Regularity for a doubly nonlinear equation

Prototypes of parabolic equations

(1) Evolutionary p-Laplace (or p-parabolic) equation

$$\frac{\partial u}{\partial t} - \mathrm{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty.$$

(DiBenedetto, Gianazza, Vespri)

(2) The porous medium equation

$$\frac{\partial u}{\partial t} - \Delta(|u|^{m-1}u) = 0, \quad 0 < m < \infty.$$

(Caffarelli, Friedman, Vazquez)

(3) The doubly nonlinear equation

$$\frac{\partial(|u|^{p-2}u)}{\partial t} - \mathrm{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty.$$

(Trudinger)

When $p = 2$ or $m = 1$ we have the standard heat equation.

Structure

Evolutionary p-Laplace equation: Can add constants but cannot scale.

The porous medium equation: Cannot add constants and cannot scale.

The doubly nonlinear equation: Cannot add constants but can scale.

The minimum of two solutions is a supersolution for all equations.

All equations are highly nonlinear: The sum of two solutions is not a solution, in general.

The plan of the talk

We consider nonnegative weak solutions of the doubly nonlinear equation.

I. Description of the context

II. Scale and location invariant Harnack estimates using Moser's iteration scheme (Kuusi)

III. Local Hölder continuity of weak solutions using a DiBenedetto type argument (Kuusi, Siljander, Urbano)

IV. Higher regularity theory using DiBenedetto-Friedman type argument (Siljander)

Motivation

(1) Everything should be folklore, but very difficult to find in the literature when $p \neq 2$.

(2) There is a lack of transparent proofs in the literature.

(3) Unexpected phenomenon: Local Hölder continuity does not follow directly from Harnack estimates, since constants cannot be added to solutions.

(4) General belief that the doubly nonlinear equation is easier than the evolutionary p-Laplace equation has turned out to be FALSE.

Possible applications

(1) A possible generalization of a theorem of Grigory'an and Saloff-Coste related to a doubling condition and the Poincaré inequality in metric measure spaces. This is related to the work of Kumagai.

(2) A possible generalization of the boundary Harnack principle for the time-independent p-Laplace equation by Lewis and Nyström. Regularity results for different measures than Lebesgue measure play a central role in their argument.

I. Description of the context

Doubling measures

The Borel measure μ is doubling, if there exists a constant $D_0 \geq 1$ such that

$$\mu(B(x, 2r)) \leq D_0 \mu(B(x, r))$$

for every $x \in \mathbb{R}^n$ and $r > 0$.

If μ is doubling and $r < R$, then

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C \Big(\frac{R}{r}\Big)^{d_\mu},$$

where

$$d_\mu = \log_2 D_0$$

is a dimension related to the measure and C is an absolute constant.

(Coifman and Weiss)

The Poincaré inequality

The measure is said to support a $(1,p)$-Poincaré inequality if there exist constants $P_0 > 0$ such that

$$\begin{aligned} &\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \\ &\qquad \le P_0 r \Big(\fint_{B(x,r)} |Du|^p \, d\mu \Big)^{1/p}, \end{aligned}$$

for every $u \in C^\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $r > 0$, where

$$\begin{aligned} u_{B(x,r)} &= \fint_{B(x,r)} u \, d\mu \\ &= \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu \end{aligned}$$

denotes the integral average.

Doubling and Poincaré implies Sobolev

There is a constant $C > 0$ such that

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^{\kappa p} \, d\mu \right)^{1/\kappa p} \leq Cr \left(\fint_{B(x,r)} |Du|^p \, d\mu \right)^{1/p}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where

$$\kappa = \begin{cases} \dfrac{d_\mu}{d_\mu - p}, & 1 < p < d_\mu, \\ 2, & p \geq d_\mu. \end{cases}$$

is the Sobolev conjugate exponent (Hajłasz and Koskela, 1995).

Here d_μ is the dimension related to the measure and the constant $C = C(p, D_0, P_0)$.

A recent result of Keith and Zhong (Ann. of Math, 2008) shows that the exponent on the right hand side is self improving as well.

General assumptions

From now on we assume that the measure μ is doubling and supports a $(1, p)$-Poincaré inequality.

Moreover, we assume that the measure is nontrivial in the sense that the measure of every nonempty open set is strictly positive and measure of every bounded set is finite.

These are standard assumptions in analysis on Riemannian manifolds and on more general metric measure spaces, but they are NOT well understood.

Example. Muckenhoupt's A_p-weights satisfy these assumptions (Fabes, Kenig and Serapioni, 1982).

