Discrete time Riccati equation and the invariant subspaces of linear operators

JARMO MALINEN†

† Department of Mathematics, Helsinki University of Technology,
P.O.BOX 1100, 02015 HUT, Finland, Jarmo.Malinen@hut.fi

Abstract. Let H and Y be separable Hilbert spaces, U finite dimensional. Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(U,H)$, $C \in \mathcal{L}(H,Y)$, $D \in \mathcal{L}(U,Y)$, and suppose that the open loop transfer function $\mathcal{D}(z) := D + zC(I - zA)^{-1}B \in H^\infty(U,Y)$. Let $J \geq 0$ be a cost operator. We study a subset of self adjoint solutions P of the discrete time algebraic Riccati equation (DARE)

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

where $\Lambda_P, \Lambda_P^{-1} \in \mathcal{L}(U)$ and $K_P \in \mathcal{L}(H;U)$. We further assume that a critical solution P^{crit} of DARE exists, such that $\mathcal{X}(z) := I - zK_{P^{crit}}(I - zA_{P^{crit}})^{-1}B \in H^\infty(U,Y)$ is an outer factor of $\mathcal{D}(z)$. Here $A_{P^{crit}} := A + BK_{P^{crit}}$.

We study connections between the nonnegative solutions of DARE and the invariant subspace structure of $(A^{crit})^*$.

Key Words. Operator DARE, nonnegative solution, partial ordering, invariant subspace, operator model

1. Introduction

In this paper, we consider the connection between the solution set of a discrete time (H^∞) Riccati equation (DARE) and the invariant subspaces of a linear operator.

Let us introduce first some notions and definitions. The basic object of this work is an operator valued H^∞ transfer function. A state space realization of this transfer function is a *discrete time linear system* (DLS) ϕ. As is well know, it can be given by the system of difference equations:

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ y_j = Cx_j + Du_j, \quad j \geq 0, \end{cases} \tag{1}$$

where $u_j \in U$, $x_j \in H$, $y_j \in Y$, and A, B, C, D are bounded linear operators between appropriate (separable) Hilbert spaces. We call the ordered quadruple $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *a DLS in difference equation form.* The three Hilbert spaces are as follows: U is the input space, H is the state space and Y is the output space of ϕ. There is also another equivalent form for DLS, *DLS in I/O form.* It consists of four linear operators in the ordered quadruple

$$\Phi := \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}.$$

The operator $A \in \mathcal{L}(H)$ is the *semi-group generator* of Φ, and it is the same operator as in equation (1). $\mathcal{B} : \ell^2(Z_-;U) \supset \mathrm{dom}(\mathcal{B}) \to H$ is the *controllability map* that maps the past input into the present state. $\mathcal{C} : H \supset \mathrm{dom}(\mathcal{C}) \to \ell^2(\mathbf{Z}_+;Y)$ is the *observability map* that maps the present state into the future outputs. The last operator $\mathcal{D} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ is the *I/O map* that maps the input into output in a causal and shift invariant way. We remark that the same DLS is denoted in I/O-form by the capital Φ, and in the difference equation form by ϕ.

If $\mathcal{B}$ or $\mathcal{C}$ is bounded, we say that Φ is *input stable* or *output stable*, respectively. If $\mathcal{D}$ is bounded then Φ is I/O-stable and the transfer function $\mathcal{D}(z) \in H^\infty(\mathcal{L}(U;Y))$. If $\lim_{j\to\infty} A^j x_0 = 0$ for all $x_0 \in H$, then the semi-group generator A is strongly stable.

Let Φ be an I/O-stable and output stable DLS, and $J \in \mathcal{L}(Y)$ a nonnegative, self adjoint cost operator. The symbol $Ric(\Phi, J)$ denotes the associated *discrete time Riccati equation*, given by

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

where Λ_P^{-1} is required to be bounded. If P is a self adjoint solution of $Ric(\Phi, J)$, we write $P \in Ric(\Phi, J)$. Several subsets of the solution set

$Ric(\Phi, J)$ are defined and studied in [10, Sections XXX]. The operator Λ_P is the *indicator*, and K_P is the *feedback operator* of P

For fixed Φ and J, two additional DLS's are associated to each solution $P \in Ric(\Phi, J)$

$$\phi_P := \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}, \quad \phi^P := \begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix}, \qquad (2)$$

where $A_P := A + BK_P$, $C_P = C + DK_P$. The objects ϕ_P, (ϕ^P) is called the *lower DLS* (*upper DLS*, respectively), centered at P. The algebraic and partial ordering properties of DARE can be conveniently described with the aid of these DLS's, see [10]. It also appears that ϕ_P and ϕ^P are fundamental notions in the infinite dimensional inner-outer factorization theory, as developed [11].

