Solutions of the Riccati equation for H^∞ Discrete Time Systems

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Abstract

Let H and Y be separable Hilbert spaces, and U finite dimensional. Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(U,H)$, $C \in \mathcal{L}(H,Y)$, $D \in \mathcal{L}(U,Y)$, and suppose that the open loop transfer function $\mathcal{D}(z) := D + zC(I-zA)^{-1}B \in \mathrm{H}^\infty(\mathbf{D};\mathcal{L}(U,Y))$, where $\mathbf{D}$ is the open unit disk.

We consider a subset of self adjoint solutions P of the discrete time algebraic operator Riccati equation (DARE)

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

where $J = J^* \in \mathcal{L}(Y)$ is a cost operator, and $\Lambda_P^{-1} \in \mathcal{L}(U)$.

Under further assumptions, we obtain the following results. To solutions of the DARE, we associate a coanalytic-analytic factorization of the Popov function $\mathcal{D}(z)^*J\mathcal{D}(z)$. To each nonnegative solution of the DARE, we associate a (partial) inner-outer factorization of the transfer function $\mathcal{D}(z)$ (if $J \geq 0$). We conclude that the natural partial ordering of the (adjoints of the) inner factors of $\mathcal{D}_P(z)$ is consistent with the partial ordering of the solutions P, as self adjoint operators. We obtain a characterization of the critical solution as the maximal nonnegative solution (if $J \geq 0$). Finally, generalizations of these results are indicated.

1 Introduction

In this paper, we consider factorization and partial orderings of operator-valued H^∞ transfer functions, via solutions of an associated discrete time algebraic Riccati equation. This work is a presentation of results given in [9] and [10], where full proofs are given.

Let us first introduce some notions and definitions. The basic object of this work is an operator-valued H^∞ transfer function. A state space realization of this transfer function is a *discrete time linear system* (DLS) ϕ. It is given by the system of difference equations

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ y_j = Cx_j + Du_j, \quad j \geq 0, \end{cases}$$

where $u_j \in U$, $x_j \in H$, $y_j \in Y$, and A, B, C, D are bounded linear operators between appropriate (separable) Hilbert spaces. We call the ordered quadruple $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *a DLS in difference equation form.* The three Hilbert spaces are as follows: U is the input space, H is the state space and Y is the output space of ϕ.

There is also another equivalent *I/O form* for a DLS. It consists of four linear operators in the ordered quadruple

$$\Phi := \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}.$$

The operator $A \in \mathcal{L}(H)$ is the *semi-group generator* of Φ, and it is the same operator as in equation (1). $\mathcal{B} : \ell^2(\mathbf{Z}_-;U) \supset \mathrm{dom}(\mathcal{B}) \to H$ is the *controllability map* that maps the past input into the present state. $\mathcal{C} : H \supset \mathrm{dom}(\mathcal{C}) \to \ell^2(\mathbf{Z}_+;Y)$ is the *observability map* that maps the present state into the future outputs. The last operator $\mathcal{D} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ is the *I/O map* that maps the input into output in a causal and shift invariant way. The operator τ is the bilateral forward shift in $\ell^2(\mathbf{Z};U)$, and $\bar{\pi}_+$, π_- are the orthogonal projections to the future and past, respectively. We denote the same DLS in I/O-form by the capital letter Φ, and in difference equation form by ϕ.

If $\mathcal{B}$ or $\mathcal{C}$ is bounded, then we say that Φ is *input stable* or *output stable*, respectively. If $\mathcal{D}$ is bounded then Φ is I/O-stable and the transfer function $\mathcal{D}(z) \in \mathrm{H}^\infty(\mathcal{L}(U;Y))$. If $\lim_{j\to\infty} A^j x_0 = 0$ for all $x_0 \in H$, then the semi-group generator A is strongly stable.

Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a DLS, and let $J = J^* \in \mathcal{L}(Y)$ be a cost operator. The symbol $Ric(\Phi, J)$ denotes the associated *discrete time Riccati equation*, given by

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases} \tag{1}$$

where Λ_P^{-1} is required to be bounded. If $P = P^*$ solves $Ric(\Phi, J)$, then we write $P \in Ric(\Phi, J)$. Several subsets of the solution set $Ric(\Phi, J)$ are defined and studied in [9]. The operator Λ_P is the *indicator*, and K_P is the *feedback operator* of P.

