Conservativity and time-flow invertibility of boundary control systems

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Overview

In this talk, we

(i) explain the connection between boundary control systems (as defined below) and operator/system nodes;

(ii) give sufficient and necessary conditions for such a boundary control system to define a (scattering) conservative system node (notion that has been defined in earlier literature); and

(iii) present a PDE example involving the wave equation in $\mathbb{R}^n$ for $n \geq 2$.

Boundary nodes (1)

Boundary control systems are described by the following equations

$$\begin{cases} \dot{z}(t) = Lz(t) & \text{(state dynamics)}, \\ Gz(t) = u(t) & \text{(input)}, \\ y(t) = Kz(t) & \text{(output)}, \end{cases}$$

for $t \geq 0$ where the operators

$$L \in \mathcal{L}(\mathcal{Z};\mathcal{X}), \quad G \in \mathcal{L}(\mathcal{Z};\mathcal{U}) \quad \text{and} \quad K \in \mathcal{L}(\mathcal{Z};\mathcal{Y})$$

and the Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ satisfy...

Boundary nodes (2)

(i) $\mathcal{Z} \subset \mathcal{X}$ with a dense, continuous inclusion;

(ii) $\mathcal{U} = \mathrm{Ran}\ G$, and $\mathrm{Ker}\ G$ is dense in $\mathcal{X}$;

(iii) $(\alpha - L)\mathrm{Ker}\ G = \mathcal{X}$, and $\mathrm{Ker}\ (\alpha - L) \cap \mathrm{Ker}\ G = \{0\}$ for some $\alpha \in \overline{\mathbb{C}_+}$.

The triple $\Xi = (G, L, K)$ is called a boundary node.

If $L|\mathrm{Ker}\ G$ generates a C_0-semigroup, we say that Ξ is internally well-posed.

There are many (essentially) equivalent definitions.

Connection to system nodes

Internally well-posed boundary nodes $\Xi = (G, L, K)$ are in one-to-one correspondence with system nodes

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \quad \text{on spaces} \quad (\mathcal{U}, \mathcal{X}, \mathcal{Y})$$

whose input operator B is injective and strictly unbounded:

$$\mathrm{Ker}\ G = \{0\} \quad \text{and} \quad B\mathcal{U} \cap \mathcal{X} = \{0\}.$$

Such system nodes S are said to be of boundary control type.

Given $\Xi = (G, L, K)$...

...you get the corresponding $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ from equations $A\&B \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] := A_{-1}x + Bu$ and $C\&D \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] := Kx$ where

(i) $\mathrm{dom}\,(A) := \mathrm{Ker}\ G$ and $A := L|\mathrm{dom}\,(A)$;

(ii) $\mathcal{X}_{-1} := \mathrm{dom}\,(A^*)^d$ using $\mathcal{X}$ as the pivot space, and the usual Yoshida extension $A_{-1} : \mathcal{X} \to \mathcal{X}_{-1}$;

(iii) $BGz := Lz - A_{-1}z \quad$ for all $z \in \mathcal{Z}$;

(iv) and $\mathrm{dom}\,(S) := \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] \mathcal{Z}$.

(Don't worry. You need not memorize them right now.)

The Cauchy problem (1)

Assume: Boundary node $\Xi = (G, L, K)$ is internally well-posed; $u \in C^2([0,\infty);\mathcal{U})$ and $z_0 \in \mathcal{Z}$ satisfy the compatibility condition $Gz_0 = u(0)$.

Then: the equations for $t \geq 0$

$$\dot{z}(t) = Lz(t), \quad Gz(t) = u(t), \quad y(t) = Kz(t),$$

have a unique solution $z(\cdot) \in C([0,\infty);\mathcal{Z}) \cap C^1([0,\infty);\mathcal{X})$, such that $z(0) = z_0$ and $y(\cdot) \in C([0,\infty);\mathcal{Y})$;

The Cauchy problem (2)

And also: the same functions $u(\cdot)$, $z(\cdot)$ and $y(\cdot)$ satisfy

$$\dot{z}(t) = A_{-1}z(t) + Bu(t), \quad y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix},$$

for $t \geq 0$. Here the system node

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$$

corresponds to the boundary node $\Xi = (G, L, K)$ in the way described above.

