 3.2.]. Since we are dealing with the equal parameters case, as in e.g. [1], and the parameters behave as in [23, Section 2.8.] upon completion, the result follows. □

3.3. **The universal centralizer.** In the above, we have defined the partial resolution $\widetilde{\mathfrak{C}}_{\check{G}}$ using Proj construction, and have limited understanding of its geometry outside of type A. Nevertheless, in this subsection we define an affine open chart in $\widetilde{\mathfrak{C}}_{\check{G}}$ and prove that it coincides with the trigonometric version of the *universal centralizer* of [7, 67]. It also appears as a Coulomb branch for zero matter, and will be used later in Section 8.

We let G be arbitrary for now. Let $\Delta := \prod_{\alpha\in\Phi} y_\alpha \in A_G$ be the Vandermonde determinant.

Definition 3.17. Let

$$U_\Delta \subset \widetilde{\mathfrak{C}}_{\check{G}}$$

be defined as the distinguished open subset given by the element $\Delta\in A_G$. By definition, U_Δ is the affine variety whose coordinate ring is the degree zero part of the localization of $\oplus_{d=0}^\infty {}_0\mathcal{A}_d$ in Δ.

Remark 3.18. Note that this is different from the preimage of of $\{\Delta\neq 0\}$ under the natural composition of maps $\widetilde{\mathfrak{C}}_{\check{G}} \to T^*T^\vee/W \to \mathfrak{t}/W$. On this locus the first map is an isomorphism.

We now describe this chart. In [7], two trigonometric versions of the universal centralizer are studied. The one of interest to us is defined as follows, see *loc. cit.* for more details.

Definition 3.19. The universal centralizer of $\check{G}$ is the variety

$$\mathfrak{B}^{\check{G}}_{\check{\mathfrak{g}}} := \{(g,s)\in \check{G}\times\check{\mathfrak{g}}^* | \mathrm{ad}_g(s) = s,\ g \text{ is regular }\} /\!\!/ \check{G}$$

Remark 3.20. In [7], this variety is denoted $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$. There is also another version of the trigonometric universal centralizer $\mathfrak{Z}^{\check{\mathfrak{g}}}_{\check{G}}$ with the roles of $\mathfrak{g}, G$ swapped. It has the nicer geometric property of being symplectically isomorphic to $T^*(T^\vee/W)$ when G is adjoint (so $\check{G}$ is simply connected).

In [7] explicit description of the coordinate ring of $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ is given. We also have the following Coulomb branch description of $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$.

Theorem 3.21. $\mathbb{C}[\mathfrak{B}_{\check{\mathfrak{g}}}^{\check{G}}] \cong H_*^{G(\mathcal{O})}(\mathrm{Gr}_G)$*, which is also the Coulomb branch for* $(G,0)$ *(see Section 5.)*

We now state and prove the main theorem of this section. The reader may want to skip the proof at a first reading and see Section 5 first.

Theorem 3.22. *We have*

$$\mathbb{C}[\mathfrak{B}_{\check{\mathfrak{g}}}^{\check{G}}] \cong \left(\left(\bigoplus_d {}_0\mathcal{A}_d^{\hbar=0} \right)_{\Delta} \right)_0 .$$

In particular, there is a natural isomorphism $\mathfrak{B}_{\check{\mathfrak{g}}}^{\check{G}} \cong U_{\Delta}$.

Proof. By [9, 4(vi)] there is a natural inclusion of the Coulomb branch algebra for $(G,\mathfrak{g})$ to the Coulomb branch algebra for $(G,0)$. Using the same geometric argument, which is essentially the the pullback map in BM homology under $\mathrm{Gr}_G \hookrightarrow {}_i\mathcal{R}_j$, we also get a map ${}_i\mathcal{A}_j^{\hbar=0} \hookrightarrow H_*^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ for any i,j.

Both the left and right hand sides are equipped with a Bruhat filtration coming from the interpretation as Hom-spaces in the category of line operators, cf. Section 5 and Theorem 3.21. Both of these spaces have bases over $H^*_{\check{G}}(pt)$ indexed by $\lambda \in X_*(T)$, which up to leading order are given by the formulas in Theorem 6.16 and [9, 3(x)(a) and Prop. 6.6.]. Since the generalized root hyperplanes in the adjoint case are just the root hyperplanes, it is clear that upon localization outside these hyperplanes the two spaces have the same bases.

But we have even more control over the pole orders at the root hyperplanes. Namely, the coefficient of u^{λ} in $[\mathcal{R}_{G,0}^{\leq\lambda}]$, meaning the class associated to λ for zero matter in the analog of Lemma 6.7 is just $\frac{1}{\prod_\alpha y_\alpha^{\max(0,\alpha(\lambda))-1}}$, and the coefficient in front of u^{λ} in the analogous expression for adjoint matter only has y_α vanishing to order at most d in the numerator. Dividing elements of ${}_0\mathcal{A}_d^{\hbar=0}$ by Δ^d for $d \geq 0$ then gives an injective map to the zero matter Coulomb branch algebra, and multiplying by $f \in H_G^*(pt)$ if necessary, we see that we get all the $H_G^*(pt)$-basis elements of $H^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$, so that this is an isomorphism. □

Remark 3.23. Alternatively, one can use the blow-up description of $\mathfrak{B}_{\check{\mathfrak{g}}}^{\check{G}}$ [7, Section 4] to get a more geometric proof of this result.

Remark 3.24. We can interpret the proof of this Theorem as follows. As stated in the proof, there are natural embeddings

$${}_j\mathcal{A}_i^{\hbar=0} \hookrightarrow H_*^{G(\mathcal{O})}(\mathrm{Gr}_G)$$

see for example Section 6.2 or [9, Lemma 5.11]. These embeddings realize rational functions of the form $f(x,y)/\Delta^{j-i}$, where $f \in {}_j\mathcal{A}_i^{\hbar=0}$, as functions on the open chart U_Δ.

When $G = GL_n$, this construction is closely related to the construction of the open chart "$U_{(1^n)}$" on $\mathrm{Hilb}^n(\mathbb{C}^2)$ given by Haiman in [41, Corollary 2.7.].

3.4. **Explicit antisymmetric polynomials.** In Theorem 6.21 we will need an explicit construction of a $\mathbb{C}$-basis in the space A of antisymmetric (S_n-antiinvariant) polynomials for $G = GL_n$, in order to compare our Coulomb branch construction with the one above. The exposition follows ideas of Haiman in [41]. The reader is advised to skip this section on a first reading.

We denote by Alt the action of the antisymmetric projector $\mathbf{e}_-$ on polynomials. Let $S = \{(a_1,b_1),\ldots(a_n,b_n)\}$ be an arbitrary n-element subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$. We define

$$\Delta_S(y_1,u_1,\ldots,y_n,u_n) = \mathrm{Alt}\left(y_1^{a_1}u_1^{b_1}\cdots y_n^{a_n}u_n^{b_n}\right) = \frac{1}{n!}\det\left(y_i^{a_j}u_i^{b_j}\right).$$

For a composition α with $\sum \alpha_i = n$, we can consider the set

$$S_\alpha = \{(0,0),\ldots,(\alpha_1-1,0),(0,1),\ldots(\alpha_2-1,1),\ldots\}$$

and denote $\Delta_{S_\alpha} = \Delta_\alpha$. In particular,

$$\Delta = \prod_{i<j}(y_i - y_j) = \Delta_{(n)}.$$

Given a composition α, write $\lambda(\alpha) = (0^{\alpha_1}, 1^{\alpha_2}, \ldots)$. More generally, for any subset $S = \{(a_i,b_i)\} \subset \mathbb{Z}_{\geq 0}\times\mathbb{Z}$, let $\lambda(S) = \mathrm{sort}(b_i)$ where we sort in non-decreasing order. Clearly, $\lambda(S_\alpha) = \lambda(\alpha)$. Furthermore, we define a collection of subsets

$$S_k = \{a_i \,:\, (a_i,k) \in S\} = \left\{a_{i_1^{(k)}} < \ldots < a_{i_{r_k}^{(k)}}\right\}$$

and a partition

$$\mu_k(S) = (a_{i_1^{(k)}}, a_{i_2^{(k)}} - 1, \ldots, a_{i_{r_k}^{(k)}} - r_k + 1).$$

Finally, we define $\mathbf{y}_k = (y_i \,:\, \lambda_i = k)$.

Lemma 3.25. *(a) The determinants Δ_S span the vector space A.*

(b) We have the following formula for the determinant Δ_α:

$$\Delta_\alpha = c \cdot \mathrm{Alt}\left[u^\lambda \prod_{r<s,\lambda_r=\lambda_s} (y_r - y_s)\right].$$

where $\lambda = \lambda(\alpha)$.

(c) More generally, we have the following formula for the determinant Δ_S:

$$\Delta_S = c \cdot \mathrm{Alt}\left[\prod_k s_{\mu_k}(\mathbf{y}_k) u^\lambda \prod_{r<s,\lambda_r=\lambda_s} (y_r - y_s)\right]$$

where s_{μ_k} are Schur polynomials, $\lambda = \lambda(S)$ and c is some nonzero scalar factor depending on the size of stabilizer of λ.

Proof. (a) The space $\mathbb{C}[T^*T^\vee]$ is spanned by the monomials $y_1^{a_1}u_1^{b_1}\cdots y_n^{a_n}u_n^{b_n}$, so $A = \mathbf{e}_-\mathbb{C}[T^*T^\vee]$ is spanned by their antisymmetrizations (recall that u_i are invertible, so b_i are allowed to be negative). If some of pairs (a_j, b_j) coincide, then the antisymmetrization vanishes, so it is sufficient to assume that (a_j, b_j) are pairwise distinct and form an n-element subset S.

Clearly, (b) follows from (c) since for $S = S_\alpha$ we have $S_k = 0, 1 \ldots, \alpha_k - 1$ and $\mu_k = (0)$.

To prove (c), observe that the function Δ_S is antisymmetric and all possible monomials in u are in the S_n-orbit of λ, so it is sufficient to compute the coefficient at u^λ. This coefficient is proportional to

$$\mathrm{Alt}_{\mathrm{Stab}(\lambda)}(y_1^{a_1}\cdots y_n^{a_n}) = \prod_k \mathrm{Alt}_{S_k}\left[\prod_{\lambda_i=k} y_i^{a_i}\right] = \prod_k \left[s_{\mu_k}(\mathbf{y}_k) \cdot \prod_{r<s,\lambda_r=\lambda_s=k} (y_r - y_s)\right].$$

□

Example 3.26. For $\alpha = (1,2,1)$ we have $S = \{(0,0),(0,1),(1,1),(0,2)\}, \lambda = (0,1,1,2)$ and

$$\Delta_S = \mathrm{Alt}\left[(y_2 - y_3)u_1^0 u_2^1 u_3^1 u_4^2\right]$$

For $S = \{(5,0),(3,1),(7,1),(2,2)\}$ we have $\lambda = (0,1,1,2), \mu_0 = (5), \mu_1 = (3,6), \mu_2 = (2)$ and

$$\Delta_S = \mathrm{Alt}\left[s_5(y_1)s_{6,3}(y_2,y_3)s_2(y_4)(y_2 - y_3)u_1^0 u_2^1 u_3^1 u_4^2\right].$$

Note that $s_5(y_1) = y_1^5, s_2(y_4) = y_4^2$ and

$$s_{6,3}(y_2,y_3) = \frac{\begin{vmatrix} y_2^7 & y_3^7 \\ y_2^3 & y_3^3 \end{vmatrix}}{y_2 - y_3} = -\frac{\mathrm{Alt}(y_2^3 y_3^7)}{y_2 - y_3}.$$

4. Trigonometric Cherednik algebra

4.1. **Definitions.** We define the extended torus $\widetilde{T} = T \times \mathbb{G}_m^*$ and the corresponding Lie algebra $\widetilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{C}_\hbar$. The extended affine Weyl group $\widetilde{W} := W \ltimes X_*(T)$ is generated by the affine Weyl group $W^{\mathrm{aff}} = W \ltimes Q^\vee$ and an additional abelian group $\Omega = X_*(T)/Q^\vee$ where $Q^\vee$ is the coroot lattice. We use the affine action of $W \ltimes X_*(T)$ on the cocharacter lattice $X_*(T)$ depending on $\hbar$. The action of $w \in \widetilde{W}$ on $\xi \in X_*(T)$ will be denoted by ${}^w\xi$. We will denote the longest element in W by w_0.

Definition 4.1. The *trigonometric DAHA* of G is the $\mathbb{C}[\hbar, c]$-algebra, which as a vector space looks like

$$\mathbb{H}_G = \mathbb{H}_{c,\hbar} = \mathbb{C}[\widetilde{W}] \otimes \mathbb{C}[X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}] \otimes \mathbb{C}[c] \otimes \mathbb{C}[\hbar]$$

and the algebra structure is determined as follows:

(1) Each of the tensor factors is a subalgebra, and $c, \hbar$ are central. We denote by σ_i the simple reflections in the copy of $\widetilde{W} \subset \mathbb{H}_G$.

(2) $\sigma_i\xi - {}^{s_i}\xi\sigma_i = c\langle \xi, \alpha_i^\vee \rangle$ for all simple reflections $\sigma_i \in \widetilde{W}$ and $\xi \in X^*(T)$.

(3) For any $\omega \in \Omega \subset \widetilde{W}$, $\omega\xi = {}^\omega\xi\omega$

Note that here the pairing $\langle , \rangle : \widetilde{\mathfrak{t}} \times \widetilde{\mathfrak{t}} \to \mathbb{C}[\hbar]$ depends on $\hbar$.

Example 4.2. For $G = GL_n$ the group $\widetilde{W}$ is generated by simple reflections $\sigma_1, \dots, \sigma_{n-1}$ (which generate W) and an additional element π (which generates Ω). The lattice part of $\widetilde{W}$ is generated by

$$X_i = \sigma_{i-1} \cdots \sigma_1 \pi \sigma_{n-1} \cdots \sigma_i.$$

The algebra $\mathbb{H}_{c,\hbar}$ is generated by σ_i, π and commuting variables $y_1, \dots, y_n$, with the following relations:

$$\sigma_i y_i = y_{i+1}\sigma_i - c, \sigma_i y_{i+1} = y_i \sigma_i + c,\ \sigma_i y_j = y_j \sigma_i\ (j \neq i, i+1),$$
$$\pi y_i = y_{i+1}\pi (1 \leq i \leq n-1),\ \pi y_n = (y_1 + \hbar)\pi.$$

Remark 4.3. We will also use specializations of this algebra, to be called trigonometric DAHAs as well when there is no risk of confusion. Let us explain how this relates to parameter conventions in the literature. In [72], the trigonometric DAHA is defined as above but with parameters $\delta = \hbar, c = u$. These correspond to the generators of $H^*_{\mathbb{G}_m^{\mathrm{rot}}}(*)$ and $H^*_{\mathbb{G}_m^{\mathrm{dil}}}(*)$, respectively. In [72], the relations $c + \nu\delta = c + \nu\hbar = 0$ and $\hbar = 1$ are imposed for $\nu \in \mathbb{C}$. This specialization of parameters is often called "the" trigonometric DAHA with parameter ν. We will denote it by $\mathbb{H}_\nu$. If we want to emphasize the role of G instead of the parameters, we will write $\mathbb{H}_G$ for any of the specializations of the Cherednik algebra.

Remark 4.4. It is common in Cherednik algebra literature to specialize $\hbar$ to 1 as above, so that the algebra $\mathbb{H}^{\hbar=1}_{c,\hbar}$ admits a natural filtration in powers of $\hbar$, making the full $\mathbb{H}_{c,\hbar}$ the Rees construction for this filtration. In this language, $\mathbb{H}^{c=\hbar=0}_{c,\hbar}$ is the associated graded of $\mathbb{H}^{\hbar=1}_{c,\hbar}$, since $\mathbb{H}_{c,\hbar}$ is flat over $\mathbb{C}[c,\hbar]$. With this in mind, we will use the specialization $c = \hbar = 0$ and the associated graded interchangeably.

Remark 4.5. Later on, we shall be interested in the family of Cherednik algebras $\mathbb{H}_{c+i\hbar,\hbar}$ for $i \leq 0$ as well, and the specializations $c + (\nu + i)\hbar = 0, \hbar = 1$, i.e. $c = -\frac{m+in}{n}$.

We introduce the symmetrizer $\mathbf{e} = \frac{1}{|W|}\sum_{w\in W} w$ and the antisymmetrizer $\mathbf{e}_- = \frac{1}{|W|}\sum_{w\in W}(-1)^{\ell(w)}w$ in the group algebra of W. We define the *spherical* and *antispherical* subalgebras in $\mathbb{H}_G$ as $\mathbf{e}\mathbb{H}_G\mathbf{e}$ and $\mathbf{e}_-\mathbb{H}_G\mathbf{e}_-$.

Note that in the specialization $c = \hbar = 0$ the structure of the algebra simplifies dramatically: $\mathbb{H}_G^{c=\hbar=0} = \mathbb{C}[W] \ltimes \mathbb{C}[T^\vee \times \mathfrak{t}]$, so

$$\mathbf{e}\mathbb{H}_G^{c=\hbar=0}\mathbf{e} \cong \mathbb{C}[T^\vee \times \mathfrak{t}]^W.$$

We will refer to this as the *commutative limit*, although this only gives the limit of the spherical subalgebra the structure of a commutative algebra.

The algebra $\mathbb{H}_G$ has a representation

$$\mathbb{H}_{c,\hbar} \hookrightarrow \mathrm{Diff}_\hbar(\mathfrak{t}^{\mathrm{reg}}) \rtimes \mathbb{C}[W] \tag{11}$$

defined e.g. in [18, Section 2.13]. Here $\mathrm{Diff}_\hbar(\mathfrak{t}^{\mathrm{reg}})$ is the algebra of $\hbar$−difference operators on the Lie algebra $\mathfrak{t}$, possibly with poles along the root hyperplanes. In this representation, the generators of $\mathbb{H}_G$ corresponding to simple reflections act by

$$\sigma_i = s_i + \frac{c}{y_{\alpha_i}}(s_i - 1)$$

and Ω acts by difference operators in a standard way.

Example 4.6. For $G = GL_n$, we get $\sigma_i = s_i + \frac{c}{y_i - y_{i+1}}(s_i - 1)$ and $\pi \cdot f(y_1, \dots, y_n) = f(y_2, \dots, y_n, y_1 + \hbar)$.

4.2. **Shift isomorphism.** We will need several involutions on the algebra $\mathbb{H}_c$.

Lemma 4.7. *The map* $\Phi : w \mapsto (-1)^{\ell(w)} w_0 w^{-1} w_0, \xi \mapsto {}^{w_0}\xi, \hbar \to -\hbar$ *for* $w \in \widetilde{W}$ *defines an involutive anti-automorphism of* $\mathbb{H}_G$.

Proof. Suppose that w_0 sends the simple root α_i to $-\alpha_j$ for some j, then $w_0 s_i = s_j w_0$. The map Φ sends σ_i to $-\sigma_j$, so we get: $\Phi(\sigma_i \xi) = -{}^{w_0}\xi\sigma_j$, $\Phi({}^{s_i}\xi\sigma_i) = -\sigma_j{}^{w_0 s_i}\xi$ and

$$\Phi(\sigma_i\xi - {}^{s_i}\xi\sigma_i) = \sigma_j{}^{s_j w_0}\xi - {}^{w_0}\xi\sigma_j$$

while $\langle \xi, \alpha_i^\vee\rangle = \langle {}^{s_j w_0}\xi, \alpha_j^\vee\rangle$, so the equation (2) is preserved. For the equation (3), observe that $w_0\omega^{-1}w_0 = \omega$ since we assume that $\hbar \mapsto -\hbar$. ☐

Example 4.8. For $G = GL_n$ we have

$$\Phi(\sigma_i) = -\sigma_{n-i},\ \Phi(y_i) = y_{n+1-i},\ \Phi(\pi) = \pi,\ \Phi(\hbar) = -\hbar$$

Observe that $\Phi(\mathbf{e}) = \mathbf{e}_-$ and $\Phi(\mathbf{e}_-) = \mathbf{e}$. In particular, Φ exchanges spherical subalgebra $\mathbf{e}\mathbb{H}_c\mathbf{e}$ with the antispherical subalgebra $\mathbf{e}_-\mathbb{H}_c\mathbf{e}_-$.