For fixed Φ and J, two additional DLSs are associated to each solution $P \in Ric(\Phi, J)$, namely

$$\phi_P := \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}, \quad \phi^P := \begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix},$$

where $A_P := A + BK_P$, $C_P = C + DK_P$. We call the object ϕ_P (ϕ^P) the *lower DLS* (*respectively upper DLS*), centered at P. The algebraic and partial ordering properties of the solutions of the DARE are conveniently described with the aid of these DLSs, see [9]. The DLSs ϕ_P and ϕ^P are fundamental notions in the factorization theory, presented in this paper.

Some solutions of the DARE are more interesting that others.

Definition 1. *Let Φ is output stable and I/O-stable.*

(i) If $P \in Ric(\Phi, J)$ is such that the lower DLS ϕ_P is I/O-stable and output stable, then we call P an H^∞-solution, *and write $P \in ric(\Phi, J)$.*

(ii) If the residual cost operator $L_{A,P}$ exists and satisfies

$$L_{A,P} := \mathrm{s} - \lim_{j\to\infty} A^{*j}PA^j = 0,$$

then the strong residual cost condition *is satisfied, and we write $P \in Ric_0(\Phi, J)$.*

(iii) The set of regular solutions *is denoted by $ric_0(\Phi, J) := Ric_0(\Phi, J) \cap ric(\Phi, J)$.*

A solution $P^{\mathrm{crit}} \in Ric(\Phi, J)$ is *critical* if the transfer function $\mathcal{D}_{\phi_{P^{\mathrm{crit}}}}(z)$ of the lower DLS $\phi_{P^{\mathrm{crit}}}$ is in H^∞ together with its inverse, and a certain residual cost condition is satisfied. The equivalent conditions for the existence of such a P^{crit} are discussed in [6], under quite general assumptions. The I/O-map of the corresponding $\phi_{P^{\mathrm{crit}}}$ is the outer factor (with a bounded inverse) in the $(J, \Lambda_{P^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, induced by the critical P^{crit}. The solution P^{crit}, when it exists, can be replaced by another, regular critical $P_0^{\mathrm{crit}} \in ric_0(\Phi, J)$, given by $P_0^{\mathrm{crit}} = (\mathcal{C}^{\mathrm{crit}})^* J\mathcal{C}^{\mathrm{crit}}$. Here $\mathcal{C}^{\mathrm{crit}} := (I - \bar{\pi}_+\mathcal{D}(\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+)^{-1}\bar{\pi}_+\mathcal{D}^*J)\mathcal{C}$ is the critical closed loop observability map.

A detailed treatment of fundamental notions of DLSs (such as the state feedback structure and various stability notions) is given in [7]. For a less general, introductory presentation, see the introduction in [9]. For associated cost optimization problems, spectral factorization problems, and critical solutions of DARE under weak assumptions, see [5] and [6]. See also the early discrete time paper [3]. For factorizations of the Popov operator and the I/O-map via nonnegative solutions of an associated DARE, see [9] and [10]. Nonnegative solutions of CAREs are considered in [2] which has a considerable intersection with our work [10]; however, [2] contains deeper control theoretic considerations. See also the references in [2], in particular [1]. Related results for the continuous time *stable well-posed linear system* are given in [11], [12], [13], [14], [15]. For the theory of matrix CARE and DARE, see [4].

After these preliminary considerations, we continue to discuss this work. In Section 2, we consider the factorization of the Popov operator. Non-negativity of the cost operator J or solution P is not yet required. In Section 3, the I/O-map is inner-outer-factorized, by means of the nonnegative solutions of the DARE. Because of the Liapunov equation techniques, we now assume $J \geq 0$. Some control theoretic interpretation of the inner-outer factorization is given in Section 4. Moreover, this factorization has some order theoretic implications, considered in Section 5. In the final Section 6, we indicate various generalizations of these results.

To clarify the presentation, the following simplifying assumptions are used throughout this paper.

- The basic DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ is an output stable and I/O-stable DLS, such that $\mathrm{dom}(\mathcal{C}) := \{x \in H \mid \mathcal{C}x \in \ell^2(\mathbf{Z}_+; Y)\} = H$. The input space U is finite dimensional, and the output space Y is separable.
- The DARE $Ric(\Phi, J)$ has a non-negative critical regular solution $P_0^{\mathrm{crit}} = (\mathcal{C}^{\mathrm{crit}})^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$.

These assumptions can be significantly relaxed, as mentioned in the final section.

