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RESISTANCE CONDITIONS, POINCARÉ INEQUALITIES, THE LIP-LIP CONDITION AND HARDY'S INEQUALITIES

Communicated by M. Bożejko

Abstract. This note investigates weaker conditions than a Poincaré inequality in analysis on metric measure spaces. We discuss two resistance conditions which are stated in terms of capacities. We show that these conditions can be characterized by versions of Sobolev–Poincaré inequalities. As a consequence, we obtain so-called Lip-lip condition related to pointwise Lipschitz constants. Moreover, we show that the pointwise Hardy inequalities and uniform fatness conditions are equivalent under an appropriate resistance condition.

1. Introduction

Rather standard assumptions in analysis on a metric measure space (X, d, μ) are that the measure is doubling and that the space supports a Poincaré inequality, see [3] and [5]. The space is said to support a weak $(1, p)$-Poincaré inequality with $1 \leq p < \infty$, if there exist $c_P > 0$ and $\sigma \geq 1$ such that for any $x \in X$ and $r > 0$, and for every locally integrable function f in X,

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq c_P r \left(\fint_{B(x,\sigma r)} (\operatorname{lip} f)^p \, d\mu \right)^{1/p},$$

where

$$f_{B(x,r)} = \fint_{B(x,r)} f \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu$$

denotes the integral average over the ball $B(x, r)$ and $\operatorname{lip} f$ is the pointwise Lipschitz constant of f. The precise definitions will be given later. These

conditions imply, for example, the Sobolev embedding theorem, which is a central tool in analysis on metric measure spaces, see [3] and [5].

The goal of this note is to consider weaker conditions than the Poincaré inequality. More precisely, the weak $(1, p)$-Poincaré inequality implies so-called resistance condition

$$(1.1) \qquad \frac{1}{c_R} \frac{\mu(B(x,r))}{r^p} \leq \text{cap}_{\text{Lip},p}(\overline{B}(x,r), B(x,2r)) \leq c_R \frac{\mu(B(x,r))}{r^p},$$

for every $x \in X$ and $r > 0$ with a uniform constant $c_R \geq 1$. Here we consider the capacity defined as

$$\text{cap}_{\text{Lip},p}(\overline{B}(x,r), B(x,2r)) = \inf \int_X (\text{lip}\, f)^p \, d\mu,$$

where the infimum is taken over all Lipschitz continuous functions f in X with $f \geq 1$ in $\overline{B}(x,r)$ and $f = 0$ in $X \backslash B(x,2r)$. The resistance condition is considerably weaker than the Poincaré inequality. Even in the case when the space is complete, the resistance condition does not imply quasiconvexity of the space and, as a consequence, it is not equivalent with the Poincaré inequality, see [9]. A similar condition has been previously employed, for example in [1] and [4] in connection with the Dirichlet forms on metric measure spaces.

Several versions of the resistance condition are available and it is not obvious which is the best approach. In this note, we discuss two conditions called the p-resistance conductor condition and the p-strong resistance conductor condition. These conditions seem to be stronger than (1.1) and, as we shall see, they can be characterized by versions of Sobolev–Poincaré inequalities in the same way as in [11]. For results in metric measure spaces, see also [8]. Using the results of [7], we conclude that if X is a metric space with μ doubling and that satisfies a p-strong resistance conductor condition, then so-called Lip-lip condition related to pointwise Lipschitz constants holds true. Moreover, we show that the pointwise Hardy inequalities and uniform fatness conditions are equivalent in our context. This is closely related to results of [10] .

2. Preliminaries

From now on, let (X, d, μ) be a metric measure space. Here μ is a doubling measure, that is, there exists $c_D \geq 1$, called the doubling constant of μ, such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r)),$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$.

Let us recall that a function $f : X \to \mathbb{R}$ is said to be Lipschitz continuous if there exists $c > 0$ such that

$$(2.1) \qquad |f(x) - f(y)| \leq c d(x, y),$$

for every $x, y \in X$. In this case the Lipchitz constant of f is defined to be the infimum over all constants $c > 0$ for which (2.1) holds and $\mathrm{Lip}(X)$ denotes the class of Lipschitz functions on X.

We denote
$$\mathrm{D}_\varepsilon f(x) = \sup_{y \in B(x,\varepsilon)} \frac{|f(x) - f(y)|}{\varepsilon},$$
where $x \in X$. If $f \in \mathrm{Lip}(X)$, then for every $x \in X$ the lower local Lipschitz constant of u is defined by

$$(2.2) \qquad\qquad \mathrm{lip}\, f(x) = \liminf_{\varepsilon \to 0} \mathrm{D}_\varepsilon f(x)$$

and the upper local Lipschitz constant by

$$(2.3) \qquad\qquad \mathrm{Lip}\, f(x) = \limsup_{\varepsilon \to 0} \mathrm{D}_\varepsilon f(x).$$

REMARK 2.4. It is useful to note that for $f \in \mathrm{Lip}(X)$, we have
$$\mathrm{Lip}\, f(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x,y)}$$
for every $x \in X$, see [7, Remark 4.2.2].

