# Lipschitz truncation and applications in PDEs

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in collaboration with

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June the 11th, 2018 Institute Mittag Leffler

# Cutting gradients

**Objective:**
Approximate a Sobolev function $u$ by *Lipschitz continuous* approximations $u_\lambda$ such that $|\{u_\lambda \neq u\}| \xrightarrow{\lambda\to\infty} 0$.

Theorem (Acerbi, Fusco '84)

*Let $u \in W^{1,1}(\Omega)$ and $\lambda > 0$ there is $u^\lambda \in W^{1,\infty}(\Omega)$ with*

- $u = u^\lambda$ *on* $\{M(\nabla u) \leq \lambda\}$
- $\|\nabla u^\lambda\|_\infty \leq c\lambda$

Here:

$$(Mf)(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy.$$

The bad set $\mathcal{O}_\lambda$ can be controlled via $|\mathcal{O}_\lambda| \leq \frac{c}{\lambda}\|\nabla u\|_1$.

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# Further development and applications

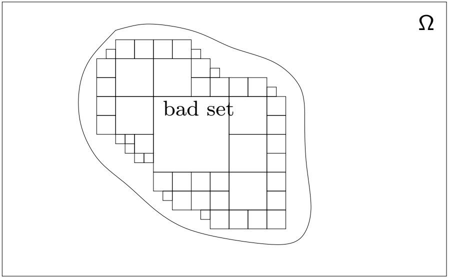
1. Calculus of variations [Acerbi-Fusco '84]: weak lower semicontinuity for Lipschitz functions imply weak lower semicontinuity for Sobolev functions.
   Justification of linear approximations of non-linear functionals by Γ-convergence: [Diening, Fornasier, Wank, '17].
2. A-harmonic approximation and partial regularity [Duzaar, Mingione '04], [Diening, Stroffolini, Verde '12 + Lengeler '12], [Bögelein, Duzaar, Mingione '13], [Diening, Sch, Stroffolini, Verde '17].
3. Very weak solutions. A-priori estimates for $p$-Laplacian [Lewis '93], [Kinnunen, Lewis '02]. Existence and Uniqueness issues [Diening, Buliček, Sch '16], [Buliček, Sch '16], and for non-linear flows [Buliček, Burczak, Sch '16].
4. Fluid dynamics. Existence of non-Newtonian fluids. [Frehse, Málek, Steinhauer '03], [Diening, Málek, Steinhauer '08], [Breit, Diening, Fuchs '12], [Breit, Diening, Sch '13].

# Lipschitz truncation stationary

For $\mathbf{u} \in W^{1,1}(\Omega)$ define

$$\mathbf{u}_\lambda := \begin{cases} \mathbf{u} & \text{on good set } \{M(\nabla u) \leq \lambda\} \\ \sum_j \varphi_j \mathbf{u}_j & \text{on bad set } \{M(\nabla u) > \lambda\} \end{cases}$$

- $\mathbf{u}_j = ⨍_{Q_j} \mathbf{u}\, dx$;
- $(\varphi_j)_j$ partition of unity;
- $\|\nabla \mathbf{u}_\lambda\|_{L^\infty(Q_j)} \lesssim ⨍_{4Q_j} |\nabla u|\, dx$.



**Lipschitz property:**

$|\nabla u_\lambda(x)| \lesssim \lambda$.

**Stability and convergence:**

For $q \geq 1$: $\int_\Omega |\nabla(u - u_\lambda)|^q \, dx \lesssim c \int_{\{M(\nabla u) > \lambda\}} |\nabla u|^q \, dx$.

For $\omega \in A_q$: $\int_\Omega |\nabla(u - u_\lambda)|^q \omega \, dx \lesssim c \int_{\{M(\nabla u) > \lambda\}} |\nabla u|^q \omega \, dx$.

**Boundary values:**

If $u \in W_0^{1,1}(\Omega)$, then $u_\lambda \in W_0^{1,\infty}(\Omega)$.

**Solenoidality:**

If $\operatorname{div} u = 0$, then $\operatorname{div} u_\lambda = 0$.

**Relative Truncation:**

Modify $u \in W_0^{1,p}(\Omega)$ on a set $\mathcal{O}$ relative to a right hand side, such that $u_\mathcal{O} \in W_0^{1,p}(\Omega)$.

