Regularity of the discrete maximal operators on metric measure spaces

Juha Kinnunen, Aalto University, Finland

http://math.aalto.fi/∼jkkinnun/

Plan of the talk

Introduction: A quick review of the Euclidean case.

Aim: Construct a maximal function that is smoother than the standard Hardy-Littlewood maximal function.

Spaces: Sobolev, Hölder, Morrey, Campanato and BMO.

Context: Metric measure spaces.

Tools: Approximations of unity and discrete convolution.

Application: Lebesgue points for Sobolev functions (NO extension theorems, NO representation formulas).

References

T. Heikkinen, J. Kinnunen, J. Korvenpää and H. Tuominen, *Regularity of the local maximal function*, Ark. Mat 53 (2015), 127-154

T. Heikkinen, J. Kinnunen and H. Tuominen, *Mapping properties of the discrete fractional maximal operator in metric spaces*, Kyoto J. Math. 53 (2013), 693-712

D. Aalto and J. Kinnunen, *The discrete maximal operator in metric spaces*, J. Anal. Math. 111 (2010), 369-390

J. Kinnunen and H. Tuominen, *Pointwise behaviour of Sobolev functions*, Math. Z., 257 (2007), 613-630

J. Kinnunen and E. Saksman, *Regularity of the fractional maximal function*, Bull. London Math. Soc., 35 (2003), 529-535

J. Kinnunen and V. Latvala, *Lebesgue points for Sobolev functions on metric spaces*, Rev. Mat. Iberoamericana, 18 (2002), 685-700

J. Kinnunen and P. Lindqvist, *The derivative of the maximal function*, J. reine angew. Math., 503 (1998), 161-167

J. Kinnunen, *The Hardy-Littlewood maximal function of a Sobolev function*, Israel J. Math., 100 (1997), 117-124

Hardy-Littlewood maximal function

The centered Hardy-Littlewood maximal function is defined as

$$Mu(x) = \sup_{r>0} \fint_{B(x,r)} |u(y)|\, dy.$$

NOTE:

$$\begin{aligned}\fint_{B(x,r)} |u(y)|\, dy &= \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)|\, dy \\ &= (|u| * \phi_r)(x),\end{aligned}$$

with

$$\phi_r(x) = \frac{\chi_{B(0,r)}(x)}{|B(0,r)|}.$$

Maximal function of a Lipschitz function

Suppose that

$$|u_h(y) - u(y)| = |u(y + h) - u(y)| \le L|h|$$

for every $y, h \in \mathbb{R}^n$, where

$$u_h(y) = u(y + h).$$

Since the maximal function commutes with translations and the maximal operator is sublinear, we have

$$\begin{aligned}
&|(Mu)_h(x) - Mu(x)| \\
&\quad = |M(u_h)(x) - Mu(x)| \\
&\quad \le M(u_h - u)(x) \\
&\quad = \sup_{r>0} \fint_{B(x,r)} |u_h(y) - u(y)|\, dy \\
&\quad \le L|h|.
\end{aligned}$$

Maximal function on Sobolev spaces

Let $1 < p \le \infty$. Then there is $c = c(n,p)$ such that

$$\|Mu\|_{1,p} \le c\|u\|_{1,p}.$$

Moreover, $|DMu| \le M|Du|$ almost everywhere.

Proof:

$$\begin{aligned} \|(Mu)_h - Mu\|_p &= \|M(u_h) - Mu\|_p \\ &\le \|M(u_h - u)\|_p \\ &\le c\|u_h - u\|_p \\ &\le c\|Du\|_p|h|. \end{aligned}$$

(Kinnunen, 1997)

REMARK: Maximal operator is also continuous in Sobolev spaces. (Luiro, 2007)

Remark

The Hardy-Littlewood maximal operator does not preserve higher order regularity.

An open question

What happens in $W^{1,1}(\mathbb{R}^n)$ or $BV(\mathbb{R}^n)$, when $n \geq 2$?