3. Sobolev–Poincaré inequalities and resistance conditions

Let $1 \leq p < \infty$ and (E, G) be a pair of sets in X, where E is a μ-measurable subset of an open set G. We define the capacity of (E, G) in X as
$$\mathrm{cap}_{\mathrm{Lip},p}(E, G) = \inf \int_X (\mathrm{lip}\, f)^p \, d\mu,$$
where the infimum is taken over all $f \in \mathrm{Lip}(X)$ with $f \geq 1$ in E and $f = 0$ in $X \backslash G$.

Let Ω be an open and bounded subset of X. We want to consider Poincaré inequalities for functions that are not necessary zero on the boundary of the domain. To this end, we shall need the concept of conductivity. Let G be an open subset of Ω and $E \subset G$ a μ-measurable set. Then
$$\mathrm{cap}_{\mathrm{Lip},p}(E, G; \Omega) = \inf \int_\Omega (\mathrm{lip}\, f)^p \, d\mu,$$
where the infimum is taken over all $f \in \mathrm{Lip}(\Omega)$ with $f \geq 1$ in E and $f = 0$ in $\Omega \backslash G$. We also denote $\mathrm{cap}_{\mathrm{Lip},p}(E, G; X) = \mathrm{cap}_{\mathrm{Lip},p}(E, G)$.

DEFINITION 3.1. The space X satisfies the p-resistance conductor condition, if there exists $c_R \geq 1$ such that for any $x \in X$, $0 < r < \mathrm{diam}(X)/2$ and $E \subsetneq B(x,r)$, we have

$$(3.2) \qquad\qquad \frac{1}{c_R} \frac{\mu(E)}{r^p} \leq \mathrm{cap}_{\mathrm{Lip},p}(E, B(x,r)).$$

The following capacitary strong type estimate will be useful later.

LEMMA 3.3. *Let* $x \in X$, $0 < r < \operatorname{diam}(X)/2$ *and assume that* $f \in$ $\operatorname{Lip}(B(x, 2r))$ *with* $f = 0$ *in* $B(x, 2r) \backslash B(x, r)$. *Then*

$$\int_0^\infty t^{p-1} \operatorname{cap}_{\operatorname{Lip},p}(E_t, B(x, r)) \, dt \leq 2^{2p-1} \int_{B(x,r)} (\operatorname{lip} f)^p \, d\mu,$$

where $E_t = \{z \in B(x, r) : |f(z)| \geq t\}$.

Proof. If $E_t = \emptyset$ for every $t > 0$, then there is nothing to prove. Hence, we may assume that $E_t \neq \emptyset$ for some $t > 0$. In this case

$$(3.4) \quad \int_0^\infty t^{p-1} \operatorname{cap}_{\operatorname{Lip},p}(E_t, B(x, r)) \, dt$$

$$= \sum_{j=-\infty}^{\infty} \int_{2^{j-1}}^{2^j} t^{p-1} \operatorname{cap}_{\operatorname{Lip},p}(E_t, B(x, r)) \, dt$$

$$\leq \sum_{j=-\infty}^{\infty} 2^{j(p-1)}(2^j - 2^{j-1}) \operatorname{cap}_{\operatorname{Lip},p}(E_{2^{j-1}}, B(x, r))$$

$$\leq \frac{1}{2} \sum_{j=-\infty}^{\infty} 2^{jp} \operatorname{cap}_{\operatorname{Lip},p}(E_{2^{j-1}}, B(x, r)).$$

Note that $E_t \subsetneq B(x, r)$ for every $t > 0$. We define f_j by

$$f_j(z) = \min\left\{ \frac{|f(z)| - 2^{j-1}}{2^{j-1}}, 1 \right\}_+$$

and we have $\{f_j \neq 0\} \subset E_{2^{j-1}} \subset B(x, r)$. Since $\operatorname{lip} f_{j-1}$ is zero in a set where f_{j-1} is constant, we conclude that

$$(3.5) \quad \operatorname{cap}_{\operatorname{Lip},p}(E_{2^{j-1}}, B(x, r)) \leq \int_{E_{2^{j-2}}} (\operatorname{lip} f_{j-1})^p \, d\mu$$

$$= \int_{E_{2^{j-2}} \backslash E_{2^{j-1}}} (\operatorname{lip} f_{j-1})^p \, d\mu \leq 2^{(2-j)p} \int_{E_{2^{j-2}} \backslash E_{2^{j-1}}} (\operatorname{lip} f)^p \, d\mu.$$

Hence, by (3.4) and (3.5), we arrive at

$$\int_0^\infty t^{p-1} \operatorname{cap}_{\operatorname{Lip},p}(E_t, B(x, r)) \, dt \leq \frac{1}{2} \sum_{j=-\infty}^{\infty} 2^{jp} 2^{(2-j)p} \int_{E_{2^{j-2}} \backslash E_{2^{j-1}}} (\operatorname{lip} f)^p \, d\mu$$

$$\leq 2^{2p-1} \int_{B(x,r)} (\operatorname{lip} f)^p \, d\mu. \quad \blacksquare$$

The next result shows that the p-resistance conductor condition can be characterized by a Sobolev type inequality for functions vanishing on a relatively large set.