Elliptic Sobolev spaces

The Sobolev space $H^{1,p}(\mathbb{R}^n, \mu)$ is defined to be the completion of $C^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{1,p} = \Big(\int_{\mathbb{R}^n} |u|^p \, d\mu \Big)^{1/p} + \Big(\int_{\mathbb{R}^n} |Du|^p \, d\mu \Big)^{1/p}.$$

The definition of the local Sobolev space

$$H^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mu)$$

is clear.

Parabolic Sobolev space

We denote by $L^p(-\infty,\infty;H^{1,p}(\mathbb{R}^n))$ the space of functions $u = u(x,t)$ such that for almost every $t \in \mathbb{R}$ the function $x \mapsto u(x,t)$ belongs to $H^{1,p}(\mathbb{R}^n,\mu)$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \Big(|u|^p + |Du|^p\Big)\, d\mu\, dt < \infty.$$

Here

$$Du = \Big(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\Big)$$

is the spatial gradient.

Notice that the time derivative u_t is deliberately avoided.

The definition for the space

$$L^p_{\text{loc}}(-\infty,\infty;H^{1,p}_{\text{loc}}(\mathbb{R}^n,\mu))$$

is clear.

Weak solutions

Let $1 < p < \infty$. A nonnegative function

$$u \in L^p_{\text{loc}}(-\infty, \infty; H^{1,p}_{\text{loc}}(\mathbb{R}^n, \mu))$$

is a weak solution of

$$\frac{\partial(u^{p-1})}{\partial t} - \text{div}(|Du|^{p-2} Du) = 0$$

in $\mathbb{R}^{n+1}$ if

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(|Du|^{p-2} Du \cdot D\varphi - u^{p-1} \frac{\partial \varphi}{\partial t} \right) d\mu \, dt = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$.

If the integral ≥ 0 for all with $\varphi \geq 0$, then u is a supersolution. If the integral ≤ 0, then u is a subsolution.

Variational approach

Let $K \geq 1$. A nonnegative function

$$u \in L^p(-\infty, \infty; H^{1,p}(\mathbb{R}^n, \mu))$$

is a parabolic quasiminimizer if

$$\begin{aligned}\frac{1}{p}\int_{\mathbb{R}}\int_{\mathbb{R}^n} |Du|^p \, d\mu \, dt - \int_{\mathbb{R}}\int_{\mathbb{R}^n} u^{p-1}\frac{\partial \varphi}{\partial t}\, d\mu\, dt \\ \leq \frac{K}{p}\int_{\mathbb{R}}\int_{\mathbb{R}^n} |D(u+\varphi)|^p \, d\mu \, dt\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ (Wieser).

If $K = 1$, then we have weak solutions.

Observe: This definition makes sense also in a metric measure space.

The Barenblatt solution

The function

$$u(x,t) = t^{\frac{-n}{p(p-1)}} \exp\left(-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}\right),$$

where $x \in \mathbb{R}^n$ and $t > 0$, is a solution of the doubly nonlinear equation with the Lebesgue measure in the upper half space.

Observe that

$$u(x,t) > 0$$

for every $x \in \mathbb{R}^n$ and $t > 0$. This indicates an infinite speed of propagation for disturbancies.

A delicate point

Methods seem to be very sensitive for the precise form of the equation: Substitution

$$v = u^{p-1}$$

gives equation of the form

$$\frac{\partial v}{\partial t} - \operatorname{div}(v^{2-p}|Dv|^{p-2}Dv) = 0,$$

which has the same homogenuity, but is linear with respect to the time derivative.

Since the function spaces are different, it is NOT clear that the weak solutions are the same as for the doubly nonlinear equation.

(Fornaro, Ivanov, Porzio, Sosio and Vespri)

II. Scale and location invariant Harnack estimates

Parabolic geometry

A natural geometry that respects the scaling is that r in the spatial direction corresponds to r^p in the time direction.

Let $0 < \sigma < 1$ and $\tau \in \mathbb{R}$. We denote

$$\begin{aligned} Q &= B(x, r) \times (\tau - r^p, \tau + r^p), \\ \sigma Q^+ &= B(x, \sigma r) \times \\ &\quad \left(\tau + \frac{1}{2} r^p - \frac{1}{2}(\sigma r)^p, \tau + \frac{1}{2} r^p + \frac{1}{2}(\sigma r)^p\right) \end{aligned}$$

and

$$\begin{aligned} \sigma Q^- &= B(x, \sigma r) \times \\ &\quad \left(\tau - \frac{1}{2} r^p - \frac{1}{2}(\sigma r)^p, \tau - \frac{1}{2} r^p + \frac{1}{2}(\sigma r)^p\right). \end{aligned}$$

Observe: There is a time lag already in the case $p = 2$.