Capacity

The Sobolev p-capacity of the set $E \subset \mathbb{R}^n$ is

$$\mathrm{cap}_p(E) = \inf_{u \in \mathcal{A}(E)} \int_{\mathbb{R}^n} \Big(|u|^p + |Du|^p \Big)\, dx,$$

where

$$\mathcal{A}(E) = \Big\{ u \in W^{1,p}(\mathbb{R}^n) : u \geq 1 \text{ on a neighbourhood of } E \Big\}.$$

Maximal function as a test function

Let $u \in W^{1,p}(\mathbb{R}^n)$, suppose that $\lambda > 0$ and denote

$$E_\lambda = \{x \in \mathbb{R}^n : Mu(x) > \lambda\}.$$

Then E_λ is open,

$$\frac{Mu}{\lambda} \in \mathcal{A}(E_\lambda)$$

and

$$\begin{aligned} \text{cap}_p(E_\lambda) &\leq \frac{1}{\lambda^p} \int_{\mathbb{R}^n} \left(|Mu|^p + |DMu|^p\right) dx \\ &\leq \frac{c}{\lambda^p} \int_{\mathbb{R}^n} \left(|u|^p + |Du|^p\right) dx. \end{aligned}$$

This weak type inequality can be used in studying the pointwise behaviour of Sobolev functions by the standard methods.

Fractional maximal function

Let $0 \leq \alpha \leq n$. The fractional maximal function is defined as

$$M_\alpha u(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |u(y)|\, dy.$$

REMARK: When $0 < \alpha < n$, there is a close connection to the Riesz potential

$$I_\alpha u(x) = \int_{\mathbb{R}^n} \frac{|u(y)|}{|x-y|^{n-\alpha}}\, dy.$$

Indeed,

$$M_\alpha u(x) \leq c I_\alpha u(x)$$

for every $x \in \mathbb{R}^n$ and

$$c^{-1}\|M_\alpha u\|_p \leq \|I_\alpha u\|_p \leq c\|M_\alpha u\|_p$$

for every $1 < p < \infty$.

Behaviour in Sobolev spaces

Let $1 < p < \infty$ and let $0 \leq \alpha < n/p$. Then there is $c = c(n, p, \alpha)$ such that

$$\|M_\alpha u\|_{1,q^*} \leq c\|u\|_{1,p}$$

with

$$q^* = \frac{np}{n - \alpha p}.$$

Proof: The same as for the Hardy-Littlewood maximal operator together with the Sobolev inequality.

(Saksman and Kinnunen, 2003)

A smoothing property

Let $u \in L^p(\mathbb{R}^n)$ with $1 < p < n$ and $1 \le \alpha < n/p$. Then there is $c = c(n, p, \alpha)$ such that

$$\|M_\alpha u\|_{q^*} \le c\|u\|_p$$

and

$$\|DM_\alpha u\|_q \le c\|u\|_p$$

with

$$q^* = \frac{np}{n - \alpha p} \quad \text{and} \quad q = \frac{np}{n - (\alpha - 1)p}.$$

(Saksman and Kinnunen, 2003)

(Heikkinen, Kinnunen, Korvenpää and Tuominen, 2015)

Conclusion

The fractional maximal operator does not only preserve the first order Sobolev spaces, but it also maps L^p spaces boundedly into certain first order Sobolev spaces.

This smoothing property is similar to the Riesz potential.

An unexpected problem

The standard Hardy-Littlewood maximal function does NOT preserve smoothness of the functions in more general metric measure spaces.

It may happen, that the maximal function of a continuous function is NOT continuous.

(Buckley, 1999)

Metric measure space

(X, d, μ) is a metric measure space.

The measure is doubling, if there exists a constant $c_D \geq 1$ such that

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r))$$

for all $x \in X$ and $r > 0$.

(Coifman and Weiss, 1971)

Upper gradient

A nonnegative Borel function g on X is an upper gradient of an extended real valued function u on X if for all $x, y \in X$ and for all paths γ joining x and y in X,

$$|u(x) - u(y)| \leq \int_\gamma g\, ds.$$

(Heinonen and Koskela, 1998)

Sobolev spaces

For $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all upper gradients of u.

The Sobolev space on X is

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if

$$\|u - v\|_{N^{1,p}(X)} = 0.$$

(Shanmugalingam, 2000)

Poincaré inequality

The space X supports a Poincaré inequality, if there exist constants $c_P > 0$ such that for all balls $B(x,r)$ of X, all locally integrable functions u on X and for all upper gradients g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le c_P r \fint_{B(x,r)} g \, d\mu,$$

where

$$\begin{aligned} u_{B(x,r)} &= \fint_{B(x,r)} u \, d\mu \\ &= \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu. \end{aligned}$$

Assumptions

From now on, we assume the measure is doubling and that the space supports the Poincaré inequality.