Conservativity of system nodes

The system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is (scattering) energy preserving if for any $u(\cdot) \in C^2(\mathbb{R}_+; \mathcal{U})$ and any (compatible) initial state $z(0) = z_0$, the solution of

$$\dot{z}(t) = A_{-1}z(t) + Bu(t), \quad y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}$$

satisfies the energy balance equation

$$\frac{d}{dt}\|x(t)\|_{\mathcal{X}}^2 = \|u(t)\|_{\mathcal{U}}^2 - \|y(t)\|_{\mathcal{Y}}^2.$$

S is conservative, if both S and the dual node S^d are energy preserving.

Why is this definition "the right one"?

This definition of conservativity can be defended from several directions:

(i) It is a generalization from the finite dimensions;

(ii) By the Cayley transform, it is equivalent to the usual discrete time definition;

(iii) It is equivalent to the old definition of the operator colligation by Brodskiĭ, Livšic, Sz.-Nagy &al. in the theory of Hilbert space contractions;

Why is this definition... (cont'd)

(iv) System theoretically, it is a very "happy class" – e.g. a strong form of the state space isomorphism theorem holds.

(v) As this work shows, it relates in the right way to the time-flow invertibility – an important property of hyperbolic linear PDEs.

(vi) As our newer work shows, it relates (after a translation to "impedance setting") in the right way to the abstract boundary spaces, used for extensions of symmetric operators in Russian literature.

How about conservative boundary nodes?

Question: How to characterize those conservative boundary nodes $\Xi = (G, L, K)$ that correspond to conservative system nodes as described above?

Practical problems:

(i) The translation of the data $\Xi = (G, L, K)$ to an operator node S is cumbersome (especially if Ξ comprises partial differential operators!)

(ii) The dual system S^d need not be of boundary control type, even if S is; $\Rightarrow$ the direct, pure translation of the definition to boundary nodes is impossible!

Characterization of conservative $\Xi = (G, L, K)$

The triple $\Xi = (G, L, K)$ is a doubly boundary node, if both Ξ and $\Xi^{\leftarrow} := (K, -L, G)$ are boundary nodes.

Theorem 1: Let $\Xi = (G, L, K)$ be a doubly boundary node, and by $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ denote the associated operator node. Then S is conservative if and only if

(i) $2\Re \langle x, Lx \rangle_{\mathcal{X}} = -\|Kx\|^2_{\mathcal{Y}}$ for all $x \in \mathrm{Ker}\ G$,

(ii) $\langle z, Lx \rangle_{\mathcal{X}} + \langle Lz, x \rangle_{\mathcal{X}} = \langle Gz, Gx \rangle_{\mathcal{U}}$ for all $z \in \mathcal{Z}$ and $x \in \mathrm{Ker}\ K$.

"Childrens version"

There is another variant whose formulation is more beautiful.

Theorem 2: Let $\Xi = (G, L, K)$ be a doubly boundary node, and by $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ denote the associated operator node.

Then S is conservative if and only if the Green–Lagrange identity

$$2\Re \langle z_0, Lz_0 \rangle_{\mathcal{X}} = \|Gz_0\|^2_{\mathcal{U}} - \|Kz_0\|^2_{\mathcal{Y}}$$

holds for all $z_0 \in \mathcal{Z}$.

References to the proofs

The proof of Theorem 1. is based on the characterization of conservative system nodes among time-flow invertible system nodes [Malinen; (2004, 2005)], in combination with the main theorem of [Malinen, Staffans, Weiss; (2003, 2005)] on "tory" systems.

The proof of the slightly weaker Theorem 2. can be carried out alternatively by a direct argument, see [Malinen, Staffans; (2005)].

Theorem 1. can be also concluded from Theorem 2. by using the main theorem of [Malinen, Staffans, Weiss; (2003, 2005)].

The scattering conservative wave equation (1)

Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open bounded set with C^2-boundary $\partial\Omega$.

We assume that $\partial\Omega$ is the union of two sets Γ_0 and Γ_1 with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

In the same PDE example, the sets Γ_1 and Γ_0 are allowed to have zero distance in [Weiss, Tucsnak; (2003)]. This is possible because stronger "background results" from [Rodrigues-Bernal, Zuazua; (1995)] are used there.