THEOREM 3.6. *The space X satisfies the p-resistance conductor condition if and only if for any $x \in X$, $0 < r < \mathrm{diam}(X)/2$ and $f \in \mathrm{Lip}(B(x, 2r))$, for which $f = 0$ in $B(x, 2r) \backslash B(x, r)$, we have*

$$\left(\int_{B(x,r)} |f|^p \, d\mu \right)^{1/p} \leq cr \left(\int_{B(x,r)} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p}.$$

Proof. If $E_t = \{z \in B(x, r) : |f(z)| \geq t\} = \emptyset$ for every $t > 0$, then the inequality is trivial. Assume then that there exists $t > 0$ such that $\emptyset \neq E_t \subsetneq B(x, r)$. The Cavalieri principle and the previous lemma imply that

$$\left(\int_{B(x,r)} |f|^p \, d\mu \right)^{1/p} = \left(p \int_0^\infty t^{p-1} \mu(E_t) \, dt \right)^{1/p}$$

$$\leq (pc_R)^{1/p} r \left(\int_0^\infty t^{p-1} \mathrm{cap}_{\mathrm{Lip},p}(E_t, B(x,r)) \, dt \right)^{1/p}$$

$$\leq (pc_R)^{1/p} r \, 2^{2-1/p} \left(\int_{B(x,r)} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p}.$$

Conversely, let $x \in X$, $0 < r < \mathrm{diam}(X)/2$ and $E \subsetneq B(x, r)$. For any $f \in \mathrm{Lip}(B(x, 2r))$ such that $f = 0$ in $B(x, 2r) \backslash B(x, r)$ and $f \geq 1$ in E, we have that

$$cr \left(\int_{B(x,r)} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p} \geq \left(\int_{B(x,r)} |f|^p \, d\mu \right)^{1/p} \geq \mu(E)^{1/p}.$$

Raising both sides to the power p and taking infimum over all such functions, we arrive at

$$\mathrm{cap}_{\mathrm{Lip},p}(E, B(x, r)) \geq \frac{\mu(E)}{cr^p}. \quad \blacksquare$$

The proof of the following capacitary strong type estimate is similar to Lemma 3.3.

LEMMA 3.7. *Let $x \in X$, $0 < r < \mathrm{diam}(X)/2$ and $f \in \mathrm{Lip}(B(x, 2r))$ such that $f = 0$ in $B(x, 2r) \backslash G$, where G is an open set with $G \subsetneq B(x, 2r)$. Then*

$$\int_0^\infty t^{p-1} \mathrm{cap}_{\mathrm{Lip},p}(E_t, G; B(x, 2r)) \, dt \leq 2^{2p-1} \int_G (\mathrm{lip}\, f)^p \, d\mu,$$

where $E_t = \{z \in B(x, r) : |f(z)| \geq t\}$ for $t > 0$.

Next, we introduce another resistance condition.

Definition 3.8. The space X satisfies the p-strong resistance conductor condition if there exists $c_R \geq 1$ such that for any $x \in X$, $0 < r < \operatorname{diam}(X)/2$ and $E \subsetneq B(x,r)$, we have

$$\frac{1}{c_R} \frac{\mu(E)}{r^p} \leq \inf \left\{ \operatorname{cap}_{\operatorname{Lip},p}(E, B(x,r)), \operatorname{cap}_{\operatorname{Lip},p}(E, G; B(x,2r)) \right\},$$

where the infimum is taken over all open sets G such that $E \subset G \subsetneq B(x,2r)$ and $G \cap (B(x,2r) \backslash B(x,r)) \neq \emptyset$.

We obtain a similar characterization of the p-strong resistance conductor condition as in Theorem 3.6.