# Application 1: Nonlinear elliptic systems

Assume $A$ to hold

$$
\begin{aligned}
C_1|\eta|^p - C_2 \leq A(x,\eta)\cdot\eta &\leq C_2(1+|\eta|^p),\\
0 &\leq (A(x,\eta_1) - A(x,\eta_2))\cdot(\eta_1-\eta_2)
\end{aligned}
$$

find $u : \Omega \to \mathbb{R}^N$ such that

$$
\boxed{\begin{aligned}
-\operatorname{div} A(x,\nabla u(x)) &= -\operatorname{div} |f(x)|^{p-2} f(x) \quad &&\text{in } \Omega,\\
u &= 0 &&\text{on } \partial\Omega.
\end{aligned}}
$$

- For $f \in L^p(\Omega;\mathbb{R}^{d\times N})$ there exists a weak solution $u \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. If $A$ is strictly monotone then the solution is unique (within $W_0^{1,p}$).
- Question: Does for $f \in L^q(\Omega;\mathbb{R}^{d\times N})$ a (unique) $u \in W_0^{1,q}(\Omega;\mathbb{R}^N)$ exist?

# Negative answer in general

- Šverák and Yan: If $p = 2$, $N \neq 1$ and $d \geq 5$, there exists $A$ uniformly monotone, smooth (independent of $x$) and a smooth $f$ such that the unique weak solution is unbounded. I.e. $u \notin W^{1,d}(\Omega)$ but $f \in L^d$.
- Serrin: If $p = 2$, $N = 1$ and $q \in (1, 2)$, there exists $A(x)$ uniformly monotone, but only measurable with respect to $x$, such that $v \in W_0^{1,q}(B_1(0))$ solves

$$\begin{aligned} -\operatorname{div}(A(x)\nabla v(x)) &= 0 && \text{in } B_1(0), \\ v &= 0 && \text{on } \partial B_1(0). \end{aligned}$$

  Serrin called such $v$ pathological solution.
- To avoid such difficulties one needs some structural assumptions on $A$ w.r.t. $\nabla u$ and certain smoothness of $A$ w.r.t. $x$.

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# Positive answers

- Whenever $q \geq p$ there exists a solution $u \in W_0^{1,p}(\Omega)$
- (**Bulíček**) For $p = 2$ and $A$ (measurable in $x$, Lipschitz in $z$) for $s \in (2-\varepsilon, 2+\varepsilon)$ there exists a weak solution that satisfies

$$\|\nabla u\|_q \leq c(1 + \|f\|_q).$$

This results follows from reverse Hölder inequality (**Gehring**)

- In case we have uniformly monotone $A$ with Uhlenbeck structure

$$A(x,\eta) = a(|\eta|)\eta$$

we have that for all $s \in [p,\infty)$ there holds (Uhlenbeck, Iwaniec)

$$\|\nabla u\|_s \leq C(p,\Omega,C_1,C_2)(1 + \|f\|_s).$$

- More results available in the scalar case with measure valued r.h.s. (**Boccardo, Murat, Acerbi, Malý, ...**)

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# Existence for $p$-Laplace

Assume $A$ is of $p$-growth, coercivity and monotonicity conditions and

$$-\operatorname{div}(A(x,\nabla u)) = -\operatorname{div}(|f|^{p-2}f) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

**Theorem (Iwaniec '92,Lewis '93)**

*There is an $\varepsilon$ depending on the p-growth, such that for all $q \in [p-\varepsilon, p]$ the following holds. If $f \in L^q(\Omega)$ and $u \in W_0^{1,q}(\Omega)$ is a solution, then*

$$\|\nabla u\|_{L^q(\Omega)} \leq c\|f\|_{L^q(\Omega)}$$

**Theorem (Bulíček, Sch 16')**

*There is an $\varepsilon$ depending on the p-growth, such that for all $q \in [p-\varepsilon, p]$ the following holds. If $f \in L^q(\Omega)$, then there exists $u \in W_0^{1,q}(\Omega)$ which is a distributional solution.*

# Construction of a solution

- For $n \in \mathbb{N}$ consider the problem

$$-\operatorname{div} A(x, \nabla u^n(x)) = -\operatorname{div} |\min\{f, n\}|^{p-2} \min\{f, n\}.$$

Since $\min\{f, n\}$ is bounded, there exists a solution $u^n \in W_0^{1,p}(\Omega)$.