Some solution of DARE are more interesting that others. If $P \in Ric(\Phi, J)$ is such that the lower DLS ϕ_P is I/O-stable and output stable, we call P an H^∞*-solution*, and write $P \in ric(\Phi, J)$. By relating $P \in Ric(\Phi, J)$ to an associated cost optimization problem (see [7]), it makes sense to classify the solutions in terms of the residual costs (in the infinite future). If the residual cost operator $L_{A,P}$ exists and satisfies

$$L_{A,P} := s - lim_{j\to\infty} A^{*j} P A^j = 0,$$

then the *strong residual cost condition* is satisfied, and we write $P \in Ric_0(\Phi, J)$. Furthermore, $Ric_{uw}(\Phi, J)$ is the set of such solutions P that satisfy the *ultra weak residual cost condition*: $\langle P\mathcal{B}\tau^{*j}\tilde{u}, \mathcal{B}\tau^{*j}\tilde{u}\rangle \to 0$ for all $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ where τ^* denotes the backward shift. The set of *regular solutions* is defined as $ric_{reg}(\Phi, J) := Ric_0(\Phi, J) \cap Ric_{uw}(\Phi, J) \cap ric(\Phi, J)$.

A solution $P^{crit} \in Ric_{uw}(\Phi, J)$ is *critical* if the transfer function $\mathcal{D}_{\phi_{P^{crit}}}(z)$ of the lower DLS $\phi_{P^{crit}}$ is in H^∞, together with its inverse. The equivalent conditions for the existence of such P^{crit} are discussed in [7], under quite general assumptions. The I/O-map of such $\phi_{P^{crit}}$ is the outer factor (with a bounded inverse) in the $(J, \Lambda_{P^{crit}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, induced by the critical P^{crit}. P^{crit}, when it exists, can be replaced by another critical $P_0^{crit} \in ric_{reg}(\Phi, J)$, given by $P_0^{crit} = (\mathcal{C}^{crit})^* J \mathcal{C}^{crit}$. Here $\mathcal{C}^{crit} := (I - \bar{\pi}_+\mathcal{D}(\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+)^{-1}\bar{\pi}_+\mathcal{D}^* J)\mathcal{C}$ is the critical closed loop observability map.

A detailed treatment of fundamental notions of DLS's (such as state feedback structure and various stability notions) is given in [8]. For a less general, introductory presentation, see the introduction in [10]. For associated cost optimization problems, spectral factorization problems and critical solutions of DARE under weak assumptions, see [6] and [7]. See also the early discrete time paper [5]. For factorizations of the Popov operator and the I/O-map via nonnegative solutions of an associated DARE, see [10] and [11]. Nonnegative solutions of CARE are considered in [2] which has a considerable intersection with our work [11]; however, the [2] contains deeper control theoretic considerations. See also the references in [2], in particular the earlier [1]. A more complete presentation of the results of this paper are given in [9]. The above notions are closely related to the concept of a continuous time *stable well-posed linear system* by O. Staffans in [13], [14] and G. Weiss in [16], [17].

After these preliminary considerations, we continue to discuss this work. Our starting point is the following Lemma 1. It relates the natural partial ordering of the nonnegative solutions $P \in Ric_{uw}(\Phi, J)$ to the partial ordering of certain (normalized) chains of partial inner factors of the I/O-map $\mathcal{D}$.

Lemma 1. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O-stable and output stable DLS. Assume that the input space U is finite dimensional and that a critical solution P^{crit} exists.*

If $P_1, P_2 \in Ric_{uw}(\Phi, J)$ satisfy $0 \leq P_1 \leq P_2$, then $\mathrm{range}(\widetilde{\mathcal{N}}^\circ_{P_1}\bar{\pi}_+) \subset \mathrm{range}(\widetilde{\mathcal{N}}^\circ_{P_2}\bar{\pi}_+)$.

