# Harnack's inequality for parabolic nonlocal equations
# Workshop on Nonlinear Parabilic PDE
# Mittag-Leffler Institute

Martin Strömqvist

Uppsala University

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We consider solutions to parabolic nonlocal equations ot the type

$$\partial_t u(x,t) + \mathcal{L}u(x,t) = 0 \quad \text{in } \Omega \times (0,T), \tag{1}$$

where

$$\mathcal{L}u(x,t) = \text{P.V.} \int_{\mathbb{R}^n} (u(x,t) - u(y,t))K(x,y,t)dy.$$

We assume that $K$ is symmetric with respect to $x$ and $y$ and satisfies, for some $\Lambda \geq 1$ and $s \in (0,1)$, the ellipticity condition

$$\frac{\Lambda^{-1}}{|x-y|^{n+2s}} \leq K(x,y,t) \leq \frac{\Lambda}{|x-y|^{n+2s}}, \tag{2}$$

uniformly in $t \in (0,T)$. When

$$K(x,y,t) = \frac{C(n,s)}{|x-y|^{n+2s}},$$

for appropriate choice of $C(n,s)$, $\mathcal{L}$ is the fractional Laplacian and (1) is called the fractional heat equation. Equations of the type (1) appear for instance in the study of Levy processes as well as in signal and image processing.

We think of $\mathcal{L}$ as an operator in divergence form and define weak solutions as functions in $L^2(0,T;H^s(\mathbb{R}^n)) \cap L^\infty_{\text{loc}}(0,T;L^2(\Omega))$ that satisfy

$$\int_I \int_{\mathbb{R}^n} u(x,t)\partial_t \phi(x,t)dxdt$$
$$= \int_I \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x,t)-u(y,t))(\phi(x,t)-\phi(y,t))K(x,y,t)dxdydt,$$

for any $\phi \in C_c^\infty(\Omega \times I)$.

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# GOAL: Prove Harnack's inequality.

# Elliptic equations

Let us first consider nonnegative solutions $u : \mathbb{R}^n \to \mathbb{R}$ of the fractional Laplace's equation in a ball:

$$(-\Delta)^s u = 0 \quad \text{in } B_1.$$

In this situation the there is a Poisson integral representation formula for the solution:

$$u(x) = (PIu)(x) = \int_{\mathbb{R}^n \setminus B_1} P(x,y)u(y)dy = c_{n,s} \int_{\mathbb{R}^n \setminus B_1} \left( \frac{1-|x|^2}{|y|^2-1} \right)^s \frac{u(y)}{|x-y|^n} dy$$

If $x, z \in B_{1/2}$ we have for $C = C_{n,s} \geq 1$,

$$C^{-1} \leq \frac{P(x,y)}{P(z,y)} \leq C.$$

If $u \geq 0$ in $\mathbb{R}^n$, this leads to the standard Harnack inequality

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u.$$

# What happens if only $u \geq 0$ in $B_r$?

($u$ is allowed to take negative values in $\mathbb{R}^n \setminus B_r$ ) Kassmann 07', 11', showed that the usual Harnack inequality fails in this situation. In fact, if $(-\Delta)^s u = 0$ in $B_r(x_0)$ and $u \geq 0$ in $B_R(x_0)$, $R \geq r$, then

$$\sup_{B_{r/2}} u \leq C \left( \inf_{B_{r/2}} u + \left(\frac{r}{R}\right)^{2s} \mathrm{Tail}(u_-; x_0, R) \right), \tag{3}$$

where

$$\mathrm{Tail}(u_-; x_0, R) = R^{2s} \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{u_- \, dx}{|x - x_0|^{n+2s}}.$$

- This formulation of the Harnack inequality first appeared in a paper by Di-Castro-Kuusi-Palatucci 14'.
- The proof for the fractional Laplacian can based on the Poisson kernel representation formula.
- This formulation allows you to deduce Hölder regularity in the classical way.