- **First step:** We know that
$\|u^n\|_{1,q} \le C(1 + \|f\|_q)$.
Hence for a subsequence

$$u^n \rightharpoonup u \quad \text{weakly in } W_0^{1,q}(\Omega),$$

$$A(\cdot, \nabla u^n) \rightharpoonup \overline{A} \quad \text{weakly in } L^{\frac{q}{p-1}}(\Omega)$$

$\implies \int_\Omega \overline{A} \cdot \nabla v \, dx = \int_\Omega |f|^{p-2} f \cdot \nabla v \, dx \qquad \text{for all } v \in W_0^{1,(q+1-p)/q}(\Omega).$

- **Second step:** Show that

$$\boxed{\overline{A}(x) = A(x, \nabla u(x))} \qquad \text{a.e. in } \Omega.$$

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# The second step.

Assume $u \in W^{1,p}$, then

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega A(\cdot, \nabla u^n) \cdot \nabla u^n &= \limsup_{n\to\infty} \int_\Omega \min\{f, n\} \cdot \nabla u^n \\ &= \int_\Omega f \cdot \nabla u = \int_\Omega \overline{A} \cdot \nabla u\end{aligned}.$$

This implies that for all $B \in \mathbb{R}^{d\times N}$

$$0 \le \langle A(\cdot, \nabla u^n) - A(B), \nabla u^n - B\rangle \rightharpoonup \langle \overline{A} - A(B), \nabla u - B\rangle$$

Now, monotone operator theory (Minty trick) leads to

$$\boxed{\overline{A}(x) = A(x, \nabla u(x)) \qquad \text{a.e. in } \Omega.}$$

# What to do if $q < p$?

- Try to get $|\Omega \setminus \Omega_\varepsilon| \leq \varepsilon$, such that

$$\limsup_{n\to\infty} \int_{\Omega_\varepsilon} A(\cdot, \nabla u^n) \cdot \nabla u^n \leq \int_{\Omega_\varepsilon} \overline{A} \cdot \nabla u$$

- Minty trick then leads to

$$\boxed{\overline{A}(x) = A(x, \nabla u(x)) \qquad \text{a.e. in } \Omega_\varepsilon}$$

- letting $\varepsilon \to 0_+$ we have also that

$$\boxed{\overline{A}(x) = A(x, \nabla u(x)) \qquad \text{a.e. in } \Omega}$$

- Is it possible to chose sets $\Omega_\varepsilon$ such that

$$\sup_n \int_{\Omega_\varepsilon} |\nabla u^n|^p < \infty?$$

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# Estimates revisited

- We need $p$-integrability of $\nabla u$
- Estimates hold if they are true for linear problem.
- Consider $f \in L^q(\Omega)$ and

$$-\operatorname{div} |\nabla u|^{p-2} \nabla u = -\operatorname{div} |f|^{p-2} f$$

  Then heuristically

$$\begin{aligned} |\nabla u| &\sim |f| \\ |\nabla u|^q &\sim |f|^q \\ |\nabla u|^p |f|^{q-p} &\sim |f|^q \end{aligned}$$

- Is the last claim true?

# Estimates revisited

- If the estimate is true then in particular

$$\int_{\mathbb{R}^d} |\nabla u|^2 |f|^{q-2} \leq C \int_{\mathbb{R}^d} |f|^q, \tag{0.1}$$

whenever $-\Delta u = -\operatorname{div} f$ in $\mathbb{R}^d$?

