PERRON'S METHOD FOR THE POROUS MEDIUM EQUATION

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Abstract. This work extends Perron's method for the porous medium equation in the slow diffusion case. The main result shows that nonnegative continuous boundary functions are resolutive in a general cylindrical domain.

1. Introduction

The porous medium equation

$$\frac{\partial u}{\partial t} - \Delta u^m = 0$$

is an important prototype of a nonlinear parabolic equation and it is by now well understood. See the monographs [7], [13] and [16] for more on this topic. However, little is known about the boundary behaviour of solutions in irregular domains and with general boundary values, except for the case $m = 1$, when we have the classical heat equation [14]. We shall consider this challenging question. Our main objective is to apply the method, introduced for harmonic functions by Perron [10], to this fascinating nonlinear equation. We focus on the slow diffusion case $m > 1$ in cylindrical domains. For simplicity, we only consider nonnegative and bounded boundary functions, in which case the solutions are nonnegative and bounded as well, by the comparison principle. However, it is of utmost importance to allow solutions to attain the value zero, so that moving boundaries, such as those exhibited by the Barenblatt solution, are not excluded.

We consider the boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in} \quad \Omega_T, \\ u = g \quad \text{on} \quad \partial\Omega \times [0, T), \\ u(x, 0) = g(x, 0), \end{cases}$$

2010 *Mathematics Subject Classification.* Primary 35K55, Secondary 35K65, 35K20, 31C45.

Key words and phrases. Perron method, Porous medium equation, comparison principle, obstacles.

The research is supported by the Academy of Finland. Part of this paper was written during the authors' stay at the Institut Mittag-Leffler in Djursholm.

in a bounded open space-time cylinder $\Omega_T = \Omega \times (0, T)$ in $\mathbb{R}^{N+1}$. The precise definitions of the solution and the boundary conditions will be given later. For a given boundary value function g, Perron's method produces two functions: the upper solution $\overline{H}_g$ and the lower solution $\underline{H}_g$ with $\underline{H}_g \leq \overline{H}_g$. Our first main result is that the upper and lower Perron solutions are indeed weak solutions of the porous medium equation. However, the upper and lower solutions may still take the wrong boundary values. The construction can be performed not only for space-time cylinders but also for more general domains in $\mathbb{R}^{N+1}$.

A central question in this theory is to determine when the upper and lower solutions are the same function. A classical result in this direction is *Wiener's resolutivity theorem* for harmonic functions: if the boundary value function is continuous, the upper and lower Perron solutions coincide, see [15]. Our second main result extends this to the porous medium equation. More precisely, nonnegative continuous boundary functions are resolutive for the porous medium equation in general cylindrical domains in the slow diffusion case $m > 1$. No regularity assumptions on the base of the space-time cylinder are needed. As far as we know, the corresponding result for more general domains in $\mathbb{R}^{N+1}$ remains open.

Perron's method requires a parabolic comparison principle so that the upper and lower Perron solutions can be defined consistently. Our first step is to establish a comparison principle in general space-time cylinders. To prove the resolutivity theorem we first reduce the situation to smooth boundary values by approximation. The key step in the proof for smooth boundary values is constructing super- and subsolutions which are sufficiently regular in the time direction. We use a penalized problem related to the obstacle problem for the porous medium equation for this purpose, see [4]. Delicate approximation results and energy estimates play a pivotal role in the argument. We hope that these results will have other applications as well. It is likely that our results and methods also apply to more general equations of the type

$$\frac{\partial u}{\partial t} - \Delta A(u) = 0,$$

see [7] and [13].

2. Weak solutions and weak supersolutions

In this section, we discuss notion on which the construction of Perron solutions will be based. First, we introduce some notation.

Let Ω be an open and bounded subset of $\mathbb{R}^N$, and let $0 < t_1 < t_2 < T$. We denote space-time cylinders by $\Omega_T = \Omega \times (0, T)$ and $U_{t_1,t_2} = U \times (t_1, t_2)$, where $U \subset \Omega$ is an open set. We call a cylinder U_{t_1,t_2} *regular* if the boundary of the base set U is smooth. The *parabolic* *boundary* of a space-time cylinder U_{t_1,t_2} is the set

$$\partial_p U_{t_1,t_2} = (\overline{U} \times \{t_1\}) \cup (\partial U \times [t_1, t_2]),$$

i.e. only the initial and lateral boundaries are taken into account.

We use the notation $H^1(\Omega)$ for the Sobolev space consisting of functions u in $L^2(\Omega)$ such that the weak gradient exists and also belongs to $L^2(\Omega)$. The Sobolev space with zero boundary values $H^1_0(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The parabolic Sobolev space $L^2(0,T;H^1(\Omega))$ consists of measurable functions $u : \Omega_T \to [-\infty, \infty]$ such that $x \mapsto u(x,t)$ belongs to $H^1(\Omega)$ for almost all $t \in (0,T)$, and

$$\iint_{\Omega_T} \left(|u|^2 + |\nabla u|^2\right) \,\mathrm{d}x\,\mathrm{d}t < \infty.$$

The definition of the space $L^2(0,T;H^1_0(\Omega))$ is similar. We say that $u \in L^2_{loc}(0,T;H^1_{loc}(\Omega))$ if u belongs to the parabolic Sobolev space for all $U_{t_1,t_2} \Subset \Omega_T$. The symbol $\Subset$ means that the set is compactly containd in the bigger set.

Definition 2.1. Assume that $m > 1$. A nonnegative function $u : \Omega_T \to \mathbb{R}$ is a *weak solution* of the porous medium equation

$$\frac{\partial u}{\partial t} - \Delta u^m = 0 \tag{2.2}$$

in Ω_T, if $u^m \in L^2_{loc}(0,T;H^1_{loc}(\Omega))$ and

$$\iint_{\Omega_T} \left(-u\frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi\right) \mathrm{d}x\,\mathrm{d}t = 0 \tag{2.3}$$

for all smooth test functions φ compactly supported in Ω_T. We define *weak supersolutions* by requiring that the integral in (2.3) is nonnegative for nonnegative test functions φ.