Theorem 4.9. *a) Let λ be a minuscule dominant coweight, then*

$$E_\lambda := \mathbf{e}X^\lambda\mathbf{e} = \mathbf{e}\prod_{\langle\lambda,\alpha\rangle=1}\frac{y_\alpha - c}{y_\alpha}u^\lambda\mathbf{e}$$

and

$$F_\lambda := \mathbf{e}X^{-\lambda}\mathbf{e} = \mathbf{e}\prod_{\langle\lambda,\alpha\rangle=1}\frac{y_\alpha + c}{y_\alpha}u^{-\lambda}\mathbf{e}$$

Here u^λ is the translation by $\hbar\lambda$.

b) For arbitrary dominant coweight λ we get

$$E_\lambda = \mathbf{e}\prod_{\alpha(\lambda)>0}\prod_{\ell=0}^{\alpha(\lambda)-1}\frac{(y_\alpha + \ell\hbar - c)}{(y_\alpha + \ell\hbar)}u^\lambda\mathbf{e} + \text{lower order terms.}$$

Proof. We prove (b) using geometric realization of $\mathbb{H}_G$ as the Coulomb branch algebra, see Proposition 6.10 below. The operator E_λ corresponds (up to lower order terms) to a certain class $z^*\iota_*^{-1}\operatorname{gr}[\mathcal{R}^{\le\lambda}]$ which agrees with the formula in (b) after localization to fixed points. See Proposition 6.10 for all details.

Alternatively, one can prove (b) using a tedious but explicit computation in the affine Hecke algebra, see [49, Example 5.4,Theorem 5.9] and [52, Proposition 5.13]. The translation X^λ can be written as a product of elements of Ω and simple reflections σ_i in some order, and one can control the leading term of each factor. This leads to a formula for the leading term for X^λ, and its symmetrization.

In part (a) the coweight λ is minuscule, and the formula from (b) simplifies. First, we have either $(\alpha^\vee,\lambda) = 1$ or $(\alpha^\vee,\lambda) = 0$ for all $\alpha^\vee$, so $\ell = 0$ for all nontrivial factors. Second, λ is minimal in Bruhat order, so there cannot be any lower order terms and the formula is exact. □

Lemma 4.10. *Suppose that λ is an (anti)dominant coweight which is minimal in the Bruhat order. Then*

$$\Phi(E_{\lambda,c})\Delta = \Delta E_{\lambda,c-\hbar}.$$

Remark 4.11. If λ is minuscule, then it is indeed minimal in the Bruhat order. The converse is not true: indeed, there are no minuscule coweights at all for root systems E_8, F_4, G_2.

Proof. Since λ is minimal in Bruhat order, the formula for E_λ in Theorem 4.9(b) is exact. Now we write:

$$\Phi(E_\lambda) = \mathbf{e}_- u^{{}^{w_0}\lambda}\prod_{\alpha^\vee(\lambda)>0}\prod_{\ell=0}^{\alpha(\lambda)-1}\frac{(y_{\overline{\alpha}} - \ell\hbar - c)}{(y_{\overline{\alpha}} - \ell\hbar)}\mathbf{e}_-,$$

where we denote $\overline{\alpha} = {}^{w_0}\alpha$. By replacing λ by λ^{w_0} (since we symmetrize anyway) and α by ${}^{w_0}\alpha$ (since we take product over all roots α), we can rewrite this product as

$$\Phi(E_\lambda) = \mathbf{e}_- u^\lambda\prod_{\alpha(\lambda)>0}\prod_{\ell=0}^{\alpha(\lambda)-1}\frac{(y_\alpha - \ell\hbar - c)}{(y_\alpha - \ell\hbar)}\mathbf{e}_-.$$

Now $u^\lambda y_\alpha = (y_\alpha + \alpha(\lambda)\hbar)u^\lambda$, therefore

$$\Phi(E_\lambda) = \mathbf{e}_-\prod_{\alpha(\lambda)>0}\prod_{\ell=0}^{\alpha(\lambda)-1}\frac{(y_\alpha - \ell\hbar - c + \alpha(\lambda)\hbar)}{(y_\alpha - \ell\hbar + \alpha(\lambda)\hbar)}u^\lambda\mathbf{e}_- =$$

$$\mathbf{e}_- u^\lambda\prod_{\alpha(\lambda)>0}\prod_{\ell=1}^{\alpha(\lambda)}\frac{(y_\alpha + \ell\hbar - c)}{(y_\alpha + \ell\hbar)}u^\lambda\mathbf{e}_-.$$

In the last step we changed index of summation from ℓ to $\alpha(\lambda) - \ell$. Now we can compute

$$\Phi(E_\lambda)\Delta = \mathbf{e}_-\prod_{\alpha(\lambda)>0}\prod_{\ell=1}^{\alpha(\lambda)}\frac{(y_\alpha + \ell\hbar - c)}{(y_\alpha + \ell\hbar)}u^\lambda\Delta\mathbf{e} =$$

$$\mathbf{e}_-\prod_{\alpha(\lambda)>0}\prod_{\ell=1}^{\alpha(\lambda)}\frac{(y_\alpha + \ell\hbar - c)}{(y_\alpha + \ell\hbar)}\frac{(y_\alpha + \alpha(\lambda)\hbar)}{y_\alpha}\Delta u^\lambda\mathbf{e} =$$

$$\mathbf{e}_-\Delta\prod_{\alpha(\lambda)>0}\prod_{\ell=0}^{\alpha(\lambda)-1}\frac{(y_\alpha + \ell\hbar - (c-\hbar))}{(y_\alpha + \ell\hbar)}u^\lambda\mathbf{e} = \Delta E_{\lambda,c-\hbar}.$$

□

Theorem 4.12. *There is a filtered algebra isomorphism* $\mathbf{e}\mathbb{H}_c\mathbf{e} \cong \mathbf{e}_-\mathbb{H}_{c-\hbar}\mathbf{e}_-$.

Proof. The spherical subalgebra $\mathbf{e}\mathbb{H}_{c-\hbar}\mathbf{e}$ is generated by the elements

$$E_\lambda[f] = \mathbf{e}f(w)X^\lambda\mathbf{e},\ \ F_\lambda[f] = \mathbf{e}f(w)X^{-\lambda}\mathbf{e}$$

where λ is a dominant coweight which is minimal in the Bruhat order, and $f(w)$ is a polynomial in w_i (see [9, Proposition 6.8.]). Similarly to Lemma 4.10, one can check that

$$\Phi(E_{\lambda,c}[f])\Delta = \Delta E_{\lambda,c-\hbar}[f'],\ \Phi(F_{\lambda,c}[f])\Delta = \Delta E_{\lambda,c-\hbar}[f'] \tag{12}$$

where f' is related to f by a certain shift by $\hbar$.

Let G be an operator in $\mathbf{e}\mathbb{H}_{c-\hbar}\mathbf{e}$, then we can consider the operator $\Delta G\Delta^{-1}$ acting on antisymmetric polynomials. By (12) the operator $\Delta G\Delta^{-1}$ belongs to $\mathbf{e}_-\mathbb{H}_c\mathbf{e}_-$. Since Φ exchanges the spherical and antispherical subalgebras, the operators $\Delta G\Delta^{-1}$ generate $\mathbf{e}_-\mathbb{H}_c\mathbf{e}_-$.

□

Remark 4.13. In [43, 74] a similar isomorphism between the spherical and antispherical subalgebras was obtained using Dunkl representation by differential-difference operators. It is natural to ask if the two isomorphisms are the same. They are not, for the isomorphism in *loc. cit.* is given by conjugation by the "Vandermonde in X", in other words $\prod_{\alpha\in\Phi^+}(1-\alpha^\vee) \in \mathbb{C}[T^\vee]$, which acts by identity on the operators $E_{\lambda,c}[1]$ since $X \mapsto X$ in the differential Dunkl representation. The two isomorphisms are related by the Harish-Chandra transform of [18]. This is similar to the fact that in the rational case, there are two Dunkl embeddings, to $\mathrm{Diff}(\mathfrak{h}^{reg}) \rtimes W$ and $\mathrm{Diff}((\mathfrak{h}^*)^{reg}) \rtimes W$ in which one gets similar shift isomorphisms by either conjugation by $\prod_{\alpha\in\Phi^+} y_\alpha$ or respectively by $\prod x_{\alpha^\vee}$ [4], and the two isomorphisms are related by Cherednik's Fourier transform.

4.3. $\mathbb{Z}$**-algebras.** We now recall the definition of $\mathbb{Z}$-algebras, as explained e.g. in [35, Section 5]. Note that our conventions are exactly opposite to those of *loc. cit.* because it makes the Springer action in Section 7 a bit more natural.

Definition 4.14. An associative (non-unital) algebra $B = \bigoplus_{i\leq j} B_{ij}$ is a $\mathbb{Z}$*-algebra* if $B_{ij}B_{jk} \subseteq B_{ik}$ for all $i \leq j \leq k$, $B_{ij}B_{lk} = 0$ if $j \neq l$, and each B_{ii} is unital such that $1_ib_{ij} = b_{ij} = b_{ij}1_{ij}$ for all $b_{ij} \in B_{ij}$.

The above definition ensures that B_{ii} is a unital associative algebra for all i, and B_{ij} is a (B_{ii}, B_{jj})-bimodule. The $\mathbb{Z}$-algebra multiplication factors though the convolution of bimodules:

$$\begin{array}{ccc} B_{ij}\otimes_{\mathbb{C}} B_{jk} & \longrightarrow & B_{ik} \\ & \searrow \quad \nearrow & \\ & B_{ij}\otimes_{B_{jj}} B_{jk} & \end{array}$$

The simplest example of $\mathbb{Z}$-algebras comes from $\mathbb{Z}$-graded algebras.

Example 4.15. Suppose that $S = \oplus_d S_d$ is an associative $\mathbb{Z}$-graded algebra with multiplication $S_dS_{d'} \to S_{d+d'}$. Define $B_{ij} = S_{j-i}$ for all i and j, then $B(S) = \bigoplus_{i\leq j} B_{ij}$ is a $\mathbb{Z}$*-algebra*. Note that in this example the algebras B_{ii} are all isomorphic to S_0.

Our main source of $\mathbb{Z}$-algebras will be a filtered deformation of Example 4.15. We say that a $\mathbb{Z}$-algebra B is **of graded type** if it has an algebra filtration (which we omit from the notations) such that $\mathrm{gr}\, B = B(S)$ for some commutative graded algebra S. Unpacking this definition, we get the following properties of $\mathrm{gr}\, B$:

- $S_0 := \mathrm{gr}\, B_{ii}$ is a commutative algebra which does not depend on i up to isomorphism
- $S_{j-i} := \mathrm{gr}\, B_{ij}$ depends only on the difference $j - i$ up to isomorphism
- For all i, j, k we have a commutative square

$$\begin{array}{ccc} \mathrm{gr}\, B_{ij}\otimes \mathrm{gr}\, B_{jk} & \longrightarrow & \mathrm{gr}\, B_{ik} \\ \downarrow & & \downarrow \\ S_{j-i}\otimes S_{k-j} & \longrightarrow & S_{k-i} \end{array}$$

- The left and right actions of $S_0 \simeq \mathrm{gr}\, B_{ii} \simeq \mathrm{gr}\, B_{jj}$ on the bimodule $\mathrm{gr}\, B_{ij}$ agree.

The last bullet point is related to the Harish-Chandra property for the bimodules B_{ij}, see [78, 55]. Namely, *a Harish-Chandra bimodule* for a filtered algebra A is a bimodule B with an exhaustive filtration s.t. $[A_{\le i}, B_{\le j}] \subseteq B_{i+j-d}$ s.t. $\operatorname{gr} B$ is finitely generated. The commutator condition implies that the left and right actions of $\operatorname{gr} A$ on $\operatorname{gr} B$ agree.

Also note that we can associate a pair of schemes to a $\mathbb{Z}$-algebra of graded type: the affine scheme $\operatorname{Spec} S_0$ and the scheme $\operatorname{Proj} S$. We have a natural morphism $\operatorname{Proj} S \to \operatorname{Spec} S_0$.

Next, we define modules over a $\mathbb{Z}$-algebra B. A graded vector space $M = \oplus M_i$ is a B-module if for all i and j we have multiplication maps $B_{ij} \otimes M_j \to M_i$ such that we have a commutative diagram

$$\begin{array}{ccc} B_{ij} \otimes B_{jk} \otimes M_k & \longrightarrow & B_{ij} \otimes M_j \\ \downarrow & & \downarrow \\ B_{ik} \otimes M_k & \longrightarrow & M_i. \end{array}$$

In particular, M_i is a module over the algebra B_{ii} for all i. If B is of graded type and M admits a filtration compatible with a filtration on B then $\operatorname{gr} M$ is graded S-module for the graded algebra S. In particular, $\operatorname{gr} M$ defines a quasicoherent sheaf on $\operatorname{Proj} S$.

4.4. **$\mathbb{Z}$-algebras from Cherednik algebras.** We now turn to defining a $\mathbb{Z}$-algebra $\mathcal{B} = {}_\bullet\mathcal{B}^\hbar_\bullet$ as follows. The component ${}_i\mathcal{B}^\hbar_i$ is the spherical Cherednik algebra $\mathbf{e}\mathbb{H}_{c+i\hbar}\mathbf{e}$ with parameter $c + i\hbar$. The component ${}_i\mathcal{B}^\hbar_{i+1}$ is the *shift bimodule*

$${}_i\mathcal{B}^\hbar_{i+1} = \mathbf{e}\mathbb{H}_{c+(i+1)\hbar,\hbar}\mathbf{e}_-$$

over the algebras ${}_{i+1}\mathcal{B}^\hbar_{i+1} = \mathbf{e}\mathbb{H}_{c+(i+1)\hbar,\hbar}\mathbf{e}$ and

$${}_i\mathcal{B}^\hbar_i = \mathbf{e}\mathbb{H}_{c+i\hbar,\hbar}\mathbf{e} \simeq \mathbf{e}_-\mathbb{H}_{c+(i+1)\hbar,\hbar}\mathbf{e}_-.$$

The last isomorphism is given by Theorem 4.12. Finally, for more general $i < j$ we define the shift bimodules

$${}_i\mathcal{B}^\hbar_j = {}_i\mathcal{B}^\hbar_{i+1} \cdots {}_{j-1}\mathcal{B}^\hbar_j$$

where $\cdot$ denotes the appropriate tensor product.

Lemma 4.16. *At $\hbar = c = 0$ one has ${}_i\mathcal{B}^{\hbar=0}_j = A^{j-i}$, where A is the subspace of diagonally antisymmetric polynomials in $\mathbb{C}[T^*T^\vee]$, and this is compatible with the multiplication. When $i = j$ this is the subspace of diagonally symmetric polynomials.*

Proof. Let us prove that ${}_i\mathcal{B}^{\hbar=0}_{i+1} = A$. Indeed, ${}_i\mathcal{B}^{\hbar=0}_{i+1} = \mathbf{e}_-\mathbb{H}\mathbf{e} \cong \mathbf{e}_-\mathbb{C}[T^*T^\vee]$ is the space of antisymmetric polynomials in $\mathbb{C}[T^*T^\vee]$. Similarly, ${}_i\mathcal{B}^{\hbar=0}_i \cong \mathbf{e}\mathbb{C}[T^*T^\vee] = \mathbb{C}[T^*T^\vee]^W$. Now

$${}_i\mathcal{B}^{\hbar=0}_j = A \otimes_{\mathbb{C}[T^*T^\vee]^W} \otimes \cdots \otimes_{\mathbb{C}[T^*T^\vee]^W} A = A^{j-i}.$$

□

Example 4.17. Consider the trigonometric Cherednik algebra for $G = GL_2$. For $\nu = 1/2$ it has a 1-dimensional representation $L_{1/2}(\mathrm{triv})$ with invariant part $\mathbf{e}L_{1/2}(\mathrm{triv}) \cong \mathbf{e}_-L_{3/2}(\mathrm{triv})$. Using this isomorphism, the bimodule $\mathbf{e}\mathbb{H}_{3/2}\mathbf{e}_-$ sends $\mathbf{e}\mathbb{H}_{3/2}\mathbf{e}_- \otimes_{\mathbf{e}_-\mathbb{H}_{3/2}\mathbf{e}_-} \mathbf{e}_-L_{3/2}(\mathrm{triv}) \cong \mathbf{e}L_{3/2}(\mathrm{triv})$. More generally,

$$\mathbf{e}\mathbb{H}_{(2k+1)/2}\mathbf{e}_- \otimes_{\mathbf{e}_-\mathbb{H}_{(2k+1)/2}\mathbf{e}_-} \mathbf{e}_-L_{(2k+1)/2}(\mathrm{triv}) \cong \mathbf{e}L_{(2k+1)/2}(\mathrm{triv})$$

and the direct sum

$$\bigoplus_{k\ge 0} \mathbf{e}L_{(2k+1)/2}(\mathrm{triv})$$

is a module for the $\mathbb{Z}$-algebra $\mathcal{B}$.

4.5. **$\mathbb{Z}$-algebra for GL_n.** Consider now the $\mathbb{Z}$-algebra as introduced above. We have

Theorem 4.18. *For all $i \le j$ the $\mathbb{C}[\hbar] \otimes \mathbb{C}[y_1, \ldots, y_n]^{S_n}$-module ${}_i\mathcal{B}^\hbar_j$ is free.*

Proof. The idea of the proof is to replace the trigonometric Cherednik algebra $\mathbb{H}_{c,\hbar} = H^{\mathrm{trig}}_{c,\hbar}$ with the rational Cherednik algebra $H^{\mathrm{rat}}_\hbar$ [39]. The algebra $H^{\mathrm{rat}}_\hbar$ is the quotient of $\mathbb{C}[X_1, \ldots, X_n] \otimes \mathbb{C}[z_1, \ldots, z_n] \rtimes S_n$ modulo the relations:

$$[X_i, z_i] = \hbar + \sum_{j\ne i} \sigma_{ij} \; i = 1, \ldots, n,$$

$$[X_i, z_j] = -\sigma_{ij}, i \ne j,$$

where $\sigma_{ij} \in S_n$ is the transposition (the generators z_i are usually called y_i in the rational Cherednik algebra literature).

For the rational Cherednik algebra the corresponding $\mathbb{Z}$–algebra ${}_\bullet\mathcal{B}_\bullet^{\mathrm{rat},\hbar=\hbar_0}$ was constructed by Gordon and Stafford [35, 36] who defined a filtration on ${}_i\mathcal{B}_j^{\mathrm{rat},\hbar=\hbar_0}$ for any specialization of $\hbar$ and proved that $\mathrm{gr}\, {}_i\mathcal{B}_j^{\mathrm{rat},\hbar=\hbar_0} \simeq A_{\mathrm{rat}}^{j-i}$ using Haiman's results. Note that this was achieved without relying on Haiman's results in [28]. This implies that ${}_i\mathcal{B}_j^{\mathrm{rat},\hbar}$ is free over $\hbar$. The freeness over $\mathbb{C}[y_1,\ldots,y_n]^{S_n}$ is e.g. [35, Lemma 6.11(2)]. We remark that this freeness uses results of [4] about Morita equivalence of Cherednik algebras.

Now the trigonometric case is obtained by Ore localization in the central element $X_1\cdots X_n$, which commutes with the action of $\hbar$ and y_i. This follows from

Lemma 4.19. *There is a natural map*

$$ {}_i\mathcal{B}_j^{\mathrm{rat},\hbar=\hbar_0} \to {}_i\mathcal{B}_j $$

which becomes an isomorphism upon localization in $\prod X_i$*:* $({}_i\mathcal{B}_j^{\mathrm{rat},\hbar=\hbar_0})_{\prod X_i} \cong {}_i\mathcal{B}_j$.