If $(-\Delta)^s u = 0$ in $B_1(0)$ and $u \geq 0$ in $B_2(x_0)$,

$$u(x) = (PIu)(x)$$

If $(-\Delta)^s u = 0$ in $B_1(0)$ and $u \geq 0$ in $B_2(x_0)$,

$$\begin{aligned} u(x) &= (PIu)(x) \leq (PIu_+)(x) \leq C(PIu_+)(z) \\ &= C(PIu)(z) + C(PIu_-)(z) = Cu(z) + C(PIu_-)(z) \\ &\leq Cu(z) + C\int_{\mathbb{R}^n \setminus B_2} P(z,y)u_-(y)dy \\ &\leq Cu(z) + C\,\mathrm{Tail}(u_-;0,2). \end{aligned}$$

This Harnack inequality was obtained for much more general nonlocal equations of p-Laplace type by Di Castro-Kuusi and Palatuchi 14', '16. Here the operator is given by

$$Lu = P.V. \int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))dy,$$

with

$$\frac{\lambda}{|x-y|^{n+sp}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+sp}}.$$

They established that if $Lu = 0$ in $B_r(x_0)$ and $u \geq 0$ in $B_R(x_0)$, $R \geq r$, then

$$\sup_{B_{r/2}} u \leq C\left(\inf_{B_{r/2}} u + \left(\frac{r}{R}\right)^{2s} \mathrm{Tail}(u_-; x_0, R)\right), \tag{4}$$

with

$$\mathrm{Tail}(u_-; x_0, R) = \left(R^{2s} \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{u_-^{p-1}dx}{|x-x_0|^{n+2s}} dx\right)^{\frac{1}{p-1}}.$$

# Parabolic equations

The weak Harnack inequality for globally positive local weak solutions to

$$\partial_t u + Lu = 0 \quad \text{in } B_r \times (-r^{2s}, r^{2s}), \quad u \geq 0 \text{ in } \mathbb{R}^n \times (-r^{2s}, r^{2s})$$

was proved by Felsinger-Kassmann 13', Kassmann Schwab 14'.

$$\int_{U^-(r)} u(x,t)dxdt \leq C \inf_{U^+(r)} u.$$

$$U^-(r) = B_{r/2} \times (-r^{2s}, -r^{2s} + \left(\frac{r}{2}\right)^{2s}), \quad U^+(r) = B_{r/2} \times (r^{2s} - \left(\frac{r}{2}\right)^{2s}, r^{2s})$$

Leads to Hölder continuity under the additional assumption of global boundedness of the solution.

- In the probability community Harnack inequalities have been studied in connection to heat kernal estimates for Levy processes. Barlow-Bass-Chen-Kassmann 09', Barlow-Bass Kumagai 06'. In particular the relation between Harnack inequalities and Heatkernel estimates has been studied. (The case of Harnack inequalities for globally positive solutions)
- Boundedness and Hölder continuity for the Cauchy problem in $\mathbb{R}^{n+1}_+$ by Caffarelli-Chan-Vasseur 11'.
- Harnack's inequality (without time-lag!) for fractional heat equation in $\mathbb{R}^{n+1}_+$. Bonforte-Sire-Vazquez 17' (Optimal existence and uniquness theory) .

# The Cauchy problem for $\partial_t u + (-\Delta)^s u = 0$ in $\mathbb{R}^n \times \mathbb{R}_+$.

- Solutions can by represented as

$$u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)u_0(y)dy,$$

  where

$$P_t(x) \approx t\left(t^{\frac{1}{s}} + |x|^2\right)^{-\frac{n+2s}{2}}.$$

- $$P_t(x) \le C\frac{t}{\tau}\left(\frac{t^{\frac{1}{s}} + |x|^2}{\tau^{\frac{1}{s}} + |z|^2}\right)^{-\frac{n+2s}{2}} P_\tau(z).$$

This leads to a Harnack inequality of elliptic type (for globally nonnegative solutions): If $r^{2s} \leq t, \tau \leq 2r^{2s}$ and $|x-z| \leq r$, then

$$u(x,t) \leq Cu(z,\tau).$$

# Back to equation (1) $\partial_t + \mathcal{L}u = 0$ in $\Omega \times (0, T)$

Assume either $\Omega = \mathbb{R}^n$ or $u = 0$ in $\mathbb{R}^n \setminus \Omega \times (0, T)$.

