Existence for evolutionary problems with linear growth by stability methods

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The Cauchy-Dirichlet problem

Basic data:

- Dimensions $n, N \in \mathbb{N}$.
- $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $0 < T < \infty$, $\Omega_T := \Omega \times (0, T)$.
- Time dependent boundary values $g\colon \Omega_T \to \mathbb{R}^N$.
- Integrand $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying a linear growth condition, convex with respect to the gradient variable.

Cauchy-Dirichlet problem: Find $u\colon \Omega_T \to \mathbb{R}^N$ such that

$$\begin{cases} \partial_t u - \operatorname{div}(D_\xi f(x, Du)) = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial_{par}\Omega_T. \end{cases}$$

The integrand

Assumptions:

- Borel measurable.
- Linear growth and coercivity condition

$$\nu|\xi| \leq f(x,\xi) \leq L(1+|\xi|)$$

  for all $x \in \Omega$, $\xi \in \mathbb{R}^{N\times n}$ with constants $0 < \nu \leq L$.
- $\xi \mapsto f(x,\xi)$ convex for a.e. $x \in \Omega$.
- Continuity condition (explained later).

Functionals with linear growth – Choice of function space

Consider the elliptic functional

$$\mathbf{F}[u] := \int_\Omega f(x, Du)\,\mathrm{d}x.$$

- $\mathbf{F}$ is finite on $W^{1,1}(\Omega, \mathbb{R}^N)$.
- Under the conditions above or even reasonable extra assumptions, $\mathbf{F}$ does not attain its minimum in any Dirichlet class $W^{1,1}_{u_o}(\Omega, \mathbb{R}^N)$.
- Therefore, extend $\mathbf{F}$ to $\mathrm{BV}(\Omega, \mathbb{R}^N)$.

Functionals with linear growth – Boundary values

Solid Dirichlet boundary values:

- The trace operator is not continuous with respect to weak* convergence in $\mathrm{BV}(\Omega, \mathbb{R}^N)$.
- Therefore, dealing with boundary values is delicate.
- Consider a reference set Ω^* compactly containing Ω.
- For a reference function $u_o \in \mathrm{BV}(\Omega^*, \mathbb{R}^N)$ define $\mathrm{BV}_{u_o}(\Omega, \mathbb{R}^N)$ as the space of functions $u \in \mathrm{BV}(\Omega^*, \mathbb{R}^N)$, which satisfy $u = u_o$ a.e. on $\Omega^* \setminus \overline{\Omega}$.

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Extended integrand: Borel measurable function $f \colon \Omega^* \times \mathbb{R}^{N\times n} \to [0, \infty)$ such that

- $\nu|\xi| \le f(x, \xi) \le L(1 + |\xi|)$ for all $x \in \Omega^*$, $\xi \in \mathbb{R}^{N\times n}$,
- $\xi \mapsto f(x, \xi)$ convex for a.e. $x \in \Omega^*$.

Functionals with linear growth – Recession function

Definition (Recession function)

The recession function $f^\infty\colon \overline{\Omega^*} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is defined by

$$f^\infty(x,\xi) := \liminf_{\substack{\tilde{x}\to x, \tilde{\xi}\to\xi \\ t\downarrow 0}} t f\big(\tilde{x}, t^{-1}\tilde{\xi}\big) \qquad \text{for } (x,\xi) \in \overline{\Omega^*} \times \big(\mathbb{R}^{N\times n} \setminus \{0\}\big),$$

and $f^\infty(x,0) := 0$ for $x \in \overline{\Omega^*}$.

- Takes into account the jumps of BV functions.

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and $f^\infty(x,0) := 0$ for $x \in \overline{\Omega^*}$.

- Takes into account the jumps of BV functions.
- Continuity assumption: For every $(x,\xi) \in \overline{\Omega^*} \times \big(\mathbb{R}^{N\times n} \setminus \{0\}\big)$,

$$\lim_{\substack{\tilde{x}\to x, \tilde{\xi}\to\xi \\ t\downarrow 0}} t f\big(\tilde{x}, t^{-1}\tilde{\xi}\big) \text{ exists in } \mathbb{R}.$$

This condition ensures that f^∞ is continuous on $\overline{\Omega^*} \times \mathbb{R}^{N\times n}$.