- Precise representation: $u := f * K$ where $K$ is the Calderón-Zygmund kernel. (0.1) is true $\Leftrightarrow$ the Maximal function is bounded in $L^2_{|f|^{q-2}}$, i.e.

$$\int_{\mathbb{R}^d} |Mg|^2 |f|^{q-2} \leq C \int_{\mathbb{R}^d} |g|^2 |f|^{q-2}.$$

- This holds **if and only if** $|f|^{q-2}$ is a Muckenhoupt $\mathcal{A}_2$ weight.
**Definition:** $\omega \in \mathcal{A}_2$, if for all balls $B_R$

$$\left(\frac{1}{|B_R|}\int_{B_R} \omega\right)\left(\frac{1}{|B_R|}\int_{B_R} \omega^{-1}\right) \leq C$$

# Weighted theory for linear operators

**Theorem (Bulíček, Diening, Sch)**

*Let $\Omega$ be a $\mathcal{C}^1$ domain, $\tilde{A} \in \mathcal{C}(\overline{\Omega})$ and $\omega \in \mathcal{A}_q$.*
*Then for $f \in L^q_\omega(\Omega)$ there exists a unique $u \in W^{1,1}_0(\Omega)$ solving*

$$\operatorname{div}(\tilde{A}\nabla u) = \operatorname{div} f$$

*in the distributional sense, fulfilling*

$$\int_\Omega |\nabla u|^q \omega \le C \int_\Omega |f|^q \omega. \tag{0.2}$$

We say that $\omega \in \mathcal{A}_q$ if and only if

$$\left(\frac{1}{|B_R|}\int_{B_R} \omega\right)\left(\frac{1}{|B_R|}\int_{B_R} \omega^{-(q'-1)}\right)^{\frac{1}{q'-1}} \le C.$$

# Estimates revisited

- For $q < p$ Can we find a proper weight such that any $f \in L^q$ belongs to $L^p_\omega$?
- Yes: $\omega := (M|f|)^{q-p}$

**Theorem (Bulíček, Sch)**

*There is an $\varepsilon$ depending on the p-growth such that for all $q \in [p - \varepsilon, p]$ the following holds. If $f \in L^q(\Omega)$, then there exists $u \in W^{1,q}_0(\Omega)$ which is a distributional solution to*

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*Moreover,*

$$\int_\Omega |\nabla u|^p \,(M(f+1))^{q-p} \,\mathrm{d}x \le c \int_\Omega |f|^q \,\mathrm{d}x + c.$$

For the estimate we needed to develop a relative Truncation.

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# Identification of a weak limit-Div Curl lemma

Theorem (Weighted–div–curl–biting Lemma; Bulíček, Diening, Sch)

*Let $\omega \in \mathcal{A}_p$. Assume that $a^k \rightharpoonup a$, $b^k \rightharpoonup b$ weakly in $L^1(\Omega;\mathbb{R}^d)$ and*

$$\sup_{k\in\mathbb{N}} \int_\Omega |a^k|^p \omega + |b^k|^{p'} \omega \,\mathrm{d}x < \infty.$$

*and*

$$\begin{aligned} &\operatorname{div} b^k && \textit{is precompact in } (W_0^{1,\infty}(\Omega))^* \\ &\nabla a^k - (\nabla a^k)^T && \textit{is precompact in } (W_0^{1,\infty}(\Omega))^*. \end{aligned}$$

*Then there exists a sequence of subsets $E_j \subset \Omega$ with $|\Omega \setminus E_j| \to 0$ as $j \to \infty$ such that (for a subsequence)*

$$a^k \cdot b^k \omega \rightharpoonup a \cdot b\,\omega \qquad \textit{weakly in } L^1(E_j) \qquad \forall j \in \mathbb{N}.$$

# Use of weighted–div–curl–biting lemma

- Set $a^n := \nabla u^n$, $b^n := A(\cdot, \nabla u^n)$

$$\int_\Omega |a^n|^p \omega + |b^n|^{p'} \omega \le C \int_\Omega |\nabla u^n|^p \omega \le C$$

- Check the compactness:

$$\begin{aligned} \operatorname{div} b^n &= \operatorname{div} |\min\{f, n\}|^{p-2} \min\{f, n\} \quad && \text{compact} \\ \nabla a^n - (\nabla a^n)^T &= \nabla(\nabla u^n) - (\nabla(\nabla u^n))^T \equiv 0 && \text{compact} \end{aligned}$$

- $\Longrightarrow$ there exists $E_j \subset \Omega$ such that $|\Omega \setminus E_j| \to 0$ as $j \to \infty$ and

$$A(\cdot, \nabla u^n) \cdot \nabla u^n \omega = b^n \cdot a^n \omega \stackrel{\text{in } L^1(E_j)}{\rightharpoonup} b \cdot a\, \omega = \overline{A} \cdot \nabla u \omega$$

- Since $\omega \neq 0$ a.e. the Minty trick $\Longrightarrow$ $\overline{A}(x) = A(x, \nabla u(x))$ a.e. in $E_j$ and consequently also in $\Omega$.

