Tustin's method for time discretization of conservative dynamical systems

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What is Tustin's method? (1)

Consider the finite-dimensional continuous time system

$$S: \begin{cases} z'(t) & = Az(t) + Bu(t), \\ y(t) & = Cz(t) + Du(t), \end{cases}$$

where A is a $n \times n$ matrix, and the input $u(\cdot)$ and the output $y(\cdot)$ are, for simplicity, scalar.

The mapping

$$\mathbf{G} : u(\cdot) \mapsto y(\cdot)$$

is the I/O map of S, and its Laplace transform is the transfer function $\widehat{\mathbf{G}}(s) = C(s - A)^{-1}B + D$.

Please, don't panic.

This is supposed to be a relatively non-technical talk!

(Albeit on a quite technical matter.)

For your continued pleasure, details such as initial conditions and times, various norms and Hilbert spaces, etc., are intentionally and almost completely omitted.

What is Tustin's method? (2)

Define the operators

$$A_\sigma := (\sigma + A)(\sigma - A)^{-1},$$
$$B_\sigma := \sqrt{2\sigma}(\sigma - A)^{-1}B,$$
$$C_\sigma := \sqrt{2\sigma}C(\sigma - A)^{-1},$$
$$D_\sigma := \widehat{\mathbf{G}}(\sigma),$$

where we define $\sigma := 2/h$ and $h > 0$ is called the time discretization parameter.

What is Tustin's method? (3)

The Tustin transform of S is a family of discrete time systems

$$\phi_\sigma : \begin{cases} z_j^{(h)} &= A_\sigma z_{j-1}^{(h)} + B_\sigma u_j^{(h)}, \\ y_j^{(h)} &= C_\sigma z_{j-1}^{(h)} + D_\sigma u_j^{(h)}, \end{cases}$$

for any $\sigma > 0$.

The mapping

$$\mathbf{D}_\sigma : \{u_j^{(h)}\}_{j \in \mathbb{Z}} \mapsto \{y_j^{(h)}\}_{j \in \mathbb{Z}}$$

is the I/O map of ϕ_σ, and its Z-transform is the transfer function $\widehat{\mathbf{D}}_\sigma(z) = C_\sigma z(I - zA_\sigma)^{-1}B_\sigma + D_\sigma$.

What is Tustin's method? (4)

To approximate S with Tustin transforms ϕ_σ, we need to connect the continuous and discrete signals somehow.

We define $\{u_j^{(h)}\}_{j\in\mathbb{Z}} = T_\sigma u$ where the discretising (or sampling) operator T_σ, $\sigma = 2/h > 0$, is given by

$$u_j^{(h)} = \frac{1}{\sqrt{h}} \int_{(j-1)h}^{jh} u(t)\, dt \quad \text{for all} \quad j \in \mathbb{Z}.$$

The interpolation (or hold) operator T_σ^* is the L^2-adjoint of T_σ. It maps discrete signals back to continuous signals.

Where is Tustin's method used?

Simulation of linear dynamical systems.

(i) Sure, there are more efficient numerical algorithms for, e.g., parabolic distributed parameter systems...

(ii) ...but there is a need for a time discretization method that is easy to implement, theoretically simple, well-known in **finite** dimensions, and it preserves energy conservativity / dissipativity.

Tustin's method is the **Leatherman multi-tool** for time discretization in numerical analysis.

But does it converge as $h \to 0+$?

Yes, it does

if S is stable and has a finite-dimensional state space.

The keyword is Crank-Nicolson in numerics literature.

Unfortunately, the convergence is far from clear when the state space of S is infinite-dimensional.

Motivation of the inf.-dim. problem

Take any infinite-dimensional Distr. Param. System.

Computers solve fin.-dim. problems obtained by a spatial discretization such as FEM or FDM.

For these fin.-dim. problems, the convergence of the Tustin's method for temporal discretization is clear.

But: Increasing the spatial resolution, we may run into deep trouble if the original inf.-dim. system itself does not behave well under Tustin's method.

Now, what do we mean by infinite-dimensional systems?

Infinite dimensional systems (1)

Many linear PDEs define dynamics whose state space is an inf.-dim., separable Hilbert space.

Think of the wave equation, linear elasticity, or why not the generalized Webster's PDE with curvature and boundary dissipation for $\bar{\phi} = \bar{\phi}(s,t)$

$$\frac{1}{A(s)}\frac{\partial}{\partial s}\left(A(s)\frac{\partial \bar{\phi}}{\partial s}\right) - \frac{2\pi\alpha W(s)}{A(s)}\frac{\partial \bar{\phi}}{\partial t} - \frac{1}{c^2\Sigma(s)^2}\frac{\partial^2 \bar{\phi}}{\partial t^2} = F$$

defining a dissipative, well-posed boundary control system (with the right boundary ctrl/obs operators).