Proof. In [79] it is shown that

$$ \imath(w) = w, \quad \imath(X_i) = X_i, \quad \imath(z_i) = X_i^{-1}(y_i - \sum_{1\le j\le i} \sigma_{ji}), $$

extends to the algebra homomorphism $\imath : \mathbb{H}_\hbar^{\mathrm{rat}} \to \mathbb{H}_{1,\hbar}^{\mathrm{trig}}$ that becomes an isomorphism after localization by $X_1\cdots X_n$. By hitting with $\mathbf{e}$ on both sides, this implies the statement on the level of the spherical subalgebras. (To match parameters, we observe that $H_{c,\hbar}^{\mathrm{trig}} \simeq H_{\lambda c,\lambda\hbar}^{\mathrm{trig}}$ for any $\lambda \in \mathbb{C}^*$.)

For the one-step bimodules, ${}_i\mathcal{B}_{i+1}^{\mathrm{rat}} = \mathbf{e}\mathbb{H}_{c+i\hbar,\hbar}^{\mathrm{rat}}\mathbf{e}_-$ by definition, so the result is true for $j = i+1$ as well. Finally,

$$ {}_i\mathcal{B}_j^{\mathrm{rat}} = {}_i\mathcal{B}_{i+1}^{\mathrm{rat}} \cdots {}_{j-1}\mathcal{B}_j^{\mathrm{rat}} $$

and by standard properties of localization and tensor product we get the result.

□

Now since tensoring with $\mathbb{C}[X_1,\ldots,X_n]_{\prod X_i}$ is faithfully flat, we deduce that since ${}_i\mathcal{B}_j^{\mathrm{rat}}$ is free over

$$ \mathbb{C}[y_1,\ldots,y_n]^{S_n}, $$

so is ${}_i\mathcal{B}_j$. This finishes the proof of Theorem 4.18. □

More geometrically, the bimodule ${}_i\mathcal{B}_j^{\mathrm{rat},\hbar}$ quantizes the line bundle $\mathcal{O}(j-i)$ on the Hilbert scheme of points on $\mathbb{C}^2$ while ${}_i\mathcal{B}_j^\hbar$ quantizes its restriction to the Hilbert scheme of $\mathbb{C}^* \times \mathbb{C}$.

Corollary 4.20. *The* $\mathbb{Z}$*-algebra* $\mathcal{B}$ *is of graded type and for* $G = GL_n$*,* $\mathrm{gr}\,\mathcal{B}$ *corresponds to the graded algebra* $S = \oplus_{d=0}^\infty A^d$*. The corresponding algebraic varieties are* $\mathrm{Proj}\, S = \mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C})$ *and* $\mathrm{Spec}\, S_0 = (\mathbb{C}^*\times\mathbb{C})^n/S_n$.

5. Coulomb branches and $\mathbb{Z}$-algebras

In this section, we explain half of the main construction of the paper, namely the construction of a $\mathbb{Z}$-algebra associated to the Coulomb branch of the $3d$ $\mathcal{N}=4$ theory with adjoint matter, or in other words the spherical trigonometric DAHA. Most of the results work in greater generality, and are stated as such wherever possible. In Section 6 we specialize these general constructions to the case of adjoint representation.

The other half of the main construction, consisting of a generalized affine Springer theory for this $\mathbb{Z}$-algebra, is treated in Section 7.

5.1. **Coulomb branches.** Let $1 \to G \to \widetilde{G} \to G_F \to 1$ be an extension of algebraic groups, where G is reductive and G_F is diagonalizable. Let N be an algebraic representation of $\widetilde{G}$, $\mathbf{P} \subset G(\mathcal{O}) \subset G_\mathcal{K}$ be a (standard) parahoric subgroup and $N_\mathbf{P}$ a lattice in $N_\mathbf{P}$ stable under $\mathbf{P}$. We will only be interested in the case where $N = \mathrm{Ad}$, $N_\mathbf{P} = \mathrm{Lie}(\mathbf{P})$.

Let $\mathcal{R}_\mathbf{P} := \mathcal{R}_{G,N,\mathbf{P},N_\mathbf{P}}$ be the *parahoric BFN space of triples* as in [26]. More precisely, we have

Definition 5.1. $\mathcal{R}_{G,N,\mathbf{P},N_\mathbf{P}}$ is the fpqc sheaf on $Sch/\mathbf{k}$ associating to S the groupoid of tuples $(\mathcal{P},\varphi,s,\mathcal{P}_\mathbf{P})$ where $\mathcal{P}$ is a G-bundle on $S\times\mathrm{Spec}\,\mathcal{K}$, $\mathcal{P}_\mathbf{P}$ is a $\mathbf{P}$-reduction of $\mathcal{P}$ over $S\times\mathrm{Spec}\,\mathcal{O}$, and φ is a trivialization over $S\times\mathrm{Spec}\,\mathcal{O}$ compatible with the $\mathbf{P}$- structure. Moreover, s is a section of the associated N-bundle of $\mathcal{P}_\mathbf{P}$ such that $\varphi\circ s(t) \in N_\mathbf{P}$.

Remark 5.2. Dropping the condition that $\varphi \circ s(t) \in N_{\mathbf{P}}$, we get the space $\mathcal{T}_{G,N,\mathbf{P},N_{\mathbf{P}}} = G_{\mathcal{K}} \times_{\mathbf{P}} N_{\mathbf{P}}$, which is an (ind-)vector bundle over the partial affine flag variety $\mathrm{Fl}_{\mathbf{P}}$. In particular, if $N = \mathrm{Ad}$, this can be thought of as the cotangent bundle of $\mathrm{Fl}_{\mathbf{P}}$.

We recall the following definitions and theorems as motivation for the following sections. From [9, 26, 42], we have

Theorem 5.3. *$H_*^{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times}(\mathcal{R}_{\mathbf{P}}) =: \mathcal{A}^\hbar_{\mathbf{P}}$ is an associative algebra with unit. It is a flat deformation of $\mathcal{A}_{\mathbf{P}} := H_*^{\widetilde{\mathbf{P}}}(\mathcal{R}_{\mathbf{P}})$. When $\mathbf{P} = G_{\mathcal{O}}$, the algebra $\mathcal{A}^\hbar_{G_{\mathcal{O}}}$ is a filtered quantization of $\mathcal{A}_{G_{\mathcal{O}}}$, which is commutative. The spectrum of $\mathcal{A}_{G_{\mathcal{O}}}$ is called the* Coulomb branch *of (G, N).*

Remark 5.4. The group $\widetilde{\mathbf{P}}$ above equals $\mathbf{P}$ if the flavor group G_F is trivial. Otherwise, we define

$$\widetilde{\mathbf{P}} := \mathrm{ev}_0^{-1}(G_F \mathrm{ev}_0(\mathbf{P}))$$

where $\mathrm{ev}_0 : \widetilde{G}_{\mathcal{O}} \to \widetilde{G}$ is the map sending $t \mapsto 0$ or more generally modding out by the nilradical. In general, we refer by $\widetilde{\ }$ to flavor-deformed objects.

Theorem 5.5 ([42, 26])**.** *Let $L_v \subset \widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times$ be the stabilizer of v in $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{\mathrm{rot}}$, where $\widetilde{\mathbf{P}}$ is the preimage of $G_F \mathrm{ev}_0(\mathbf{P})$ under $\mathrm{ev}_0 : \widetilde{G}_{\mathcal{O}} \to \widetilde{G}$. The algebra $\mathcal{A}^\hbar_{\mathbf{P}}$ acts on $H_*^{L_v}(M_v^{\mathbf{P}})$ via natural cohomological correspondences, provided the group L_v is compact in the t-adic topology.*

5.2. **A category of line defects.** Heuristically, the equivariant BM homologies of the spaces of triples above are endomorphisms of objects in a "category of line operators" [21, 86, 85] which is something like $G_{\mathcal{K}}$-equivariant D-modules on $N_{\mathcal{K}}$.

We won't stipulate on the definition of the actual category (see however [3] in the adjoint case), but this category should contain objects coming from $\eta = (U, \mathbf{P})$, where $U \subset N_{\mathcal{K}}$ is a $\mathbf{P}$-stable lattice and $\mathbf{P}$ is a parahoric subgroup of $G_{\mathcal{K}}$. We will simply *define* $\mathrm{Hom}(\eta, \eta') = H_*^{\mathbf{P}'\rtimes\mathbb{C}^\times}({}_\eta\mathcal{R}_{\eta'})$, where

$${}_\eta\mathcal{R}_{\eta'} = \{[g, s] \in G_{\mathcal{K}} \times^{\mathbf{P}'} U' | gs \in U\}.$$

We will use the notation $U = N_{\mathbf{P}}$ to emphasize $N_{\mathbf{P}}$ is a $\mathbf{P}$-stable lattice. By abuse of notation, we will also write $\mathrm{Hom}(\eta, \eta')$ for the flavor- or loop-rotation deformed versions of these spaces.

Theorem 5.6. *There is an associative multiplication $\mathrm{Hom}(\eta, \eta') \otimes_{\mathbb{C}} \mathrm{Hom}(\eta', \eta'') \to \mathrm{Hom}(\eta, \eta'')$ via the following modification of the BFN convolution product.*

$$\begin{array}{ccccc}
{}_\eta\mathcal{R}_{\eta'} \times {}_{\eta'}\mathcal{R}_{\eta''} & \xleftarrow{p} & p^{-1}({}_\eta\mathcal{R}_{\eta'} \times {}_{\eta'}\mathcal{R}_{\eta''}) & \xrightarrow{q} & q(p^{-1}({}_\eta\mathcal{R}_{\eta'} \times {}_{\eta'}\mathcal{R}_{\eta''})) \\
\downarrow i & & \downarrow j & & \downarrow m \\
\mathcal{T}_{\eta'} \times {}_{\eta'}\mathcal{R}_{\eta''} & \xleftarrow[p]{} & G_{\mathcal{K}} \times {}_{\eta'}\mathcal{R}_{\eta''} & & {}_\eta\mathcal{R}_{\eta''}
\end{array}$$

Here the maps p, q, m send

$$p : (g_1, [g_2, s]) \mapsto ([g_1, g_2 s], [g_2, s]), \quad q : (g_1, [g_2, s]) \mapsto [g_1, [g_2, s]],$$
$$m : [g_1, [g_2, s]] \mapsto [g_1 g_2, s]$$

and i, j are inclusions of closed subvarieties.

Proof. This can be proved using a straightforward modification of the proof of associativity in [9, Section 3]. Similar results for $\eta = (N_{\mathbf{P}}, \mathbf{P})$ where $\mathbf{P}$ is an Iwahori subgroup are mentioned in [85]. □

Corollary 5.7. *For any η the space $\mathrm{Hom}(\eta, \eta)$ is an associative algebra, and $\mathrm{Hom}(\eta, \eta')$ is a bimodule over $\mathrm{Hom}(\eta, \eta)$ and $\mathrm{Hom}(\eta', \eta')$. Given η, η' and η'' we have a natural morphism of bimodules over $\mathrm{Hom}(\eta, \eta)$ and $\mathrm{Hom}(\eta'', \eta'')$:*

$$\mathrm{Hom}(\eta, \eta') \bigotimes_{\mathrm{Hom}(\eta', \eta')} \mathrm{Hom}(\eta', \eta'') \to \mathrm{Hom}(\eta, \eta'').$$

Proof. We need to prove the morphism from Theorem 5.6 is bilinear over $\mathrm{Hom}(\eta', \eta')$. The only axiom of a tensor product we need to show is $m \cdot r \otimes n = m \otimes r \cdot n$ for $m \in \mathrm{Hom}(\eta, \eta'), n \in \mathrm{Hom}(\eta', \eta'')$ and $r \in \mathrm{Hom}(\eta', \eta')$, which is clear from the associativity of the construction. □

The following is a generalization of [9, Lemma 5.3].

Theorem 5.8. *${}_\eta\mathcal{A}_{\eta'}$ is flat as a left $\mathbb{C}[\mathfrak{t}^*][\hbar, c] = H_*^{\widetilde{T}}(pt)$-module.*

Proof. The associated graded for the Bruhat filtration is free by equivariant formality. On the other hand, filtered colimits of free modules are flat. □

5.3. **$\mathbb{Z}$-algebras and the flavor deformation.** Taking a sequence $\eta_0, \eta_{-1}, \dots$ it is clear from Theorem 5.6 that we get a $\mathbb{Z}$-algebra by taking

$$\mathcal{A} = \bigoplus_{i\leq j} {}_i\mathcal{A}_j$$

where ${}_i\mathcal{A}_j$ denotes ${}_{\eta_i}\mathcal{A}_{\eta_j}$. That is,

Theorem 5.9. *The algebra $\oplus_{i\leq j}\, {}_i\mathcal{A}^\hbar_j$ is a $\mathbb{Z}$-algebra.*

In the setup of BFN Coulomb branches, nontrivial $\mathbb{Z}$-algebras are most easily obtained via a *flavor deformation* of G, i.e. by letting G_F be nontrivial. We now explain this procedure for $G_F = \mathbb{G}_m$ and associate to $(\widetilde{G}, N)$ a $\mathbb{Z}$-algebra.

Specifically, we can consider a sequence $\eta_i = (t^{-i}U, G_\mathcal{O})$ for some fixed lattice U (for example, $U = N_\mathcal{O}$). If we set $\hbar = 0$, it is easy to see that $S_{j-i} := {}_i\mathcal{A}^{\hbar=0}_j$ depends only on $j - i$ and the algebra $\mathcal{A}^{\hbar=0}$ is of graded type as in Section 4.3. In particular, at $\hbar = 0$ all commutative algebras ${}_i\mathcal{A}^{\hbar=0}_i$ are isomorphic to the commutative Coulomb branch algebra $S_0 = \mathcal{A}_{G_\mathcal{O}}$. In particular, this construction yields a map $\operatorname{Proj} S \to \operatorname{Spec} S_0$, which is a variant of the construction of a partial resolution of the Coulomb branch in [11].

5.4. **$\mathbb{Z}$-algebras in the abelian case.** Since it might be of independent interest and is used for computations below, we now work out the $\mathbb{Z}$-algebras for the cases when $G = T$ is a diagonalizable algebraic group and $\{\eta_i\}_{i=0}^\infty$ is given by $\eta_i = (T_\mathcal{O}, t^{i\phi}N_\mathcal{O})$ for some (flavor) cocharacter $\phi : \mathbb{G}_m \to T$. Note that when $N = 0$, the $\mathbb{Z}$-algebra collapses to $\mathcal{A}^\hbar_{T,0}[c]$ where c is the flavor parameter (the generalization to more flavors is straightforward).

Under $\mathcal{T}_j \hookrightarrow \mathrm{Gr}_T \times N_\mathcal{K}$ the image is naturally identified with

$$\bigsqcup_{\lambda\in\mathrm{Gr}_T} \{t^\lambda\} \times t^j t^\lambda N_\mathcal{O}$$

and similarly

$${}_i\mathcal{R}_j \cong \bigsqcup_{\lambda\in\mathrm{Gr}_T} \{t^\lambda\} \times (t^j t^\lambda N_\mathcal{O} \cap t^i N_\mathcal{O}).$$

Now let ${}_ir^\lambda_j$ be the preimage of $\lambda \in \mathrm{Gr}_G$ under the projection ${}_i\mathcal{R}_j \to \mathrm{Gr}_T$. Suppose also N is the direct sum of the characters $\xi_1, \dots, \xi_n$ as a T-representation.

Theorem 5.10. *Under the convolution product in Theorem 5.6, we have for all $i, j, k \in \mathbb{Z}$ that*

$${}_ir^\lambda_j\, {}_jr^\mu_k = \prod_{\ell=1}^n A_\ell(i,j,k,\lambda,\mu)\, {}_ir^{\lambda+\mu}_k$$

where

$$A_\ell(i,j,k,\lambda,\mu) = \prod_{a=\max(\lambda+i,k-\mu)+1}^{\max(\lambda+i,k-\mu,j)} (\xi_\ell + c + (a + \xi_\ell(\lambda))\hbar) \prod_{b=\min(\lambda+i,k-\mu,j)+1}^{\min(\lambda+i,k-\mu)} (\xi_\ell + c + (b + \xi_\ell(\lambda))\hbar)$$

Proof. We may restrict to the case where the rank of T is 1. In this case, the computation is essentially [85, Proposition 3.10], generalizing [9, Theorem 4.1]. In the notation of *loc. cit.* we have

$${}_ir^\lambda_j = y_\lambda r(\lambda + i, j),\ {}_jr^\mu_k = y_\mu r(\mu + j, k)$$

so we get

$$\begin{aligned}{}_ir^\lambda_j\, {}_jr^\mu_k &= y_\lambda r(\lambda + i, j) r(j, k - \mu) y_\mu \\ &= y_\lambda e\left(\frac{t^{\lambda+i}N_\mathcal{O} \cap t^{k-\mu}N_\mathcal{O}}{t^{\lambda+i}N_\mathcal{O} \cap t^{k-\mu}N_\mathcal{O} \cap t^j N_\mathcal{O}}\right) e\left(\frac{t^{\lambda+i}N_\mathcal{O} + t^{k-\mu}N_\mathcal{O} + t^j N_\mathcal{O}}{t^{\lambda+i}N_\mathcal{O} + t^{k-\mu}N_\mathcal{O}}\right) y_\mu\end{aligned}$$

And we compute the Euler classes

$$e\left(\frac{t^{\lambda+i}N_\mathcal{O} \cap t^{k-\mu}N_\mathcal{O}}{t^{\lambda+i}N_\mathcal{O} \cap t^{k-\mu}N_\mathcal{O} \cap t^j N_\mathcal{O}}\right) = \prod_{a=\max(\lambda+i,k-\mu)+1}^{\max(\lambda+i,k-\mu,j)} (\xi_\ell + c + a\hbar)$$

$$e\left(\frac{t^{\lambda+i}N_\mathcal{O} + t^{k-\mu}N_\mathcal{O} + t^j N_\mathcal{O}}{t^{\lambda+i}N_\mathcal{O} + t^{k-\mu}N_\mathcal{O}}\right) = \prod_{b=\min(\lambda+i,k-\mu,j)+1}^{\min(\lambda+i,k-\mu)} (\xi_\ell + c + b\hbar)$$

From the relation $y_\lambda \chi = (\chi + \hbar\langle\chi, \lambda\rangle) y_\lambda$ for $\chi \in \mathfrak{t}^*$ we get that

$${}_ir^\lambda_j\, {}_jr^\mu_k = A_\ell(i,j,k,\lambda,\mu)\, {}_ir^{\lambda+\mu}_k$$

□

Remark 5.11. When $\hbar = c = 0$, the above becomes

$$ {}_ir_j^\lambda {}_jr_k^\mu = \prod_{\ell=1}^n \frac{\xi_\ell^{\max(\lambda+i,k-\mu,j)}}{\xi_\ell^{\max(\lambda+i,k-\mu)}} \cdot \frac{\xi_\ell^{\min(\lambda+i,k-\mu)}}{\xi_\ell^{\min(\lambda+i,k-\mu,j)}} {}_ir_k^{\lambda+\mu} $$

Lemma 5.12. *All algebras in question are naturally graded with*

$$ \deg \xi_\ell = 2, \ \deg({}_ir_j^\lambda) = |\lambda + i - j|. $$

Proof. Observe that

$$ |a-b| + |b-c| - |a-c| = 2\left(\max(a,b,c) - \max(a,c) + \min(a,c) - \min(a,b,c)\right). \tag{13} $$

Indeed, both sides of (13) are symmetric in a and c and vanish if b is between a and c. If $b < a < c$ then we get

$$ (a-b) + (c-b) - (c-a) = 2(a-b) = 2(c-c+a-b), $$

while for $a < c < b$ we get

$$ (b-a) + (b-c) - (c-a) = 2(b-c) = 2(b-c+a-a). $$

By substituting $a = \lambda + i, b = j, c = k - \mu$ we can verify that the defining equations are homogeneous since

$$ \deg({}_ir_j^\lambda) = |\lambda+i-j|, \ \deg({}_jr_k^\mu) = |k-\mu-j| = |j+\mu-k|, \ \deg({}_ir_k^{\lambda+\mu}) = |(\lambda+i)-(k-\mu)| = |\lambda+\mu+i-k|. $$

□

As in [10, Appendix] and [9], the inclusion $\mathrm{Gr}_T \hookrightarrow \mathcal{T}_j$ as a subbundle gives rise to an injective map z^* in equivariant Borel-Moore homology:

$$ {}_iz_j^* : {}_i\mathcal{A}_j^\hbar \hookrightarrow H_*^{\widetilde{T}_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_T). $$

If $u_\lambda \in H_*^{\widetilde{T}_\mathcal{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_T)$ is the class of the cocharacter λ, or more algebraically the $\hbar$-difference operator on $\mathfrak{t}$ acting on $f \in k[\mathfrak{t}]$ by $f(x) \mapsto f(x + \hbar\lambda)$, we have

Theorem 5.13. *Under ${}_iz_j^*$, we have*

$$ {}_iz_j^*({}_ir_j^\lambda) = e(t^\lambda t^i N_\mathcal{O} / t^\lambda t^i N_\mathcal{O} \cap t^j N_\mathcal{O}) u_\lambda $$

Note that the injectivity of ${}_iz_j^*$ is clear from the above.