Hardy-Littlewood maximal function

The centered Hardy-Littlewood maximal function on X is defined as

$$Mu(x) = \sup_{r>0} \fint_{B(x,r)} |u| \, d\mu.$$

PROPERTIES:

I. Strong type estimate

$$\|Mu\|_{L^p(X)} \le c\|u\|_{L^p(X)}, \quad 1 < p \le \infty.$$

II. Weak type estimate

$$\mu(\{Mu > \lambda\}) \le \frac{c}{\lambda} \int_X |u| \, d\mu, \quad 0 < \lambda < \infty.$$

Coverings by balls

Let $r > 0$. There is a family of balls $B(x_i, r)$, $i = 1, 2, \dots$, such that

$$X = \bigcup_{i=1}^{\infty} B(x_i, r)$$

and

$$\sum_{i=1}^{\infty} \chi_{B(x_i, 6r)} \le c < \infty.$$

This means that the dilated balls $B(x_i, 6r)$ have bounded overlap.

The constant c depends only on the doubling constant and, in particular, it is independent of the scale r.

Partition of unity

There are functions ϕ_i, $i = 1, 2, \dots$, such that

(i) $0 \le \phi_i \le 1$,

(ii) $\phi_i = 0$ on $X \setminus B(x_i, 6r)$,

(iii) $\phi_i \ge c$ on $B(x_i, 3r)$,

(iv) ϕ_i is Lipschitz with constant c/r_i with c depending only on the doubling constant, and

(v)

$$\sum_{i=1}^{\infty} \phi_i = 1$$

in X.

Construction

Let

$$\varphi_i(x) = \begin{cases} 1, & x \in B(x_i, 3r), \\ 2 - \dfrac{d(x, x_i)}{3r}, & x \in B(x_i, 6r) \setminus B(x_i, 3r), \\ 0, & x \in X \setminus B(x_i, 6r) \end{cases}$$

and

$$\phi_i(x) = \frac{\varphi_i(x)}{\sum_{j=1}^{\infty} \varphi_j(x)}.$$

It is not difficult to see that the functions satisfy the required properties.

Discrete convolution

Define an approximation of $u \in L^1_{\mathrm{loc}}(X)$ at the scale of $3r$ by setting

$$u_r(x) = \sum_{i=1}^{\infty} \phi_i(x) u_{B(x_i,3r)}$$

for every $x \in X$.

The function u_r is called the discrete convolution of u.

(Coifman and Weiss 1971, Macías and Segovia 1979)

Properties

I. The discrete convolution is Lipschitz continuous.

II. Suppose that $u \in L^p(X)$ with $1 \le p \le \infty$ and let $r > 0$. Then $u_r \in L^p(X)$ and

$$\|u_r\|_{L^p(X)} \le c\|u\|_{L^p(X)}.$$

The constant c depends only on the doubling constant.

Moreover,

$$\|u_r - u\|_{L^p(X)} \to 0$$

as $r \to 0$, when $1 \le p < \infty$.

Sketch of a proof

The case $p = \infty$:

$$
\begin{aligned}
|u_r(x)| &\le \Big| \sum_{i=1}^{\infty} \phi_i(x) u_{B(x_i,3r)} \Big| \\
&\le \sum_{i=1}^{\infty} \phi_i(x) |u_{B(x_i,3r)}| \\
&\le \|u\|_{L^\infty(X)} \sum_{i=1}^{\infty} \phi_i(x) \\
&= \|u\|_{L^\infty(X)}
\end{aligned}
$$

for every $x \in X$.

The case $1 \le p < \infty$ and the L^p-convergence are sligthly more involved.

The discrete maximal function

Let r_j, $j = 1, 2, \dots$, be an enumeration of the positive rationals. For every radius r_j we take a covering by balls $B(x_i, r_j)$, $i = 1, 2, \dots$.

The discrete maximal function related to the coverings $B(x_i, r_j)$, $i, j = 1, 2, \dots$, is defined by

$$M^* u(x) = \sup_j |u|_{r_j}(x)$$

for every $x \in X$.

Observations

I. The discrete maximal operator depends on the chosen coverings. This is not a serious matter, since the estimates are independent of the coverings.

II. As a supremum of continuous functions, the discrete maximal function is lower semicontinuous and hence measurable.

III.

$$M^*(\alpha u)(x) = |\alpha| M^* u(x).$$

IV.

$$M^*(u + v)(x) \leq M^* u(x) + M^* v(x).$$

Discrete $\approx$ Hardy-Littlewood

There is a constant $c \geq 1$, which depends only on the doubling constant, such that

$$c^{-1} Mu(x) \leq M^{*}u(x) \leq cMu(x)$$

for every $x \in X$.