Also a two-way result on the partial ordering is given in [11], but we do not use it here. To explain what the Toeplitz operator $\widetilde{\mathcal{N}}^\circ_P\bar{\pi}_+$ is, consider the following DLS's

$$\phi_{P,P_0^{crit}} := \begin{pmatrix} A^{crit} & B \\ K^{crit} - K_P & I \end{pmatrix}, \qquad (3)$$
$$\widetilde{\phi_{P,P_0^{crit}}} = \begin{pmatrix} (A^{crit})^* & (K^{crit} - K_P)^* \\ B^* & I \end{pmatrix},$$

where $K^{crit} := K_{P_0^{crit}}$, and $A^{crit} := A + BK^{crit}$. So, the DLS $\widetilde{\phi_{P,P^{crit}}}$ is the adjoint of $\phi_{P,P^{crit}}$. The I/O-map of $\phi_{P,P^{crit}}$ equals $\mathcal{N}_P$, where $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$ is the $(\Lambda_P, \Lambda_{P^{crit}})$-inner-outer factorization, see [10]. Normalize and adjoin to obtain $\widetilde{\mathcal{N}}^\circ_P := \Lambda_{P^{crit}}^{-\frac{1}{2}}\widetilde{\mathcal{N}}_P\Lambda_P^{\frac{1}{2}}$, where $\widetilde{F(z)} := F(\bar{z})^*$ and analogously for the I/O-maps, DLS's and DARE's. Now the transfer function $\widetilde{\mathcal{N}}^\circ_P(z)$ is inner $\mathcal{L}(U)$-valued analytic function in $\mathbf{D}$, having unitary nontangential boundary limits $\widetilde{\mathcal{N}}^\circ_P(e^{i\theta})$ a.e. $e^{i\theta} \in \mathbf{T}$. So as to the range spaces $\mathrm{range}(\widetilde{\mathcal{N}}^\circ_P\bar{\pi}_+)$, the reader will immediately notice that this situation is described by the Beurling-Lax-Halmos-theorem of forward shift invariant subspaces.

By looking at the realization (3), we see that the semi-group generator $\phi_{P,P_0^{crit}}$ is independent of P. Moreover, the operator $K^{crit} - K_P$ in a sense, "measures the distance" of P from the critical P_0^{crit}. The extreme case appears when $P = P_0^{crit}$, and the state space is no longer visible from the output of $\phi_{P,P_0^{crit}}$. This gives us the idea that for a given P, only the "visible part" of $(A^{crit})^*$ is responsible for the structure of the corresponding I/O-map, namely $\widetilde{\mathcal{N}}_P^{\circ}$ (normalization is here immaterial). Furthermore, it is reasonable to expect that these "parts" of $(A^{crit})^*$ were "ordered" in the same sense as the operators $\widetilde{\mathcal{N}}_P^{\circ}$. If that were the case, then the partial ordering of Lemma 1 could be carried from the solution set $Ric_{uw}(\Phi, J)$ to the "parts" of the semigroup $(A^{crit})^*$. With even more luck, we might be able to connect the function theoretic structure of $\widetilde{\mathcal{N}}_P^{\circ}(z)$ to the operator theoretic structure of the corresponding part of $(A^{crit})^*$, in a general, truly infinite dimensional manner. This is the battle plan of this paper.

For a rigorous treatment, we first assume that the I/O-map of $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is $(J, \Lambda_{P^{crit}})$-inner, see Section 2. For such inner systems, the partial ordering of $Ric(\Phi, J)$ is related to certain A^*-invariant subspaces H^P.

In Section 3, we present the characteristic DLS $\phi(P)$, which is a reduced, observable version of ϕ_P. The adjoint DLS $\widetilde{\phi(P)}$ has the semi-group generator $(A^P)^* := \Pi_P A^* | H^P$, a compression of the original A (Π_P is the orthogonal projection onto H^P). Now, a normalized $\widetilde{\phi^{\circ}(P)}$ is approximately controllable, has the semi-group generator $(A^P)^*$ and the I/O-map $\widetilde{\mathcal{N}}_P^{\circ}$. We proceed to connect the function theoretic properties of the inner function $\widetilde{\mathcal{N}}_P^{\circ}(z)$ to the operator theoretic properties of $(A^P)^*$.