2 Factorization of the Popov operator

The Popov operator refers to the Toeplitz operator $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$ or the shift invariant operator $\mathcal{D}^*J\mathcal{D}$. We remark that the following factorization result does not require the cost operator $J = J^* \in \mathcal{L}(Y)$ to be nonnegative.

Theorem 2. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an I/O-stable and output stable DLS.*

(i) For each solution $P \in ric_0(\Phi, J)$, the Popov operator has the factorization

$$\mathcal{D}^*J\mathcal{D} = \mathcal{D}_{\phi_P}^*\Lambda_P\mathcal{D}_{\phi_P},$$

where ϕ_P is the lower DLS (of Φ and J), centered at P.

(ii) Assume, in addition that $\overline{\mathrm{range}(\mathcal{B})} = H$. Assume that the Popov operator has a factorization of form

$$\mathcal{D}^* J \mathcal{D} = \mathcal{D}^*_{\phi'} \Lambda \mathcal{D}_{\phi'},$$

where $\phi' := \left(\begin{smallmatrix} A & B \\ -K & I \end{smallmatrix}\right)$ is an I/O-stable and output stable DLS, with $K \in \mathcal{L}(H, U)$, and $\Lambda = \Lambda^*, \Lambda^{-1} \in \mathcal{L}(U)$. Then $\phi' = \phi_P$ and $\Lambda = \Lambda_P$ for some $P \in ric_0(\Phi, J)$.

If $J \geq 0$, we need not *a priori* assume that P is a H^∞-solution in claim (i) because then $ric_0(\Phi, J) = Ric_0(\Phi, J)$, see [9]. For an analogous but somewhat different discrete time result, see [3, Theorem 4.6].

3 Factorization of the I/O-map

In this section, we assume that the cost operator J is nonnegative. The operator $\mathcal{N}_P$ denotes the $(\Lambda_P, \Lambda_{P^{\mathrm{crit}}})$-inner factor of $\mathcal{D}_{\phi_P}$. Its existence follows from the assumed existence of a critical solution of the DARE $ric(\Phi, J)$, as shown in [9].

Theorem 3. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O-stable and output stable DLS. Let $P \in Ric_0(\Phi, J)$, $P \geq 0$. Then both ϕ_P and $J^{\frac{1}{2}}\phi^P$ are output stable and I/O-stable. We have the factorization*

$$J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{D}_{\phi_P} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{N}_P \cdot \mathcal{X},$$

where all factors are I/O-stable. Here $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is (I, Λ_P)-inner, $\mathcal{N}_P$ is $(\Lambda_P, \Lambda_{P^{\mathrm{crit}}})$-inner, and $\mathcal{X}$ is outer with a bounded inverse.

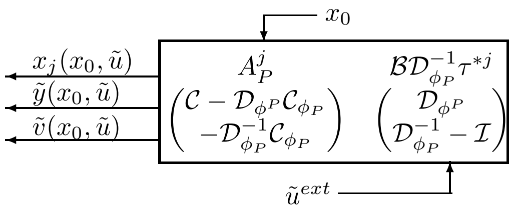
Realizations for all these factors can be given. Note that P is not *a priori* required to be a H^∞-solution; this is a part of the conclusion. By applying this result recursively, we see that the increasing chains of nonnegative H^∞-solutions (of finite length) will give corresponding chains of inner factors, see [10]. We remark that H^∞-equations (1) generally have such nontrivial chains of solutions.

Theorem 4. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O-stable and output stable DLS. Assume that $\overline{\mathrm{range}(\mathcal{B})} = H$. Let $P \in ric_0(\Phi, J)$. Then $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is I/O-stable if and only if $P \geq 0$.*

The result of the previous theorem can be translated to a number of partial ordering results, some of which are presented Section 5. In this translation, the notions of upper and lower DLS are used, as a convenient technical tool.

4 Control theoretic interpretation of the factorization

We first extend the original DLS Φ by the state feedback operators, associated to $P \in Ric(\Phi, J)$, $J \geq 0$. By closing the feedback loop we obtain the closed loop DLS



where $\tilde{u}^{ext}$ denotes an external disturbance signal into the feedback loop, $\tilde{u}$ is the internal input signal to the open loop system, and $x_0 \in H$ is the initial state. We remark that the above system makes sense even if Φ is unstable. In this case any solution $P \geq 0$ makes the DLS $J^{\frac{1}{2}}\phi^P$ output stable, and the transfer function $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in H^2(\mathbf{D}; \mathcal{L}(U;Y))$ because $\dim U < \infty$. Under stronger structural conditions on the semi-group, we would have $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in \mathrm{H}^\infty(\mathbf{D}; \mathcal{L}(U;Y))$; i.e. I/O-stability of $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$.