6. Coulomb branches and $\mathbb{Z}$-algebras in the adjoint case

6.1. **From Coulomb branch to Cherednik algebra.** We now discuss the adjoint case. For arbitrary G and $N = \mathrm{Ad}$, the construction of [9] yields a noncommutative resolution of $T^*T^\vee/W$ in the sense of [84]. Instead of the spherical case, we focus on the Iwahori case as well as the resulting $\mathbb{Z}$-algebras.

First of all, we claim Theorem 5.6 for $\eta = (\mathbf{I}, \mathrm{Lie}(\mathbf{I}))$ gives a realization of the dDAHA (as conjectured in many places, including [9]) and that the resulting action on the affine Springer fibers coincides with Yun's action. The goal of this section is to prove these claims, and to show that for $\eta = (G_\mathcal{O}, \mathrm{Lie}(G_\mathcal{O}))$ we similarly get the spherical dDAHA, as expected in [9, 7] and other places.

We now state the main theorem of this section.

Theorem 6.1. *The Iwahori-Coulomb branch algebra*

$$ \widetilde{\mathcal{A}}_{G,\mathbf{I}}^\hbar = H_*^{\widetilde{\mathbf{I}} \rtimes \mathbb{C}^\times}(\mathcal{R}_{N_\mathbf{I},\mathbf{I}}) $$

is naturally isomorphic to $\mathbb{H}_{c,\hbar}$.

Proof. There are essentially two subalgebras to find in $\widetilde{\mathcal{A}}_{G,\mathbf{I}}^\hbar$, $\mathbb{C}[\widetilde{W}]$ and $\mathbb{C}[\mathfrak{t}^*]$. The latter is identified with $H_T^*(pt)$, and the elements of this subalgebra act as capping by Chern classes of the associated line bundles on Fl_G. Ditto for the equivariant parameters $c, \hbar$.

We denote by $\mathcal{R}_{N_\mathbf{I},\mathbf{I}}^{\le w}$ the preimage of the Schubert cell $\overline{\mathrm{Fl}_G^{\le w}}$ under the projection $\mathcal{R}_{N_\mathbf{I},\mathbf{I}} \to \mathrm{Fl}_G$. The algebra $\widetilde{\mathcal{A}}_{G,\mathbf{I}}^\hbar$ is spanned by $[\mathcal{R}_{N_\mathbf{I},\mathbf{I}}^{\le w}]$ for $w \in \widetilde{W}$. We use the maps m, p, q as in Theorem 5.6. Let us study $[\mathcal{R}_{N_\mathbf{I},\mathbf{I}}^{\le w}][\mathcal{R}_{N_\mathbf{I},\mathbf{I}}^{\le w'}]$. Since

$$ mqp^{-1}(\overline{\mathrm{Fl}_G^{\le w}} \times \overline{\mathrm{Fl}_G^{\le w'}}) \subset \overline{\mathrm{Fl}_G^{\le ww'}} $$

is codimension zero (see e.g. [63, Lemma 4.4]) and N is the adjoint representation,

$$ mpq^{-1}(\mathcal{R}_{N_\mathbf{I},\mathbf{I}}^{\le w} \times \mathcal{R}_{N_\mathbf{I},\mathbf{I}}^{\le w'}) \subset \mathcal{R}_{N_\mathbf{I},\mathbf{I}}^{ww'} $$

is codimension zero, the algebra $\widetilde{\mathcal{A}}_{G,\mathbf{I}}$ is a filtered flat deformation of

$$\mathrm{gr}_{\mathrm{Bruhat}}\widetilde{\mathcal{A}}^{\hbar}_{G,\mathbf{I}} = \bigoplus_{w\in\widetilde{W}} H_*^{\widetilde{\mathbf{I}}\rtimes\mathbb{C}^\times}(\mathcal{R}^w_{N_\mathbf{I},\mathbf{I}})$$

such that

$$[\mathcal{R}^{\le w}_{N_\mathbf{I},\mathbf{I}}][\mathcal{R}^{\le w'}_{N_\mathbf{I},\mathbf{I}}] = [\mathcal{R}^{\le ww'}_{N_\mathbf{I},\mathbf{I}}] + \text{ lower order terms in Bruhat order}$$

Note that each w has a reduced expression, say $w = s_{i_1}\cdots s_{i_j}$. By taking the associated graded we see we must have

$$[\mathcal{R}^{\le s_{i_1}}_{N_\mathbf{I},\mathbf{I}}]\cdots[\mathcal{R}^{\le s_{i_j}}_{N_\mathbf{I},\mathbf{I}}] = [\mathcal{R}^{\le s_{i_1}\cdots s_{i_j}}_{N_\mathbf{I},\mathbf{I}}] + \text{ lower order terms.}$$

This implies that the classes above the one-dimensional $\mathbf{I}$-orbits in Fl_G generate $\widetilde{\mathcal{A}}^{\hbar}_{G,\mathbf{I}}$ together with the equivariant parameters. So it remains to check these satisfy the right relations. Similar to (7) we have

$$\begin{array}{ccc} \mathcal{R}_{N_\mathbf{I},\mathbf{I}} & \xrightarrow{\varphi'} & [\widetilde{\mathfrak{g}}/G] = [\mathfrak{b}/B] \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ \mathcal{R}_{N_\mathcal{O},G_\mathcal{O}} & \xrightarrow{\varphi} & [\mathfrak{g}/G] \end{array} \tag{14}$$

On the right column, the Springer action is classically identified with the convolution action by correspondences on the Steinberg variety. Similar to [88, Proposition 5.2.1.], the pullback of these correspondences is identified with the correspondence on (the equivariant homology of) $\mathcal{R}_{N_\mathbf{I},\mathbf{I}}$ given by $*[\mathcal{R}^{\le s_i}_{N_\mathbf{I},\mathbf{I}}]$. Letting $G_\mathcal{O}, N_\mathcal{O}$ be replaced by parahorics of rank two subgroups of $\widetilde{W}$, it is clear the relations of $\mathbb{C}[\widetilde{W}]$ are satisfied.

We still need to check relation (2) from Definition 4.1. This is identical to [88, Lemma 6.4.1].

Since we have a morphism preserving the Bruhat filtration, inducing an isomorphism on associated graded objects, $\widetilde{\mathcal{A}}^{\hbar}_{G,\mathbf{I}} \cong \mathbb{H}_G$. □

Remark 6.2. In K-theory, a similar proposition is proven for the full DAHA in [82, Section 2.5].

Corollary 6.3. *The Coulomb branch algebra $\mathcal{A}^{\hbar}_{G,G_\mathcal{O}}$ of Braverman-Finkelberg-Nakajima is isomorphic to the spherical subalgebra of $\mathbb{H}_G$.*

Proof. Let $\mathbf{e} = |W|^{-1}\sum_{w\in W} w$. Then from (14) and the definitions it follows that

$$\mathcal{A}^{\hbar}_{G,G_\mathcal{O}} = \mathbf{e}\mathcal{A}^{\hbar}_{G,\mathbf{I}}\mathbf{e}$$

and by Theorem 6.1 we get the result. □

Remark 6.4. This proves the speculation in [9, Remark 6.20]. For $G = GL_n$ this was proved by Kodera and Nakajima in [52].

Finally, we give a geometric realization of the "shift" bimodules of the trigonometric Cherednik algebra using line operators.

Let $\eta = (t^i\,\mathrm{Lie}(\mathbf{I}),\mathbf{I})$ and $\eta' = (t^i\,\mathrm{Lie}(\mathbf{I}\cap tG(\mathcal{O})),\mathbf{I})$, where $\mathrm{Lie}(\mathbf{I}\cap tG(\mathcal{O}))$ is the pronilpotent radical of $\mathrm{Lie}(\mathbf{I})$. In the notations of 5.2 denote

$$_{i+1}\widetilde{\mathcal{R}}_i := {}_\eta\mathcal{R}_{\eta'}$$

and

$$_{i+1}\widetilde{\mathcal{A}}_i = H_*^{\widetilde{\mathbf{I}}\rtimes\mathbb{C}^\times}({}_{i+1}\widetilde{\mathcal{R}}_i)$$

Theorem 6.5. *There are natural isomorphisms of graded bimodules*

$$_{i+1}\mathcal{A}_i \cong \mathbf{e}_{i+1}\widetilde{\mathcal{A}}_i\mathbf{e} \cong \mathbf{e}_{-i}\widetilde{\mathcal{A}}_i\mathbf{e}$$

Proof. Let $_{i+1}\widetilde{\mathcal{R}}_i \hookrightarrow {}_i\widetilde{\mathcal{R}}_i$ be the natural inclusion. $(g,s)\in{}_i\widetilde{\mathcal{R}}_i$ belongs to $_{i+1}\widetilde{\mathcal{R}}_i$ exactly when $gs\in\mathrm{Lie}(\mathbf{I})\cap t\mathfrak{g}(\mathcal{O})$, or in other words when it is in the kernel of the map

$$_i\widetilde{\mathcal{R}}_i \to [\mathfrak{b}/B]$$

sending (g,s) to gs mod t. We also have the cartesian square

$$\begin{array}{ccc} {}_i\widetilde{\mathcal{R}}_i & \xrightarrow{\varphi'} & [\widetilde{\mathfrak{g}}/G] = [\mathfrak{b}/B] \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ {}_i\mathcal{R}_i & \xrightarrow{\varphi} & [\mathfrak{g}/G] \end{array} \tag{15}$$

similar to (7). Now, note that by finite-dimensional Springer theory

$$H_*^{\widetilde{\mathbf{I}}\rtimes\mathbb{C}^\times}({}_i\mathcal{R}_i) = {}_i\widetilde{\mathcal{A}}_i\mathbf{e}$$

and

$$H_*^{\widetilde{G}_{\mathcal{O}}\rtimes\mathbb{C}^\times}({}_i\mathcal{R}_i) = \mathbf{e}_i\widetilde{\mathcal{A}}_i\mathbf{e}$$

Finally ${}_{i+1}\mathcal{A}_i \cong \mathbf{e}_{i+1}\widetilde{\mathcal{A}}_i\mathbf{e}$ and

$$\mathbf{e}_{-i}\widetilde{\mathcal{A}}_i\mathbf{e}[2\dim G/B] \cong \mathbf{e}_{i+1}\widetilde{\mathcal{A}}_i\mathbf{e}$$

again similarly to the proof of Lemma 2.2.

□

6.2. **Localization of the spherical algebra in the adjoint case.** We now analyze the $\mathbb{Z}$-algebra introduced in the previous section via localization to fixed points. In particular, we may deduce results about the associated graded of the Bruhat filtration for the convolution algebras, using an "abelianization" procedure appearing e.g. in [14]. We should note that similar fixed-point analysis does not apply to the Springer action itself unless we are in a situation similar to [72, 83, 26], but we are still able to deduce many results about the convolution action on general grounds in Section 7.

We let G and N be arbitrary for now. Suppose $\mathbf{P} = G_{\mathcal{O}}$. The spaces ${}_i\mathcal{R}_j$ have natural closed embeddings to ${}_i\mathcal{R}_j \hookrightarrow G_{\mathcal{K}} \times^{G_{\mathcal{O}}} t^j N_{\mathcal{O}}$. Moreover, there is the embedding of the zero-section

$$z : \mathrm{Gr}_G \hookrightarrow G_{\mathcal{K}} \times^{G_{\mathcal{O}}} t^j N_{\mathcal{O}}$$

and an inclusion of "the fibers above fixed points"

$$\iota : {}_i\mathcal{R}_j|_{\mathrm{Gr}_T} \hookrightarrow {}_i\mathcal{R}_j.$$

The latter map gives rise to an equivariant pushforward ι_* (see [9, Remark 5.23]). The map z for $G = T$ gives the maps ${}_iz_j^*$ from Theorem 5.13. Similarly to [9, 10] we then have

Proposition 6.6. *We have an embedding*

$${}_iz_j^*(\iota_*)^{-1} : {}_i\mathcal{A}_j^\hbar \hookrightarrow \mathcal{A}_{T,0}^\hbar[\hbar^{-1}, (\textit{generalized roots} + m\hbar + nc)^{-1}|m, n \in \mathbb{Z}].$$

Note that this is not *a ring homomorphism unless $i = j$, but a bimodule homomorphism, as in Theorem 5.13.*

Let $\pi : {}_i\mathcal{R}_j \to \mathrm{Gr}_G$ be the projection. We use the Cartan decomposition of the affine Grassmannian into $G_{\mathcal{O}}$ orbits:

$$\mathrm{Gr}_G = \bigsqcup_{\lambda\in X_*^+} \mathrm{Gr}_G^\lambda, \quad \mathrm{Gr}_G^\lambda = G_{\mathcal{O}}t^\lambda G_{\mathcal{O}}/G_{\mathcal{O}}$$

The closures of these orbits will be denoted by $\mathrm{Gr}_G^{\le\lambda} = \overline{\mathrm{Gr}_G^\lambda}$. Then the subvariety ${}_i\mathcal{R}_j^{\le\lambda} := \pi^{-1}(\mathrm{Gr}_G^{\le\lambda})$ gives rise to a class in equivariant Borel-Moore homology as in [9, Section 2].

In particular, we have the following localization formula.

Lemma 6.7. *For a minuscule cocharacter λ, we have*

$$ {}_iz_j^*(\iota_*)^{-1}f \cap [{}_i\mathcal{R}_j^{\le\lambda}] = \sum_{\lambda'=w\lambda\in W\lambda} \frac{wf \times e(t^{\lambda'}t^jN_{\mathcal{O}}/t^{\lambda'}t^jN_{\mathcal{O}} \cap t^iN_{\mathcal{O}})}{e(T_{\lambda'}\mathrm{Gr}_G^\lambda)} u_{\lambda'} \tag{16}$$

Proof. We are using Borel-Moore homology, so results of Brion [13] apply and the formula follows from Theorem 5.13. For the case $i = j = 0$, see [9, Proposition 6.6].

□

If λ is not minuscule, the corresponding Schubert variety is not smooth and there is no nice formula for $[{}_i\mathcal{R}_j^{\le\lambda}]$. Still, the right hand side of the equation (6.7) yields the formula for the associated graded with respect to the Bruhat filtration, see e.g. [10, Eq. (6.3)]. We first consider the case $G = GL_n$.

Theorem 6.8. *Let $G = GL_n$, then the following hold:*

(a) For $c = \hbar = 0$, arbitrary cocharacter λ and a function $f(y)$ which is symmetric under the stabilizer of λ we have the following:

$${}_jz_i^*\iota_*^{-1}\,\mathrm{gr}[{}_j\mathcal{R}_i^\lambda][f] = \sum_{\lambda'\in W\lambda} f' \prod_{s\neq r} \frac{(y_r - y_s)^{\max(\lambda'_r-\lambda'_s+i,j)}}{(y_r-y_s)^{\lambda'_r-\lambda'_s+i}(y_r-y_s)^{\max(\lambda'_r-\lambda'_s,0)}} u^{\lambda'}$$

Here f' is the image of f under any permutation in W which sends λ to λ'. If λ is minuscule, the formula is exact without taking associated graded.

(b) For general $c, \hbar$ we have

$$ {}_j z_i^* \iota_*^{-1} \operatorname{gr}[{}_j\mathcal{R}_i^\lambda][f] = \sum_{\lambda' \in W\lambda} f' \frac{\prod_{\lambda'_r - \lambda'_s + i < j} \prod_{\ell=0}^{j-(\lambda'_r-\lambda'_s+i)-1} (y_r - y_s + (\lambda'_r - \lambda'_s + i + \ell) + c)}{\prod_{s \neq r} \prod_{\ell=0}^{\max(\lambda'_r - \lambda'_s, 0)} (y_s - y_r + \ell\hbar)} u^{\lambda'} $$

where the notations are as above.

Proof. We compute the right hand side in the equation (16). If $\lambda = (\lambda_1, \dots, \lambda_n) \in X_*(T) \subset X_*(GL_n)$ we get

$$ t^\lambda . t^i N_{\mathcal{O}} = \begin{pmatrix} t^i N_{\mathcal{O}} & t^{\lambda_1-\lambda_2+i} N_{\mathcal{O}} & t^{\lambda_1-\lambda_3+i} N_{\mathcal{O}} & \cdots & t^{\lambda_1-\lambda_n+i} N_{\mathcal{O}} \\ t^{\lambda_2-\lambda_1+i} N_{\mathcal{O}} & t^i N_{\mathcal{O}} & t^{\lambda_2-\lambda_3+i} N_{\mathcal{O}} & \cdots & t^{\lambda_2-\lambda_n+i} N_{\mathcal{O}} \\ t^{\lambda_3-\lambda_1+i} N_{\mathcal{O}} & t^{\lambda_3-\lambda_2+i} N_{\mathcal{O}} & t^i N_{\mathcal{O}} & \cdots & t^{\lambda_3-\lambda_n+i} N_{\mathcal{O}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{\lambda_n-\lambda_1+i} N_{\mathcal{O}} & t^{\lambda_n-\lambda_2+i} N_{\mathcal{O}} & t^{\lambda_n-\lambda_3+i} N_{\mathcal{O}} & \cdots & t^i N_{\mathcal{O}} \end{pmatrix} $$

Whence we compute the Euler class at $c = \hbar = 0$:

$$ e(t^\lambda t^i N_{\mathcal{O}} / t^\lambda t^i N_{\mathcal{O}} \cap t^j N_{\mathcal{O}}) = \prod_{s \neq r} \frac{(y_r - y_s)^{\max(\lambda_r - \lambda_s + i, j)}}{(y_r - y_s)^{\lambda_r - \lambda_s + i}} $$

For general $c, \hbar$ the factors with $\lambda'_r - \lambda'_s + i \geq j$ still contribute 1, and the formula for the Euler class reads as

$$ \prod_{\lambda'_r - \lambda'_s + i < j} \prod_{\ell=0}^{j-(\lambda'_r-\lambda'_s+i)-1} (y_r - y_s + (\lambda'_r - \lambda'_s + i + \ell) + c). $$

It is well known that the tangent space $T_\lambda \operatorname{Gr}^\lambda$ is naturally identified with $\frac{N_{\mathcal{O}}}{N_{\mathcal{O}} \cap t^\lambda . N_{\mathcal{O}}}$, from which we get

$$ e(T_\lambda \operatorname{Gr}_G^\lambda) = \prod_{s \neq r} \prod_{\ell=0}^{\max(\lambda_r - \lambda_s, 0)} (y_s - y_r + \ell\hbar). $$

□

Remark 6.9. The above formula makes sense even if f is not symmetric with respect to the stabilizer of λ. In this case, we first symmetrize with respect to the stabilizer of λ and then symmetrize with respect to the whole group $W = S_n$.