The scattering conservative wave equation (2)

We are interested in the system node S that (hopefully) is described by the exterior problem

$$\begin{cases} z_{tt}(t,\xi) = \Delta z(t,\xi) \quad \text{for } \xi \in \Omega \text{ and } t \geq 0, \\ -z_t(t,\xi) - \frac{\partial z}{\partial \nu}(t,\xi) = \sqrt{2}\, u(t,\xi) \quad \text{for } \xi \in \Gamma_1 \text{ and } t \geq 0, \\ \sqrt{2}\, y(t,\xi) = -z_t(t,\xi) + \frac{\partial z}{\partial \nu}(t,\xi) \quad \text{for } \xi \in \Gamma_1 \text{ and } t \geq 0, \\ z(t,\xi) = 0 \quad \text{for } \xi \in \Gamma_0 \text{ and } t \geq 0, \text{ and} \\ z(0,\xi) = z_0(\xi), \quad z_t(0,\xi) = w_0(\xi) \quad \text{for } \xi \in \Omega. \end{cases}$$

Note that Γ_0 is the reflecting part of $\partial\Omega$.

The scattering conservative wave equation (3)

We discover the boundary node $\Xi = (G, L, K)$ by

$$z_{tt} = \Delta z \quad \hat{=} \quad \frac{d}{dt}\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\Delta & 0 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix}.$$

The spaces $\mathcal{Z}$, $\mathcal{X}$ and and operator L are defined by

$$L := \left[\begin{smallmatrix} 0 & -1 \\ -\Delta & 0 \end{smallmatrix}\right] : \mathcal{Z} \to \mathcal{X} \text{ with}$$

$$\mathcal{Z} := \mathcal{Z}_0 \times H^1_{\Gamma_0}(\Omega) \text{ and } \mathcal{X} := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$$

where $\mathcal{Z}_0 := \left\{ z \in H^1_{\Gamma_0}(\Omega) \cap H^{3/2}(\Omega) : \Delta z \in L^2(\Omega) \right\}$.

The scattering conservative wave equation (4)

The norm of $\mathcal{Z}_0$ is given by

$$\|z_0\|^2_{\mathcal{Z}_0} := \|z_0\|^2_{H^1(\Omega)} + \|z_0\|^2_{H^{3/2}(\Omega)} + \|\Delta z_0\|^2_{L^2(\Omega)}.$$

For the state space $\mathcal{X}$, we use the energy norm

$$\left\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix} \right] \right\|^2_{\mathcal{X}} := \||\nabla z_0|\|^2_{L^2(\Omega)} + \|w_0\|^2_{L^2(\Omega)}.$$

The scattering conservative wave equation (5)

Define the input and output spaces by setting $\mathcal{U} = \mathcal{Y} := L^2(\Gamma_1)$, together with

$$G\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] := \frac{1}{\sqrt{2}}\left(-\frac{\partial z_0}{\partial \nu}|\Gamma_1 + w_0|\Gamma_1\right) \text{ and}$$

$$K\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] := \frac{1}{\sqrt{2}}\left(\frac{\partial z_0}{\partial \nu}|\Gamma_1 + w_0|\Gamma_1\right).$$

We have now the triple of operators $\Xi = (G, L, K)$, together with the Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$.

The scattering conservative wave equation (6)

Proposition 3: The triple of operators $\Xi = (G, L, K)$ defined above is a doubly boundary node on spaces $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$.

The proof requires well-known properties of the Sobolev spaces (like the Poincaré inequality), standard results on Dirichlet and Neumann traces, and elliptic regularity theory.

We now know that there exists a unique system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ associated to Ξ.

The scattering conservative wave equation (7)

Proposition 4: Let the boundary node $\Xi = (G, L, K)$ be defined as above. Use the energy norm

$$\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix} \right] \|_{\mathcal{X}}^2 := \| |\nabla z_0| \|_{L^2(\Omega)}^2 + \|w_0\|_{L^2(\Omega)}^2.$$

for the state space $\mathcal{X}$. Then the system node S associated to Ξ is conservative.

Indeed, the conditions of Theorem 2. can be checked by using a generalized Greens formula.

A numerical example will be given later by V. Havu.

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That's all of it, folks!

Have a nice day.