Functionals with linear growth – Extended functional

Notation:

- $D^a u$ is the absolutely continuous part of the Lebesgue decomposition of Du with respect to $\mathcal{L}^n$.
- $D^s u$ is the singular part of the Lebesgue decomposition of Du with respect to $\mathcal{L}^n$.
- ∇u denotes the Radon-Nikodym density of $D^a u$ with respect to $\mathcal{L}^n$.

Extended functional: Define $\boldsymbol{\mathcal{F}}\colon \mathrm{BV}(\Omega, \mathbb{R}^N) \to [0, \infty)$ by

$$\boldsymbol{\mathcal{F}}[u] := \int_{\Omega^*} f(x, \nabla u)\,\mathrm{d}x + \int_{\Omega^*} f^\infty\left(x, \frac{D^s u}{|D^s u|}\right)\mathrm{d}|D^s u|.$$

Parabolic function spaces related to $\mathrm{BV}(\Omega, \mathbb{R}^N)$

- Note that $\mathrm{BV}(\Omega^*, \mathbb{R}^N)$ is not separable. Therefore, we have problems with the Bochner measurability condition of $L^1(0, T; \mathrm{BV}(\Omega^*, \mathbb{R}^N))$.
- Use $L^1_{w*}(0, T; \mathrm{BV}(\Omega^*, \mathbb{R}^N))$, the space of weakly* measurable maps $u\colon (0, T) \to \mathrm{BV}(\Omega^*, \mathbb{R}^N)$ with $t \mapsto \|u(t)\|_{\mathrm{BV}(\Omega^*, \mathbb{R}^N)} \in L^1(0, T)$.

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- For $g \in L^1_{w*}(0,T;\mathrm{BV}(\Omega^*, \mathbb{R}^N))$, $g + L^1_{w*}(0,T;\mathrm{BV}_0(\Omega, \mathbb{R}^N))$ denotes the affine subspace of functions $u \in L^1_{w*}(0,T;\mathrm{BV}(\Omega^*, \mathbb{R}^N))$ that satisfy $u(t) \in g(t) + \mathrm{BV}_0(\Omega, \mathbb{R}^N)$ for a.e. $t \in (0,T)$.

Assumptions on the boundary values

- $g \in L^1(0,T;W^{1,1}(\Omega^*,\mathbb{R}^N))$;
- $\partial_t g \in L^1(0,T;L^2(\Omega^*,\mathbb{R}^N))$;
- $g_o := g(0) \in L^2(\Omega^*,\mathbb{R}^N)$.

Variational solutions

Definition (Variational solutions)

Assume that the integrand f, the functional $\boldsymbol{\mathcal{F}}$ and the boundary values g are as above. A function

$$u \in L^\infty\big(0,T;L^2(\Omega^*,\mathbb{R}^N)\big) \cap \big(g + L^1_{w*}\big(0,T;\mathrm{BV}_0(\Omega,\mathbb{R}^N)\big)\big)$$

is a variational solution associated with f and g if and only if the variational inequality

$$\int_0^\tau \boldsymbol{\mathcal{F}}[u]\,\mathrm{d}t \le \iint_{\Omega^*_\tau} \partial_t v \cdot (v-u)\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \boldsymbol{\mathcal{F}}[v]\,\mathrm{d}t - \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega^*,\mathbb{R}^N)} + \tfrac{1}{2}\|v(0)-g_o\|^2_{L^2(\Omega^*,\mathbb{R}^N)}$$

holds true for a.e. $\tau \in [0,T]$ and any comparison map $v \in g + L^1_{w*}(0,T;\mathrm{BV}_0(\Omega,\mathbb{R}^N))$ with $\partial_t v \in L^1(0,T;L^2(\Omega^*,\mathbb{R}^N))$ and $v(0) \in L^2(\Omega^*,\mathbb{R}^N)$.