For a general group G and $N = \operatorname{Ad}$, we write can write the formula as follows.

Proposition 6.10. *For arbitrary G and arbitrary coweight λ we have:*

$$ {}_j z_i^* \iota_*^{-1} \operatorname{gr}[{}_j\mathcal{R}_i^{\leq\lambda}] = \sum_{\lambda' \in W\lambda} \frac{\prod_{\alpha(\lambda')+i<j} \prod_{\ell=0}^{i-\alpha(\lambda')-j-1} (y_\alpha + (\alpha(\lambda') + j + \ell)\hbar + c)}{\prod_{\alpha \in \Phi} \prod_{\ell=0}^{\max(0, \alpha(\lambda'))-1} (y_\alpha + \ell\hbar)} u_{\lambda'} $$

Proof. The proof of Theorem 6.8 is naturally adopted to arbitrary root systems. We will not need it below and leave the details for the reader. □

Lemma 6.11. *For $G = GL_n$ and $N = \operatorname{Ad}$ and the minuscule coweight $\omega_m = (1, \dots, 1, 0, \dots, 0)$ and $i \geq j$, we have*

$$ z^* \iota_*^{-1} [{}_i\mathcal{R}_i^{\leq\omega_m}] = \sum_{I \subset [n], |I| = m} \prod_{r \in I, s \notin I} \frac{y_s - y_r + (i-1)\hbar + c}{y_r - y_s} u^I \tag{17} $$

$$ z^* \iota_*^{-1} [{}_{i+1}\mathcal{R}_i^{\leq\omega_m}] = \sum_{I \subset [n], |I| = m} \frac{(\prod_{r \in I, s \notin I} (y_s - y_r + (i-1)\hbar + c)(y_s - y_r + i\hbar + c))(\prod_{r \in I, s \in I \text{ or } r \notin I, s \notin I} y_r - y_s + i\hbar + c)}{\prod_{r \in I, s \notin I} y_r - y_s} u^I \tag{18} $$

$$ z^* \iota_*^{-1} [{}_j\mathcal{R}_i^{\leq\omega_m}] = \prod_{k=i+1}^{j-1} \prod_{r,s} (y_r - y_s + k\hbar + c) \cdot z^* \iota_*^{-1} [{}_{i+1}\mathcal{R}_i^{\leq\omega_m}] \tag{19} $$

for $j \geq i + 2$.

Proof. This is a direct application of Theorem 6.8, recall that since ω_m is minuscule we do not need to take associated graded. A symmetrization of ω_m leads to a weight

$$ \lambda' = (\lambda'_1, \dots, \lambda'_n), \ \lambda'_r = \begin{cases} 1 & \text{if } r \in I \\ 0 & \text{otherwise} \end{cases} $$

for some m-element subset I. In this case $\max(\lambda'_r - \lambda'_s, 0)$ equals 1 if $r \in I, s \notin I$ and 0 otherwise which gives the denominator. For the numerator, we observe

$$\lambda'_r - \lambda'_s + i = \begin{cases} i-1 & \text{if } r \notin I, s \in I \\ i & \text{if } r \in I, s \in I \text{ or } r \notin I, s \notin I \\ i+1 & \text{if } r \in I, s \notin I \end{cases}$$

and the result follows. □

More generally, we have the formula for arbitrary G and minuscule λ.

Lemma 6.12. *For $N = \mathrm{Ad}$ and λ minuscule,*

$$z^*\iota_*^{-1}[{}_i\mathcal{R}_i^{\le\lambda}][f] = \sum_{\lambda'=w\lambda\in W\lambda} wf \times \prod_{\alpha(\lambda')=1} \frac{-y_\alpha+(i-1)\hbar+c}{y_\alpha} u^{\lambda'} \tag{20}$$

$$z^*\iota_*^{-1}[{}_{i+1}\mathcal{R}_i^{\le\lambda}] = \sum_{\lambda'=w\lambda\in W\lambda} wf \times \frac{\left(\prod_{\alpha(\lambda')=1}(y_\alpha+(i-1)\hbar+c)(y_\alpha+i\hbar+c)\right)\left(\prod_{\alpha(\lambda')=0} y_\alpha+i\hbar+c\right)}{y_\alpha} u^{\lambda'} \tag{21}$$

$$z^*\iota_*^{-1}[{}_j\mathcal{R}_i^{\le\lambda}] = \prod_{k=i+1}^{j-1}\prod_{\alpha\in\Phi}(y_\alpha+k\hbar+c)\cdot z^*\iota_*^{-1}[{}_{i+1}\mathcal{R}_i^{\le\lambda}] \tag{22}$$

Lemma 6.13. *Let $\varepsilon(x) = \max(x+i, j) - (x+i) - \max(x, 0)$, then*

$$\varepsilon(x) + \varepsilon(-x) = \begin{cases} j-i & \text{if } |x| \ge |j-i|, \\ (j-i) + |j-i| - |x| & \text{if } |x| \le |j-i|. \end{cases}$$

Proof. Let us first prove that for arbitrary x, d one has

$$\max(x,d) + \max(-x,d) = d + \max(|x|,|d|) = \begin{cases} d+|x| & \text{if } |x| \ge |d|, \\ d+|d| & \text{if } |x| \le |d|. \end{cases} \tag{23}$$

Clearly, $\max(x,d) + \max(-x,d) = \max(|x|,d) + \max(-|x|,d)$. For $d \ge 0$ we get $\max(-|x|,d) = d$ and (23) is clear. For $d < 0$ we can rewrite

$$\max(|x|,d) + \max(-|x|,d) = |x| - \min(|x|,|d|) = d + \max(|x|,|d|).$$

Now we can prove lemma, by letting $d = j - i$. Note that $\max(x+i,j) = i + \max(x, j-i)$, therefore

$$\varepsilon(x) + \varepsilon(-x) = \max(x+i,j) - (x+i) - \max(x,0) + \max(-x+i,j) - (-x+i) - \max(-x,0) =$$

$$\max(x, j-i) - \max(x,0) + \max(-x, j-i) - \max(-x,0).$$

Now we can use (23) with $d = j - i$. □

Corollary 6.14. *Let $G = GL_n$. At $c = \hbar = 0$ we get*

$${}_jz_i^*\,\mathrm{gr}\,\iota_*^{-1}[{}_j\mathcal{R}_i^\lambda][f] = \pm\mathrm{Sym}\left(f\cdot\Delta^{j-i}\prod_{r<s,|\lambda_r-\lambda_s|<|j-i|}(y_r-y_s)^{|j-i|-|\lambda_r-\lambda_s|}u^\lambda\right)$$

Proof. Consider a pair $r < s$. In the right hand side of Theorem 6.8 we get

$$(y_r-y_s)^{\varepsilon(\lambda'_r-\lambda'_s)}(y_s-y_r)^{\varepsilon(\lambda'_s-\lambda'_r)} = \pm(y_r-y_s)^{\varepsilon(\lambda'_r-\lambda'_s)+\varepsilon(\lambda'_s-\lambda'_r)}.$$

By Lemma 6.13, the result follows. □

Example 6.15. Again for $G = GL_n$, assume that $j = i+1$, then at $c = \hbar = 0$ we get

$${}_{i+1}z_i^*\,\mathrm{gr}\,\iota_*^{-1}[{}_{i+1}\mathcal{R}_i^\lambda][f] = \pm\Delta\mathrm{Alt}\left(f\cdot\prod_{r<s,\lambda_r=\lambda_s}(y_r-y_s)u^\lambda\right)$$

For arbitrary groups and $\hbar = c = 0$ we get a similar formula.

Corollary 6.16. *For arbitrary G and λ we have*

$${}_jz_i\iota_*^{-1}\,\mathrm{gr}[{}_j\mathcal{R}_i^\lambda][f] = \mathrm{Sym}_W\left(f\Delta^{j-i}\prod_{\alpha\in\Phi^+,|\alpha(\lambda)|<|j-i|}(y_\alpha)^{|j-i|-|\alpha(\lambda)|}u^\lambda\right)$$

Proof. The proof follows from setting $\hbar = c = 0$ in Theorem 6.10 in exactly the same way as Theorem 6.8 and Corollary 6.14. □

6.3. **Factorization of bimodules.**

Lemma 6.17. *Suppose that λ is an arbitrary integral coweight for GL_n and $d > 0$. Then there exist d coweights $\mu^{(0)}, \ldots, \mu^{(d-1)}$ such that $\mu^{(0)} + \ldots + \mu^{(d-1)} = \lambda$ and for all i and j the following holds:*
1) If $|\lambda_i - \lambda_j| < d$ then

$$d - |\lambda_i - \lambda_j| = \sum_{k, \mu_i^{(k)} = \mu_j^{(k)}} 1.$$

2) If $|\lambda_i - \lambda_j| > d$ then $\mu_i^{(k)} \neq \mu_j^{(k)}$ for all k.

Proof. We define $\mu^{(k)}$ by "dividing λ by d with remainder". More precisely, let $\lambda_i = dq_i + r_i$ where $0 \leq r_i < d$. We define

$$\mu_i^{(k)} = \begin{cases} q_i + 1 & \text{for } k < r_i \\ q_i & \text{for } k \geq r_i \end{cases}$$

Clearly, $\mu_i^{(0)} + \ldots + \mu_i^{(d-1)} = \lambda_i$. Without loss of generality, we can assume that $\lambda_j \geq \lambda_i$. We have the following cases:
1) $\lambda_j = dq_i + r_j, r_j \geq r_i$. In this case $\mu_i^{(k)} = \mu_j^{(k)}$ for $k < r_i$ and $k \geq r_j$, so

$$\sum_{k, \mu_i^{(k)} = \mu_j^{(k)}} 1 = d - (r_j - r_i) = d - (\lambda_j - \lambda_i).$$

2) $\lambda_j = d(q_i + 1) + r_j, r_j < r_i$. In this case $\mu_i^{(k)} = \mu_j^{(k)}$ for $r_j \leq k < r_i$ and

$$\sum_{k, \mu_i^{(k)} = \mu_j^{(k)}} 1 = r_i - r_j = d - (\lambda_j - \lambda_i).$$

3) If $\lambda_j > d(q_i + 1) + r_j$ then $\mu_j^{(k)} \geq q_i + 2$ for $k < r_i$ and $\mu_j^{(k)} \geq q_i + 1$ for $k \geq r_i$, so $\mu_i^{(k)} \neq \mu_j^{(k)}$ for all k. □

Example 6.18. Suppose that $d = 2$, then we split $\lambda = \mu^{(0)} + \mu^{(1)}$ as follows. If $\lambda_i = 2k$ is even, we set $\mu_i^{(0)} = \mu_i^{(1)} = k$; if $\lambda_i = 2k + 1$ is odd, we set $\mu_i^{(0)} = k + 1$ and $\mu_i^{(1)} = k$. Clearly, if $\lambda_i = \lambda_j$ then both $\mu_i^{(0)} = \mu_j^{(0)}$ and $\mu_i^{(1)} = \mu_j^{(1)}$. If $|\lambda_i - \lambda_j| = 1$, it is not hard to see that exactly one of equations $\mu_i^{(0)} = \mu_j^{(0)}$ and $\mu_i^{(1)} = \mu_j^{(1)}$ holds.

Corollary 6.19. *Suppose that $G = GL_n$, $j - i = d$ and $c = \hbar = 0$. For an arbitrary coweight λ and $\mu^{(k)}$ as in Lemma 6.17 we have*

$$ {}_j z_i^* \iota_*^{-1} \operatorname{gr}[{}_j\mathcal{R}_i^\lambda] = \pm \prod_{k=0}^{d-1} {}_{i+k+1} z_{i+k}^* \iota_*^{-1} \operatorname{gr}\left[{}_{i+k+1}\mathcal{R}_{i+k}^{\mu^{(k)}}\right] + \text{lower order terms.} \tag{24}$$

Proof. By Lemma 6.17 we get

$$\prod_{r<s, |\lambda_r - \lambda_s| < d} (y_r - y_s)^{d - |\lambda_r - \lambda_s|} = \prod_k \prod_{r<s, \mu_r^{(k)} = \mu_s^{(k)}} (y_r - y_s).$$

By Corollary 6.14, the left hand side of Eq. (24) is a symmetric polynomial with leading term (in the dominance order on the u_λ)

$$\pm\Delta^d \prod_{r<s, |\lambda_r - \lambda_s| < d} (y_r - y_s)^{d - |\lambda_r - \lambda_s|} u^{\operatorname{sort}(\lambda)}$$

while the right hand side is a product of d symmetric polynomials with leading terms

$$\pm\Delta \prod_{r<s, \mu_r^{(k)} = \mu_s^{(k)}} (y_r - y_s) u^{\operatorname{sort}(\mu^{(k)})}$$

It is easy to see that in the above construction $\operatorname{sort}(\lambda) = \operatorname{sort}(\mu^{(0)}) + \ldots + \operatorname{sort}(\mu^{(d-1)})$, so the result follows. □

Remark 6.20. It seems reasonable to conjecture analogs of Lemma 6.17 and Corollary 6.19 for other groups, at least for simply laced groups. This would have the consequence that the isomorphism constructed in Theorem 6.21 would hold for other groups, showing for instance that the global sections of the line bundle we construct equal A_G^d. Since the line bundle is not expected to be ample outside the simply laced case (see [53]) we do not expect the result to hold in general.

We also note that the combinatorics appearing in the Lemma are closely related to the *root-system chip-firing* of [24]. It would be interesting to make the connection more precise. The second author thanks Pavel Galashin for correspondence regarding this point.

6.4. **The geometric $\mathbb{Z}$-algebra for the adjoint representation.** We are ready to prove the main result of this section.

Theorem 6.21. *When $G = GL_n$, the $\mathbb{Z}$-algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. For general G, there is an injection $\mathcal{A} \hookrightarrow \mathcal{B}$ inducing ${}_{j-1}\mathcal{A}_j \cong {}_{j-1}\mathcal{B}_j$ and ${}_j\mathcal{A}_j \cong {}_j\mathcal{B}_j$.*

Proof. We need to prove the following facts:

(a) ${}_i\mathcal{A}_i \cong {}_i\mathcal{B}_i$ as algebras
(b) ${}_i\mathcal{A}_{i+1} \cong {}_i\mathcal{B}_{i+1}$ as bimodules over ${}_i\mathcal{A}_i$ (resp. ${}_i\mathcal{B}_i$) and ${}_{i+1}\mathcal{A}_{i+1}$ (resp. ${}_{i+1}\mathcal{B}_{i+1}$)
(c) ${}_i\mathcal{A}_{i+1} \cdots {}_{j-1}\mathcal{A}_j \hookrightarrow {}_i\mathcal{A}_j$ and this is an isomorphism for $G = GL_n$. Note that ${}_i\mathcal{B}_j \cong {}_i\mathcal{B}_{i+1} \cdots {}_{j-1}\mathcal{B}_j$ by definition.

Part (a) follows from Theorem 6.1. Part (b) follows from Theorem 6.5.

In type A, it is instructive to review what part (b) says in order to prove part (c). We can compute the bases in the associated graded spaces on both sides: $\mathrm{gr}\, {}_i\mathcal{B}_{i+1} = A$ is the space of antisymmetric polynomials in $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}, y_1, \ldots, y_n]$ and by Lemma 3.25(a) it has a vector space basis Δ_S parametrized by all n-element subsets of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$. On the other hand, in $\mathrm{gr}\, {}_i\mathcal{A}_{i+1}$ we have a basis $\mathrm{gr}[{}_j\mathcal{R}_i^{\lambda}][f]$ parametrized by a weight λ and a function f. By Example 6.15 and Lemma 3.25(c) these can be explicitly identified by setting f to be the product of Schur polynomials. Finally, having a filtered homomorphism inducing an isomorphism on associated graded spaces gives an isomorphism.

Let us prove part (c) for G arbitrary. By Corollary 5.7 the convolution product gives a natural map

$$ {}_i\mathcal{A}_{i+1} \bigotimes_{{}_{i+1}\mathcal{A}_{i+1}} \cdots \bigotimes_{{}_{j-1}\mathcal{A}_{j-1}} {}_{j-1}\mathcal{A}_j \to {}_i\mathcal{A}_j, \tag{25} $$

To check that (25) is injective, it is sufficient to check that it becomes an isomorphism after localization in the multiplicative set generated by $\{y_\alpha + n\hbar + mc | \alpha \in \Phi, m, n \in \mathbb{Z}\}$ which we get from [9, Remark 3.24]. Finally, we need to prove that it is an isomorphism for $G = GL_n$. To prove that it is surjective, we first consider the commutative limit $\hbar = c = 0$ and take the associated graded with respect to the Bruhat filtration. Then surjectivity follows from Corollary 6.19.

Next, we use parts (a) and (b) of the theorem to rewrite the left hand side of (25) as

$$ {}_i\mathcal{B}_{i+1} \bigotimes_{{}_{i+1}\mathcal{B}_{i+1}} \cdots \bigotimes_{{}_{j-1}\mathcal{B}_{j-1}} {}_{j-1}\mathcal{B}_j = {}_i\mathcal{B}_j. $$

By Theorem 4.18 this is free over $\mathbb{C}[y_1, \ldots, y_n]^{S_n}$. Since the space ${}_i\mathcal{R}_{j+1}$ is equivariantly formal, the bimodule ${}_i\mathcal{A}_{j+1}$ is free over $\mathbb{C}[\hbar] \otimes \mathbb{C}[y_1, \ldots, y_n]^{S_n}$ as well. Therefore (25) is surjective for general $c, \hbar$.

□

Remark 6.22. For $G = GL_n$, Simental [78] classified Harish-Chandra bimodules for the *rational* Cherednik algebra and proved that the shift bimodule is the unique Harish-Chandra bimodule which sends polynomial representation to the polynomial representation. In particular, this implies an analogue of Theorem 6.21 for the rational Cherednik algebra.

It would be interesting to know if the methods of [78] can be generalized to the trigonometric case to give an alternate proof of Theorem 6.21.

Combining the above result with the Proj construction we get

Corollary 6.23. *When $G = GL_n$, the graded algebra $\bigoplus_i {}_{j-i}\mathcal{A}_j^{\hbar=c=0}$ is naturally isomorphic to the homogeneous coordinate ring of $\mathrm{Hilb}^n(\mathbb{C} \times \mathbb{C}^*)$ for any j.*

Proof. Specialize the above theorem for $c = \hbar = 0$ and use Theorem 4.18. □

Remark 6.24. One should also compare this to the results in [11] which essentially show ${}_i\mathcal{A}_j \cong \mathcal{O}(j-i)$ in the case $G = GL_n, N = \mathrm{Ad} \oplus V^\ell$ for $\ell \geq 1$, using factorization and results about the Hilbert schemes on $A_{\ell-1}$-resolutions.

6.5. **A flag $\mathbb{Z}$-algebra.** In this section, we sketch to what extent the construction of ${}_i\mathcal{A}_j$ extends to the flag level, i.e. when we replace $\mathfrak{g}(\mathcal{O})$ by the standard Iwahori subalgebra and $G(\mathcal{O})$ by the Iwahori subgroup $\mathbf{I}$. This gives a Springer-theoretic construction of the "one-step" shift bimodule ${}_{i-1}\mathcal{A}_i$. On the level of affine Springer fibers, the analogous geometry is discussed in Section 7.4.