Throughout the work we assume that $m > 1$. It is an interesting question whether corresponding results can be proved also when $m < 1$. Our results and methods also apply to solutions with a changing sign, but we have chosen to consider only nonnegative solutions for simplicity. However, it is important to allow solutions to attain the value zero.

Weak solutions are locally Hölder continuous after a possible redefinition on a set of $(N+1)$-dimensional measure zero; see [6, 8, 16] or [13, Chapter 7]. Thus, without loss of generality, we may assume that solutions are continuous. Moreover, weak supersolutions are lower semicontinuous after a redefinition on a set of $(N+1)$-dimensional measure zero, see [1].

Besides a local notion of weak solutions, we need a concept of weak solutions to the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in} \quad \Omega_T, \\ u = g \quad \text{on} \quad \partial\Omega \times [0,T), \\ u(x,0) = g(x,0), \end{cases}$$

where g is a positive, continuous function defined on $\overline{\Omega}_T$ with $g^m \in L^2(0,T;H^1(\Omega))$. The lateral boundary condition is interpreted in the Sobolev sense, meaning that $u^m - g^m \in L^2(0,T;H^1_0(\Omega))$. The initial condition is incorporated into the weak formulation by requiring that

$$\iint_{\Omega_T} \left(-u\frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \right) \mathrm{d}x\,\mathrm{d}t = \int_\Omega g(x,0)\varphi(x,0)\,\mathrm{d}x \tag{2.4}$$

for smooth test functions φ vanishing at the time T and with compact support in space. With this definition, solutions to the initial-boundary value problem are unique. This follows by an application of Oleĭnik's test function, see the proof of Lemma 3.2 below. It is straightforward to check that the initial values are attained in this sense if and only if

$$\lim_{t\to 0} \int_\Omega u(x,t)\eta(x)\,\mathrm{d}x = \int_\Omega g(x,0)\eta(x)\,\mathrm{d}x \tag{2.5}$$

for all $\eta \in C_0^\infty(\Omega)$.

3. Viscosity supersolutions

We will employ the notion of *viscosity supersolutions* to (2.2), following [9]. The term "viscosity" is used here purely as a label. In the case $m = 1$ this definition gives supertemperatures, see [14]. For the more common definition of viscosity solutions using pointwise touching test functions, we refer to [3] and [5]. It is an interesting question whether the two definitions give the same class of functions.

Definition 3.1. A function $u : \Omega_T \to [0,\infty]$ is a *viscosity supersolution*, if

(1) u is lower semicontinuous,
(2) u is finite in a dense subset of Ω_T, and
(3) the following comparison principle holds: Let $U_{t_1,t_2} \Subset \Omega$, and let h be a solution to (2.2) which is continuous in $\overline{U_{t_1,t_2}}$. If $h \le u$ on $\partial_p U_{t_1,t_2}$, the $h \le u$ in U_{t_1,t_2}.

The definition of *viscosity subsolutions* is similar; they are upper semicontinuous, and the inequalities in the comparison principle are reversed.

Observe that these functions are defined at *every* point. A similar definition was introduced by F. Riesz [11] for the Laplacian. The fundamental example of a viscosity supersolution in the sense of Definition

3.1 is the Barenblatt solution [2, 17], which is given by the formula

$$\mathcal{B}_m(x,t) = \begin{cases} t^{-\lambda}\left(C - \frac{\lambda(m-1)}{2mn}\frac{|x|^2}{t^{2\lambda/n}}\right)_+^{1/(m-1)}, & t>0,\\ 0, & t \le 0,\end{cases}$$

where

$$\lambda = \frac{n}{n(m-1)+2}.$$

The constant C is usually chosen so that

$$\int_\Omega \mathcal{B}_m(x,t)\,\mathrm{d}x = 1$$

for all $t > 0$. It is a viscosity supersolution, but not a weak supersolution. This is due to the lack of integrability of the gradient. For other examples, see [13]. However, *bounded* viscosity supersolutions are weak supersolutions. In particular, their mth power belongs to $L^2_{loc}(0,T;H^1_{loc}(\Omega))$. Weak supersolutions are viscosity supersolutions, provided that they are lower semicontinuous, see [1] and [9]. In the present work, it is enough for the reader to consider lower semicontinuous weak supersolutions that are defined at each point in their domain.

Some properties are immediate consequences of the definition. The pointwise minimum of a finite number of viscosity supersolutions is a viscosity supersolution. In particular, the truncations $\min\{u,k\}$, $k = 1,2,\dots$, of a viscosity supersolution u are viscosity supersolutions. The fact that an increasing limit of viscosity supersolutions is a viscosity supersolution, provided that the limit is finite in a dense subset, also follows directly from the definition.

Our main interest is Perron solutions with continuous boundary values in irregular domains. In this context, the situation does not change if one only considers bounded viscosity super- and subsolutions, and we shall do so from now on.

We begin with the definition of the *Poisson modification* of a viscosity supersolution. Let u be a bounded viscosity supersolution and $U_{t_1,t_2} \Subset \Omega_T$ be a regular space-time cylinder. We define

$$P(u, U_{t_1,t_2}) = \begin{cases} u & \text{in} \quad \Omega_T \setminus U_{t_1,t_2},\\ h & \text{in} \quad U_{t_1,t_2},\end{cases}$$

where h is the solution in U_{t_1,t_2} with boundary values u. The function h is constructed as follows: by semicontinuity, we find an increasing sequence (φ_k) of smooth functions converging to u pointwise in U_{t_1,t_2} as $k \to \infty$. Let h_k be the solution with values φ_k on the parabolic boundary of U_{t_1,t_2}. Then $h_k \le u$ and the sequence (h_k) is increasing by the comparison principle. It follows that $h = \lim h_k$ is a solution in U_{t_1,t_2}. Further, it is easy to verify that $P(u, U_{t_1,t_2})$ is a viscosity supersolution in Ω_T, see [9, pp. 157–158].