Approximation of the Cauchy-Dirichlet problem

Approximation of the integrand:

- For $p > 1$, consider f^p.
- Standard p- growth and coercivity condition

$$\nu^p|\xi|^p \leq f^p(x,\xi) \leq 2^p L^p(1+|\xi|^p)$$

  for all $x \in \Omega^*$, $\xi \in \mathbb{R}^{N\times n}$.
- $\xi \mapsto f^p(x,\xi)$ convex for a.e. $x \in \Omega^*$.

Assumptions on the boundary values:

- $g_p \in L^p(0,T;W^{1,p}(\Omega^*,\mathbb{R}^N))$;
- $\partial_t g_p \in L^1(0,T;L^2(\Omega^*,\mathbb{R}^N))$;
- $g_{p,o} := g_p(0) \in L^2(\Omega^*,\mathbb{R}^N)$.

Variational solutions for $p > 1$

Definition (Variational solutions, $p > 1$)

Assume that $p > 1$ and that the integrand f and boundary values g_p are as above. A function

$$u \in C^0\big([0,T]; L^2(\Omega,\mathbb{R}^N)\big) \cap \big(g_p + L^p\big(0,T; W_0^{1,p}(\Omega,\mathbb{R}^N)\big)\big)$$

is a variational solution associated with f^p and g_p if and only if the variational inequality

$$\iint_{\Omega_\tau} f^p(x,Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_\tau} \partial_t v \cdot (v-u)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_\tau} f^p(x,Dv)\,\mathrm{d}x\mathrm{d}t - \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac{1}{2}\|v(0)-g_{p,o}\|^2_{L^2(\Omega,\mathbb{R}^N)}$$

holds true for any $\tau \in [0,T]$ and any comparison map $v \in g_p + L^p(0,T; W_0^{1,p}(\Omega,\mathbb{R}^N))$ with $\partial_t v \in L^1(0,T; L^2(\Omega,\mathbb{R}^N))$ and $v(0) \in L^2(\Omega,\mathbb{R}^N)$.

Existence of variational solutions for $p > 1$

- Existence result for $\partial_t g \in L^2(\Omega_T, \mathbb{R}^N)$ has already been established.
- Refined existence result for $\partial_t g \in L^1(0, T; L^2(\Omega, \mathbb{R}^N))$ by approximation.

Further assumptions

Exponents:

- $p_i > 1$ for $i \in \mathbb{N}$,
- $p_i \downarrow 1$ as $i \to \infty$.

Convergence assumptions on $g_i := g_{p_i}$:

- $g_i \to g$ in $L^1\big(0,T;W^{1,1}(\Omega^*,\mathbb{R}^N)\big)$;
- $g_i \overset{*}{\rightharpoonup} g$ weakly* in $L^\infty\big(0,T;L^2(\Omega^*,\mathbb{R}^N)\big)$;
- $\partial_t g_i \to \partial_t g$ in $L^1\big(0,T;L^2(\Omega^*,\mathbb{R}^N)\big)$;
- $\displaystyle\lim_{i\to\infty}\iint_{\Omega_T^*}|Dg_i|^{p_i}\,\mathrm{d}x\mathrm{d}t = \iint_{\Omega_T^*}|Dg|\,\mathrm{d}x\mathrm{d}t.$

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Variational solution associated with f^{p_i} and g_i:

$$u_i \in C^0\big([0,T];L^2(\Omega,\mathbb{R}^N)\big) \cap \Big(g_i + L^{p_i}\big(0,T;W_0^{1,p_i}(\Omega,\mathbb{R}^N)\big)\Big).$$

The main result

Theorem (Existence and stability result)