# Application to the Ladyzhenskaya & Lions model

One (ambitious) aim is to built a theory for the shear depending viscosity model: $\operatorname{div} u = 0$ and

$$\operatorname{div}(u \otimes u) - \operatorname{div}(\nu_0 + \nu_1 |\varepsilon u|^{p-2})\varepsilon u + \nabla \pi = -\operatorname{div} f$$

$\nu_0, \nu_1$ constant and $p \in (1, \infty)$.

**Existence theory** in the natural space: $(u, \pi) \in W^{1,p}_{0,\mathrm{div}}(\Omega) \times L^{p'}_0(\Omega)$, if $f \in L^{p'}(\Omega)$ Ladyzhenskaya '69, Lions '69 $p > 9/5$ (inst. $p > 11/5$), Frehse–Málek–Steinhauer '00–'03 $p > 6/5$ (inst. $p \geq \frac{8}{5}$), Diening–Ruzicka–Wolf '10, Breit–Diening–Schw. '13 (inst. $p > 6/5$).

**Question:** $f \notin L^p(\Omega)$? Quite open. We have a result for $A$ with 2-growth satisfying: `Asymptotic Uhlenbeck condition`:
$\limsup_{|\eta| \to \infty} \operatorname*{ess\,sup}_{x \in \Omega} \frac{|A(x,\eta) - \nu_0 \eta|}{|\eta|} = 0.$

For uniqueness: `Strong asymptotic Uhlenbeck condition`:
$\limsup_{|\eta| \to \infty} \operatorname*{ess\,sup}_{x \in \Omega} \left| \frac{\partial A(x,\eta)}{\partial \eta} - \nu_0 Id \right| = 0.$

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**Existence theory** in the natural space: $(u, \pi) \in W^{1,p}_{0,\mathrm{div}}(\Omega) \times L^{p'}_0(\Omega)$, if $f \in L^{p'}(\Omega)$ Ladyzhenskaya '69, Lions '69 $p > 9/5$ (inst. $p > 11/5$), Frehse–Málek–Steinhauer '00–'03 $p > 6/5$ (inst. $p \geq \frac{8}{5}$), Diening–Ruzicka–Wolf '10, Breit–Diening–Schw. '13 (inst. $p > 6/5$).

**Question:** $f \notin L^p(\Omega)$? Quite open. We have a result for $A$ with 2-growth satisfying: `Asymptotic Uhlenbeck condition`:

$\limsup_{|\eta|\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} \frac{|A(x,\eta) - \nu_0 \eta|}{|\eta|} = 0.$

For uniqueness: `Strong asymptotic Uhlenbeck condition`:

$\limsup_{|\eta|\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} \left| \frac{\partial A(x,\eta)}{\partial \eta} - \nu_0 Id \right| = 0.$

# Application to the Ladyzhenskaya & Lions model

Theorem (Bulíček, Burczak, Sch '16)

*Assume that $A$ satisfies the asymptotically Uhlenbeck condition. Then for $f \in L^q(\Omega)$ with $q \in (1, \infty)$ there exists $(u, \pi) \in W^{1,q}_{0,\mathrm{div}}(\Omega) \times L^q_0(\Omega)$ solving $-\operatorname{div} A(\cdot, \varepsilon u) + \nabla\pi = -\operatorname{div} f$ in $\Omega$, and*

$$\int_\Omega |\nabla u|^q + |\pi|^q \, dx \le c \int_\Omega |f|^q \, dx + c.$$

*Moreover, if $A$ is strictly monotone and satisfies the strong asymptotic Uhlenbeck condition, then the solution is unique.*

The following example is included: $A(x, \eta) = \nu_0 \, |\varepsilon u + 1|^{p-2} \, \varepsilon u$, for $p \in (1, 2]$.

It behaves Newtonian at large shear speeds.