This is done by the tools of Sections 4 and 5. Here, a special case of the Sz.Nagy-Foias operator model (for C_{00}-contractions) is introduced. These tools are applied in Section 6. More precisely, the characteristic function $\widetilde{\mathcal{N}}_P^{\circ}(z)$ is connected to the restriction of the backward shift S^* onto the (closure of the) range of the observability map $\mathcal{C}_{\widetilde{\phi^{\circ}(P)}}$. Then the basic identity $S^* \mathcal{C}_{\widetilde{\phi^{\circ}(P)}} = \mathcal{C}_{\widetilde{\phi^{\circ}(P)}} (A^P)^*$ is used to connect the restricted shift (and simultaneously $\widetilde{\mathcal{N}}_P^{\circ}(z)$) to the semi-group $(A^P)^*$.

In the final Section 7, we indicate how the approach is valid for a general non-inner DLS Φ. This is done by making a preliminary state feedback, associated to the critical solution P_0^{crit}. The upper (closed loop) DLS $\phi^{P_0^{crit}}$ has $(J, \Lambda_{P^{crit}})$-inner I/O-map, and the solution sets of DARE's coincide: $Ric(\Phi, J) = Ric(\phi^{P_0^{crit}}, J)$. Because we must require extra structure from the useful solutions $P \in Ric(\phi^{P_0^{crit}}, J)$, it follows that not all $P \in Ric(\phi, J)$ are relevant for our purposes. When A is strongly stable, and ϕ is both input and output stable, then the relevant solutions are conveniently characterized and presented here. The general case (when A is not strongly stable) is more complicated.

To clarify the presentation, the following standing assumptions are used throughout this paper.

- $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_{\tau^{*j}} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is an input stable, output stable and I/O-stable DLS, such that $\mathrm{dom}(\mathcal{C}) := \{x \in H \mid \mathcal{C}x \in \ell^2(\mathbf{Z}_+; Y)\} = H$. Also $\overline{\mathrm{range}(\mathcal{B})} = H$ is assumed.
- The cost operator J is nonnegative, and the input space U is finite dimensional.
- The DARE $Ric(\Phi, J)$ has a nonnegative critical regular solution $P_0^{crit} = (\mathcal{C}^{crit})^* J \mathcal{C}^{crit} \in ric_{reg}(\Phi, J)$.
- The I/O-map $\mathcal{D}$ of Φ is assumed to be $(J, \Lambda_{P^{crit}})$-inner, except in the last section. This means that in the $(J, \Lambda_{P^{crit}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, induced by a critical P^{crit}, the outer part is an identity operator.
- All the solutions $P \in Ric(\Phi, J)$ are considered to be regular, unless explicitly otherwise stated.

These assumptions can be significantly relaxed, as mentioned in the final section. Eventually, Φ is required to be exactly controllable; i.e. $\mathrm{range}(\mathcal{B}) = H$.

2. DLS with inner I/O-map

We consider an I/O-stable and output stable DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_{\tau^{*j}} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ and a cost operator $J \geq 0$, such that $\mathcal{D}$ is $(J, \Lambda_{P^{crit}})$-inner.

Lemma 2. *Let $P \in ric_{reg}(\Phi, J)$, $P \geq 0$ be arbitrary.*

(i) $\ker(\mathrm{P_0^{crit}} - \mathrm{P}) = \ker(\mathcal{C}_{\phi_{\mathrm{P}}})$. *In particular,* $\ker(\mathrm{P_0^{crit}} - \mathrm{P})$ *is A-invariant.*

(ii) The feedback operators satisfy $K_{P_0^{crit}} = 0$ and $K_P = -\Lambda_P^{-1} B^* (P_0^{crit} - P) A$. Also $A^{crit} = A_{P_0^{crit}} = A$. The lower DLS at P satisfies

$$\phi_P = \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix} = \begin{pmatrix} A^{crit} & B \\ K^{crit} - K_P & I \end{pmatrix}.$$

(iii) The upper DLS $\phi^{P_0^{crit}}$ equals the original Φ. For the lower lower DLS: $\phi_P = \phi_{P, P_0^{crit}}$.

The DLS with $(J, \Lambda_{P^{crit}})$-inner I/O-map is itself its own closed loop system (if $\overline{\mathrm{range}(\mathcal{B})} = H$), when the same cost operator J is used to define the cost optimization problem. This is not surprising; you cannot further optimize what is already optimal.

We introduce the following notation: $H_P := \ker(\mathcal{C}_{\phi_{\mathrm{P}}}) \subset H$, its orthogonal complement in H is H^P. Denote by Π_P the orthogonal projection onto H^P. Clearly, if $0 \leq P_1 \leq P_2$, then

$\{0\} \subset H^{P_1} \subset H^{P_2} \subset H$. This connects the partial ordering of $ric_{reg}(\Phi, J) \subset Ric(\Phi, J)$ to the partial ordering of A^* -invariant subspaces H^P.