Transmission graphs. Delay equations. Etc.

Infinite dimensional systems (2)

All these and many other linear systems can be represented in the dynamical form

$$\begin{bmatrix} z'(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}$$

where the system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an operator-valued generalization of the block matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

The Tustin's method can be extended to system nodes.

Indeed, the resulting DLS ϕ_σ is known as the Cayley or bilinear transform of S in Mathematical System Theory.

Approximation of the I/O mapping

We say that S is I/O stable if $\widehat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$

For I/O stable S and for any $u \in L^2(\mathbb{R})$ we have

$$\|T_\sigma^* \mathbf{G}_\sigma T_\sigma u - \mathbf{G}u\|_{L^2(\mathbb{R})} \to 0$$

as $\sigma = 2/h \to \infty$.

This was proved by Havu and Malinen in 2005.

An extension to multivariate signals is an easy piece.

An unpublished version of the result exists for S whose input and output are in a separable Hilbert space.

Convergence in the state space?

So, Tustin's method "saves the phenomena" as they appear outside the state space.

How about the convergence of the corresponding state trajectories of S and ϕ_σ as $\sigma \to \infty$?

Can we approach this problem using system nodes?

System nodes are a very big class, containing wildly different kinds of dynamics. **Too general as such.**

Energy conservativity appears to be a key factor in getting somewhere.

A reminder of conservativity

Conservativity is defined by energy balances; the solutions satisfy in continuous time for S

$$\frac{d}{dt}\|z(t)\|_X^2 = |u(t)|^2 - |y(t)|^2 \quad \text{for } t \in \mathbb{R}$$

and in discrete time for ϕ_σ

$$\|z_j^{(h)}\|_X^2 - \|z_{j-1}^{(h)}\|_X^2 = |u_j^{(h)}|^2 - |y_j^{(h)}|^2 \quad \text{for } j \in \mathbb{Z}.$$

We use an energy norm $\|\cdot\|_X$ in the common state (Hilbert) space X of S and ϕ_σ. **Physics!**

Dissipativity is defined by replacing the equalities above by inequalities $\leq$.

Approximation of the final state (1)

"Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a conservative system node with scalar input and output, whose semigroup is completely non-unitary, and transfer function $\widehat{\mathbf{G}}(\cdot)$ is inner."

This jargon plainly means that

(i) energy is neither created or dissipated by S,

(ii) all energy that goes into S also comes out, and

(iii) there is no externally invisible subspace in the state space X.

Approximation of the final state (2)

Take a twice continuously differentiable input signal $u(t)$ for $t \in [-T, 0]$, and discretize it

$$\{u_j^{(h)}\}_{j=-J_h,\ldots,0} = T_\sigma u$$

using time step $h = 2/\sigma > 0$ where $J_h \approx T/h$.

Obtain the final state $z(0)$ of S using u as the input with $z(-T) = 0$. Similarly, get the final state $z_0^{(h)}$ of ϕ_σ using $\{u_j^{(h)}\}$ as the input and $z_{-J_h}^{(h)} = 0$. Then

$$\lim_{h \to 0+} \|z_0^{(h)} - z(0)\|_X = 0.$$

Comments on the proof

To prove this theorem,

- we first prove the same result for a canonical Hankel range realization Σ_G of the transfer function $\widehat{\mathbf{G}}$ of S,

- we then use the State Space Isomorphism Theorem to show that Σ_G and S are actually the same thing — apart from a unitary (but otherwise very sick) equivalence of their state spaces...

- ... and finally note that the result does not depend on the unitary change of coordinates in the state.

Comments on generalizations

There is an extension to vector-valued signals u and y. This was proved by Havu and Malinen in 2010.

It seems possible to remove the assumption that $\widehat{\mathbf{G}}(\cdot)$ is inner by using a more complicated canonical realization than Hankel range.

It seems plausible that adding dissipation to S should make it “better behaved” but it is not at all clear how to prove it.

Mathematical System Theory...

... is a theory of classes and generalities.

MST results are typically not the best possible for any particular application or instance.

Instead, MST results and concepts should be general wisdom that is easy to communicate:

Tustin's method converges as expected even for a wide and relevant class of infinite-dimensional state space systems.

Convergence speed estimates are not possible on this level of generality.

Most Important References

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That's all, folks!

Have a nice day.