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# The parabolic challenge

**The setting:**

$$\partial_t u = \operatorname{div}(G) \text{ in } [0,T] \times \Omega \text{ with } G \in L^q([0,T] \times \Omega).$$

If $q \in (1, \infty)$ this is equivalent to $\partial_t u \in L^{q'}([0,T], W^{-1,q'}(\Omega))$.
For example:

$$\partial_t u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) \text{ with } \nabla u \in L^p([0,T] \times \Omega)$$

How can we construct a Lipschitz truncation?
Modify the bad set $\{M^\alpha(\nabla u) > \lambda\} \cup \{\alpha M^\alpha(G) > \lambda\} =: \mathcal{O}^\alpha_\lambda$.

Here

$$M^\alpha(g)(t,x) = \sup_{r>0} \fint_{t-\alpha r^2}^{t} \fint_{B_r(x)} |g| \, dx \, dt.$$

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## Theorem (Parabolic Lip-Trunc; Diening, Sch, Stroffolini, Verde)

*Let* $G \in L^{p'}([0,T]\times\Omega)$ *and* $u \in L^p([0,T], W_0^{1,p}(\Omega))$ *satisfy* $\partial_t u_\lambda = \operatorname{div} G$. *Then there exists an approximation* $u_\lambda \in L^p([0,T], W_0^{1,p}(\Omega))$ *with the following properties:*

1. $u_\lambda$ *is Lipschitz continuous with respect to the scaled, parabolic metric, i.e.*

$$|u_\lambda(t,x) - u_\lambda(s,y)| \le c\,\lambda \max\left\{ \frac{|t-s|^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}}, |x-y| \right\}$$

   *for all* $(t,x),(s,y) \in (0,T]\times\Omega$.

2. *for all* $\eta \in C_0^\infty((0,T))$ *it holds:*

$$\langle \partial_t u, u_\lambda \eta\rangle = \frac{1}{2}\int_{[0,T]\times\Omega} (|u_\lambda|^2 - 2u\cdot u_\lambda)\partial_t\eta dz + \int_{\mathcal{O}_\lambda^\alpha} (\partial_t u_\lambda)(u_\lambda - u)\eta dz.$$

Theorem (Almost caloric; Diening, Sch, Stroffolini, Verde)

*Let $Q = [a,b] \times B$ be a time space cylinder. Let $\sigma, \theta \in (0,1)$ and $q \in [1,\infty)$. Then, for $\varepsilon > 0$ there exists a $\delta > 0$ s.t. for $u \in L^p([a,b], W_0^{1,p}(B))$, $G \in L^{p'}(Q)$ and $u_t = \operatorname{div} G$. We find that if*

$$\Big| \fint_Q -u\partial_t \xi + |\nabla u|^{p-2}\nabla u \cdot \nabla \xi dz \Big| \leq \delta \Big[ \fint_Q |\nabla u|^p + |G|^{p'}\, dz + \|\nabla \xi\|_\infty^p \Big],$$

*for all $\xi \in C_0^\infty(Q)$, then for $V(z) = |z|^{\frac{p-2}{2}} z$ and*

$$\Big( \fint_a^b \Big( \fint_B \frac{|u-h|^{2\sigma}}{|b-a|^\sigma} dx \Big)^{\frac{q}{\sigma}} dt \Big)^{\frac{1}{q}} + \Big( \fint_Q \Big| |\nabla u|^{\frac{p-4}{2}} \nabla u - |\nabla h|^{\frac{p-4}{2}} \nabla h \Big|^{2\theta} dz \Big)^{\frac{1}{\theta}}$$

$$\leq \varepsilon \fint_Q |\nabla u|^p + |G|^{p'}\, dz,$$

*where $\partial_t h - \operatorname{div}(|\nabla h|^{p-2} \nabla h) = 0$ in $Q$with $h = u$ on $\partial_p Q$.*

# Outlook and WIP

- Very weak instationary parabolic solutions: weighted estimates (WIP with M. Bulíček and J. Burczak).
- Existence and regularity for very weak solutions of power law fluids: Solenoidal Lipschitz/ Relative Truncation with 0 trace (WIP with C. Mîndrilă).
- Instationary fluids: The parabolic solenoidal Lipschitz truncation with 0 trace.