We need an auxiliary result to bypass the fact that we may not add constants to solutions.

Lemma 3.2. *Assume that* $g \in C(\overline{\Omega}_T)$ *is a function such that* $g^m \in L^2(0,T;H^1(\Omega))$ *and* $0 \le g \le M$. *Define the function* g_ε *by*

$$g_\varepsilon = (g^m + \varepsilon^m)^{1/m},$$

where $0 \le \varepsilon \le 1$. *Let* u *and* u_ε *be the unique weak solutions in the sense of* (2.4) *to the initial-boundary value problems*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in} \quad \Omega_T, \\ u^m - g^m \in L^2(0,T;H^1_0(\Omega)), \\ u(x,0) = g(x,0), \end{cases}$$

and

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon^m = 0 \quad \text{in} \quad \Omega_T, \\ u_\varepsilon^m - g_\varepsilon^m \in L^2(0,T;H^1_0(\Omega)), \\ u_\varepsilon(x,0) = g_\varepsilon(x,0), \end{cases}$$

respectively. Then we have

$$\iint_{\Omega_T} (u_\varepsilon - u)(u_\varepsilon^m - u^m)\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon^m|\Omega_T|(M+1) + \varepsilon|\Omega_T|(M+1)^m.$$

Proof. We use the so-called Oleĭnik's test function. The function $u_\varepsilon^m - u^m - \varepsilon^m$ has zero boundary values on the lateral boundary in Sobolev's sense. Thus

$$\eta(x,t) = \begin{cases} \int_t^T (u_\varepsilon^m - u^m - \varepsilon^m)\,\mathrm{d}s, & 0 < t < T, \\ 0, & t \ge T, \end{cases}$$

is an admissible test function for the equations satisfied by u and u_ε. This gives

$$\iint_{\Omega_T} \left(-u\frac{\partial \eta}{\partial t} + \nabla u^m \cdot \nabla \eta \right) \mathrm{d}x\,\mathrm{d}t = \int_\Omega u(x,0)\eta(x,0)\,\mathrm{d}x,$$

and

$$\iint_{\Omega_T} \left(-u_\varepsilon\frac{\partial \eta}{\partial t} + \nabla u_\varepsilon^m \cdot \nabla \eta \right) \mathrm{d}x\,\mathrm{d}t = \int_\Omega u_\varepsilon(x,0)\eta(x,0)\,\mathrm{d}x.$$

Since we have

$$\eta_t = -(u_\varepsilon^m - u^m) + \varepsilon^m \quad \text{and} \quad \nabla \eta = \int_t^T \nabla(u_\varepsilon^m - u^m)\,\mathrm{d}s,$$

we obtain

$$\iint_{\Omega_T} \Big((u_\varepsilon - u)(u_\varepsilon^m - u^m - \varepsilon^m) + \nabla(u_\varepsilon^m - u^m) \cdot \int_t^T \nabla(u_\varepsilon^m - u^m)\,\mathrm{d}s \Big)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (g_\varepsilon(x,0) - g(x,0)) \left(\int_0^T (u_\varepsilon^m - u^m - \varepsilon^m)\,\mathrm{d}s \right) \mathrm{d}x$$

by subtracting the equations. Integration with respect to the variable t shows that the triple integral equals to

$$\frac{1}{2} \int_\Omega \left(\int_0^T (\nabla u_\varepsilon^m - \nabla u^m)\,\mathrm{d}s \right)^2 \mathrm{d}x,$$

which is a positive quantity. Thus we get the estimate

$$\iint_{\Omega_T} (u - u_\varepsilon)(u^m - u_\varepsilon^m)\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon^m \iint_{\Omega_T} (u_\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega (g_\varepsilon(x,0) - g(x,0)) \left(\int_0^T (u_\varepsilon^m - u^m)\,\mathrm{d}s \right) \mathrm{d}x - \varepsilon^m T \int_\Omega (g_\varepsilon(x,0) - g(x,0))\,\mathrm{d}x.$$

The last term on the right-hand side is negative, since $g_\varepsilon \ge g$, and we simply discard it. Furthermore, by the definition of g_ε, we have

$$g_\varepsilon - g = (g^m + \varepsilon^m)^{1/m} - g \le \varepsilon$$

and, by the maximum principle, we conclude that $u \le M$ and $u_\varepsilon \le M + 1$. The required estimate follows, since

$$u_\varepsilon - u \le M + 1 \quad \text{and} \quad u_\varepsilon^m - u^m \le (M+1)^m. \qquad \square$$

We conclude by a comparison principle between viscosity sub- and supersolutions. The essential feature here is that the base Ω of the space-time cylinder Ω_T may be an arbitrary bounded open set. For the comparison principle in regular cylinders, see [7, pp. 10-12] or [13, pp. 132-134].

Theorem 3.3 (Comparison Principle)**.** *Let u be a bounded viscosity subsolution and v a bounded viscosity supersolution such that*

$$\limsup_{z \to z_0} u(z) \le \liminf_{z \to z_0} v(z) \tag{3.4}$$

for all $z_0 \in \partial_p \Omega_T$. Then $u \le v$ in Ω_T.