Proof of the first inequality

For each $x \in X$ there exists $i = i_x$ such that $x \in B(x_i, r_j)$. This implies that

$$B(x, r_j) \subset B(x_i, 2r_j)$$

and hence

$$\begin{aligned} \fint_{B(x,r_j)} |u|\, d\mu &\le c \fint_{B(x_i,3r_j)} |u|\, d\mu \\ &\le c\phi_i(x) \fint_{B(x_i,3r_j)} |u|\, d\mu \\ &\le cM^*u(x). \end{aligned}$$

In the second inequality we used the fact that $\phi_i \ge c$ on $B(x_i, r_j)$. The claim follows by taking the supremum on the left side.

Proof of the second inequality

Let $x \in X$ and r_j be a positive rational number. Since $\phi_i = 0$ on $X \setminus B(x_i, 6r_j)$ and

$$B(x_i, 3r_j) \subset B(x, 9r_j)$$

for every $x \in B(x_i, 6r_j)$, we have by the doubling condition that

$$\begin{aligned} |u|_{r_j}(x) &= \sum_{i=1}^{\infty} \phi_i(x) |u|_{B(x_i,3r_j)} \\ &\leq \sum_{i=1}^{\infty} \phi_i(x) \frac{\mu(B(x, 9r_j))}{\mu(B(x_i, 3r_j))} \fint_{B(x,9r_j)} |u| \, d\mu \\ &\leq cMu(x), \end{aligned}$$

where c depends only on the doubling constant. The second inequality follows by taking the supremum on the left side.

L^p **bounds**

I. Strong type estimate

$$\|M^*u\|_{L^p(X)} \le c\|Mu\|_{L^p(X)} \le c\|u\|_{L^p(X)},$$

where $1 < p \le \infty$. The constant c depends only on the doubling constant and p.

II. Weak type estimate

$$\mu(\{M^*u > \lambda\}) \le \mu(\{Mu > c\lambda\}) \le \frac{c}{\lambda}\int_X |u|\,d\mu,$$

where $0 < \lambda < \infty$. The constant c depends only on the doubling constant.

Sobolev spaces

If $u \in N^{1,p}(X)$ with $p > 1$, then $M^*u \in N^{1,p}(X)$ with a norm bound.

In addition, the function cM^*g_u is an upper gradient of M^*u whenever g_u is an upper gradient of u. The constant c depends only on the doubling constant.

Application: Pointwise bahaviour of the Sobolev functions on metric measure spaces.

(Kinnunen and Latvala, 2002)

Sketch of a proof

Let $r > 0$. Then

$$
\begin{aligned}
|u|_r(x) &= \sum_{i=1}^{\infty} \phi_i(x) |u|_{B(x_i,3r)} \\
&= |u(x)| + \sum_{i=1}^{\infty} \phi_i(x) \Big(|u|_{B(x_i,3r)} - |u(x)| \Big).
\end{aligned}
$$

This implies that

$$
g_r = g_u + \sum_{i=1}^{\infty} \left(\frac{c}{r} \Big| |u| - |u|_{B(x_i,3r)} \Big| + g_u \right) \chi_{B(x_i,6r)}
$$

is an upper gradient of $|u|_r$.

Let $x \in B(x_i, 6r)$. Then $B(x_i, 3r) \subset B(x, 9r)$ and

$$\begin{aligned}
&\Big||u(x)| - |u|_{B(x_i,3r)}\Big| \\
&\quad \le \Big||u(x)| - |u|_{B(x,9r)}\Big| + \Big||u|_{B(x,9r)} - |u|_{B(x_i,3r)}\Big|.
\end{aligned}$$

We estimate the second term on the right side by the Poincaré inequality and the doubling condition as

$$\begin{aligned}
&\Big||u|_{B(x,9r)} - |u|_{B(x_i,3r)}\Big| \\
&\quad \le \fint_{B(x_i,3r)} \Big||u| - |u|_{B(x,9r)}\Big| \, d\mu \\
&\quad \le c \fint_{B(x,9r)} \Big||u| - |u|_{B(x,9r)}\Big| \, d\mu \\
&\quad \le cr \fint_{B(x,9r)} g_u \, d\mu.
\end{aligned}$$

The first term on the right side is estimated by a standard telescoping argument. Since μ-almost every point is a Lebesgue point for u, we have

$$
\begin{aligned}
&\Big| |u(x)| - |u|_{B(x,9r)} \Big| \\
&\quad \le \sum_{j=0}^{\infty} \Big| |u|_{B(x,3^{2-j}r)} - |u|_{B(x,3^{1-j}r)} \Big| \\
&\quad \le c \sum_{j=0}^{\infty} \fint_{B(x,3^{2-j}r)} \Big| |u| - |u|_{B(x,3^{2-j}r)} \Big| \, d\mu \\
&\quad \le c \sum_{j=0}^{\infty} 3^{2-j} r \fint_{B(x,3^{2-j}r)} g_u \, d\mu \le c r M g_u(x).
\end{aligned}
$$

for μ-almost every $x \in X$.