3. Characteristic DLS

Let Φ and $J \geq 0$ be as before. Let $P \in ric_{reg}(\Phi, J)$. The characteristic DLS $\phi(P)$ (of pair (Φ, J)), centered at P is defined by

$$\phi(P) = \begin{pmatrix} A^P & B^P \\ -K^P & I \end{pmatrix} := \begin{pmatrix} \Pi_P A|H^P & \Pi_P B \\ -K_P|H^P & I \end{pmatrix},$$

where $A^P \in \mathcal{L}(H^P)$, $B^P \in \mathcal{L}(U; H^P)$ and $C^P \in \mathcal{L}(H^P; Y)$. The state space of $\phi(P)$ is H^P. The DLS $\phi(P)$ is just a reduced version of ϕ_P, where the null space $\ker(\mathcal{C}_{\phi_P})$ has been "divided away" from the state space. Actually, the adjoint DLS $\widetilde{\phi(P)}$ is more interesting to us. Its properties are given by

Lemma 3. *Let $P \in ric_{reg}(\Phi, J)$, $P \geq 0$ be arbitrary. Then the following holds:*

(i) $\widetilde{\phi(P)}$ is I/O-stable, input stable, and $\overline{\mathrm{range}(\mathcal{B}_{\widetilde{\phi(P)}})} = H^P$. The I/O-map of $\widetilde{\phi(P)}$ satisfies $\mathcal{D}_{\widetilde{\phi(P)}} = \mathcal{D}_{\widetilde{\phi_P}} = \widetilde{\mathcal{N}}_P$.

(ii) If, in addition, Φ is input stable, then $\widetilde{\phi(P)}$ is output stable.

The DLS $\widetilde{\phi(P)}$ is interesting because the ranges of their I/O-maps $\widetilde{\mathcal{N}}_P \bar{\pi}_+$ obey the partial ordering of the solutions $P \in Ric_{uw}(\Phi, J)$, by Lemma 1.

Suppose we know a solution P_0 of DARE. Then we know the restricted subspace $H^P = \ker(\mathrm{P_0^{crit}} - \mathrm{P})^\perp$, and the projetion Π_P onto it. Because the semi-group generator $(A^P)^* := A^*|H^P$ of $\widetilde{\phi(P)}$ is now known, we can connect the structure of the (partial inner factor) transfer function $\widetilde{\mathcal{N}}_P(z)$ to this part of operator A^*.

4. Ranges for Toeplitz and Hankel operators

In the rest of this paper, we normalize the characteristic DLS as follows.

$$\widetilde{\mathcal{N}}_P^\circ := \Lambda_{P^{crit}}^{-\frac{1}{2}} \widetilde{\mathcal{N}}_P \Lambda_P^{\frac{1}{2}}, \quad \widetilde{\phi^\circ(P)} := \Lambda_{P^{crit}}^{-\frac{1}{2}} \widetilde{\phi(P)} \Lambda_P^{\frac{1}{2}}.$$

Then $\widetilde{\mathcal{N}}_P^\circ(z)$ is an inner $\mathcal{L}(U)$-valued function. It follows that the range of the Toeplitz operator $\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+$ and the closure of the range of $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ are orthogonal, and fill up the whole space $\ell^2(\mathbf{Z}_+; U)$.

Lemma 4. *Let Φ be an input stable, output stable and I/O-stable DLS. Assume that a critical P^{crit} exists, and let $P \in ric_{reg}(\Phi, J)$ be arbitrary. Then*

$$\ell^2(\mathbf{Z}_+; U) = \mathrm{range}(\widetilde{\mathcal{N}}_{\mathrm{P}}^\circ \bar{\pi}_+) \oplus \overline{\mathrm{range}(\mathcal{C}_{\widetilde{\phi^\circ(\mathrm{P})}})}.$$

5. Shift operator model

As usual, $\Theta(z) \in H^\infty(\mathcal{L}(U))$ is called inner, if the boundary trace function $\Theta(e^{i\theta})$ is unitary a.e. $e^{i\theta} \in \mathbf{T}$. For an inner $\Theta(z)$, define the Hilbert subspace

$$K_\Theta := H^2(\mathbf{T}; U) \ominus \Theta H^2(\mathbf{T}; U). \tag{1}$$

We consider the restriction $S^*|K_\Theta$ and its adjoint, the compression $P_\Theta S|K_\Theta$, where P_Θ is the orthogonal projection onto K_Θ. $S^*|K_\Theta$ is a contraction on $K_\Theta \subset H^2(\mathbf{T}; U)$. It is well known that properties of $S^*|K_\Theta$ are coded into the characteristic function $\Theta(e^{i\theta})$ of $S^*|K_\Theta$. This is a particular case of the famous Sz.Nagy-Foias operator model for contractions, see [3, Chapter IX, Section 5], [12], [15]. For compressions of the shifts in control theory, see also [4].