Proof. Let $\varepsilon_j = 1/j$, $j = 1, 2, 3, \ldots$. By (3.4), we can find regular cylinders $Q_j = U_j \times (t_j, T)$, with $U_j \Subset \Omega$, such that

$$u^m \le v^m + \varepsilon_j^m \quad \text{in} \quad \Omega_T \setminus Q_j.$$

Let w_j be the weak solution in Q_j with boundary values given by v on $\partial_p Q_j$, and let $\widetilde{w}_j$ be the weak solution with boundary values $(v^m + \varepsilon_j^m)^{1/m}$ on $\partial_p Q_j$ in the sense of (2.4). Define the functions h_j and $\widetilde{h}_j$ by

$$h_j = \begin{cases} v & \text{in } \ \Omega_T \setminus Q_j, \\ w_j & \text{in } \ Q_j, \end{cases}$$

and

$$\widetilde{h}_j = \begin{cases} (v^m + \varepsilon_j^m)^{1/m} & \text{in } \ \Omega_T \setminus Q_j, \\ \widetilde{w}_j \ \text{ in } \ Q_j. \end{cases}$$

Recall that Q_j is a regular cylinder. An application of the comparison principle on Q_j shows that

$$h_j \leq v \quad \text{and} \quad u \leq \widetilde{h}_j \quad \text{in } \Omega_T.$$

Now

$$0 \leq (u-v)_+(u^m - v^m)_+ \leq \begin{cases} (\widetilde{h}_j - h_j)(\widetilde{h}_j^m - h_j^m) & \text{in } \ Q_j, \\ \varepsilon_j^m (u-v)_+ \ \text{ in } \ \Omega_T \setminus Q_j. \end{cases}$$

We integrate this estimate, apply Lemma 3.2 and let $j \to \infty$ to get

$$\iint_{\Omega_T} (u-v)_+(u^m - v^m)_+ \, \mathrm{d}x \, \mathrm{d}t = 0.$$

The claim follows. □

4. Perron solutions

The following definition of Perron solutions is based on the comparison principle.

Definition 4.1. Let $g : \partial_p \Omega_T \to \mathbb{R}$ be given. The *upper class* $\mathfrak{U}_g$ consists of the viscosity supersolutions v which are locally bounded from below, and satisfy

$$\liminf_{z \to \xi} v(z) \geq g(\xi)$$

for all $\xi \in \partial_p \Omega_T$. The *upper Perron solution* is defined as

$$\overline{H}_g(z) = \inf_{v \in \mathfrak{U}_g} v(z).$$

The *lower class* $\mathfrak{L}_g$ consists of all viscosity subsolutions u that are locally bounded from above, and satisfy

$$\limsup_{z \to \xi} u(z) \leq g(\xi)$$

for all $\xi \in \partial_p \Omega_T$. The *lower Perron solution* is

$$\underline{H}_g(z) = \sup_{u \in \mathfrak{L}_g} u(z).$$

If there exists a function $h \in C(\overline{\Omega}_T)$ solving the boundary value problem in the classical sense, then

$$h = \overline{H}_g = \underline{H}_g.$$

To see this, simply note that the function h belongs to both the upper class and the lower class. As we will see, both $\overline{H}_g$ and $\underline{H}_g$ are local weak solutions to the equation.

A central issue in this theory is the question about when $\underline{H}_g = \overline{H}_g$. If this happens, the boundary function is called *resolutive* and we denote the common function by H_g. An immediate consequence of the comparison principle (Theorem 3.3) is that if $u \in \mathfrak{L}_g$ and $v \in \mathfrak{U}_g$, then $u \leq v$. Thus

$$\underline{H}_g \leq \overline{H}_g \tag{4.2}$$

for bounded boundary functions g. Our main result (Theorem 5.1) shows that continuous functions are resolutive. It should be noticed that even when the solutions coincide, they may attain wrong boundary values. If the boundary function g is smooth enough, then the weak solutions defined in (2.4) and the Perron solutions coincide, see Theorem 5.8. The main purpose of the definition above is to allow the boundary function g to be general. In particular, it is not assumed that $g^m \in L^2(0, T; H^1(\Omega))$.

So far, the domain was a space-time cylinder Ω_T. The definition of upper and lower Perron solutions given above makes sense in an arbitrary bounded open set Υ in $\mathbb{R}^{N+1}$. Further, Lemma 4.3 and Theorem 4.6 below continue to hold, since their proofs are purely local. However, a comparison principle with the boundary values taken over the whole topological boundary of Υ is not known for the porous medium equation. In particular, we do not know whether (4.2) remains true in this generality.

Before addressing the resolutivity question in the next section, we establish some basic properties of the lower and upper Perron solutions.

Lemma 4.3. *If g is bounded, then $\overline{H}_g$ and $\underline{H}_g$ are continuous in Ω_T.*

Proof. We prove the claim for $\overline{H}_g$, the other case being similar. Take cylinders $U_{t_1,t_2} \Subset V_{\sigma_1,\sigma_2} \Subset \Omega_T$, and points z_1, $z_2 \in U_{t_1,t_2}$. Given a positive number ε, we will show that

$$\overline{H}_g(z_1) - \overline{H}_g(z_2) < 2\varepsilon,$$

provided that U_{t_1,t_2} is sufficiently small. We can find functions v_i^1 and v_i^2 from the upper class such that

$$\lim_{i\to\infty} v_i^1(z_1) = \overline{H}_g(z_1) \quad \text{and} \quad \lim_{i\to\infty} v_i^2(z_2) = \overline{H}_g(z_2).$$

Then also $v_i = \min\{v_i^1, v_i^2\}$ is in the upper class, and we have

$$\lim_{i\to\infty} v_i(z_1) = \overline{H}_g(z_1) \quad \text{and} \quad \lim_{i\to\infty} v_i(z_2) = \overline{H}_g(z_2).$$