We conclude that

$$g_r(x) \le c M g_u(x)$$

for μ-almost every $x \in X$.

FACT: Suppose that u_i, $i = 1, 2, \ldots$, are functions and g_i are upper gradients of u_i, $i = 1, 2, \ldots$, respectively. Let

$$u = \sup_i u_i \quad \text{and} \quad g = \sup_i g_i.$$

If $u < \infty$ $\mu-$almost everywhere, then g is an upper gradient of u.

Since

$$M^* u(x) = \sup_j |u|_{r_j}(x)$$

and $cM^* g_u$ is an upper gradient of $|u|_{r_j}$ for every j, we conclude that it is an upper gradient of $M^* u$ as well.

The claim follows from the maximal function theorem.

Dimension related to the measure

The doubling condition implies that

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \le c\Big(\frac{R}{r}\Big)^Q$$

for every $0 < r \le R$. Here $c > 0$ depends only on the doubling constant and

$$Q = \log_2 c_D.$$

We say that μ satisfies the measure lower bound condition, if

$$\mu(B(x,r)) \ge cr^Q$$

for every $r > 0$ and $x \in X$.

From now on, we assume that this condition holds.

Fractional maximal function

Let $0 \leq \alpha \leq Q$. The fractional maximal function of u is

$$M_\alpha u(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |u| \, d\mu.$$

The discrete fractional maximal function

The discrete fractional maximal function related to the coverings $B(x_i, r_j)$, $i, j = 1, 2, \dots,$ is defined by

$$M^*_\alpha u(x) = \sup_j r_j^\alpha |u|_{r_j}(x)$$

for every $x \in X$, where $|u|_{r_j}$ is the discrete convolution as before.

(Heikkinen, Kinnunen, Nuutinen and Tuominen, 2013)

Properties

I. There is a constant $c \geq 1$, which depends only on the doubling constant, such that

$$c^{-1} M_\alpha u(x) \leq M_\alpha^* u(x) \leq c M_\alpha u(x)$$

for every $x \in X$.

II. Let $p > 1$ and $0 \leq \alpha < Q/p$. Then there is a constant c, depending only on the the doubling constant, constant in the measure lower bound, p and α, such that

$$\|M_\alpha^* u\|_{L^{p^*}(X)} \leq c \|u\|_{L^p(X)},$$

for every $u \in L^p(X)$ with $p^* = Qp/(Q - \alpha p)$.

III. Let $0 < \alpha < Q$. Then there is a constant $c > 0$, depending only on the the doubling constant, the constant in the measure lower bound and α, such that

$$\mu(\{M_\alpha^* u > \lambda\}) \leq c \Big(\frac{\|u\|_1}{\lambda} \Big)^{Q/(Q-\alpha)},$$

for every $u \in L^1(X)$.

Sobolev spaces

Let $u \in N^{1,p}(X)$ and that $0 < \alpha < Q/p$. Then there is a constant $c > 0$, depending only on the doubling constant, the constant in the measure lower bound, p and α, such that

$$\|M^*_\alpha u\|_{N^{1,p^*}(X)} \le c\|u\|_{N^{1,p}(X)}$$

with

$$p^* = \frac{Qp}{Q - \alpha p}.$$

Conclusion

The discrete fractional maximal function preserves the first order Sobolev spaces.

QUESTION: Does it smoothen the functions as in the Euclidean case?

A smoothing property

Let $u \in L^p(X)$ with $1 < p < Q$ and $1 \le \alpha < Q/p$. Then there is a constant c, depending only on the doubling constant, the constant in the measure lower bound, p and α, such that $cM^*_{\alpha-1}u$ is an upper gradient of $M^*_\alpha u$.