Assume that the sequence $(p_i)_{i\in\mathbb{N}}$*, the integrand* f*, the functional* $\boldsymbol{\mathcal{F}}$*, the boundary values* g *and* g_i *and the variational solutions* u_i *are as above. Then, there exists a subsequence* $(u_{i_k})_{k\in\mathbb{N}}$ *and*

$$u \in L^\infty\big(0,T;L^2(\Omega^*,\mathbb{R}^N)\big) \cap \big(g + L^1_{w^*}(0,T;\mathrm{BV}_0(\Omega,\mathbb{R}^N))\big)$$

such that

$$\begin{cases} u_{i_k} \to u & \text{in } L^1(\Omega_T,\mathbb{R}^N), \\ u_{i_k} \xrightarrow{*} u & \text{weakly* in } L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big) \end{cases}$$

as $k\to\infty$*. The limit function* u *is a variational solution associated with* f *and* g*.*

Known existence results for the total variation flow

- Andreu, Ballester, Caselles & Mazón (2001):
  - Notion of entropy solutions.
  - Cauchy-Dirichlet problem with initial datum in L^1 and time independent boundary values.
  - Proof by nonlinear semigroup theory.
- Andreu, Mazón & Moll (2005):
  - Nonlinear boundary condition.
  - For initial data in L^2, entropy solutions are strong solutions.
  - Proof of the existence result by nonlinear semigroup theory.
- Bögelein, Duzaar & Scheven (2016):
  - Notion of variational solutions.
  - Cauchy-Dirichlet problem with initial datum in L^2 and time dependent boundary values.
  - Proof via method of minimizing movements.

Known existence results for other equations

- Lichnewsky & Temam (1978):
  - Time dependent minimal surface problem.
  - Notion of variational solutions.
  - Cauchy-Dirichlet problem with time independent boundary values.
  - Proof by parabolic regularization.

- Andreu, Caselles & Mazón (2002):
  - Equations of the type $\partial_t u - \operatorname{div}(D_\xi f(x, Du)) = 0$, where f satisfies a linear growth condition, $\xi \mapsto f(x, \xi)$ is convex and in $C^1(\mathbb{R}^n)$ and f^∞ is continuous.
  - This excludes the total variation flow.
  - Notion of entropy solutions.
  - Cauchy-Dirichlet problem with time independent boundary values.

Known stability results

- Tölle (2011):
  - Total variation flow.
  - Cauchy-Dirichlet problem with zero boundary values and Cauchy-Neumann problem.
  - Convergence of solutions strongly in $L^\infty(0,T;L^2(\Omega))$.
  - Proof by Mosco convergence of the associated functionals.
- Gianazza & Klaus (2017):
  - Total variation flow.
  - Notion of variational solutions.
  - Cauchy-Dirichlet problem with time independent boundary values.
  - Proof relies on a density result.

Possible extensions

- Free boundary values.
- Equations including a lower order term, i.e.

$$\partial_t u - \operatorname{div}(D_\xi f(x, Du)) = -D_u g(x, u).$$

- Strong convergence in $L^\infty(0, T; L^2(\Omega, \mathbb{R}^N))$.

Proof sketch – Convergence of variational solutions I

- Without loss of generality, assume that $p_i \le 2$ for all $i \in \mathbb{N}$.
- By suitable energy bounds, $(u_i)_{i\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\Omega,\mathbb{R}^N)) \cap L^1(0,T;W^{1,1}(\Omega,\mathbb{R}^N))$.
- $u_i \xrightarrow{*} u$ weakly* in $L^\infty(0,T;L^2(\Omega^*,\mathbb{R}^N))$ for a (not relabelled) subsequence.

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- $u_i \xrightarrow{*} u$ weakly* in $L^\infty(0,T;L^2(\Omega^*,\mathbb{R}^N))$ for a (not relabelled) subsequence.
- $\tilde{u}_i$ denotes the extension of u_i to Ω^* by g_i.
- By a lemma concerning the regularity of the limit map and since $g_i \xrightarrow{*} g$ in $L^\infty(0,T;L^2(\Omega^*,\mathbb{R}^N))$ as $i \to \infty$, conclude that

$$\tilde{u}_i \xrightarrow{*} u \text{ weakly* in } L^\infty(0,T;L^2(\Omega^*,\mathbb{R}^N))$$

  for a limit map

$$u \in L^\infty(0,T;L^2(\Omega^*,\mathbb{R}^N)) \cap \big(g + L^1_{w*}(0,T;\mathrm{BV}_0(\Omega,\mathbb{R}^N))\big).$$