Let
$$w_i = P(v_i, V_{\sigma_1,\sigma_2}) \in \mathfrak{U}_g.$$
Then $\overline{H}_g \le w_i \le v_i$, and we have
$$v_i(z_1) < \overline{H}_g(z_1) + \varepsilon \quad \text{and} \quad v_i(z_2) < \overline{H}_g(z_2) + \varepsilon$$
for sufficiently large i. From the above facts and the local Hölder continuity of w_i, it follows that
$$\begin{aligned}\overline{H}_g(z_1) - \overline{H}_g(z_2) \le & w_i(z_1) - w_i(z_2) + \varepsilon \\ \le & \operatorname*{osc}_{U_{t_1,t_2}} w_i + \varepsilon \le 2\varepsilon\end{aligned}$$
by choosing U_{t_1,t_2} in a suitable way. Observe that the assumption on the boundedness of g implies that the modulus of continuity of w_i is independent of i. By exchanging the roles of z_1 and z_2, we have
$$|\overline{H}_g(z_1) - \overline{H}_g(z_2)| \le 2\varepsilon,$$
which completes the proof. □

To prove that Perron solutions are indeed weak solutions to the porous medium equation, we need some auxiliary results. The first of them is a Caccioppoli estimate. The proof can be found in [9, Lemma 2.15].

Lemma 4.4. *Let u be a weak supersolution in Ω_T such that $u^m \in L^2(0,T;W^{1,2}(\Omega))$ and $0 \le u \le M$. Then*
$$\iint_{\Omega_T} \eta^2 |\nabla u^m|^2 \,\mathrm{d}x\,\mathrm{d}t \le 16M^{2m}T \int_\Omega |\nabla \eta|^2 \,\mathrm{d}x + 6M^{m+1} \int_\Omega \eta^2 \,\mathrm{d}x$$
for all nonnegative functions $\eta \in C_0^\infty(\Omega)$.

The preceding lemma implies a convergence result in a straightforward manner, see for example [9, Proof of Theorem 3.2].

Proposition 4.5. *Let $0 \le u_j \le M$, $j = 1, 2, \dots,$ be weak solutions that converge pointwise almost everywhere to a function u. Then u is also a weak solution.*

Theorem 4.6. *If g is bounded, $\overline{H}_g$ and $\underline{H}_g$ are local weak solutions in Ω_T.*

Proof. We give the proof for $\overline{H}_g$, the case of $\underline{H}_g$ being again symmetrical. Let q_n, $n = 1, 2, 3, \dots$, be an enumeration of the points in Ω_T with rational coordinates. The first aim is to construct functions in the upper class converging to $\overline{H}_g$ at the points q_n. To accomplish this, let $v_i^n \in \mathfrak{U}_g$ be such that
$$\overline{H}_g(q_n) \le v_i^n(q_n) < \overline{H}_g(q_n) + \frac{1}{i}, \quad i = 1, 2, 3, \dots,$$
and define
$$w_i = \min\{v_1^1, v_2^1, \dots, v_i^1, v_1^2, v_2^2, \dots, v_i^2, \dots, v_1^i, v_2^i, \dots, v_i^i\}.$$

Then $w_i \in \mathfrak{U}_g$, $w_1 \geq w_2 \geq w_3 \ldots$, and

$$\overline{H}(q_n) \leq w_i(q_n) \leq v_i^n(q_n),$$

for $i \geq n$. It follows that

$$\lim_{i\to\infty} w_i(q_n) = \overline{H}(q_n)$$

at each point q_n. Let $U_{t_1,t_2} \Subset \Omega_T$ be an arbitrary regular cylinder and denote

$$W_i = P(w_i, U_{t_1,t_2}).$$

Then $\overline{H}_g \leq W_i \leq w_i$, the sequence (W_i) is decreasing, and its limit W is a solution in U_{t_1,t_2} by Proposition 4.5. At every point q_n we have

$$W(q_n) = \lim_{i\to\infty} W_i(q_n) = \overline{H}(q_n).$$

Both W and $\overline{H}_g$ are continuous in U_{t_1,t_2}, and they coincide on a dense subset; hence they must coincide everywhere. Since W is a solution in U_{t_1,t_2}, so is $\overline{H}_g$. The property of being a solution is local, so the proof is complete. □

5. Resolutivity

The following theorem is our main result. It states that *continuous functions are resolutive.*

Theorem 5.1 (Resolutivity)**.** *If $g : \partial_p\Omega_T \to \mathbb{R}$ is continuous, then*

$$\overline{H}_g = \underline{H}_g.$$

To prove the resolutivity theorem, by approximation we first reduce the situation to smooth boundary values. For smooth boundary values, we need to construct functions belonging to the upper class $\mathfrak{U}_g$ that are sufficiently smooth in time and attain the correct boundary and initial values in the weak sense. We do this by solving a penalized equation. For this purpose, assume the function g to be continuously differentiable in $\overline{\Omega}_T$ and such that $g^m \in C^2(\overline{\Omega}_T)$. Then

$$\Psi = \frac{\partial g}{\partial t} - \Delta g^m$$

is bounded. We will use the positive part $\Psi_+ = \max\{\Psi, 0\}$ below. Choose a number $\delta > 0$, and let $\zeta_\delta : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $0 \leq \zeta_\delta(s) \leq 1$, $\zeta_\delta(s) = 1$ for all $s \geq 0$, $\zeta_\delta(s) = 0$ for all $x \leq -\delta$, and $|\zeta_\delta'(s)| \leq 2/\delta$. We have the following existence result, see [4].

Proposition 5.2. *Let g be continuously differentiable in $\overline{\Omega}_T$ and such that $g^m \in C^2(\overline{\Omega}_T)$. Then there exists a bounded weak solution u such* *that* $u^m \in L^2(0,T;H^1(\Omega))$ *to the boundary value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = \zeta_\delta(g^m - u^m)\Psi_+ \quad \text{in} \quad \Omega_T, \\ u^m - g^m \in L^2(0,T;H_0^1(\Omega)), \\ u(x,0) = g(x,0), \end{cases}$$

satisfying the inequality $u \geq g$ *in* Ω_T.