Parabolic Harnack inequality

Let $1 < p < \infty$ and assume that the measure μ is doubling and supports a weak $(1, p)$-Poincaré inequality. Let $u \geq 0$ be a weak solution and let $0 < \sigma < 1$. Then we have

$$\operatorname*{ess\,sup}_{\sigma Q^-} u \leq C \operatorname*{ess\,inf}_{\sigma Q^+} u,$$

where the constant C depends only on p, D_0, P_0 and σ.

Observe: The constant C is independent of the scale and location.

(For $p = 2$ this was proved by Moser and Trudinger in 1960's.)

Challenges

(1) In the case $p = 2$ it is known that if $u > 0$ is a solution, then

$$\log u$$

is a subsolution to the same equation. However, if $p \neq 2$, then $\log u$ is NOT a subsolution to the same equation. Instead it is a subsolution to a more complicated equation of a p-parabolic type.

(2) Parabolic BMO is delicate in the case when $p \neq 2$.

(3) Measure is NOT translation invariant and does NOT scale as the Lebesgue measure.

Ingredients of the proof

(1) Homogeneous Caccioppoli type energy estimates (OK).

(2) Sobolev embedding (OK).

(3) The Moser iteration scheme (OK).

(4) Parabolic BMO (replaced with Bombieri's real analysis lemma).

Caccioppoli inequality

Suppose that $u \geq 0$ is a weak solution. Then there exists a constant $C = C(p)$ such that

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}^n} |Du|^p \varphi^p \, d\mu \, dt + \operatorname*{ess\,sup}_{t \in \mathbb{R}} \int_{\mathbb{R}^n} u^p \varphi^p \, d\mu \\
\leq C \int_{\mathbb{R}} \int_{\mathbb{R}^n} u^p |D\varphi|^p \, d\mu \, dt \\
+ C \int_{\mathbb{R}} \int_{\mathbb{R}^n} u^p \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| d\mu \, dt
\end{aligned}$$

for every nonnegative $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Proof. Choose the test function

$$\varphi = u \varphi^p.$$

Bombieri's lemma

Let ν be a Borel measure and θ, A and γ be positive constants, $0 < \delta < 1$ and $0 < q \leq \infty$. Let U_σ be bounded measurable sets with $U_{\sigma'} \subset U_\sigma$ for $0 < \delta \leq \sigma' < \sigma \leq 1$. Moreover, if $q < \infty$, we assume that the doubling condition $\nu(U_1) \leq A\nu(U_\delta)$ holds. Let f be a positive measurable function on U_1 which satisfies the reverse Hölder inequality

$$\left(\fint_{U_{\sigma'}} f^q \, d\nu \right)^{1/q} \leq \left(\frac{A}{(\sigma - \sigma')^\theta} \fint_{U_\sigma} f^s \, d\nu \right)^{1/s}$$

with $0 < s < q$. Assume further that f satisfies

$$\nu(\{x \in U_1 | \log f > \lambda\}) \leq \frac{A\nu(U_\delta)}{\lambda^\gamma}$$

for all $\lambda > 0$. Then

$$\left(\fint_{U_\delta} f^q \, d\nu \right)^{1/q} \leq C,$$

where C depends only on θ, δ, γ, q and A.

III. Local Hölder continuity of weak solutions

The standard methods

Problem: Constants cannot be added to solutions.

(1) De Giorgi's and DiBenedetto's argument is based on estimating the distribution sets by using Caccioppoli estimates where instead of u we have $(u-k)_{\pm}$. However, this kind of energy estimates are not directly available.

(2) Moser's argument fails, because Harnack's inequality does not seem to directly imply the Hölder continuity.

DiBenedetto's argument

This applies to the p-parabolic equation when we can add constants.

Reduction of oscillation: The idea is to show that the oscillation in a cylinder is reduced by a controlled factor when the cylinder is shrinked by some factor.

Two alternatives:

(1) If the set where the solution is small is small, then the solution is small almost everywhere in a subcylinder and this information is then forwarded in time.

(2) If the set where the solution is small is large, then the set where the solution is above some threshold level is small also at later times. Then a De Giorgi type iteration scheme shows that the supremum of the solution in a subcylinder gets strictly smaller.