Theorem 3.9. *The space X satisfies the p-strong resistance conductor condition if and only if, for any $x \in X$, $0 < r < \operatorname{diam}(X)/2$ and $f \in \operatorname{Lip}(B(x,2r))$ with $f = 0$ in $B(x,2r) \backslash G$ for G is an open set in $B(x,2r)$, we have*

$$\left(\int_{B(x,r)} |f|^p \, d\mu \right)^{1/p} \leq cr \left(\int_G (\operatorname{lip} f)^p \, d\mu \right)^{1/p}.$$

Proof. Let us start with the sufficiency. If $f = 0$ in $B(x,r)$, there is nothing to prove. Hence, we may assume that $E_t = \{ z \in B(x,r) : |f(z)| \geq t \} \neq \emptyset$ for some $t > 0$. If $G \subseteq B(x,r)$, then the result follows from Theorem 3.6. If not, $G \cap (B(x,2r) \backslash B(x,r)) \neq \emptyset$ and by the p-strong resistance conductor condition,

$$\mu(E_t) \leq c_R r^p \operatorname{cap}_{\operatorname{Lip},p}(E_t, G; B(x,2r)).$$

Hence, by the Cavalieri principle and Lemma 3.7, we have

$$\left(\int_{B(x,r)} |f|^p \, d\mu \right)^{1/p} = \left(p \int_0^\infty t^{p-1} \mu(E_t) \, dt \right)^{1/p}$$

$$\leq \left(c_R p \int_0^\infty r^p t^{p-1} \operatorname{cap}_{\operatorname{Lip},p}(E_t, G; B(x,2r)) \, dt \right)^{1/p}$$

$$\leq cr \left(\int_G (\operatorname{lip} f)^p \, d\mu \right)^{1/p}.$$

Conversely, let $x \in X$, $0 < r < \operatorname{diam}(X)/2$, $E \subsetneq B(x,r)$ and G be an open set such that $E \subset G \subsetneq B(x,2r)$ and $G \cap (B(x,2r) \backslash B(x,r)) \neq \emptyset$. Given $\delta > 0$, there exists $f \in \operatorname{Lip}(B(x,2r))$ such that $f = 0$ in $B(x,2r) \backslash G$, $f \geq 1$ in E and

$$\operatorname{cap}_{\operatorname{Lip},p}(E, G; B(x,2r)) + \delta \geq \int_G (\operatorname{lip} f)^p \, d\mu \geq \frac{1}{cr^p} \int_{B(x,r)} |f|^p \, d\mu$$

$$\geq \frac{1}{cr^p} \int_E 1 \, d\mu = \frac{\mu(E)}{cr^p}.$$

By letting $\delta \to 0$, we arrive at

$$(3.10) \qquad \frac{\mu(E)}{cr^p} \leq \inf \operatorname{cap}_{\mathrm{Lip},p}(E, G; B(x, 2r)),$$

where the infimum is taken over all open G that $E \subset G \subsetneq B(x, 2r)$, $G \cap (B(x, 2r) \backslash B(x, r)) \neq \emptyset$. Moreover, taking $G = B(x, r)$, by Theorem 3.6, the space X satisfies the p-resistance conductor condition. Therefore, the fact that X satisfies this condition, together with (3.10), implies that the space X satisfies the p-strong resistance conductor condition. ∎

THEOREM 3.11. *If X satisfies the p-strong resistance conductor condition, then for any $f \in \operatorname{Lip}(B(x, 2r))$, $x \in X$ and $0 < r < \operatorname{diam}(X)/2$, we have*

$$\left(\fint_{B(x,r)} |f - f_{B(x,r)}|^p \, d\mu \right)^{1/p} \leq cr \left(\fint_{B(x,2r)} (\operatorname{lip} f)^p \, d\mu \right)^{1/p},$$

where c depends only on p and c_R.

Proof. Let $f \in \operatorname{Lip}(B(x, 2r))$ and take $\alpha \in \mathbb{R}$ such that

$$(3.12) \qquad \mu(\{z \in B(x, 2r) : f(z) \geq \alpha\}) \geq \frac{\mu(B(x, 2r))}{2}$$

and

$$\mu(\{z \in B(x, 2r) : f(z) > \alpha\}) \leq \frac{\mu(B(x, 2r))}{2}.$$

Set $v = (f - \alpha)_+$. Then $v \in \operatorname{Lip}(B(x, 2r))$, $v = (f - \alpha)_+ = 0$ in $B(x, 2r) \backslash G$, where $G = \{z \in B(x, 2r) : f(z) > \alpha\}$ and $\mu(G) \leq \mu(B(x, 2r))/2$. Theorem 3.9 implies that

$$\left(\fint_{B(x,r)} (f - \alpha)_+^p \, d\mu \right)^{1/p} \leq cr \left(\fint_G (\operatorname{lip} v)^p \, d\mu \right)^{1/p} = cr \left(\fint_G (\operatorname{lip} f)^p \, d\mu \right)^{1/p}.$$

Set $g = (\alpha - f)_+$. Then $g \in \operatorname{Lip}(B(x, 2r))$, $g = (\alpha - f)_+ = 0$ in $B(x, 2r) \backslash H$, where $H = \{z \in B(x, 2r) : \alpha > f(z)\}$ and by (3.12) we have

$$\mu(H) = \mu(B(x, 2r)) - \mu(\{z \in B(x, 2r) : f(z) \geq \alpha\}) \leq \mu(B(x, 2r))/2.$$