The contraction $S^*|K_\Theta$ has a number of useful properties:

Proposition 5. *Let $\Theta(z)$ be a contractive analytic function. $\Theta(z)$ is inner (from both sides) if and only if $S^*|K_\Theta \in C_{00}$, where C_{00} is the class of contractions T on a Hilbert space, such that*

$$s - lim_{j\to\infty} T^j = 0, \quad s - lim_{j\to\infty} T^{*j} = 0.$$

Definition 6. *The spectrum $\sigma(\Theta)$ of an inner function $\Theta(z)$ is defined to be the complement of the set of $z \in \overline{\mathbf{D}}$, such that an open neighborhood N_z of z exists with*

(i) $\Theta(z)^{-1}$ exists in $N_z \cap \overline{\mathbf{D}}$,

(ii) $\Theta(z)^{-1}$ can be analytically continued to a full neighborhood N_z.

The spectrum of $S^*|K_\Theta \in C_{00}$ is considered in the following:

Lemma 7. *Let $\Theta(z)$ be as above. Define the C_{00}-contraction $T_\Theta := P_\Theta S|K_\Theta \in \mathcal{L}(K_\Theta)$. Then*

(i) $\sigma(T_\Theta) = \sigma(\Theta)$, where $\sigma(\Theta) \subset \overline{\mathbf{D}}$ is the spectrum of the characteristic function $\Theta(z)$.

(ii) The point spectrum of T_Θ and T_Θ^* satisfies

$$\sigma_p(T_\Theta) = \{z \in \mathbf{D} \quad | \quad \ker(\Theta(\mathrm{z})) \neq \{0\}\}$$
$$\sigma_p(T_\Theta^*) = \{z \in \mathbf{D} \quad | \quad \ker(\Theta(\bar{\mathrm{z}}))^* \neq \{0\}\}$$

Because we are dealing with the DARE, we use the operator model in the time domain sequence space $\ell^2(\mathbf{Z}_+; U)$ instead of $H^2(\mathbf{T}; U)$. We adopt the following notation for a DLS ϕ' whose I/O-map $\mathcal{D}_{\phi'}$ is inner:

$$K_{\phi'} := \ell^2(\mathbf{Z}_+; U) \ominus \mathrm{range}(\mathcal{D}_{\phi'} \bar{\pi}_+),$$
$$S^* := \bar{\pi}_+ \tau^*.$$

Thus the transfer function $\mathcal{D}_{\phi'}(z)$ is the characteristic function of the contraction $S^*|K_{\phi'}$.

6. Similarity transform

By combining the contents of previous sections, we see that the properties of $\widetilde{\phi^\circ(P)}$ are as follows:

Lemma 8. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an input stable, output stable and I/O-stable DLS, such that the input space U is finite dimensional. Let $J \geq 0$ be a cost operator. Assume that there exists a critical solution $P^{crit} \in Ric_{uw}(\Phi, J)$ and the I/O-map $\mathcal{D}$ is $(J, \Lambda_{P^{crit}})$-inner. Let $P \in ric_{reg}(\Phi, J)$, $P \geq 0$ be arbitrary.*

(i) The DLS $\widetilde{\phi^\circ(P)}$ is input stable, output stable and I/O-stable. Its transfer function is inner and equals $\widetilde{\mathcal{N}}_P^\circ$. Also $\overline{\mathrm{range}(\mathcal{B}_{\widetilde{\phi^\circ(P)}})} = H^P$ and $\overline{\mathrm{range}(\mathcal{C}_{\widetilde{\phi^\circ(P)}})} = K_{\widetilde{\phi^\circ(P)}}$

(ii) The following similarity transform holds

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} (A^P)^*,$$

where $(A^P)^* := A^*|H^P$ and $H^P := \ker(P_0^{crit} - P)^\perp$ is A^*-invariant.