Proof sketch – Convergence of variational solutions II

- Show that

$$\int_0^{T-h} \|u_i(t+h) - u_i(t)\|_{W^{-\ell,2}(\Omega,\mathbb{R}^N)}\,\mathrm{d}t \le c\,\|Du_i\|^{p_i-1}_{L^{p_i}(\Omega_T,\mathbb{R}^N)} h^{\frac{1}{2}}$$

  for all $p_i \le 2$, $\ell \ge 1$ with a constant $c = c(Nn, \ell, L, |\Omega|, T)$.

- For $\ell \ge \frac{n}{2}$ apply the Jacques Simon lemma with $p = 1$ and the spaces

$$W^{1,1}(\Omega,\mathbb{R}^N) \subset L^1(\Omega,\mathbb{R}^N) \subset W^{-\ell,2}(\Omega,\mathbb{R}^N).$$

  This yields $u_i \to u$ in $L^1(\Omega_T,\mathbb{R}^N)$.

- Since $g_i \to g$ in $L^1(\Omega_T^*,\mathbb{R}^N)$, conclude that

$$\tilde{u}_i \to u \text{ in } L^1(\Omega_T^*,\mathbb{R}^N).$$

Proof sketch – Preliminary variational inequality I

- Choose $g_i + w$ for some $w \in L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^N))$ with $\partial_t w \in L^1(0,T;L^2(\Omega,\mathbb{R}^N))$ and $w(0) \in L^2(\Omega,\mathbb{R}^N)$. as comparison map in the variational inequality associated with f^{p_i} and g_i, i.e.

$$\begin{aligned}\iint_{\Omega_\tau^*} f^{p_i}(x, D\tilde{u}_i)\,\mathrm{d}x\mathrm{d}t \le& \iint_{\Omega_\tau^*} \partial_t(g_i + w)\cdot(g_i + w - \tilde{u}_i)\,\mathrm{d}x\mathrm{d}t \\ &+ \iint_{\Omega_\tau^*} f^{p_i}(x, D(g_i + w))\,\mathrm{d}x\mathrm{d}t \\ &- \tfrac{1}{2}\|(g_i + w - \tilde{u}_i)(\tau)\|^2_{L^2(\Omega^*,\mathbb{R}^N)} \\ &+ \tfrac{1}{2}\|w(0)\|^2_{L^2(\Omega^*,\mathbb{R}^N)}.\end{aligned}$$

- Aim: let $i \to \infty$.

Proof sketch – Preliminary variational inequality II

- To treat the boundary term $\frac{1}{2}\|(g_i + w - \tilde{u}_i)(\tau)\|^2_{L^2(\Omega^*,\mathbb{R}^N)}$, take the mean integral of the previous inequality over $(t_o, t_o + \delta)$ for $t_o \in (0, T)$ and $\delta < T - t_o$.
- Deduce from $\tilde{u}_i \to u$ in $L^1(\Omega^*_T, \mathbb{R}^N)$ by Reshetnyak's lower semicontinuity theorem, Fatou's lemma and Hölder's inequality that

$$\int_0^{t_o} \boldsymbol{\mathcal{F}}[u]\,\mathrm{d}t \le \left[\iint_{\Omega^*_{t_o}} f^{p_i}(x, D\tilde{u}_i)\,\mathrm{d}x\mathrm{d}t \right]^{\frac{1}{p_i}}.$$