Remark 5.3. In the proof of Theorem 5.1 below, we need to choose approximations of a given continuous function g so that the smoothness assumptions of Proposition 5.2 hold. This is accomplished by approximating a suitable smaller power g^α, $\alpha \leq 1$, of the function rather than the function g itself. Indeed, we may express the derivatives of the powers one and m in terms of the derivatives of the power α. Some simple calculations show that the choice $\alpha = \min\{1, m/2\}$ will do.

Remark 5.4. Due to our assumption that the boundary values are positive, the roles of upper and lower solutions in the proof of Theorem 5.1 are not quite symmetric. For subsolutions, we need a version of Proposition 5.2 where the solutions can change sign. See pp. 97-100 in [13] for the modifications needed to the arguments in [4].

We need an energy estimate for the time derivative of a solution to the above equation. For similar results, see [12, Proposition 13] and [13, Section 3.2.5].

Theorem 5.5. *Assume that* $f \in L^\infty(\Omega_T)$. *Let* u *be a bounded weak solution to the boundary value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = f \quad \text{in} \quad \Omega_T, \\ u^m - g^m \in L^2(0,T;H_0^1(\Omega)), \\ u(x,0) = g(x,0). \end{cases}$$

Then

$$\frac{\partial u^{(m+1)/2}}{\partial t} \in L^2(\Omega_T),$$

with the estimate

$$\begin{aligned} \iint_{\Omega_T} \left| \frac{\partial u^{(m+1)/2}}{\partial t} \right|^2 \,\mathrm{d}x\,\mathrm{d}t &+ \int_\Omega |\nabla(u^m - g^m)|^2(x,T)\,\mathrm{d}x \\ \leq c\Bigg(&\int_\Omega |(g^m)_t(x,T)u(x,T) - (g^m)_t(x,0)g(x,0)| \,\mathrm{d}x \\ &+ \iint_{\Omega_T} \big(|u|^2 + u^{m-1}(|f|^2 + |\Delta g^m|^2)\big) \,\mathrm{d}x\,\mathrm{d}t \\ &+ \iint_{\Omega_T} \big(|(g^m)_t|^2 + |\Delta g^m|^2 + |(g^m)_{tt}|^2 + |f|^2\big) \,\mathrm{d}x\,\mathrm{d}t \Bigg). \end{aligned}$$

Furthermore, we have

$$\frac{\partial u^q}{\partial t} \in L^2(\Omega_T)$$

for any $q \geq (m+1)/2$.

Proof. First we assume that u is smooth in $\overline{\Omega}_T$. This assumption may be removed by a standard approximation argument, see [13, Proof of Theorem 5.5] and [16, Section 1.3.2]. Denote $w = u^{(m+1)/2}$. Then

$$\begin{aligned}
\left|\frac{\partial w}{\partial t}\right|^2 &= \frac{(m+1)^2}{4} u^{m-1}|u_t|^2 \\
&= \frac{(m+1)^2}{4m}(u^m)_t u_t = \frac{(m+1)^2}{4m}\big((u^m - g^m)_t u_t + (g^m)_t u_t\big) \\
&= \frac{(m+1)^2}{4m}\big((u^m - g^m)_t(\Delta u^m - \Delta g^m) \\
&\quad + (g^m)_t u_t + \Delta g^m (u^m - g^m)_t + f(u^m - g^m)_t\big).
\end{aligned}$$

We focus on the first term after the last equality. To this end, we note that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^m - g^m|^2 \,\mathrm{d}x &= \int_\Omega \nabla(u^m - g^m)\cdot\nabla(u^m - g^m)_t \,\mathrm{d}x \\
&= -\int_\Omega \Delta(u^m - g^m)(u^m - g^m)_t \,\mathrm{d}x,
\end{aligned}$$

since $u^m - g^m$ has zero boundary values on the lateral boundary. Thus an integration gives

$$\begin{aligned}
\iint_{\Omega_T} \left|\frac{\partial w}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t &= -\frac{(m+1)^2}{8m}\int_0^T \frac{d}{dt}\int_\Omega |\nabla(u^m - g^m)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{(m+1)^2}{4m}\iint_{\Omega_T}\big((g^m)_t u_t + (u^m - g^m)_t \Delta g^m + f(u^m - g^m)_t\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\frac{(m+1)^2}{8m}\int_\Omega |\nabla(u^m - g^m)|^2 \,\mathrm{d}x\Big|_0^T \\
&\quad + \frac{(m+1)^2}{4m}\left(\iint_{\Omega_T} -(g^m)_{tt} u \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega (g^m)_t u \,\mathrm{d}x\Big|_0^T\right) \\
&\quad + \frac{(m+1)^2}{4m}\iint_{\Omega_T}\big((u^m)_t \Delta g^m - (g^m)_t \Delta g^m - f(g^m)_t + (u^m)_t f\big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

To proceed, we compute

$$\frac{\partial u^m}{\partial t} = u^{m-(m+1)/2}\frac{\partial w}{\partial t},$$

and apply Young's inequality to the two terms containing the time derivative of u^m to get

$$\iint_{\Omega_T} |(u^m)_t \Delta g^m| \, \mathrm{d}x \, \mathrm{d}t \le \varepsilon \iint_{\Omega_T} \left| \frac{\partial w}{\partial t} \right|^2 \mathrm{d}x \, \mathrm{d}t + c_\varepsilon \iint_{\Omega_T} u^{m-1} |\Delta g^m|^2 \, \mathrm{d}x \, \mathrm{d}t$$

and

$$\iint_{\Omega_T} |(u^m)_t f| \, \mathrm{d}x \, \mathrm{d}t \le \varepsilon \iint_{\Omega_T} \left| \frac{\partial w}{\partial t} \right|^2 \mathrm{d}x \, \mathrm{d}t + c_\varepsilon \iint_{\Omega_T} u^{m-1} |f|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