Let $ev_0^{-1}(\mathfrak{b}) = \mathfrak{i}$ be the standard Iwahori subalgebra. Then $ev_0^{-1}(0) \subset \mathfrak{i}$. Consider the sequence of subalgebras

$$\mathfrak{g}_0 := \mathfrak{g}(\mathcal{O}) \supset \mathfrak{g}_{1/2} := \mathfrak{i} \supset \mathfrak{g}_1 := t\mathfrak{g}(\mathcal{O}) \supset t\mathfrak{i} \supset t^2\mathfrak{g}(\mathcal{O}) \supset t^2\mathfrak{i} \supset \cdots$$

Then, as k-vector spaces (but importantly, *not* as Lie algebras) we have the subquotients

$$\mathfrak{g}_i/\mathfrak{g}_{i+1/2} \cong \begin{cases} \mathfrak{b}, & i \in \mathbb{Z} + 1/2 \\ \mathfrak{n}_-, & i \in \mathbb{Z} \end{cases}$$

Example 6.25. For $G = SL_2$ we have

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{m} & \mathcal{O} \end{pmatrix} \supset \begin{pmatrix} \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} \end{pmatrix} \supset \begin{pmatrix} \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m}^2 & \mathfrak{m} \end{pmatrix} \supset \cdots$$

so that $\mathfrak{g}_{1/2}/\mathfrak{g}_1 \cong \mathfrak{b},\ \mathfrak{g}_1/\mathfrak{g}_{3/2} \cong \mathfrak{n}_-$.

6.5.1. *Bimodules.* Consider now the spaces

$${}_j\widetilde{\mathcal{R}}_i := \{[g,s] \in G_{\mathcal{K}} \times^{\mathbf{I}} \mathfrak{g}_i | gs \in \mathfrak{g}_j\},\ i \in \frac{1}{2}\mathbb{Z}$$

And

$${}_j\mathcal{R}_i := \{[g,s] \in G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathfrak{g}_i | gs \in \mathfrak{g}_j\},\ i \in \mathbb{Z}.$$

Proposition 6.26. *Let $[\mathfrak{b}/B] \xrightarrow{r} [\mathfrak{g}/G]$ be the Grothendieck-Springer resolution. Then for $i \in \mathbb{Z}$ we have the cartesian diagrams*

$$\begin{array}{ccc} {}_{i+1/2}\widetilde{\mathcal{R}}_{i+1/2} & \xrightarrow{\psi} & [\mathfrak{b}/B] \\ \downarrow & & \downarrow \\ {}_i\mathcal{R}_i & \xrightarrow[\phi]{} & [\mathfrak{g}/G] \end{array}$$

In particular,

$${}_i\mathcal{A}_i \cong \mathbf{e}\ {}_{i+1/2}\mathcal{A}_{i+1/2}\ \mathbf{e}$$

by Springer theory. On the other hand, it is easy to see that

$$\phi^{-1}(0) = \{[g,s] \in G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathfrak{g}_i | gs \in t\mathfrak{g}_i\} = {}_{i+1}\mathcal{R}_i$$

and

$$(r \circ \psi)^{-1}(0) = \{[g,s] \in G_{\mathcal{K}} \times^{\mathbf{I}} \mathfrak{g}_{i+1/2} | gs \in \mathfrak{g}_{i+1}\} = {}_{i+1}\widetilde{\mathcal{R}}_{i+1/2}.$$

In particular,

$${}_{i+1}\mathcal{A}_i = \mathbf{e}_-\ {}_{i+1/2}\widetilde{\mathcal{A}}_{i+1/2}\Delta\ \mathbf{e} = \mathbf{e}\ {}_{i+1}\widetilde{\mathcal{A}}_{i+1/2}\ \mathbf{e}.$$

From what we have before,

$${}_i\widetilde{\mathcal{A}}_i$$

is the trigonometric Cherednik algebra when $i \in 1/2 + \mathbb{Z}$. The algebra for $i \in \mathbb{Z}$ is **not** a Cherednik algebra, but indeed a matrix algebra over the spherical Cherednik algebra, as in [86, 85].

7. Generalized affine Springer theory

7.1. **Generalized affine Springer fibers.** In this section we generalize the Springer action from [26, 42] to the line operators discussed above. Let $\mathbf{P}$ be a parahoric subgroup, $N_{\mathbf{P}}$ be a lattice in $N_{\mathcal{K}}$ stable under $\mathbf{P}$. Given this data, denote $\eta = (\mathbf{P}, N_{\mathbf{P}})$. Further, suppose that

$$1 \to G \to \widetilde{G} \to G_F \to 1$$

is an extension of algebraic groups and that $\widetilde{\mathbf{P}}$ is a parahoric subgroup of $\widetilde{G}_{\mathcal{K}}$ which fits into an extension

$$1 \to \mathbf{P} \to \widetilde{\mathbf{P}} \to (G_F)_{\mathcal{O}} \to 1$$

so that $\widetilde{\mathbf{P}} \cap G_{\mathcal{K}} = \mathbf{P}$. Let $\widetilde{G}_{\mathcal{K}}^{\mathcal{O}}$ be the preimage in $\widetilde{G}_{\mathcal{K}}$ of $(G_F)_{\mathcal{O}}$.

Definition 7.1. Let $v \in N_{\mathcal{K}}$. The *generalized affine Springer fiber* of v is the ind-subscheme of $\mathrm{Fl}_{\mathbf{P}}$ defined by

$$_\eta M_v := \{g \in \mathrm{Fl}_{\mathbf{P}} \, | g^{-1}.v \in N_{\mathbf{P}}\}.$$

Remark 7.2. Recall that if $N = \mathrm{Ad}$, $\mathbf{P}$ is a fixed parahoric subgroup, $N_{\mathbf{P}} = \mathrm{Lie}(\mathbf{P})$, $_\eta M_\gamma = \mathrm{Sp}_\gamma^{\mathbf{P}}$, the classical affine Springer fiber for $\mathbf{P}$.

Definition 7.3. The *orbital variety* of $\gamma \in N_{\mathcal{K}}$ and $\eta = (\mathbf{P}, N_{\mathbf{P}})$ is

$$_\eta \mathbb{O}_\gamma := \widetilde{G}_{\mathcal{K}}.\gamma \cap N_{\mathbf{P}}.$$

Remark 7.4. Note that the orbital variety only depends on the lattice $N_{\mathbf{P}}$. However, we always use it in conjunction with $\mathbf{P}$, explaining the slightly redundant notation with η.

In particular, we have

Lemma 7.5 ([26, 42])**.** *We have an isomorphism of stacks* $[_\eta \mathbb{O}_\gamma / \mathbf{P}] \cong [L_\gamma \backslash _\eta M_\gamma]$, *where* L_γ *is the stabilizer of* γ *in* $G_{\mathcal{K}}$.

Lemma 7.6. *Suppose* $_\eta M_\gamma$ *is finite-dimensional over* $\mathbb{C}$. *Then* $_\eta M_\gamma$ *admits a* L_γ*-equivariant dualizing complex* $\omega_{_\eta M_\gamma}$, *and the equivariant Borel-Moore homology* $H_*^{H_\gamma}(_\eta M_\gamma) =: H_*^{\mathbf{P}}(H_\gamma \backslash X_\gamma)$ *is well-defined for any algebraic subgroup* $H_\gamma \subseteq L_\gamma$. *Here* $X_\gamma = \{g \in \widetilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times | g^{-1}.\gamma \in N_{\mathbf{P}}\}$.

Proof. We can approximate M_γ by finite-type H_γ-stable varieties, and then take the colimit. □

Example 7.7. Suppose $N = \mathrm{Ad}$, and γ is split regular semisimple. Then L_γ is a split maximal torus in $G_{\mathcal{K}}$, in particular the loop group of a split maximal torus $T \subset G$. Let $H_\gamma \subset L_\gamma$ be the subgroup of "constant loops" of this torus. $H_*^{H_\gamma}(\mathrm{Sp}_\gamma)$ is studied e.g. in [51, 30, 1, 15].

Remark 7.8. Note that generically, L_γ is a torus in $G_{\mathcal{K}}$. We may extend the setup to the flavor-deformed equivariant version by considering $_\eta \widetilde{\mathbb{O}}_\gamma := \widetilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times.\gamma$ and and its quotient by $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times$ instead. We leave constructions of these extended notions to the reader or refer to [26].

Suppose that $\widetilde{G} = G \times \mathbb{C}^*$, so $G_F = \mathbb{C}^*$ is the flavor group above. The group $\widetilde{G}$ acts on N via $v \mapsto hg^{-1}\gamma g$. We denote the resulting GASF

$$\underline{M}_\gamma = \widetilde{G}_{\mathcal{K}}.\gamma \cap N_{\mathbf{P}} / \widetilde{\mathbf{P}}$$

Since $[h] = [t^d] \in X^*(\mathbb{C}^*) = \mathbb{Z}$ we see that $\underline{M}_\gamma$ splits into components

$$M_{t^{-d}\gamma} \cong \{g \in \mathrm{Gr}_G \, | t^d g^{-1}\gamma g = g^{-1} t^d \gamma g \in \mathfrak{g}(\mathcal{O})\} = \mathrm{Sp}_{t^{-d}\gamma}.$$

We recognize this to be the affine Springer fiber of $t^{-d}\gamma$ in $\mathrm{Fl}_{\mathbf{P}}$, or in other words that

$$\underline{M}_\gamma = \bigsqcup_{d \in \mathbb{Z}} M_{t^d \gamma}.$$

7.2. **Springer action from the Coulomb perspective.** We now define the Coulomb branch version of the Springer action, and in particular the geometric action of our $\mathbb{Z}$-algebra.

Theorem 7.9. *The following convolution diagram defines naturally associative maps*

$$H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{\mathrm{rot}}}(_\eta \mathcal{R}_{\eta'}) \otimes H_*^{L_\gamma}(_{\eta'} M_\gamma) \to H_*^{L_\gamma}(_\eta M_\gamma).$$

$$\begin{array}{ccccccc} _\eta\mathcal{R}_{\eta'} \times {_{\eta'}}\mathbb{O}_\gamma & \xleftarrow{p} & p^{-1}(_\eta\mathcal{R}_{\eta'} \times {_{\eta'}}\mathbb{O}_\gamma) & \xrightarrow{q} & q(p^{-1}(_\eta\mathcal{R}_{\eta'} \times {_{\eta'}}\mathbb{O}_\gamma)) & \xrightarrow{m} & _\eta\mathbb{O}_\gamma \\ \downarrow i & & \downarrow & & & & \\ G_{\mathcal{K}} \times_{\mathbf{P}'} N_{\mathbf{P}'} \times {_{\eta'}}\mathbb{O}_\gamma & \longleftarrow & G_{\mathcal{K}} \times {_{\eta'}}\mathbb{O}_\gamma & & & & \end{array}$$

Here $p : (g, s) \mapsto ([g, s], s)$, q *is the quotient by the diagonal action of* $\mathbf{P}'$ *and* m *is the map sending* $[g, s] \mapsto g.s$.

Proof. We explain the maps p, q, m induce in BM homology. Consider the space

$$_\eta \mathcal{P}_{\eta'} := \{(g, s) \in \widetilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times_{\mathrm{rot}} \times N_{\mathbf{P}'} | g^{-1}.s \in N_{\mathbf{P}}\}$$

and note there are maps $\pi_1 : {_\eta}\mathcal{P}_{\eta'} \to N_{\mathbf{P}'}$ and $(g, s) \mapsto s$ and $\pi_2 : {_\eta}\mathcal{P}_{\eta'} \to N_{\mathbf{P}}$ given by $(g, s) \mapsto g^{-1}.s$. Then consider $\mathcal{F}^v_{\mathbf{P}, N_{\mathbf{P}}} := \omega_{_{\eta'}\mathbb{O}_\gamma}[-2 \dim \widetilde{\mathbf{P}}]$, which is an object in the $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{\mathrm{rot}}$-equivariant derived category of $N_{\mathbf{P}}$.

First of all, we have the "pull-back with support" map p^* (see [9, Section 3(ii)])

$$
\begin{aligned}
p^* &: H^{-*}_{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{\mathrm{rot}}\times\widetilde{\mathbf{P}}'\rtimes\mathbb{C}^\times_{\mathrm{rot}}}({}_\eta\mathcal{R}_{\eta'}\times N_{\mathbf{P}}, (\omega_{{}_\eta\mathcal{R}_{\eta'}})\boxtimes({}_{\eta'}\mathcal{F}_\gamma)) \\
&= H_*^{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{\mathrm{rot}}}({}_\eta\mathcal{R}_{\eta'})\otimes H_*^{\widetilde{\mathbf{P}}'\rtimes\mathbb{C}^\times_{\mathrm{rot}}}({}_{\eta'}\mathbb{O}_\gamma)\to H^*_{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{\mathrm{rot}}\times\widetilde{\mathbf{P}}'\rtimes\mathbb{C}^\times_{rot}}({}_\eta\mathcal{P}_{\eta'},\pi_1^!({}_{\eta'}\mathcal{F}_\gamma)).
\end{aligned}
\tag{26}
$$

Further, we have a map $\pi_1^!{}_{\eta'}\mathcal{F}_\gamma\to\pi_2^!{}_\eta\mathcal{F}_\gamma$ and since $\pi_2=m\circ q$, we get

$$q_*: H^*_{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{\mathrm{rot}}\times\widetilde{\mathbf{P}}'\rtimes\mathbb{C}^\times_{\mathrm{rot}}}({}_\eta\mathcal{P}_{\eta'},\pi_1^!{}_{\eta'}\mathcal{F}_\gamma)\to H^*_{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{\mathrm{rot}}}(q({}_\eta\mathcal{P}_{\eta'}), m^!{}_\eta\mathcal{F}_\gamma)$$

Finally, m is (ind-)proper because its fibers are closed subvarieties of a partial affine flag variety, so that using the adjunction $m_!m^!\to\mathrm{id}$ we get a map

$$(m\circ q)_*: H_*^{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{\mathrm{rot}}}(q({}_\eta\mathcal{P}_{\eta'}), m^!\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}})\to H_*^{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{\mathrm{rot}}}({}_\eta\mathbb{O}_\gamma)=H_*^{L_\gamma}({}_\eta M_\gamma)$$

See [26, Theorem 4.5] and [42] for more details, for example the proof of associativity of the maps.

□

While the convolution diagram in Theorem 7.9 is rather abstract and the maps in Borel-Moore homology involved are defined sheaf-theoretically, in easy cases it is possible to analyze the action as follows. Similar to [26, Section 4.2], we define the *Hecke stack* for γ,η which has $\mathbb{C}$-points

$${}_\eta\mathcal{R}^\gamma_{\eta'}(\mathbb{C})=\{(s_2,g,s_1)\in{}_\eta\mathbb{O}_\gamma\times G_\mathcal{K}\times{}_{\eta'}\mathbb{O}_\gamma|g.s_1=s_2\}/\mathbf{P}.$$

Here the quotient is by the action $h.(s_2,g,s_1)=(s_2,gh^{-1},hs_1)$. There is a natural Schubert stratification of ${}_\eta\mathcal{R}^\gamma_{\eta'}$ inherited from ${}_\eta\mathcal{R}_\eta$, where

$${}_\eta\mathcal{R}^\gamma_{\eta'}\hookrightarrow{}_\eta\mathcal{R}_{\eta'}$$

via $[s_2,g,s_1]\mapsto[g,s_1]$. Similarly we have maps

$$
\begin{array}{ccc}
& {}_\eta\mathcal{R}^\gamma_{\eta'} & \\
\swarrow & & \searrow \\
{}_\eta M_\gamma & & {}_{\eta'}\mathbb{O}_\gamma
\end{array}
\tag{27}
$$

We will use this diagram later on in our computation of certain shift maps.

In the adjoint case, the name "Springer action" is warranted, as it coincides with the action defined by Yun, Oblomkov-Yun [72] (and Varagnolo-Vasserot [82]):

Theorem 7.10. *Let $N=\mathrm{Ad}$ and $\eta=(\mathbf{I},N_\mathbf{I})=\eta'$. Then the action of the algebra $\widetilde{\mathcal{A}}_{G,\mathbf{I}}$ on $H_*^{L_\gamma}(M_\gamma)$ defined by Theorem 7.9 coincides with the one defined in* [72] *on the equivariant homology of affine Springer fibers, under the isomorphism of Theorem 6.1*

Proof. Theorem 6.1 shows that the Springer action of simple reflections in the affine Weyl group is the same. The equivariant parameters act by Chern classes of line bundles on the affine flag variety, and that the relations are the same follows from Theorem 6.1. □

The novel feature in allowing arbitary η,η' shows the following.

Corollary 7.11. *The convolution product in Theorem 7.9 gives maps*

$${}_j\mathcal{A}_i^\hbar\otimes H_*^{L_\gamma}(M_{t^i\gamma})\to H_*^{L_\gamma}(M_{t^j\gamma})$$

that naturally assemble into an action of the $\mathbb{Z}$-algebra $B^\hbar=\bigoplus_{i\le j}{}_j\mathcal{A}_i^\hbar$ Moreover, the action in Theorem 5.6 not including loop rotation, i.e. setting $\hbar=0$, defines maps

$$H_*^{G_\mathcal{O}}(\mathcal{R}^d_{\widetilde{G},N})\times H_*^L(M_\gamma^{d'})\to H_*^L(M_\gamma^{d+d'}).$$

In particular, the above corollary gives a geometric construction of "column vector" modules for our geometric $\mathbb{Z}$-algebra $B=\bigoplus_{i\le j\le 0}{}_i\mathcal{A}_j$.

7.3. **The adjoint case.** In the case $N = \mathrm{Ad}$, the construction of these affine Springer theoretic modules is also closely related to the construction of a commutative (partial) resolution as in the previous sections, in the following way. For $G = GL_n$, by the results of [11], the commutative limit ${}_{i+d}\mathcal{A}_i^{\hbar=0}$ is identified with the sections of $\mathcal{O}(d)$ on the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$. In particular,

$$\mathrm{Proj} \bigoplus_{d\geq 0} {}_{i+d}\mathcal{A}_i^{\hbar=0} \cong \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}).$$

In general, we have the following proposition.

Proposition 7.12. *Let $\hbar = 0$. Then every $\gamma \in N_{\mathcal{K}}$ as in Theorem 7.9 gives a quasicoherent sheaf $\mathcal{F}_\gamma$ on the partial resolution of the Coulomb branch given by*

$$\mathrm{Proj} \bigoplus_{d\geq 0} {}_{i+d}\mathcal{A}_i^{\hbar=0}.$$

Corollary 7.13. *When $G = GL_n$, the above construction gives a quasicoherent sheaf on $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$ associated to $\gamma \in N_{\mathcal{K}}$.*

It is in general hard to compute which sheaf this is. In all examples we have checked, this should be a coherent sheaf for regular semisimple elements. This is a conjecture that we discuss in Section 8.

Recall that Lemma 2.13 tells us that the support of $\mathcal{F}_\gamma$ is determined by H_γ, i.e. the equivariance we consider, as well as the splitting type of γ.

Example 7.14. When $\gamma = zt^d$ as above, we get twists of the "Procesi bundle" as shown in [51] which are supported everywhere. See Proposition 8.11 for the precise statement.

When γ is elliptic, these sheaves are supported on the punctual Hilbert scheme over $(1,0) \in \mathbb{C}^\times \times \mathbb{C}$.

7.4. **Action on representations.** Let $\widetilde{\mathrm{Sp}}'_\gamma = \{g\mathbf{I} | g\gamma g^{-1} \in t\mathfrak{g}(\mathcal{O})\}$. Consider the Springer module $M_\gamma = H_*^{L_\gamma}(\mathrm{Sp}_\gamma)$. Then we have a natural map $M_\gamma \to M_{t\gamma}$ given by inclusion. There are also maps

$$\widetilde{M_\gamma} \to \widetilde{M}'_{t\gamma} \to \widetilde{M}_{t\gamma}$$

given by inclusion, or in other words convolving with the identity or point class in ${}_{1/2}\widetilde{\mathcal{A}}_1$ and then by the point class in ${}_1\widetilde{\mathcal{A}}_{3/2}$.

There are also natural Gysin maps $\widetilde{M}_{t\gamma} \to \widetilde{M}'_{t\gamma} \to \widetilde{M}_\gamma$. The first one is codimension zero, and the second one codimension $\dim G/B$. Composed, on the level of equivariant parameters, these look like the **square** of the Vandermonde determinant Δ. The issue arises from the normalization in the embedding to difference-reflection operators, in which the point class of ${}_1\widetilde{\mathcal{A}}_{3/2}$ is naturally identified with Δ (and a Cartan part), but acts as the identity on components (effectively, it cuts down the tangent spaces of the components by Δ).