Proof sketch – Preliminary variational inequality III

- By the convergence assumptions on g_i and the properties of f, infer

$$
\begin{aligned}
\int_0^{t_o} \boldsymbol{\mathcal{F}}[u]\,\mathrm{d}t \le &\, -\!\!\!\!\!\!\int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau^*} \partial_t(g+w)\cdot(g+w-u)\,\mathrm{d}x\mathrm{d}t\mathrm{d}\tau \\
&+ \iint_{\Omega_{t_o+\delta}^*} f(x, D(g+w))\,\mathrm{d}x\mathrm{d}t \\
&- -\!\!\!\!\!\!\int_{t_o}^{t_o+\delta} \tfrac{1}{2}\|(g+w-u)(\tau)\|^2_{L^2(\Omega^*,\mathbb{R}^N)}\mathrm{d}\tau \\
&+ \tfrac{1}{2}\|w(0)\|^2_{L^2(\Omega^*,\mathbb{R}^N)}
\end{aligned}
$$

  for any $w \in L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^N))$.
- Next, replace w by a function $v \in L^1_{w*}(0,T;\mathrm{BV}_0(\Omega,\mathbb{R}^N))$ with $\partial_t v \in L^1(0,T;L^2(\Omega^*,\mathbb{R}^N))$ and $v(0) \in L^2(\Omega^*,\mathbb{R}^N)$.
- To this end, consider suitable mollifications $w := M_\varepsilon[v]$.

Proof sketch – Definition of regularizations

- Inner parallel set $\Omega_\varepsilon := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.
- Cut-off function η_ε with $\eta_\varepsilon \equiv 0$ on $\mathbb{R}^n \setminus \Omega_\varepsilon$, $\eta_\varepsilon \equiv 1$ on $\Omega_{\varepsilon+\sqrt{\varepsilon}}$ and

$$\eta_\varepsilon(x) := \frac{\operatorname{dist}(x, \partial\Omega) - \varepsilon}{\sqrt{\varepsilon}} \quad \text{on } \Omega_\varepsilon \setminus \Omega_{\varepsilon+\sqrt{\varepsilon}}.$$

- Standard mollifier ϕ_ε in $\mathbb{R}^n$.
- For v as above, define $M_\varepsilon[v] := (\eta_\varepsilon v) * \phi_\varepsilon$.

Proof sketch – Some properties of the regularizations

- $M_\varepsilon[v] \in C^0([0,T]; W_0^{1,2}(\Omega, \mathbb{R}^N))$;
- $M_\varepsilon[v](0) \to v(0)$ in $L^2(\mathbb{R}^n, \mathbb{R}^N)$;
- $M_\varepsilon[v] \to v$ in $L^2(\mathbb{R}^n \times (0,T), \mathbb{R}^N)$ as $\varepsilon \downarrow 0$;
- $\|M_\varepsilon[v]\|_{L^\infty(0,T;L^2(\mathbb{R}^n,\mathbb{R}^N))} \leq \|v\|_{L^\infty(0,T;L^2(\mathbb{R}^n,\mathbb{R}^N))}$;
- $\partial_t M_\varepsilon[v] \to \partial_t v$ in $L^1(0,T; L^2(\Omega, \mathbb{R}^N))$ as $\varepsilon \downarrow 0$.

Proof sketch – First conclusions

These properties allow us to treat

- $\displaystyle -\!\!\!\!\!\!\int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau^*} \partial_t (g + M_\varepsilon[v]) \cdot (g + M_\varepsilon[v] - u) \, \mathrm{d}x\mathrm{d}t\mathrm{d}\tau;$
- $\displaystyle -\!\!\!\!\!\!\int_{t_o}^{t_o+\delta} \tfrac{1}{2} \|(g + M_\varepsilon[v] - u)(\tau)\|^2_{L^2(\Omega^*,\mathbb{R}^N)} \mathrm{d}\tau;$
- $\tfrac{1}{2} \|M_\varepsilon[v](0)\|^2_{L^2(\Omega^*,\mathbb{R}^N)}.$

Remaining term:

$$\iint_{\Omega^*_{t_o+\delta}} f(x, D(g + M_\varepsilon[v])) \, \mathrm{d}x\mathrm{d}t.$$