If, in addition, $\overline{\mathrm{range}(\mathcal{B})} = H$, then $\ker(\mathcal{C}_{\widetilde{\phi^\circ(P)}}) = \{0\}$. (However, the inverse of $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ can be unbounded.)

(iii) If Φ is exactly controllable (i.e. $\mathrm{range}(\mathcal{B}) = H$), then $\mathcal{C}_{\widetilde{\phi^\circ(P)}} : H^P \to K_{\widetilde{\phi^\circ(P)}}$ is a bounded bijection, with a bounded inverse.

Theorem 9. *Make the same assumption as in claim (iii) of previous Lemma.*

Then for all $P \in ric_{reg}(\Phi, J)$, $P \geq 0$, the restriction $(A^P)^* := A^*|H^P$ is similar to a C_{00}-contraction, whose characteristic function is $\widetilde{\mathcal{N}}_P^\circ(z)$. By adjoining, the compression $\Pi_P A|H^P$ is similar to a C_{00}-contraction, whose characteristic function is $\mathcal{N}_P^\circ(z)$.

The latter claim follows from the previous, by [12, Lemma on p.75]. In particular, this implies that the spectrum $\sigma(A^P)$ and the spectrum inner function $\sigma(\mathcal{N}_P^\circ)$ coincide. Also the point spectrum maps onto the point spectrum, in the sense of Lemma 7. A partial result concerning the point spectrum of A^P can be given, without assuming the exact controllability of Φ, see [9]. In particular, if A^P is a compact operator on H^P, then $\sigma(A^P) \setminus \{0\} = \overline{\sigma_p((A^P)^*)} = \{z \in \mathbf{D} \mid \ker(\mathcal{N}_P(\bar{z})) \neq \{0\}\}$; here bar denotes complex conjugate, and the assumption $\dim U < \infty$ is crucial. The case of compact A is always covered. Finally, the case when the complex valued inner function $\det \mathcal{N}_P^\circ(z)$ is a Blaschke product is connected to the completeness of A^P (i.e. eigenvectors of span the whole space), see [12, Lecture IV].

7. General DLS's

We explain how the results, given above for DLS with an inner I/O-map, can be extended to a general DLS Φ, such that there is a critical solution $P^{crit} \in Ric_{uw}(\Phi, J)$. Consider the closed loop H^∞-DARE, denoted by $Ric(\phi^{P_0^{crit}}, \Lambda_{P^{crit}})$.

$$\begin{cases} (A^{crit})^* P A^{crit} - \tilde{P} + (C^{crit})^* J C^{crit} = \tilde{K}_P^* \Lambda_P \tilde{K}_P \\ \Lambda_P = D^* J D + B^* P B \\ \Lambda_P \tilde{K}_P = -D^* J C^{crit} - B^* P A^{crit}, \end{cases}$$

Because $\phi^{P_0^{crit}}$ is a critical closed loop DLS, its I/O-map is $(J, \Lambda_{P^{crit}})$-inner. The full solution sets satisfy $Ric(\Phi, J) = Ric(\phi^{P_0^{crit}}, \Lambda_{P^{crit}})$. In a general case, the regular solutions of $Ric(\phi^{P_0^{crit}}, \Lambda_{P^{crit}})$ (that we consider in Theorem 9) can not be described in simple terms of the original data Φ and J. If both A and A^{crit} are strongly stable, then this problem becomes trivial.

Theorem 10. *Let $J \geq 0$ a self adjoint cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an input stable, output stable and I/O-stable DLS, such that the input space U is finite dimensional. Assume that the semi-group generator A is strongly stable:*

$$\lim_{j \to \infty} A^j x_0 = 0 \quad \textit{for all} \quad x_0 \in H.$$

Assume that the unique critical solution exists, equaling the regular solution $P_0^{crit} := (\mathcal{C}^{crit})^* J \mathcal{C}^{crit}$. Let $P \in ric_{reg}(\Phi, J)$. By $\phi(P)$ denote its characteristic DLS, and by $\mathcal{N}_P$ denote the $(\Lambda_P, \Lambda_{P_0^{crit}})$-inner factor of $\mathcal{D}_{\phi_P}$.

If $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is coercive, then

(i) the adjoint of $(A^{crit})^P := \Pi_P A^{crit}|H^P$ is similar to a C_{00}-contraction, whose characteristic function is $\mathcal{N}_P^\circ(z)$. The adjoint similarity transform is given by

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} ((A^{crit})^P)^*$$

where $\mathcal{C}_{\widetilde{\phi^\circ(P)}} : H^P \to K_{\widetilde{\phi^\circ(P)}}$ is a bounded bijection.