Note also we have **two** projections

$$\widetilde{\mathrm{Sp}}'_{t\gamma} \to \mathrm{Sp}_\gamma,\ \widetilde{\mathrm{Sp}}_\gamma \to \mathrm{Sp}_\gamma$$

the first one of which is a fibration, and the second one has fibers which are usual Springer fibers (stratified fibration). Effectively, the two line operators (for $(1/2,1)$ and $(1,3/2)$) in the flags get squeezed down to a single one on the spherical level (the one for $(0,1)$).

Lemma 7.15. *We have $\dim \mathrm{Sp}_{t\gamma} = \dim \mathrm{Sp}_\gamma + \dim G/B$.*

Proof. By a result of Bezrukavnikov [6] the dimension of Sp_γ is given by

$$\dim \mathrm{Sp}_\gamma = \frac{1}{2}\left(\nu_{\mathrm{ad}}(\gamma) - \mathrm{rk}(\mathfrak{g}) + \dim(\mathfrak{h}^w)\right), \tag{28}$$

where $w \in W$ is such that $Z(\gamma)$ is of type w, $\mathfrak{h}^w$ denotes the w-invariants in $\mathfrak{h}$ and $\nu_{\mathrm{ad}}(\gamma)$ is the valuation of

$$\det\left(\mathrm{ad}\gamma : \mathfrak{g}(K)/Z(\gamma) \to \mathfrak{g}(K)/Z(\gamma)\right)$$

It is easy to see that changing γ to $t\gamma$ does not change w. The matrix $\mathrm{ad}\gamma$ is multiplied by t which changes $\nu_{\mathrm{ad}}(\gamma)$ by $|\Phi| = 2\dim G/B$, and the result follows. □

Lemma 7.16. *Let $\pi : \widetilde{\mathrm{Sp}}_{t\gamma} \to \mathrm{Sp}_{t\gamma}$ be the natural projection. If γ is elliptic then $\pi^{-1}(\mathrm{Sp}_\gamma)$ is an irreducible component of $\widetilde{\mathrm{Sp}}_{t\gamma}$. More generally, if γ is regular semisimple, and C is an irreducible component of Sp_γ, then $\pi^{-1}(C)$ is an irreducible component of $\widetilde{\mathrm{Sp}}_{t\gamma}$.*

Proof. By the proof of Lemma 2.2 the projection $\pi^{-1}(\mathrm{Sp}_\gamma) \to \mathrm{Sp}_\gamma$ has fibers G/B at every point. Since γ is elliptic, Sp_γ is irreducible and hence $\pi^{-1}(\mathrm{Sp}_\gamma)$ is irreducible as well. If C is as in the statement of the Lemma, the same proof goes through.

Furthermore, all components of $\widetilde{\mathrm{Sp}_{t\gamma}}$ have dimension $\dim \mathrm{Sp}_{t\gamma}$. By Lemma 7.15 we have

$$\dim \pi^{-1}(\mathrm{Sp}_\gamma) = \dim \mathrm{Sp}_\gamma + \dim G/B = \dim \mathrm{Sp}_{t\gamma},$$

and the result follows. □

Lemma 7.16 allows us to construct an important correspondence between Sp_γ and $\mathrm{Sp}_{t\gamma}$. By the work of Tsai [81], there are W many irreducible components up to the centralizer action in $\widetilde{\mathrm{Sp}}_{t\gamma}$. Furthermore, we expect the following:

Conjecture 7.17. *([81, Conjecture 8.6]) The Springer action on $H_*(\widetilde{\mathrm{Sp}}_\gamma)$ yields a regular representation in top-dimensional homology spanned by the classes of the irreducible components.*

In particular, for γ elliptic there is a distinguished component $\pi^{-1}(\mathrm{Sp}_\gamma)$ and another component biregular to $\mathrm{Sp}_{t\gamma}$, and (assuming Conjecture 7.17) one can define a correspondence in Borel-Moore homology sending the former to the latter (for example, the symmetrizer $\mathbf{e}$ would suffice). More generally, fix a component C as above and note that the lattice part of the centralizer of γ acts transitively on the set of these components [47]. Now the class of $\pi^{-1}(C)$ can either be sent to the class of any lattice translate of $\pi^{-1}(C)$, or by the symmetrizer in the finite Weyl group to a one-dimensional W-invariant subspace of the BM homology of $\mathrm{Sp}_{t\gamma}$.

This leads to the following:

Proposition 7.18. *Assume γ is elliptic and G is simply-connected, and assume Conjecture 7.17 holds for γ. Consider the correspondence $\mathbf{e}[\pi^{-1}(\mathrm{Sp}_\gamma)] * -$ between Sp_γ and $\mathrm{Sp}_{t\gamma}$. The action of this correspondence in homology corresponds to the action of some class in ${}_{i+1}\mathcal{R}_i$ as in Theorem 7.9, which sends the fundamental class of Sp_γ to the fundamental class of $\mathrm{Sp}_{t\gamma}$.*

Proof. Lets construct a cycle Γ' in ${}_{i+1}\mathcal{R}_i^\gamma$ such that correspondence (27) with the class Γ' sends $[\mathrm{Sp}_\gamma]$ to $[\mathrm{Sp}_{t\gamma}]$. First, we define $\Gamma \subset {}_{i+1}\mathcal{R}_i^\gamma$ as the lift of the graph of the embedding of Sp_γ into $\mathrm{Sp}_{t\gamma}$. The lift Γ is defined as the locus of triples $(s_2, g, s_1) \in {}_{i+1}\mathcal{R}_i^\gamma$ such that $G_\mathcal{O}.s_1 = G_\mathcal{O}.s_2$.

Let $\eta = (G_\mathcal{O}, t^i\mathfrak{g}(\mathcal{O}))$, $\eta' = (G_\mathcal{O}, t^{i+1}\mathfrak{g}(\mathcal{O}))$ and $\widetilde{\eta} = (\mathbf{I}, t^i\mathfrak{g}(\mathcal{O}))$. In particular, ${}_{i+1}\mathcal{R}_i^\gamma = {}_\eta\mathcal{R}_{\eta'}^\gamma$ and on the homology of the fibers of the projection $\widetilde{\pi} : {}_\eta\mathcal{R}_{\widetilde{\eta}}^\gamma \to {}_{i+1}\mathcal{R}_i^\gamma$ there is an action of W. The push-forward along the projection $\widetilde{\pi}$ is the projection onto the W-invariant part of the homology.

Let $\widetilde{q} : {}_\eta\mathcal{R}_{\widetilde{\eta}}^\gamma \to {}_{\widetilde{\eta}}\mathbb{O}_\gamma$ be map from the corresponding diagram (27). The previous proposition implies that the map $\widetilde{q}$ restricted to $\widetilde{\Gamma} = \widetilde{\pi}^{-1}(\Gamma)$ is dominant over one of irreducible component of ${}_{\widetilde{\eta}}\mathbb{O}_\gamma$. By Conjecture 7.17 the set of irreducible components of ${}_{\widetilde{\eta}}\mathbb{O}_\gamma$ is a regular representation of W. Thus there is $w \in W$ such $w.[\widetilde{\Gamma}]$ projects dominantly onto ${}_{\eta'}\mathbb{O}_\gamma$. The class $\Gamma' = \widetilde{\pi}_*(w.[\widetilde{\Gamma}])$ satisfies required properties.

□

Corollary 7.19. *Under the assumption of the previous proposition we have the relation between the fundamental classes:*

$$[\mathrm{Sp}_{t^{i+1}\gamma}] \in {}_{-i-1}\mathcal{A}_{-i} * [\mathrm{Sp}_{t^i\gamma}]$$

8. Finite generation and examples

8.1. **Finite generation conjecture.** As we saw in Section 7, in particular Theorem 7.9, the space

$$\mathbf{F}_\gamma := \bigoplus_{k=0}^{\infty} H_*(\mathrm{Sp}_{t^k\gamma})$$

is a graded module over the graded algebra $\bigoplus_{d=0}^\infty {}_0\mathcal{A}_d$. Equivalently, $\mathbf{F}_\gamma$ defines a quasi-coherent sheaf $\mathcal{F}_\gamma$ on $\mathrm{Proj} \bigoplus_{d=0}^\infty {}_0\mathcal{A}_d$.

Conjecture 8.1. *The module $\mathbf{F}_\gamma$ is finitely generated and the sheaf $\mathcal{F}_\gamma$ is coherent.*

Note that by Theorem 2.12 the homology of $\mathrm{Sp}_{t^k\gamma}$ is finitely generated over ${}_0\mathcal{A}_0$. For $G = GL_n$ the graded algebra $\bigoplus_{d=0}^\infty {}_0\mathcal{A}_d = \bigoplus_{d=0}^\infty A^d$ is generated by ${}_0\mathcal{A}_0$ and ${}_0\mathcal{A}_1 = A$, so Conjecture 8.1 is equivalent to saying that for a given γ there exists k_0 such that $\mathbf{F}_\gamma$ is generated by $\bigoplus_{k=0}^{k_0} H_*(\mathrm{Sp}_{t^k\gamma})$ under the action of ${}_0\mathcal{A}_0$ and ${}_0\mathcal{A}_1$.

Below we prove the conjecture in some special cases.

Theorem 8.2. *Conjecture 8.1 holds for $G = GL_n$ and $\gamma = \mathrm{diag}(s_0, \dots, s_n)$ for $s_i \neq s_j$.*

Proof. This follows from Proposition 8.11 below. □

Example 8.3. Let $G = GL_2$ and

$$\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}.$$

Then

$$\mathrm{Sp}_{t^d\gamma} \cong \bigsqcup_{\mathbb{Z}} C_{d+1}$$

where each C_{d+1} is an infinite chain of $\mathbb{P}^{d+1}$, consecutive members of which intersect transversally along a $\mathbb{P}^d$. These $\mathbb{P}^d$ are Spaltenstein varieties of d-planes in $2d$-space stable under a nilpotent element with Jordan blocks of sizes (d, d), motivically equivalent to projective spaces $\mathbb{P}^d$. The inclusion maps are again embedding the chains into one another and they are regular embeddings (because they are effective Cartier divisors).

Note that the direct sum of homologies of these $\mathbb{P}^{d+1}$ surjects onto the homology of $\mathrm{Sp}_{t^d\gamma}$. Let us prove that the module $\mathbf{F}_\gamma$ is generated by the homology of Sp_γ under the action of ${}_0\mathcal{A}_0$ and ${}_0\mathcal{A}_1$.

Indeed, Sp_γ is just a discrete set of points in bijection with the affine Weyl group. Its homology is a free rank two module over the lattice of translations.

In case of elliptic γ the finite generation conjecture follows from the stable of ${}_0\mathcal{A}_0$-cyclicity of the homology of $\mathrm{Sp}_{t^k\gamma}$:

Proposition 8.4. *Let us assume that γ is elliptic, Conjecture 7.17 holds for $t^k\gamma$ and $[\mathrm{Sp}_{t^k\gamma}] \in H_*(\mathrm{Sp}_{t^k\gamma})$ is the fundamental class. Then if there exists N such that $H_*(\mathrm{Sp}_{t^k\gamma}) = \mathbb{C}[T^*T^\vee]^W \cdot [\mathrm{Sp}_{t^k\gamma}]$ for $k \geq N$ then the conjecture 8.1 holds for $\mathbf{F}_\gamma$.*

Proof. The actions of ${}_0\mathcal{A}_0 = \mathbb{C}[T^*T^\vee]^W$ and ${}_0\mathcal{A}_1$ on $\mathbf{F}_\gamma$ commute. Hence by Proposition 7.18 the submodule $\oplus_{k\geq N} H_*(\mathrm{Sp}_{t^k\gamma})$ is generated by ${}_0\mathcal{A}_0$ and ${}_0\mathcal{A}_1$ from $[\mathrm{Sp}_{t^N\gamma}]$. The module $\oplus_{k<N} H_*(\mathrm{Sp}_{t^k\gamma})$ is finite dimensional. □

The subalgebra $\mathbb{C}[\mathfrak{t}]^W$ is isomorphic to the cohomology ring $H^*(\mathrm{Gr}_G)$ it acts on $H_*(\mathrm{Sp}_\gamma)$ by cap product. It is natural conjecture that a stronger version of the condition of the previous proposition is true for $G = PGL_n$.

Conjecture 8.5. *Let $\mathfrak{g} = \mathrm{Lie}(PGL_n)$ and $\gamma \in \mathfrak{g}(\mathcal{O})$ is an elliptic regular semisimple topologically nilpotent element. Then*

$$H_*(\mathrm{Sp}_\gamma) = H^*(\mathrm{Gr}_G) \cap [\mathrm{Sp}_\gamma].$$

If $G = GL_n$ or $G = SL_n$ and $\gamma \in \mathfrak{g}(\mathcal{O})$ is an elliptic element then Sp_γ has many connected components and the group $\pi_0(G_\gamma)$ permutes the connected components. In the light of aforementioned Theorem 2.12 it is natural to propose

Conjecture 8.6. *Let $\mathfrak{g} = \mathrm{Lie}(GL_n)$ or $\mathfrak{g} = \mathrm{Lie}(SL_n)$ and $\gamma \in \mathfrak{g}(\mathcal{O})$ is an elliptic regular semisimple topologically nilpotent element. Then*

$$H_*(\mathrm{Sp}_\gamma) = \mathbb{C}[T^*T^\vee]^W[\mathrm{Sp}_\gamma].$$

Remark 8.7. The conjecture is false outside of type A since there are examples of elliptic affine Springer fibers with homology of not of type (p, p) [47, 72]. Note however that $H_*(\mathrm{Sp}_\gamma)$ is always finitely generated under $\mathbb{C}[T^*T^\vee]^W$ by [89] (see also Lemma 2.13).

For the homogeneous elements Conjecture 8.6 is known [73] and one can deduce

Theorem 8.8. *Conjecture 8.1 holds for $G = GL_n$ and equivalued $\gamma_{m,n}$ with characteristic polynomial $x^m - y^n$,* $\gcd(m, n) = 1$.

Proof. The affine Grassmanian Gr_G has $\pi_1(GL_n) = \mathbb{Z}$ connected components $\mathrm{Gr}_G = \mathrm{Gr}_G^0 \times \mathbb{Z}$. Respectively, we have $\mathrm{Sp}_\gamma = \mathrm{Sp}_\gamma^0 \times \mathbb{Z}$.

Observe that if $\gamma_{m,n}$ is equivalued with characteristic polynomial $x^m - y^n$ then $t^k\gamma_{m,n}$ is equivalued with characteristic polynomial $x^{kn+m} - y^n$. The compactified Jacobian $J_{m/n}$ of the one-point compactification of the planar curve $\{x^m - y^n\}$ is irreducible and homeomorphic to $\mathrm{Sp}^0_{\gamma_{m,n}}$ [73]. Moreover, $\mathrm{Gr}_G^0 = \mathrm{Gr}_{PGL_n}$ and the $\mathrm{Sp}^0_{\gamma_{m,n}}$ is the corresponding Springer fiber.

It is shown in [73] that for $\mathrm{Sp}^0_{\gamma_{m,n}}$ Conjectures 7.17 and 8.5 are true. The group $\pi_0(G_{\gamma_{m,n}}) = \mathbb{Z}$ acts transitively on the connected components of Sp_γ, hence Conjecture 8.6 is true for $\mathrm{Sp}_{\gamma_{m,n}}$. Thus the theorem follows from Proposition 8.4.

□

Example 8.9. For $G = GL_2$ and $\gamma = \gamma_{1,2}$ we recover the $\mathbb{Z}$-algebra module from Example 4.17.

8.2. **Examples in type A.** In the case $G = GL_n$ the sheaf $\mathcal{F}_\gamma$ can be described in terms of geometry of $\mathrm{Hilb}_n(\mathbb{C}^\times \times \mathbb{C})$ for some homogeneous γ's.

Proposition 8.10. *Let γ be homogeneous of slope $(kn+1)/n$. Then $\mathcal{F}_\gamma$ is isomorphic to the restriction of $\mathcal{O}(k)$ to the punctual Hilbert scheme at $(1,0) \in \mathbb{C}^\times \times \mathbb{C}$.*

Proof. The localized equivariant homology $H_*^{\mathbb{G}_m}(\mathrm{Sp}_\gamma)$ affords the unique finite-dimensional representation of $\mathbf{e}\mathbb{H}_{-\frac{kn+1}{n}}\mathbf{e}$ as was checked in [72, 83]. By Lemma 2.13, $\mathcal{F}_\gamma$ is supported on this punctual Hilbert scheme (i.e. the corresponding fiber of the Hilbert-Chow map). Completing our $\mathbb{Z}$-algebra at a neighborhood of the identity in $T^\vee$, we get a completion of the rational Cherednik algebra and the corresponding bimodules, for GL_n with parameters given by integral shifts of $(kn+1)/n$. The Gordon-Stafford construction then implies [36] that the corresponding sheaf on the punctual Hilbert scheme coincides with $\mathcal{O}(k)$.

□

Proposition 8.11. *Let γ be homogeneous of slope k, or more generally equivalued of valuation k. Then $\mathcal{F}_\gamma$ is isomorphic to $\mathcal{P} \otimes \mathcal{O}(k)$ where $\mathcal{P}$ is the Procesi sheaf on* $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$.

Proof. For equivalued γ of valuation k the main result of [51] identifies the equivariant Borel-Moore homology $H_*^T(\mathrm{Sp}_\gamma)$ with the space of global sections of $\mathcal{P} \otimes \mathcal{O}(k)$ on $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$ as a module over the algebra of global functions

$$ {}_0\mathcal{A}_0 = \mathbb{C}[T^*T^\vee]^W = \mathbb{C}[x_1^\pm, \ldots, x_n^\pm, y_1, \ldots, y_n]^{S_n}, $$

By the work of Haiman [40] we get:

$$ H^0(\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}), \mathcal{P} \otimes \mathcal{O}(k)) = \bigcap_{i \neq j} \langle 1 - x_i/x_j, y_i - y_j \rangle^k, \quad H^i(\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}), \mathcal{P} \otimes \mathcal{O}(k)) = 0, \quad i > 0. \tag{29} $$

By Theorem 3.5 the graded algebra ${}_0\mathcal{A}_\bullet$ is generated by the degree 1 component ${}_0\mathcal{A}_1 = A$, where A is the space of antisymmetric polynomials in $\mathbb{C}[x_1^\pm, \ldots, x_n^\pm, y_1, \ldots, y_n]$.

It is easy to see that by (29) we have a correctly defined map

$$ A \otimes H^0(\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}), \mathcal{P} \otimes \mathcal{O}(k)) \to H^0(\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}), \mathcal{P} \otimes \mathcal{O}(k+1)), $$

and it follows from [51] that this agrees with the convolution ${}_0\mathcal{A}_1 \otimes H_*^T(\mathrm{Sp}_\gamma) \to H_*^T(\mathrm{Sp}_{t\gamma})$. This completes the proof. □

For general γ elliptic of slope $\frac{m}{n}$, the situation is as follows. Whilst our construction gives a sheaf $\mathcal{F}_\gamma$, which is coherent by Proposition 8.4, we do not know how to identify this sheaf on $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C})$. Indeed, a variant of this problem already appears in [36, Problem 5.5.].

8.3. **Beyond type A.** For general G, both the computations of the cohomology of affine Springer fibers and the sheaves on $\widetilde{\mathfrak{C}}_{\check{G},\check{\mathfrak{g}}}$ are very complicated. It would be for example interesting to compute the sheaf one gets from the Bernstein-Kazhdan example of [47, Appendix]. Nevertheless, we have the following analogue of Proposition 8.11 for general G.