**Theorem (Harnack inequality, S Arxiv 18')**

*Let $0 < r < R/2$, let $t_0 > r^{2s}$ and let*

$$t_1 = t_0 + 2r^{2s} - \alpha(r/2)^{2s}, \quad \text{for some } \alpha \in (1, 2^{2s}).$$

*Suppose that $t_1 < T$ and that $u$ is a solution to (1) such that*

$$u \geq 0 \text{ in } B_R(x_0) \times (t_0 - r^{2s}, t_1).$$

*Then*

$$\sup_{U^-(x_0,t_0,r/2)} u \leq C \left( \inf_{U^-(x_0,t_1,r/2)} u + \left(\frac{r}{R}\right)^{2s} \operatorname{Tail}(u_-; x_0, R, t_0 - r^{2s}, t_1) \right),$$

*where $C$ depends on $n, s, \Lambda$ and $\alpha$.*

Here

- $$U^-(x,t,r) = B_r(x_0)\times(t_0 - r^{2s}, t_0), \quad U^+(x,t,r) = B_r(x_0)\times(t_0, t_0 + r^{2s}).$$
- The parabolic tail is defined by
$$\mathrm{Tail}(v; x_0, R, I) = r^{2s} \fint_I \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|v(x,t)|}{|x - x_0|^{n+2s}} dxdt.$$
- Note that we get the standard Harnack inequality for globally positive solutions.

The theorem is proved by running a Moser iteration, to obtain local boundedness and the weak Harnack inequality. An apriori larger version of the parabolic tail appears in the estimates.

$$\mathrm{Tail}_\infty(v; x_0, R, I) = \sup_{t\in I} \int_{\mathbb{R}^n\setminus B_R} \frac{|v(x,t)|}{|x-x_0|^{n+2s}}\, dx.$$

## Theorem (Weak Harnack inequality)

*Suppose that u is a supersolution to such that*

$$u \geq 0 \text{ in } B_R(x_0)\times(t_0-2r^{2s}, t_0+2r^2), \quad r < R/2.$$

*Then*

$$\fint_{B_r(x_0)\times(t_0-2r^{2s},t_0-r^{2s})} u\,dxdt \leq C \inf_{B_r(x_0)\times(t_0+r^{2s},t_0+2r^{2s})} u + C\left(\frac{r}{R}\right)^{2s} \mathrm{Tail}_\infty(u_-; x_0, R, t_0-2r^{2s}, t_0+2r^{2s}).$$

# Local boundedness of subsolutions.

## Theorem

*Suppose that u is a subsolution. Then for any* $x_0 \in \mathbb{R}^n$, $r > 0$, $t_0 \in (r^{2s}, T)$, $\theta \in (0,1)$ *and any* $\delta \in (0,1)$, *there exist positive constants* $C(\delta) = C(\delta, n, \Lambda, s)$ *and* $m = m(n,s)$, *such that*

$$\sup_{U^-(x_0,t_0,\theta r)} u \leq \frac{C(\delta)}{(1-\theta)^m} \fint_{U^-(x_0,t_0,r)} u_+ dxdt + \delta \, \mathrm{Tail}_\infty(u_+; x_0, r, t_0 - r^{2s}, t_0).$$

To obtain the Harnack inequality we need an inequality of the type

$$\mathrm{Tail}_\infty(u_+; x_0, r, t_0 - r^{2s}, t_0) \le C\,\mathrm{Tail}_\infty(u_-; x_0, r, t_0 - r^{2s}, t_0) + C \sup_{B_r(x_0)\times(t_1,t_2)} u.$$

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This can only be obtained for the mean value mean value version of the tail.

$$\mathrm{Tail}(u_+; x_0, r, t_0 - r^{2s}, t_0) \leq C\,\mathrm{Tail}(u_-; x_0, r, t_0 - r^{2s}, t_0) + C \sup_{B_r(x_0)\times(t_1,t_2)} u.$$

# How do tails appear?

Assume $u, \phi \geq 0$ in $B_r \times I$ and $\operatorname{supp} \phi(\cdot, t) \subset B_{r/2}$, $r < R$. Typically we use this for $u$ sub/supersolution and $\phi = u^q \eta$, for a cut-off $\eta$.

$$\int_I \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x,t) - u(y,t))(\phi(x,t) - \phi(y,t)) K dx dy$$

$$= \underbrace{\int_I \int_{B_r} \int_{B_r} (u(x,t) - u(y,t))(\phi(x,t) - \phi(y,t)) K dx dy}_{local}$$

$$+ 2 \underbrace{\int_I \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} (u(x,t) - u(y,t)) \phi(x,t) K dx dy}_{nonlocal}$$