Theorem 3.9 implies that

$$\left(\fint_{B(x,r)} (\alpha - f)_+^p \, d\mu \right)^{1/p} \leq cr \left(\fint_H (\operatorname{lip} f)^p \, d\mu \right)^{1/p}$$

$$\leq cr \left(\fint_{\{z \in B(x,2r):f(z) \leq \alpha\}} (\operatorname{lip} f)^p \, d\mu \right)^{1/p}.$$

By adding up both inequalities, we obtain

$$(3.13) \qquad \left(\fint_{B(x,r)} |f - \alpha|^p \, d\mu \right)^{1/p} \leq cr \left(\fint_{B(x,2r)} (\operatorname{lip} f)^p \, d\mu \right)^{1/p}.$$

On the other hand,

$$\inf_{a \in \mathbb{R}} \left(\int_{B(x,r)} |f - a|^p \, d\mu \right)^{1/p} \leq \left(\int_{B(x,r)} |f - f_{B(x,r)}|^p \, d\mu \right)^{1/p}$$

$$\leq 2 \inf_{a \in \mathbb{R}} \left(\int_{B(x,r)} |f - a|^p \, d\mu \right)^{1/p}.$$

Hence, by (3.13), we arrive at

$$\left(\int_{B(x,r)} |f - f_{B(x,r)}|^p \, d\mu \right)^{1/p} \leq cr \left(\int_{B(x,2r)} (\operatorname{lip} f)^p \, d\mu \right)^{1/p}. \quad \blacksquare$$

Let us show that the p-strong resistance conductor condition implies the Poincaré inequality.

COROLLARY 3.14. *Let $f \in \operatorname{Lip}(B(x, 2r))$, $x \in X$ and $0 < r < \operatorname{diam}(X)/2$. If X satisfies the p-strong resistance conductor condition, then*

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq cr \left(\fint_{B(x,2r)} (\operatorname{lip} f)^p \, d\mu \right)^{1/p},$$

where c depends only on p, c_D and c_R.

Proof. It follows from Theorem 3.11 and Hölder's inequality. ∎

To finish this section let us observe that the 1-strong resistance conductor condition implies, by Corollary 3.14 and Hölder's inequality, the $(1, p)$-Poincaré inequality for any locally Lipschitz function. That is, X satisfies a $(1, p)$-Poincaré inequality for any locally Lipschitz function in the sense of [3, Chapter 4]. Therefore, in a similar way as in [3, Corollary 4.19], it follows that X satisfies the (p, p)-Poincaré inequality for any locally Lipschitz function. Finally, as in [6], we can see that X satisfies the p-strong resistance condition.

4. Lip-lip condition, p-fatness and Hardy inequalities

It is shown in [7] that the Poincaré inequality implies the Lip-lip condition when the space is complete. In this section, we show first that the Lip-lip condition follows when the space satisfies a p-strong resistance conductor condition even without completeness. Our argument is similar to the one used in [7, Section 4.3] with minor changes.

Let $x \in X$. Since μ is doubling, there exists a measurable set A in X containing x where $\operatorname{Lip} f$ is a continuous function. Moreover, by the Lebesgue differentiation theorem (see [5, Theorem 1.8] or [7, Proposition 3.2.2]) for

μ-almost every $y \in A$,

$$(4.1) \qquad \lim_{r \to 0} \frac{\mu(B(y,r) \cap A)}{\mu(B(y,r))} = 1.$$

The proof of the next lemma follows from a straight forward adaption the argument in [7, Section 4.3] applying (4.1) and Remark 2.4.

LEMMA 4.2. *Let $\varepsilon > 0$ and $x \in X$. There exists $0 < r_\varepsilon^x \le \varepsilon$ such that for any $0 < r < r_\varepsilon^x$ and $y \in B(x, 4r)$*

$$|f(y) - f_{B(y,r)}| \le r(\mathrm{Lip}f(x) + \varepsilon).$$

As in [7, Section 4.3] applying Corollary 3.14 we obtain the following result.

THEOREM 4.3. *If X satisfies the p-strong resistance conductor condition, then for any $x \in X$ and $f \in \mathrm{Lip}(X)$,*

$$\mathrm{Lip}f(x) \le c\,\mathrm{lip}\,f(x),$$

where c depends only on p, c_D and c_R.

Next we recall two definitions.