However, even if Φ were I/O-stable, the closed loop control signal I/O-map $\mathcal{D}^{-1}_{\phi_P} - \mathcal{I}$ would generally be unstable, for $P \geq 0$. If $P \geq 0$ is power stabilizing such that $\sigma(A_P) \subset \mathbf{D}$, then the closed loop DLS is exponentially stable. For non-power stabilizing $P \geq 0$, a partial stabilization of the closed loop system (and the semi-group generator A_P) would be achieved. Partial stabilization by nonnegative solutions of CARE is considered in [2], where the Riccati equation is different.

If Φ is I/O-stable (and output stable), $P \in ric_0(\Phi, J)$, and $P \geq 0$, then the open loop control signal I/O-map $\mathcal{D}_{\phi_P}$ is I/O-stable (and output stable), by definition. It then follows, under the standing hypotheses of this paper, that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is I/O-stable, by Theorem 3. However, $\mathcal{D}^{-1}_{\phi_P}$ is I/O-stable if (and only if) P is a critical solution.

5 Correspondence of the partial ordering

The operator $\widetilde{\mathcal{N}}_P$ is the adjoint I/O-map of $\mathcal{N}$, and it is defined via transfer functions by $\widetilde{\mathcal{N}}_P(z) := \mathcal{N}_P(\bar{z})^*$.

Theorem 5. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O-stable and output stable DLS, such that $\overline{\mathrm{range}(\mathcal{B})} = H$.*

For $P_1, P_2 \in ric_0(\Phi, J)$, the following are equivalent

(i) $P_1 \leq P_2$.

(ii) $\mathrm{range}(\widetilde{\mathcal{N}}_{\mathrm{P}_1}\bar{\pi}_+) \subset \mathrm{range}(\widetilde{\mathcal{N}}_{\mathrm{P}_2}\bar{\pi}_+)$.

In other words, the mapping $P \mapsto \mathrm{range}(\widetilde{\mathcal{N}}_{\mathrm{P}}\bar{\pi}_+)$ is order preserving from the POSET $ric_0(\Phi, J)$ (partially ordered set, ordered by the natural partial ordering of self adjoint operators) into the sub-POSET $\{\mathrm{range}(\widetilde{\mathcal{N}}_{\mathrm{P}}\bar{\pi}_+)\}_{P \in ric_0(\Phi,J)}$ of the forward shift invariant subspaces of $\ell^2(\mathbf{Z}_+; U)$ (ordered by the inclusion of subspaces). This order preserving mapping is the starting point of [8].

Corollary 6. *Make the same assumptions as in Theorem 5. Denote the regular critical solution by* $P_0^{\mathrm{crit}} := (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$.

If $P_0 \in ric_0(\Phi, J)$ *is such that* $P_0^{\mathrm{crit}} \leq P_0$, *then* P_0 *is critical. If, in addition,* $\overline{\mathrm{range}(\mathcal{B})} = H$, *then* $P^{\mathrm{crit}} = P_0$.

The regular critical solution is extremal in the set of *regular* H^∞-solutions $ric_0(\Phi, J)$. We remark that if $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is an I/O-stable and output stable DLS, the solution set $Ric(\Phi, J)$ can contain nonnegative solutions that are not H^∞-solutions (provided that the semi-group generator A is "sufficiently unstable").

6 Generalizations

Several assumptions of this paper can be significantly relaxed, see [9] and [10]. The input space U can always be a separable Hilbert space, if the input operator $B \in \mathcal{L}(U; H)$ is on some occasions assumed to be a (compact) Hilbert-Schmidt operator. The solutions $P \in Ric(\Phi, J)$ need not always be regular, and weaker residual cost conditions can be introduced. The positivity of the cost operator J can be replaced by the positivity of the indicator Λ_P, for solutions P of interest. This is connected to the positivity of the Popov operator $\bar{\pi}_+ \mathcal{D}^* J \mathcal{D} \bar{\pi}_+$. Also an inertia result can be given, guaranteeing that all interesting solutions have positive indicators whenever one has.

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