We insert these inequalities into the estimate above, choose a sufficiently small ε, and absorb terms to get

$$\begin{aligned} &\iint_{\Omega_T} \left| \frac{\partial w}{\partial t} \right|^2 \mathrm{d}x \, \mathrm{d}t + \int_\Omega |\nabla (u^m - g^m)|^2 (x, T) \, \mathrm{d}x \\ &\quad \le c \Big(\int_\Omega |\nabla (u^m - g^m)|^2 (x, 0) \, \mathrm{d}x + \iint_{\Omega_T} |(g_{tt})^m u| \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad + \int_\Omega |(g^m)_t (x, T) u(x, T) - (g^m)_t (x, 0) u(x, 0)| \, \mathrm{d}x \\ &\qquad + \iint_{\Omega_T} u^{m-1} |\Delta g^m|^2 \, \mathrm{d}x \, \mathrm{d}t + \iint_{\Omega_T} |(g^m)_t \Delta g^m| \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad + \iint_{\Omega_T} |f (g^m)_t| \, \mathrm{d}x \, \mathrm{d}t + \iint_{\Omega_T} u^{m-1} |f|^2 \, \mathrm{d}x \, \mathrm{d}t \Big). \end{aligned}$$

We recall that $u^m = g^m$ at the initial time, so the required estimate follows from an application of Cauchy's inequality. □

Lemma 5.6. *Let g satisfy the smoothness assumptions of Proposition 5.2, and let v be the solution to the boundary value problem of Proposition 5.2. Let $u = P(v, D_{t_1,T})$ be the Poisson modification of v with respect to a regular space-time cylinder $D_{t_1,T}$ with $D \Subset \Omega$ and $0 < t_1 < T$. Then*

$$\begin{aligned} &\iint_{D_{t_1,T}} |\nabla u^m|^2 \, \mathrm{d}x \, \mathrm{d}t + \sup_{t_1 < t < T} \int_D u^{m+1}(x, t) \, \mathrm{d}x \\ &\quad \le c \Big(\iint_{D_{t_1,T}} \Big(|\nabla v^m|^2 + |v|^2 + \left| \frac{\partial v^m}{\partial t} \right|^2 \Big) \mathrm{d}x \, \mathrm{d}t \\ &\qquad + \sup_{t_1 < t < T} \int_D v^{m+1}(x, t) \, \mathrm{d}x \Big). \end{aligned}$$

Proof. The formal computations below are justified rigorously by a standard application of a suitable mollification in the time direction. Since u is a solution in $D_{t_1,T}$ with boundary values given by v^m, we may use $u^m - v^m$ as a test function and have

$$\iint_{D_{t_1,T}} \Big(\frac{\partial u}{\partial t} (u^m - v^m) + \nabla u^m \cdot \nabla (u^m - v^m) \Big) \mathrm{d}x \, \mathrm{d}t = 0. \tag{5.7}$$

The next goal is to eliminate the time derivative of u. We use the fact that $(u^{m+1})_t = (m+1)u_t u^m$ and integrate by parts in the other term to get

$$\begin{aligned}\iint_{D_{t_1,T}} \frac{\partial u}{\partial t}(u^m - v^m)\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{m+1}\left(\int_D u^{m+1}(x,T)\,\mathrm{d}x - \int_D u^{m+1}(x,t_1)\,\mathrm{d}x\right) \\ &\quad + \iint_{D_{t_1,T}} u\frac{\partial v^m}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_D u(x,T)v^m(x,T)\,\mathrm{d}x + \int_D u(x,t_1)v^m(x,t_1)\,\mathrm{d}x.\end{aligned}$$

This leads to the estimate

$$\begin{aligned}0 &\le \iint_{D_{t_1,T}} |\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{m+1}\int_D u^{m+1}(x,T)\,\mathrm{d}x \\ &\le \frac{1}{m+1}\int_D v^{m+1}(x,t_1)\,\mathrm{d}x + \int_D v^{m+1}(x,T)\,\mathrm{d}x - \iint_{D_{t_1,T}} u\frac{\partial v^m}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \iint_{D_{t_1,T}} \nabla u^m\cdot\nabla v^m\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

since $u \le v$. By Young's inequality, we obtain

$$\begin{aligned}\iint_{D_{t_1,T}} \nabla u^m\cdot\nabla v^m\,\mathrm{d}x\,\mathrm{d}t \\ &\le \varepsilon\iint_{D_{t_1,T}} |\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t + c_\varepsilon\iint_{D_{t_1,T}} |\nabla v^m|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We insert this into the previous estimate, and absorb terms. We arrive at

$$\begin{aligned}\int_{D_{t_1,T}} |\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{m+1}\int_D u^{m+1}(x,T)\,\mathrm{d}x \\ &\le c\biggl(\int_D v^{m+1}(x,t_1)\,\mathrm{d}x + \int_D v^{m+1}(x,T)\,\mathrm{d}x \\ &\quad + \iint_{D_{t_1,T}} |v|\left|\frac{\partial v^m}{\partial t}\right|\,\mathrm{d}x\,\mathrm{d}t + \iint_{D_{t_1,T}} |\nabla v^m|^2\,\mathrm{d}x\,\mathrm{d}t\biggr).\end{aligned}$$