DEFINITION 4.4. A set $E \subset X$ is said to be uniformly p-fat if there exists a constant $c_f \ge 1$ such that for every point $x \in E$ and for all $0 < r < \mathrm{diam}(X)/4$, we have

$$\mathrm{cap}_{\mathrm{Lip},p}(\overline{B}(x,r) \cap E, B(x,2r)) \ge c_f \mathrm{cap}_{\mathrm{Lip},p}(\overline{B}(x,r), B(x,2r)).$$

DEFINITION 4.5. The set $\Omega \subset X$ satisfies the pointwise p-Hardy inequality, if there exists $c_H < \infty$ and $L \ge 1$ such that for all $u \in \mathrm{Lip}(X)$ with $u = 0$ in $X \backslash \Omega$,

$$(4.6) \qquad \frac{|u(x)|}{d_\Omega(x)} \le c_H \left(\sup_{0 < r \le L\,d_\Omega(x)} \fint_{B(x,r)} (\mathrm{lip}\,u)^p \, d\mu \right)^{1/p}$$

$$= c_H \left(\mathcal{M}_{Ld_\Omega(x)}(\mathrm{lip}\,u)^p(x) \right)^{1/p}$$

holds for almost every $x \in \Omega$. Here $d_\Omega(x) = d(x, X \backslash \Omega)$ and $\mathcal{M}_{Ld_\Omega(x)}$ denotes the Hardy-Littlewood maximal function with the restricted radii.

The following result is a modification of the corresponding result for spaces satisfying a Poincaré inequality, see [10].

THEOREM 4.7. *Let $\Omega \subset X$ be open and let X satisfy a p-resistance conductor condition. If Ω satisfies the pointwise p-Hardy's inequality, then $X \backslash \Omega$ is uniformly p-fat.*

Proof. Let $B(x,r)$, where $x \in X \backslash \Omega$ and $0 < r < \text{diam}(X)/4$. Let

$$f(z) = \min\left\{ \frac{2r - d(z,x)}{r}, 1 \right\}_+ .$$

It is clear that f is a $1/r$-Lipschitz function such that $f = 1$ in $\overline{B}(x,r)$, $0 \le f \le 1$ and $f = 0$ in $X \backslash B(x,2r)$. We may use f as an admissible function in the definition of the capacity and obtain

$$(4.8) \qquad \text{cap}_{\text{Lip},p}(\overline{B}(x,r), B(x,2r)) \le c_D \frac{\mu(B(x,r))}{r^p}.$$

Therefore, it is enough to show that there exists $c > 0$ such that

$$(4.9) \qquad \frac{\mu(B(x,r))}{r^p} \le c \int_{B(x,2r)} (\text{lip } f)^p \, d\mu,$$

for any $f \in \text{Lip}(X)$ such that $f = 0$ in $X \backslash B(x,2r)$ and $f \ge 1$ in $\overline{B}(x,r) \cap (X \backslash \Omega)$.

Let $l = (2(L+1))^{-1}$, where L is the constant in the pointwise p-Hardy inequality. The doubling property implies that $c_D \mu(B(x,lr)) \ge l^s \mu(B(x,r))$. First let us assume that $f_B > l^s/(2c_D)$, where $B = B(x,r)$. Since $f \in \text{Lip}(B(x,4r))$ and $f = 0$ in $B(x,4r) \backslash B(x,2r)$, Theorem 3.6 and Hölder's inequality imply that

$$(4.10) \qquad \frac{l^s}{2c_D} < f_B \le \frac{c_D}{\mu(B(x,2r))} \int_{B(x,2r)} |f| \, d\mu$$

$$\le c \left(\fint_{B(x,2r)} |f|^p \, d\mu \right)^{1/p} \le cr \left(\fint_{B(x,2r)} (\text{lip } f)^p \, d\mu \right)^{1/p}.$$

Hence (4.9) holds in that case. On the other hand, if $f_B \le l^s/(2c_D)$, then we can argue as in [10] and obtain (4.9) also in that case.

By taking infimum in (4.9) over all $f \in \text{Lip}(X)$ such that $f = 0$ in $X \backslash B(x,2r)$ and $f \ge 1$ in $\overline{B}(x,r) \cap X \backslash \Omega$, we obtain by (4.8)

$$\text{cap}_{\text{Lip},p}(\overline{B}(x,r) \cap X \backslash \Omega, B(x,2r)) \ge c \frac{\mu(B(x,r))}{r^p}$$

$$\ge c \, \text{cap}_{\text{Lip},p}(\overline{B}(x,r), B(x,2r)). \quad \blacksquare$$

Recall that Corollary 3.14 states that under the p-strong resistance conductor condition, we have

$$(4.11) \qquad \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \le cr \left(\fint_{B(x,2r)} (\text{lip } f)^p \, d\mu \right)^{1/p}$$

for every $x \in X$, $0 < r < \text{diam}(X)/2$ and $f \in \text{Lip}(B(x,2r))$. Consequently, there exists $c > 0$ and $\tau \ge 1$ such that

$$(4.12) \qquad |f(x) - f_{B(x,r)}| \le cr \big(\mathcal{M}_{\tau r} \text{lip } f(x)^p \big)^{1/p}$$

for every $x \in X$.