Theorem 8.12. *Let G be arbitrary and γ equivalued of valuation k. Then we have the isomorphism of graded modules*

$$ \mathbf{F}_\gamma = \bigoplus_{j=0}^{\infty} H_*^T(\mathrm{Sp}_{t^j\gamma}) \simeq \bigoplus_{j=0}^{\infty} \bigcap_{\alpha \in \Phi^+} \langle 1 - \alpha^\vee, y_\alpha \rangle^{k+j} $$

over the graded algebra $\bigoplus {}_0\mathcal{A}(0)_d \simeq \bigoplus \mathbf{e}_d \bigcap_{\alpha \in \Phi^+} \langle 1 - \alpha^\vee, y_\alpha \rangle^d$. As a consequence, the corresponding sheaves over $\mathrm{Proj} \bigoplus {}_0\mathcal{A}(0)_d = \widetilde{\mathfrak{C}}_{\check{G}}$ are isomorphic as well.

Proof. The proof is similar to Proposition 8.11. By the main result of [51] the isomorphism holds for each j separately on the level of modules over ${}_0\mathcal{A}(0)_0 \cong \mathbb{C}[T^*T^\vee]^W$. The comparison of the action of ${}_0\mathcal{A}(0)_d$ follows from Theorem 3.8 and the constructions in [51, 26]. More precisely, the result in [51] identifies $\Delta^j H_*^T(\mathrm{Sp}_{t^j\gamma})$ with $\bigcap_{\alpha \in \Phi^+} \langle 1 - \alpha^\vee, y_\alpha \rangle^{k+j}$ inside $\mathbb{C}[T^*T^\vee] \cong H_*^T(\mathrm{Gr}_T)$ using GKM localization. The latter has a multiplication structure which coincides with convolution on the Coulomb branch for T with zero matter. The fact that the convolution action for ${}_0\mathcal{A}(0)_0$ respects the localization is [26, Proposition 4.15.]. □

Let G be quasisimple of adjoint type. Respectively, let $cox \in W$ be the Coxeter element of the Weyl group of G and n be the order of cox. For any m co-prime with n there is a regular semisimple element $\gamma_{m,n} \in \mathfrak{g}(\mathcal{O})$ which is homogeneous: $\gamma_{m,n}(\lambda \cdot t) = \lambda^{m/n} \operatorname{Ad}_{g(\lambda)} \gamma_{m,n}(t)$, $g(\lambda) \in G$. The element $\gamma_{m.n}$ is unique up to rescaling and conjugation, an explicit construction of $\gamma_{m,n}$ can found for example in [72]. The element $\gamma_{m,n}$ is equivalued of valuation m/n.

The stabilizer in $\widetilde{G}_{\mathcal{K}} \rtimes \mathbb{G}_m$ is given by $L_{\gamma_{m,n}} = \mathbb{G}_m$, and it acts naturally on $\mathrm{Sp}_{\gamma_{m,n}}$. It is shown in [72] that $\dim \mathrm{Sp}^{\mathbb{G}_m}_{\gamma_{m,n}} = 0$, the fixed points are isolated and that the localized homology $H^{\mathbb{G}_m}_*(\mathrm{Sp}_{\gamma_{m,n}}) \otimes \mathbb{C}(\hbar)$ is generated by tautological classes $H^*_{\mathbb{G}_m}(\mathrm{Gr}_G)$ from the fundamental class $[\mathrm{Sp}_{\gamma_{m,n}}]$. We expect the generation statement in the non-equivariant setting:

Conjecture 8.13. *Let G,* $\mathrm{Lie}(\mathfrak{g})$,$\gamma_{m,n} \in \mathfrak{g}(\mathcal{O})$ *are as above, then*

$$H_*(\mathrm{Sp}_{\gamma_{m,n}}) = H^*(\mathrm{Gr}_G) \cap [\mathrm{Sp}_{\gamma_{m,n}}].$$

Note also that this "Coxeter case" gives the so called spherical simple modules of the dDAHA, as first observed in [83]. More generally, the slopes with so called regular elliptic denominators yield (spherical and other) finite-dimensional modules of the dDAHA [83, 72]. Since γ elliptic implies $t\gamma$ elliptic, one sees that the tensor products by the shift bimodules ${}_{i-1}\mathcal{B}_i$ send finite-dimensional modules to finite-dimensional modules, which one could also deduce from the theory of shift functors for dDAHA like in [4]. As far as the authors are aware, this theory is still undeveloped, but would potentially give insight on the $m = 1, n = h$ case of Proposition 8.10 for other groups.

REFERENCES

[1] Boixeda Alvarez, Pablo, and Ivan Losev. Affine Springer Fibers, Procesi bundles, and Cherednik algebras. arXiv preprint arXiv:2104.09543 (2021). With an appendix by Oscar Kivinen.

[2] Birkar, Caucher; Cascini, Paolo; Hacon, Christopher D.; McKernan, James. Existence of minimal models for varieties of log general type. J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.

[3] Alexis Bouthier, David Kazhdan, and Yakov Varshavsky. Perverse sheaves on infinite-dimensional stacks, and affine Springer theory. arXiv preprint arXiv:2003.01428 (2020).

[4] Berest, Yuri, Pavel Etingof, and Victor Ginzburg. Cherednik algebras and differential operators on quasi-invariants. Duke Mathematical Journal 118.2 (2003): 279–337.

[5] Berest, Yuri, Pavel Etingof, and Victor Ginzburg. Finite-dimensional representations of rational Cherednik algebras. International Mathematics Research Notices 2003.19 (2003): 1053–1088.

[6] Bezrukavnikov, Roman. The dimension of the fixed point set on affine flag manifolds. Math. Res. Lett. 3 (1996), no. 2, 185–189.

[7] Bezrukavnikov, Roman, Michael Finkelberg, and Ivan Mirković. Equivariant homology and K-theory of affine Grassmannians and Toda lattices. Compositio Mathematica 141.3 (2005): 746–768.

[8] Bondal, A. I.; Polishchuk, A. E. Homological properties of associative algebras: the method of helices. Russian Acad. Sci. Izv. Math. 42 (1994), no. 2, 219–260

[9] Braverman, Alexander, Michael Finkelberg, and Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. Adv. Theor. Math. Phys. 22 (2018), no. 5, 1071–1147.

[10] Braverman, Alexander, Michael Finkelberg, and Hiraku Nakajima. Coulomb branches of $3d\ \mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian (with appendices by Alexander Braverman, Michael Finkelberg, Joel Kamnitzer, Ryosuke Kodera, Hiraku Nakajima, Ben Webster, and Alex Weekes). Adv. Theor. Math. Phys. 23 (2019), no. 1, 75–166.

[11] Braverman, Alexander, Michael Finkelberg, and Hiraku Nakajima. Line bundles over Coulomb branches. arXiv preprint arXiv:1805.11826 (2018).

[12] Braverman, Alexander, Michael Finkelberg, and Hiraku Nakajima. Ring objects in the equivariant derived Satake category arising from Coulomb branches (with an appendix by Gus Lonergan). Adv. Theor. Math. Phys. 23 (2019), no. 2, 253–344. (2017).

[13] Brion, Michel. Poincaré duality and equivariant (co) homology. Michigan Mathematical Journal 48.1 (2000): 77-92.

[14] Bullimore, Mathew, Tudor Dimofte, and Davide Gaiotto. The Coulomb Branch of 3d $\mathcal{N} = 4$ Theories. Communications in Mathematical Physics 354.2 (2017): 671–751.

[15] E. Carlsson, A. Mellit. GKM spaces, and the signed positivity of the nabla operator. arXiv:2110.07591

[16] Chen, Zongbin. On the local behavior of weighted orbital integrals and the affine Springer fibers. arXiv preprint arXiv:2103.15091 (2021).

[17] Chen, Tsao-Hsien, Ngô Bao Chau. Invariant theory for the commuting scheme of symplectic Lie algebras. arXiv:2102.01849.

[18] Cherednik, Ivan. Double affine Hecke algebras. Vol. 319. Cambridge University Press, 2005.

[19] Cherkis, Sergey. Instantons on gravitons. Comm. Math. Phys. 306 (2011), no. 2, 449–483.

[20] Costello, Kevin; Creutzig, Thomas; Gaiotto, Davide. Higgs and Coulomb branches from vertex operator algebras. J. High Energy Phys. 2019, no. 3, 066, 48 pp.

[21] Dimofte, Tudor; Garner, Niklas; Geracie, Michael; Hilburn, Justin Mirror symmetry and line operators. J. High Energy Phys. 2020, no. 2, 075, 147 pp.

[22] Etingof, Pavel. Reducibility of the polynomial representation of the degenerate double affine Hecke algebra. arXiv preprint arXiv:0706.4308 (2007).

[23] Etingof, Pavel. Cherednik and Hecke algebras of varieties with a finite group action. Mosc. Math. J. 17 (2017), no. 4, 635–666.
[24] Galashin, Pavel, Hopkins, Sam, McConville, Thomas and Postnikov, Alexander. Root system chip-firing I: Interval-firing. Mathematische Zeitschrift 292.3 (2019): 1337–1385.
[25] Gan, Wee Liang, Ginzburg, Victor. Almost-commuting variety, $\mathcal{D}$-modules, and Cherednik algebras. With an appendix by Ginzburg. IMRP Int. Math. Res. Pap. 2006, 1–54.
[26] Garner, Niklas, and Oscar Kivinen. Generalized affine Springer theory and Hilbert schemes on planar curves. International Mathematics Research Notices, rnac038, DOI 10.1093/imrn/rnac038. arXiv preprint arXiv:2004.15024 (2020).
[27] Ginzburg, Victor. Isospectral commuting variety, the Harish-Chandra D-module, and principal nilpotent pairs. Duke Mathematical Journal 161.11 (2012): 2023–2111.
[28] Ginzburg, Victor, Iain Gordon, and John Tobias Stafford. "Differential operators and Cherednik algebras." Selecta Mathematica 14.3 (2009): 629-666.
[29] Ginzburg, Victor, and Dmitry Kaledin. Poisson deformations of symplectic quotient singularities. Advances in Mathematics 186.1 (2004): 1–57.
[30] Goresky, Mark, Robert Kottwitz, and Robert Macpherson. Homology of affine Springer fibers in the unramified case. Duke Mathematical Journal 121.3 (2004): 509–561.
[31] Gorsky, Eugene, Andrei Neguţ, and Jacob Rasmussen. Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology. Adv. Math. 378 (2021), Paper No. 107542, 115 pp.
[32] Gorsky, Eugene, Hogancamp, Matthew. Hilbert schemes and y-ification of Khovanov-Rozansky homology. To appear in Geometry & Topology. (2017), arXiv:1712.03938.
[33] Gorsky, E., Oblomkov, A., Rasmussen, J., Shende, V. Torus knots and the rational DAHA. Duke Mathematical Journal, 163, (2014) 2709–2794.
[34] Gorsky, Eugene, Simental, José and Monica Vazirani. Parabolic Hilbert schemes via the Dunkl-Opdam subalgebra. arXiv:2004.14873
[35] Gordon, Iain, and J. Toby Stafford. Rational Cherednik algebras and Hilbert schemes. Advances in Mathematics 198.1 (2005): 222–274.
[36] Gordon, Iain, and J. Toby Stafford. Rational Cherednik algebras and Hilbert schemes, II: Representations and sheaves. Duke Mathematical Journal 132.1 (2006): 73–135.
[37] Gordon, Iain. On the quotient ring by diagonal invariants. Inventiones Mathematicae 153.3 (2003): 503–518.
[38] Griffeth, Stephen. Unitary representations of cyclotomic rational Cherednik algebras." Journal of Algebra 512 (2018): 310–356.
[39] Etingof, Pavel, Ginzburg, Victor, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. Invent. Math. 147 (2002), no. 2, 243–348.
[40] Haiman, Mark. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. J. Amer. Math. Soc. 14 (2001), no. 4, 941—1006.
[41] Haiman, Mark. t, q-Catalan numbers and the Hilbert scheme. Discrete Mathematics 193.1 (1998): 201–224.
[42] Hilburn, Justin, Joel Kamnitzer, and Alex Weekes. BFN Springer theory. arXiv preprint arXiv:2004.14998 (2020).
[43] Heckman, Gerrit J. A Remark on the Dunkl Differential–Difference Operators. Harmonic analysis on reductive groups. Birkhäuser, Boston, MA, 1991. 181–191.
[44] Hogancamp, Matthew, David EV Rose, and Paul Wedrich. Link splitting deformation of colored Khovanov–Rozansky homology. arXiv preprint arXiv:2107.09590 (2021).
[45] Intriligator, K.; Seiberg, N. Mirror symmetry in three-dimensional gauge theories. Phys. Lett. B 387 (1996), no. 3, 513–519.
[46] Joseph, Anthony. On a Harish-Chandra homomorphism. Comptes Rendus de l'Académie des Sciences-Series I-Mathematics 324.7 (1997): 759–764.
[47] Kazhdan, David, and George Lusztig. Fixed point varieties on affine flag manifolds. Israel Journal of Mathematics 62.2 (1988): 129-168.
[48] Kawamata, Yujiro, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem."Algebraic geometry, Sendai, 1985. Mathematical Society of Japan, 1987.
[49] Kirillov, Alexander A., Jr. Lectures on affine Hecke algebras and Macdonald's conjectures. Bull. Amer. Math. Soc. (N.S.) 34 (1997), no. 3, 251–292.
[50] Kivinen, Oscar. Forthcoming.
[51] Kivinen, Oscar. Unramified affine Springer fibers and isospectral Hilbert schemes. Selecta Mathematica 26.4 (2020): 1-42.
[52] Kodera, Ryosuke, and Hiraku Nakajima. Quantized Coulomb branches of Jordan quiver gauge theories and cyclotomic rational Cherednik algebras. Proc. Symp. Pure Math. Vol. 98. 2018.
[53] Losev, Ivan. Deformations of symplectic singularities and Orbit method for semisimple Lie algebras. Selecta Math. (N.S.) 28 (2022), no. 2, Paper No. 30, 52 pp.
[54] Losev, Ivan. Derived equivalences for Symplectic reflection algebras. International Mathematics Research Notices 2021.1 (2021): 442–472.
[55] Losev, Ivan. Harish-Chandra bimodules over quantized symplectic singularities. Transformation Groups (2021): 1–36.
[56] Losev, Ivan. Almost commuting varieties for symplectic Lie algebras. arXiv:2104.11000.
[57] Lusztig, George. Affine Weyl groups and conjugacy classes in Weyl groups. Transformation Groups 1.1 (1996): 83–97.
[58] Maulik, Davesh. Stable pairs and the HOMFLY polynomial. Inventiones mathematicae 204.3 (2016): 787–831.
[59] A. Malkin, V. Ostrik, M. Vybornov. The minimal degeneration singularities in the affine Grassmannians. Duke Math. J. 126 (2005), no. 2, 233–249.
[60] Maulik, Davesh, Yun, Zhiwei. Macdonald formula for curves with planar singularities. Journal für die reine und angewandte Mathematik (Crelles Journal) 694 (2014), 27–48.
[61] Migliorini, Luca, Shende, Vivek. A support theorem for Hilbert schemes of planar curves. Journal of the European Mathematical Society, 15, (2013), (6), 2353–2367.
[62] Migliorini, Luca, Vivek Shende, and Filippo Viviani. A support theorem for Hilbert schemes of planar curves, II. Compositio Mathematica 157.4 (2021): 835–882.
[63] Mirković, Ivan, and Kari Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. Annals of mathematics 166.1 (2007): 95–143.

[64] Nakajima, Hiraku. Lectures on Hilbert schemes of points on surfaces. University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999.
[65] Nakajima, Hiraku and Yuuya Takayama. Cherkis bow varieties and Coulomb branches of quiver gauge theories of affine type A. Selecta Math. (N.S.) 23 (2017), no. 4, 2553–2633.
[66] Namikawa, Yoshinori. Poisson deformations of affine symplectic varieties. Duke Mathematical Journal 156.1 (2011): 51–85.
[67] Ngô, Bao Châu. Le lemme fondamental pour les algebres de Lie. Publications Mathématiques de l'IHÉS 111.1 (2010): 1-169.
[68] Oblomkov, Alexei, Rozansky, Lev. Knot homology and sheaves on the Hilbert scheme of points on the plane. Selecta Mathematica 24.3 (2018): 2351–2454.
[69] Oblomkov, Alexei, Rozansky, Lev.Soergel bimodules and matrix factorizations. arXiv:2010.14546
[70] Oblomkov, Alexei, Rasmussen, Jacob, Shende, Vivek. (2018). The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link, with Appendix by E. Gorsky. Geometry and Topology, 22(2), 645–691.
[71] Oblomkov, Alexei, Shende, Vivek. The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link. Duke Mathematical Journal, 161(7), (2012) 1277–1303.
[72] Oblomkov, Alexei, Yun, Zhiwei. Geometric Representations of Graded and Rational Cherednik Algebras. Advances in Mathematics, 292 (2016), 601 – 706.
[73] Oblomkov, Alexei, Yun, Zhiwei. The cohomology ring of certain compactified Jacobians, arXiv:1710.05391 (2017).
[74] Opdam, Eric M. Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. Compositio Mathematica 85.3 (1993): 333–373.
[75] Piontkowski, Jens. Topology of the compactified Jacobians of singular curves. Math. Z. 255 (2007), no. 1, 195–226.
[76] Rains, Eric M. "Elliptic Double Affine Hecke Algebras." Symmetry, Integrability and Geometry: Methods and Applications 16 (2020): 111-133.
[77] Shafarevich, I., Basic algebraic geometry I. Varieties in projective space. Second edition. Translated from the 1988 Russian edition and with notes by Miles Reid. Springer-Verlag, Berlin, 1994.
[78] Simental, José. Harish-Chandra bimodules over rational Cherednik algebras. Advances in Mathematics 317 (2017): 299–349.
[79] Suzuki, Takeshi. Rational and trigonometric degeneration of the double affine Hecke algebra of type A, Int. Math. Res. Not. 2005, no. 37, 2249–2262.
[80] Takayama, Yuuya Nahm's equations, quiver varieties and parabolic sheaves. Publ. Res. Inst. Math. Sci. 52 (2016), no. 1, 1–41.
[81] Tsai, Cheng-Chiang. Components of affine Springer fibers. Int. Math. Res. Not. IMRN 2020, no. 6, 1882–1919.
[82] Varagnolo, Michela, and Eric Vasserot. Double affine Hecke algebras and affine flag manifolds, I. Affine flag manifolds and principal bundles. Springer, Basel, 2010. 233-289.
[83] Varagnolo, Michela, and Eric Vasserot. (2009). Finite-dimensional representations of DAHA and affine Springer fibers: the spherical case. Duke Mathematical Journal, Duke Math. J. 147 (2009), no. 3, 439–540.
[84] Van den Bergh, Michel. Non-commutative crepant resolutions. The legacy of Niels Henrik Abel. Springer, Berlin, Heidelberg, 2004. 749–770.
[85] Webster, Ben. Koszul duality between Higgs and Coulomb categories O. arXiv preprint arXiv:1611.06541 (2016).
[86] Weekes, Alex. Generators for Coulomb branches of quiver gauge theories. arXiv preprint arXiv:1903.07734 (2019).
[87] Weekes, Alex. Quiver gauge theories and symplectic singularities. Advances in Mathematics 396 (2022) 108185.
[88] Yun, Zhiwei. Global Springer theory. Advances in Mathematics 228.1 (2011): 266–328.
[89] Yun, Zhiwei. The spherical part of the local and global Springer actions. Mathematische Annalen 359.3-4 (2014): 557–594.
[90] Yun, Zhiwei. Lectures on Springer theories and orbital integrals. Geometry of moduli spaces and representation theory, 155–215, IAS/Park City Math. Ser., 24, Amer. Math. Soc., Providence, RI, 2017.

DEPARTMENT OF MATHEMATICS, UC DAVIS
Email address: egorskiy@math.ucdavis.edu

INSTITUTE OF MATHEMATICS, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
Email address: oscar.kivinen@epfl.ch

DEPARTMENT OF MATHEMATICS, UMASS AMHERST
Email address: oblomkov@math.umass.edu