The proof is then completed by estimating the term with $v^{m+1}(x,T)$ by a supremum over time, and by replacing T by $t_1 < \tau < T$ such that

$$\int_D v^{m+1}(x,\tau)\,\mathrm{d}x \ge \sup_{t_1<t<T}\frac{1}{2}\int_D v^{m+1}(x,t)\,\mathrm{d}x,$$

and applying Young's inequality. □

Proof of Theorem 5.1. By extension, we may assume that g is defined in the whole of $\mathbb{R}^{N+1}$. We first show that it suffices to prove that for smooth boundary values g that both the upper and lower Perron solution agree with the unique weak solution solution with boundary and initial values g, in the sense of (2.4). Let us denote $\varepsilon_j = 1/j$, $j = 1, 2, \ldots$. There exist functions φ_j satisfying the smoothness assumptions of Proposition 5.2, see Remark 5.3, converging uniformly to g and such that

$$\varphi_j^m \leq g^m \leq \varphi_j^m + \varepsilon_j^m.$$

Assuming the above conclusion for smooth functions, we get

$$H_{\varphi_j} \leq \underline{H}_g \leq \overline{H}_g \leq H_{(\varphi_j^m + \varepsilon_j^m)^{1/m}}.$$

Since

$$|H_{\varphi_j} - H_{(\varphi_j^m + \varepsilon_j^m)^{1/m}}| \to 0$$

as $j \to \infty$ by Lemma 3.2, it follows that $\underline{H}_g = \overline{H}_g$ almost everywhere. The conclusion that $\underline{H}_g = \overline{H}_g$ everywhere follows by continuity of the Perron solutions.

Let us then assume that g is smooth, and let h be the unique weak solution with initial and boundary values given by g, i.e. $h^m - g^m \in L^2(0, T; H_0^1(\Omega))$ and (2.4) holds. We need to show that $h \geq \overline{H}_g$; the problem is that we do not know whether h belongs to the upper class or not. To deal with this, let v be the solution of the the penalized boundary value problem of Proposition 5.2. Then also

$$v^m - g^m \in L^2(0, T; H_0^1(\Omega)).$$

Exhaust Ω_T by an increasing sequence of regular cylinders $Q_j = U_j \times (t_j, T)$, and let $w_j = P(v, Q_j)$, $j = 1, 2, \ldots$. Then $w_j \in \mathfrak{U}_g$, the sequence (w_j) is decreasing, and $\overline{H}_g \leq w_j$. The limit function

$$w = \lim_{j \to \infty} w_j$$

is a solution in Ω_T, and

$$w \geq \overline{H}_g$$

since w is a pointwise limit of functions in the upper class. It remains to show that w has the boundary and initial values given by g, since then by the uniqueness of weak solutions, we have

$$h = w \geq \overline{H}_g.$$

To check the lateral boundary values, note that the sequence $(w_j^m - g^m)$ is bounded in $L^2(0, T; H_0^1(\Omega))$ by Lemma 5.6. It follows that

$$w^m - g^m \in L^2(0, T; H_0^1(\Omega)),$$

since $L^2(0, T; H_0^1(\Omega))$ is a closed subspace of $L^2(0, T; H^1(\Omega))$ and weak limits must agree with pointwise limits.

We use the criterion (2.5) to show that the initial values of the limit function w are given by the function $g(x,0)$. Let $\eta \in C_0^\infty(\Omega)$ be arbitrary. Choose a time instant $0 < t < T$, and j large enough, so that $t_j < t$ and so that the support of η is contained in U_j. We have

$$\left|\int_\Omega (w(x,t) - g(x,0))\eta(x)\,\mathrm{d}x\right| \le \left|\int_\Omega (w(x,t) - w_j(x,t))\eta(x)\,\mathrm{d}x\right| + \left|\int_\Omega (w_j(x,t) - w_j(x,t_j))\eta(x)\,\mathrm{d}x\right| + \left|\int_\Omega (v(x,t_j) - g(x,0))\eta(x)\,\mathrm{d}x\right|$$

by adding and substracting suitable terms, using the triangle inequality, and the fact that $w_j(x,t_j) = v(x,t_j)$ on the support of η. The first and third terms on the right tend to zero as $j \to \infty$. To deal with the second term, we formally test the equation satisfied by w_j with $\varphi = \eta\chi_{(t_j,t)}$, where $\chi_{(t_j,t)}$ is the characteristic function of the interval (t_j,t). This can be justified by an approximation argument. We get

$$\left|\int_\Omega (w_j(x,t) - w_j(x,t_j))\eta\,\mathrm{d}x\right| = \left|\iint_{U_j\times(t_j,t)} \nabla w_j^m \cdot \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\right|.$$

We estimate the right hand side by Hölder's inequality to get

$$\left|\iint_{U_j\times(t_j,t)} \nabla w_j^m \cdot \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\right| \le |\Omega|^{1/2}|t-t_j|^{1/2}\|\nabla w_j^m\|_{L^2(\Omega_T)}\|\nabla\eta\|_{L^\infty(\Omega)},$$

where we also used the fact that $|U_j \times (t_j,t)| \le |\Omega||t-t_j|$. Since the norm of ∇w_j^m can be controlled independently of j by applying Lemma 5.6, we may use this estimate for the second term to get

$$\left|\int_\Omega (w(x,t) - g(x,0))\eta(x)\,\mathrm{d}x\right| \le ct\|\nabla\eta\|_{L^\infty(\Omega)}$$

after letting $j \to \infty$. Since η was arbitrary, letting $t \to 0$ shows that (2.5) holds for the function w, as desired.

By a similar argument using the variant of Proposition 5.2 described in Remark 5.4, we see that $h \le \underline{H}_g$, so that

$$h \le \underline{H}_g \le \overline{H}_g \le h,$$

which completes the proof. □

The second part of the previous proof gives the following uniqueness result.

Theorem 5.8. *Let g satisfy the smoothness assumptions of Proposition 5.2 and let u be the weak solution to the boundary value problem in the sense of* (2.4)*. Then $u = H_g$.*

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