Theorem 4.13. *Let $1 \le p < \infty$ and $\Omega \subsetneq X$ be an open set. In the claims (i)–(iii) we assume that X satisfies the p-strong resistance conductor condition and in the claim (iv) we assume that X satisfies the p-resistance conductor condition.*

(i) *If $X \setminus \Omega$ is uniformly p-fat, then for all balls $B(x, r)$, with $x \in X \setminus \Omega$ and $r > 0$, and $f \in \mathrm{Lip}(X)$, with $f = 0$ in $X \setminus \Omega$, it follows that*

$$\int_{B(x,r)} |f|^p \, d\mu \le c r^p \int_{B(x,2r)} (\mathrm{lip}\, f)^p \, d\mu.$$

(ii) *Let $x \in X \setminus \Omega$, $r > 0$ and $f \in \mathrm{Lip}(X)$ with $f = 0$ in $X \setminus \Omega$ such that*

$$\int_{B(x,r)} |f|^p \, d\mu \le c r^p \int_{B(x,2r)} (\mathrm{lip}\, f)^p \, d\mu.$$

   Then

(4.14)
$$|f_{B(x, d_\Omega(x))}|^p \le c d_\Omega(x)^p \fint_{B(x, 8 d_\Omega(x))} (\mathrm{lip}\, f)^p \, d\mu.$$

(iii) *If for all $x \in \Omega$ and $f \in \mathrm{Lip}(X)$ such that $f = 0$ in $X \setminus \Omega$, we have*

$$|f_{B(x, d_\Omega(x))}|^p \le c d_\Omega(x)^p \fint_{B(x, 8 d_\Omega(x))} (\mathrm{lip}\, f)^p \, d\mu,$$

   then Ω satisfies the pointwise p-Hardy inequality.

(iv) *If Ω satisfies the pointwise p-Hardy's inequality, then $X \setminus \Omega$ is uniformly p-fat.*

Proof. (i) Let $B = B(x, r)$, with $x \in X \setminus \Omega$, $0 < r < \mathrm{diam}(X)/2$ and $f \in \mathrm{Lip}(X)$ such that $f = 0$ in $X \setminus \Omega$. By the p-fatness of $X \setminus \Omega$, since X satisfies a p-resistance conductor condition

$$\mathrm{cap}_{\mathrm{Lip},p}\big(\overline{B}(x, r/2) \cap \{f = 0\}, B(x, r)\big) \ge \mathrm{cap}_{\mathrm{Lip},p}\big(\overline{B}(x, r/2) \cap X \setminus \Omega, B(x, r)\big)$$
$$\ge c_f \mathrm{cap}_{\mathrm{Lip},p}\big(B(x, r/2), B(x, r)\big)$$
$$\ge \frac{c_f}{c_D} \frac{\mu(B(x, r))}{c_R r^p}.$$

By a Maz'ya type inequality (see [11] and [2, Proposition 3.2]), we have

$$\frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f|^p \, d\mu$$
$$\le \frac{c}{\mathrm{cap}_{\mathrm{Lip},p}(B(x, r/2) \cap \{f = 0\}, B(x, r))} \int_{B(x,2r)} (\mathrm{lip}\, f)^p \, d\mu$$
$$\le \frac{c r^p}{\mu(B(x, r))} \int_{B(x,2r)} (\mathrm{lip}\, f)^p \, d\mu,$$

and then

$$\int_{B(x,r)} |f|^p \, d\mu \le cr^p \int_{B(x,2r)} (\text{lip } f)^p \, d\mu.$$

If instead $\text{diam}(X)/2 \le r \le \text{diam}(X)$, let us take $\hat{B} = B(x, \text{diam}(X)/3)$. It follows that

$$(4.15) \qquad \int_{B(x,r)} |f|^p d\mu$$

$$\le c\left(\int_{B(x,r)} |f - f_{B(x,r)}|^p \, d\mu + \int_{B(x,r)} |f_{B(x,r)} - f_{\hat{B}}|^p \, d\mu + \int_{B(x,r)} |f_{\hat{B}}|^p \, d\mu\right)$$

$$\le c\left(\int_{B(x,r)} |f - f_{B(x,r)}|^p d\mu + \mu(B(x,r))|f_{B(x,r)} - f_{\hat{B}}|^p + \mu(B(x,r))|f_{\hat{B}}|^p\right).$$