Moreover,

$$\|M^*_\alpha u\|_{L^{p^*}(X)} \le c\|u\|_{L^p(X)}$$

and

$$\|M^*_{\alpha-1} u\|_{L^q(X)} \le c\|u\|_{L^p(X)}$$

with

$$p^* = \frac{Qp}{Q-\alpha p} \quad \text{and} \quad q = \frac{Qp}{Q-(\alpha-1)p}.$$

Morrey spaces

A function $u \in L^1_{\mathrm{loc}}(X)$ belongs to the Morrey space $\mathcal{M}^{p,\beta}(X)$, if

$$\|u\|_{\mathcal{M}^{p,\beta}(X)} = \sup r^{-\beta}\Big(\fint_{B(x,r)} |u|^p \, d\mu\Big)^{1/p} < \infty,$$

where the supremum is taken over all $x \in X$ and $r > 0$.

Campanato spaces

A function $u \in L^1_{\mathrm{loc}}(X)$ belongs to the Campanato space $\mathcal{L}^{p,\beta}(X)$, if

$$\|u\|_{\mathcal{L}^{p,\beta}(X)} = \sup r^{-\beta} \left(\fint_{B(x,r)} |u - u_{B(x,r)}|^p \, d\mu \right)^{1/p} < \infty,$$

where the supremum is taken over all $x \in X$ and $r > 0$.

Properties

Morrey spaces, Campanato spaces, BMO and functions in $C^{0,\beta}(X)$ have the following connections:

I. $\mathcal{M}^{p,\beta}(X) \subset \mathcal{L}^{p,\beta}(X)$,

II. $\mathcal{L}^{p,\beta}(X) = \mathcal{M}^{p,\beta}(X)$ if $-Q/p < \beta < 0$ (here we identify functions that differ only by an additive constant),

III. $\mathcal{L}^{1,0}(X) = \mathrm{BMO}(X)$, and

IV. $\mathcal{L}^{p,\beta}(X) = C^{0,\beta}(X)$ if $0 < \beta \leq 1$.

Behaviour on Morrey spaces

Let $\alpha > 0$ and $\beta < -\alpha$. Let $u \in \mathcal{M}^{p,\beta}(X)$ with $1 < p < \infty$. Then there is a constant $c > 0$, depending only on the doubling constant, p, α and β, such that

$$\|M_\alpha u\|_{\mathcal{M}^{p/(1+\alpha/\beta),\alpha+\beta}(X)} \leq c\|u\|_{\mathcal{M}^{p,\beta}(X)}.$$

(Heikkinen, Kinnunen, Nuutinen and Tuominen, 2013)

Behaviour on Hölder spaces

Let $u \in C^{0,\beta}(X)$ with $0 < \beta \leq 1$. If $\alpha + \beta < 1$, then $M_\alpha^* u \in C^{0,\alpha+\beta}(X)$.

If $\alpha + \beta \geq 1$, then $M_\alpha^* u$ is Lipschitz continuous on each bounded subset $A \subset X$ with the Lipschitz constant $\max\{1, \operatorname{diam}(A)^{\alpha+\beta-1}\}$.

(Heikkinen, Kinnunen, Nuutinen and Tuominen, 2013)

Behaviour on Campanato spaces

Let $\alpha > 0$, $0 \leq \alpha + \beta \leq 1$ and let $u \in \mathcal{L}^{p,\beta}(X)$. Then there is a constant $c > 0$, depending only on the doubling constant p and α and β, such that

$$\|M_\alpha^* u\|_{C^{0,\alpha+\beta}(X)} \leq c\|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

(Heikkinen, Kinnunen, Nuutinen and Tuominen, 2013)

Coifman-Rochberg lemma

Let $0 < \alpha < Q$. Assume that $u \in L^1_{\mathrm{loc}}(X)$ is such that $M_\alpha u$ is finite almost everywhere. Then $M_\alpha u$ is a Muckenhoupt A_1-weight, that is,

$$\fint_B M_\alpha u \, d\mu \le c \operatorname*{ess\,inf}_B M_\alpha u$$

for every ball B in X. The constant c does not depend on u.

(Heikkinen, Kinnunen, Nuutinen and Tuominen, to appear)

Summary

I. The standard Hardy-Littlewood maximal does not preserve smoothness in metric measure spaces, but it is possible to construct a discrete maximal function which has better regularity properties.

II. The construction is based on partitions of unities and the discrete convolutions. The obtained maximal function is pointwisely equivalent to the standard maximal function.

III. The fractional maximal function does not only preserve the smoothness but also smoothens the function in the same fashion as the Riesz potential.

IV. The techniques apply to a wide range of spaces (Sobolev, Hölder, Morrey, Campanato, BMO) in the context of metric measure spaces.