The nonlocal term may be split into

$$\int_I \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} (u(x,t) - u(y,t))\phi(x,t) K dx dy$$
$$= \underbrace{\int_I \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} u(x,t)\phi(x,t) K dx dy}_{local} + \underbrace{\int_I \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} -u(y,t)\phi(x,t) K dx dy}_{nonlocal}$$

$$local \approx \int_{\mathbb{R}^n \setminus B_r} \frac{dy}{|y|^{n+2s}} \int_I \int_{B_r} u(x,t)\phi(x,t) dx dt$$
$$= C r^{-2s} \int_I \int_{B_r} u(x,t)\phi(x,t) dx dt.$$

$$nonlocal \approx \int_I \left( \int_{\mathbb{R}^n \setminus B_r} \frac{-u(y,t)dy}{|y|^{+2s}} \right) \left( \int_{B_r} \phi(x,t)dx \right) dt.$$

$$\begin{aligned} & -C \sup_{t\in I} \int_{R^n\setminus B_r} \frac{u_+(y,t)}{|y|^{n+2s}} \int_I \int_{B_r} \phi(x,t)dxdt \\ & \le nonlocal \\ & \le C \sup_{t\in I} \int_{R^n\setminus B_r} \frac{u_-(y,t)}{|y|^{n+2s}} \int_I \int_{B_r} \phi(x,t)dxdt \end{aligned}$$

# How to compare tails on different timeslices/continuity property of tails?

## Proposition (Bonforte-Vazquez 14')

*There is a nonnegative $C^2$-function $\Phi_r : \mathbb{R}^n \to \mathbb{R}$ such that $\Phi \equiv r^{-n}$ in $B_r$,*

$$\Phi_r(x) \approx \frac{r^{2s}}{|x|^{n+2s}} \quad \text{in } \mathbb{R}^n \setminus B_r,$$

$$|\mathcal{L}\Phi_r| \approx r^{-2s}\Phi_r.$$

# Use $\Phi_r$ as testfunction:

This gives us, assuming $u$ is a subsolution and $t_2 - t_1 \approx r^{2s}$:

$$\int_{t_1}^{t_2} \int_{R^n} \partial_t u(x,t)\Phi(x) dxdt \leq - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} u\mathcal{L}\Phi dxdt$$
$$\leq Cr^{-2s} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} u\Phi dxdt$$
$$\leq C\,\mathrm{Tail}(u, r, t_1, t_2) + \fint_{t_1}^{t_2} \fint_{B_r} udxdt.$$

This leads to an estimate of the form

**Lemma**

$$\mathrm{Tail}_\infty(u, r, t_1, t_2) \leq C\varepsilon^{-1}\,\mathrm{Tail}(u, r, t_1 - \varepsilon r^{2s}, t_2) + C\varepsilon^{-1} \fint_{t_1 - \varepsilon r^{2s}}^{t_2} \fint_{B_r} udxdt.$$

- We need to use the Lemma for $u_+$ and $u_-$. This is possible only when $u_+$ and $u_-$ are subsolutions in $\mathbb{R}^n \times I$.

(i) If $u$ is a solution in $\mathbb{R}^n \times I$, then $u_+$ and $u_-$ are subsolutions in $\mathbb{R}^n \times I$.

(ii) If $u$ is a solution in $\Omega \times I$ such that $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega \times I$, then $u_+$ and $u_-$ are subsolutions in $\mathbb{R}^n \times I$.

Thus the Harnack inequality is valid in the situations (i) and (ii).

# Further questions

- What about Harnack's inequality for local solutions to parabolic equations in general?
- Timelag?
- p-parabolic equations.
    - Intrinsic Harnack inequality - Joint with Kaj Nyström.
      How do we estimate

$$\sup_I \int_{\mathbb{R}^n \setminus B_r} \frac{|u(x,t)|^{p-1}}{|x|^{n+ps}} dy?$$

    - Local boundedness - S [ArXiv] 17'
    - Hölder continuity of solutions - Joint with E. Lindgren, L. Brasco - in preparation.

# THANK YOU!