Applying the previous case to $\hat{B}$, we have

$$\mu(B(x,r))|f_{\hat{B}}|^p = \mu(B(x,r))\left|\frac{1}{\mu(\hat{B})}\int_{\hat{B}} f \, d\mu\right|^p$$

$$\le \mu(B(x,r))\left|\frac{1}{\mu(\hat{B})}\left(\int_{\hat{B}} f^p \, d\mu\right)^{1/p} \mu(\hat{B})^{1-1/p}\right|^p$$

$$\le c\frac{\mu(B(x,r))}{\mu(\hat{B})}\left(\frac{\text{diam}(X)}{3}\right)^p \int_{2\hat{B}} (\text{lip } f)^p \, d\mu$$

$$\le cr^p \int_{2\hat{B}} (\text{lip } f)^p \, d\mu$$

$$\le cr^p \int_{B(x,2r)} (\text{lip } f)^p \, d\mu.$$

Moreover, by Theorem 3.11,

$$\mu(B(x,r))|f_{B(x,r)} - f_{\hat{B}}|^p$$

$$\le \mu(B(x,r))\left|\frac{1}{\mu(\hat{B})^{1/p}}\left(\int_{\hat{B}} (f(z) - f_{B(x,r)})^p \, d\mu(z)\right)^{1/p}\right|^p$$

$$\le \frac{\mu(B(x,r))}{\mu(\hat{B})}\int_{B(x,r)} |f(z) - f_{B(x,r)}|^p \, d\mu(z)$$

$$\le cr^p \int_{B(x,2r)} (\text{lip } f)^p \, d\mu,$$

and

$$\int_{B(x,r)} |f - f_{B(x,r)}|^p \, d\mu \le cr^p \int_{B(x,2r)} (\text{lip } f)^p \, d\mu.$$

The claim follows from (4.15) and the last three inequalities.

(ii) Let $x \in \Omega$ and $f \in \mathrm{Lip}(X)$ such that $f = 0$ in $X \setminus \Omega$, and $B_x = B(x, d_\Omega(x))$. Choose $w \in X \setminus \Omega$ so that $r = d(w, x) \leq 2d_\Omega(x)$, and consider $B(w, r)$. Then, it follows that

$$|f_{B_x}| \leq |f_{B_x} - f_{B(w,r)}| + |f|_{B(w,r)}.$$

By (4.11), the fact that $B(w, r) \subset 4B_x$ and $B_x \subset 2B(w, r)$, we have that

$$|f_{B_x} - f_{B(w,r)}| \leq cd_\Omega(x) \left(\fint_{8B_x} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p}.$$

Moreover, by Hölder's inequality, the hypothesis and the Maz'ya type inequality, it follows that

$$|f|_{B(w,r)} \leq \left(\fint_{B(w,r)} |f|^p \, d\mu \right)^{1/p} \leq cr \left(\fint_{B(w,2r)} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p}$$

$$\leq cd_\Omega(x) \left(\fint_{B(w,2r)} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p} \leq cd_\Omega(x) \left(\fint_{8B_x} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p}.$$

(iii) Let $f \in \mathrm{Lip}(X)$ such that $f = 0$ in $X \setminus \Omega$ and $x \in \Omega$, which is a Lebesgue point. By hyphotesis,

$$|f_{B(x,d_\Omega(x))}| \leq c^{1/p} d_\Omega(x) \left(\fint_{B(x,8d_\Omega(x))} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p},$$

and by (4.12),

$$|f(x) - f_{B_x}| \leq cd_\Omega(x) \left(\mathcal{M}_{\tau d_\Omega(x)} (\mathrm{lip}\, f(x))^p \right)^{1/p}.$$

Hence,

$$|f(x)| \leq |f(x) - f_{B_x}| + |f_{B_x}| \leq cd_\Omega(x) \left(\fint_{B(x,\max\{8,\tau\}d_\Omega(x))} (\mathrm{lip}\, f)^p \, d\mu \right)^{1/p}.$$

Since $f \in \mathrm{Lip}(X)$, any point in Ω is a Lebesgue point and the pointwise p-Hardy inequality follows.

(iv) The claim follows from Theorem 4.7. ∎

Finally, we have the following characterization.

COROLLARY 4.16. *Let X satisfy the p-strong resistance conductor condition and let $\Omega \subsetneq X$ be an open set. Then the following properties are equivalent.*

(i) *$X \setminus \Omega$ is uniformly p-fat.*

(ii) *For all $B(w, r)$, with $w \in X \setminus \Omega$, $r > 0$ and $f \in \mathrm{Lip}(X)$, $f = 0$ in $X \setminus \Omega$,*

$$\int_{B(w,r)} |f|^p \, d\mu \leq cr^p \int_{B(w,2r)} (\mathrm{lip}\, f)^p \, d\mu.$$

(iii) *For all $x \in \Omega$ and $f \in \mathrm{Lip}(X)$ such that $f = 0$ in $X \setminus \Omega$*

$$|f_{B(x,d_\Omega(x))}|^p \le c d_\Omega(x)^p \fint_{B(x,8d_\Omega(x))} (\mathrm{lip}\, f)^p \, d\mu.$$

(iv) Ω *admits the pointwise p-Hardy's inequality.*

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