(ii) $\sigma(((A^{crit})^P)) = \sigma(\widetilde{\mathcal{N}}_P^\circ) \subset \mathbf{D}$, *where the spectrum of the inner function is given in Definition 6.*

(iii) *Both* $(A^{crit})^P$ *and* $\left((A^{crit})^P\right)^*$ *are strongly stable.*

In particular, the above claims hold if $\mathrm{range}(\mathcal{B}) = H$ *(i.e.* Φ *is exactly controllable).*

Several assumptions of this paper can be significantly relaxed, see [9]. The positivity of the cost operator J can be replaced by the positivity of the indicator Λ_P for solutions P of interest. This is connected to the positivity of the Popov operator $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+$. The input space can be a separable Hilbert space throughout the work (with one exception), if the input operator $B \in \mathcal{L}(U;H)$ is assumed to be a (compact) Hilbert-Schmidt operator on some occasions. The solutions $P \in Ric(\Phi, J)$ need not always be regular, and some partial results will hold even if Φ is not even approximately controllable or input stable. Theorem 10 can be generalized in many directions. For example, the strong stability of A is not needed, but then the description of the relevant solutions in $P \in Ric(\Phi, J)$ would be more complicated.

Acknowledgement. I wish to thank Professor O. J. Staffans for several discussions.

References

[1] F. M. Callier, L. Dumortier, and J. Winkin. On the nonnegative self-adjoint solutions of the operator Riccati equation for infinite dimensional systems. *Integral equations and operator theory*, 22:162– 195, 1995.

[2] L. Dumortier. *Partially stabilizing linear-quadratic optimal control for stabilizable semigroup systems.* PhD thesis, Facultes Universitaires Notre-Dame de la Paix, 1998.

[3] C. Foias and A. E. Frazho. *The commutant lifting approach to interpolation problems*, volume 44 of *Operator Theory: Advances and applications.* Birkhäuser Verlag, 1990.

[4] C. Foias, H. Öxbay, and A. Tannenbaum. *Robust control of infinite dimensional systems, Frequency domain Methods*, volume 209. Springer Verlag, 1996.

[5] J. W. Helton. A spectral factorization approach to the distributed stable regulator problem; the algebraic Riccati equation. *SIAM Journal of Control and Optimization*, 14:639–661, 1976.

[6] J. Malinen. Minimax control of distributed discrete time systems through spectral factorization. *Proceedings of EEC97, Brussels, Belgium*, 1997.

[7] J. Malinen. Nonstandard discrete time cost optimization problem: The spectral factorization approach. *Helsinki University of Technology, Institute of mathematics, Research Report*, A385, 1997.

[8] J. Malinen. Well posed discrete time linear systems and their feedbacks. *Helsinki University of Technology, Institute of mathematics, Research Report*, A384, 1997.

[9] J. Malinen. Discrete time Riccati equations, operatore models, and invariant subspaces of linear operators. *Manuscript*, 1998.

[10] J. Malinen. Riccati equations for H^∞ discrete time systems: Part I: Factorization of the Popov operator. *Manuscript*, 1998.

[11] J. Malinen. Riccati equations for H^∞ discrete time systems: Part II: Factorization of the I/O-map. *Manuscript*, 1998.

[12] N. K. Nikolskii. *Treatise on the Shift Operator*, volume 273 of *Grundlehren der mathematischen Wissenschaften.* Springer Verlag, 1986.

[13] O. J. Staffans. Quadratic optimal control of stable systems through spectral factorization. *Mathematics of Control, Signals and Systems*, 8:167–197, 1995.

[14] O. J. Staffans. Quadratic optimal control of stable well-posed linear systems. *To appear in Transactions of American Mathematical Society*, 349:3679–3715, 1997.

[15] B. Sz.-Nagy and C. Foias. *Harmonic Analysis of Operators on Hilbert space.* North-Holland Publishing Company, 1970.

[16] G. Weiss. Regular linear systems with feedback. *Mathematics of Control, Signals, and Systems*, 7:23–57, 1994.

[17] G. Weiss. Transfer functions of regular linear systems, Part I: Characterizations of regularity. *Transactions of American Mathematical Society*, 342(2):827–854, 1994.