Proof sketch – Further properties of the regularizations

- For any function $g \in L^1(0,T; W^{1,1}(\mathbb{R}^n, \mathbb{R}^N))$ and a.e. $t \in [0,T]$

$$\begin{cases} DM_\varepsilon[v](t) \overset{*}{\rightharpoonup} Dv(t) \text{ weakly}^* \text{ in } \mathrm{RM}(\mathbb{R}^n; \mathbb{R}^{N\times n}), \\ \big|(\mathcal{L}^n, Dg(t) + DM_\varepsilon[v](t))\big|(\overline{\Omega^*}) \to \big|(\mathcal{L}^n, Dg(t) + Dv(t))\big|(\overline{\Omega^*}) \end{cases}$$

  in the limit $\varepsilon \downarrow 0$.

- This is called area-strict convergence, because

$$|(\mathcal{L}^n, \mu)|(\overline{\Omega^*}) = \int_{\Omega^*} \sqrt{1 + \mu^a}\, \mathrm{d}x + |\mu^s|(\overline{\Omega^*}).$$

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Proof sketch – Further properties of the regularizations

- For any function $g \in L^1(0,T;W^{1,1}(\mathbb{R}^n,\mathbb{R}^N))$ and a.e. $t \in [0,T]$

$$\begin{cases} DM_\varepsilon[v](t) \overset{*}{\rightharpoonup} Dv(t) \text{ weakly}^* \text{ in } \mathrm{RM}(\mathbb{R}^n;\mathbb{R}^{N\times n}), \\ \big|(\mathcal{L}^n, Dg(t) + DM_\varepsilon[v](t))\big|(\overline{\Omega^*}) \to \big|(\mathcal{L}^n, Dg(t) + Dv(t))\big|(\overline{\Omega^*}) \end{cases}$$

  in the limit $\varepsilon \downarrow 0$.

- This is called area-strict convergence, because

$$|(\mathcal{L}^n,\mu)|(\overline{\Omega^*}) = \int_{\Omega^*} \sqrt{1+\mu^a}\,\mathrm{d}x + |\mu^s|(\overline{\Omega^*}).$$

- $\sup\limits_{\varepsilon\in(0,1)} \big|DM_\varepsilon[v](t)\big|(\mathbb{R}^n) \le c(\partial\Omega)|Dv(t)|(\overline{\Omega}).$

Proof sketch – The remaining term

- By Reshetnyak's continuity theorem, deduce from the area-strict convergence property that

$$\int_{\Omega^*} f(x, D(g(t) + M_\varepsilon[v](t)))\,\mathrm{d}x \to \boldsymbol{\mathcal{F}}[g(t) + M_\varepsilon[v](t)]$$

  for a.e. $t \in [0,T]$ as $\varepsilon \downarrow 0$.

- Apply the dominated convergence theorem.

Proof sketch – Conclusion

- Conclude that

$$\int_0^{t_o} \boldsymbol{\mathcal{F}}[u]\,\mathrm{d}t \le \fint_{t_o}^{t_o+\delta} \iint_{\Omega_\tau^*} \partial_t(g+v)\cdot(g+v-u)\,\mathrm{d}x\mathrm{d}t\mathrm{d}\tau \\ + \int_0^{t_o} \boldsymbol{\mathcal{F}}[g+v]\,\mathrm{d}t \\ - \fint_{t_o}^{t_o+\delta} \tfrac{1}{2}\|(g+v-u)(\tau)\|^2_{L^2(\Omega^*,\mathbb{R}^N)}\mathrm{d}\tau \\ + \tfrac{1}{2}\|g(0)+v(0)-g_o\|^2_{L^2(\Omega^*,\mathbb{R}^N)}$$

  holds true for any $v \in L^1_{w*}(0,T;\mathrm{BV}_0(\Omega,\mathbb{R}^N))$ with $\partial_t v \in L^1(0,T;L^2(\Omega^*,\mathbb{R}^N))$ and $v(0) \in L^2(\Omega^*,\mathbb{R}^N)$.

- Let $\delta \downarrow 0$ to prove that u is a variational solution associated with f and g.

Thank you!