The doubly nonlinear equation

A dichotomy related to the equation:

(1) In large scales, when the oscillation of the solution is big, the solution behaves like the solutions of the heat equation. In this case, the reduction of oscillation follows directly from Harnack's inequality.

(2) In small scales the oscillation is small, a posteriori. In this case, the time derivative term formally looks like

$$\frac{\partial(u^{p-1})}{\partial t} = (p-1)u^{p-2}\frac{\partial u}{\partial t} \approx C\,\frac{\partial u}{\partial t}.$$

This implies a p-parabolic type behavior and hence DiBenedetto's argument applies.

Theorem. Let $1 < p < \infty$ and assume that the measure is doubling and supports a weak $(1,p)$-Poincaré inequality. Then every weak solution $u \geq 0$ of the doubly nonlinear equation in $\mathbb{R}^n$ is locally Hölder continuous, in symbols,

$$u \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^n).$$

New features of the argument

(1) **A modified Caccioppoli inequality.** We introduce a device which absorbs the nonlinearity in the time derivative.

(2) **Forwarding in time.** In the first case, when the infimum is small, the fact that in Harnack's inequality the infimum is taken at a later time than the supremum provides us a natural way to forward information in time. In the second case, new logarithmic lemmas are used. After the energy estimate and the logarithmic lemmas have been proved the second case follows DiBenedetto's argument.

Observe: Harnack estimates are used in the argument.

New Caccioppoli estimate

Let $u \geq 0$ be a locally bounded weak solution and $k \geq 0$. Then there exists a constant $C = C(p) > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}^n} |D(u-k)_\pm|^p \varphi^p \, dt \, d\mu \\
+ \operatorname*{ess\,sup}_{t \in \mathbb{R}} \int_{\mathbb{R}^n} \mathcal{J}((u-k)_\pm) \varphi^p \, d\mu \\
\leq C \int_{\mathbb{R}} \int_{\mathbb{R}^n} (u-k)_\pm^p |D\varphi|^p \, dt \, d\mu \\
+ C \int_{\mathbb{R}} \int_{\mathbb{R}^n} \mathcal{J}((u-k)_\pm) \varphi^{p-1} \left(\frac{\partial \varphi}{\partial t} \right)_+ dt \, d\mu
\end{aligned}$$

for every nonnegative $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. Here

$$\mathcal{J}((u-k)_\pm) = (p-1) \int_0^{(u-k)_\pm} (k \pm s)^{p-2} s \, ds.$$

Observe that

$$\frac{\partial}{\partial t} \mathcal{J}((u-k)_\pm) = \pm \frac{\partial (u^{p-1})}{\partial t} (u-k)_\pm.$$

IV. Higher regularity theory

Optimal regularity

As the Barenblatt solution for the doubly non-linear equation is C^∞-smooth it raises the question whether such regularity is true also for general solutions.

However, the standard stationary theory for elliptic p-Laplacian

$$\mathrm{div}(|Du|^{p-2}Du) = 0$$

shows that the spatial Hölder continuity of the gradient, in symbols

$$C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n+1}),$$

is the best possible regularity that we can have (Lewis).

Boundedness of the gradient

The first step in the $C^{1,\alpha}_{\text{loc}}$-proof is to show that the gradient is bounded and, consequently, the solution is spatially Lipschitz-continuous.

The standard methods

DiBenedetto and Friedman for the p-parabolic equation.

(1) Differentiation of the equation.

(2) Caccioppoli inequalities for the differentiated equation.

(3) Moser's iteration to show that the gradient of the solution is locally integrable to any power.

(4) Boundedness of the gradient by a De Giorgi type argument.

The doubly nonlinear equation

Problem. Nonlinearity of the time derivative term.

(1) After differentiating the equation, we will have an extra factor of u^{p-2} in front of the time derivative.

(2) The obtain estimate is non-homogeneous although the original equation admits scaling.

The differentiated equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big((p-1)u^{p-2}u_{x_i}\Big) \\
&\quad - \operatorname{div}\bigg(|Du|^{p-2}Du_{x_i} + \frac{\partial}{\partial x_i}\Big(|Du|^{p-2}\Big)Du\bigg) = 0, \\
&\qquad i = 1, 2, \dots, n
\end{aligned}
$$

The gradient bound

Let

$$r_j = \frac{r}{2} + \frac{r}{2^j}$$

and denote

$$Q_j = B(x, r_j) \times (t - u(x_0, t_0)^{m-2} r_j^2, t)$$

and

$$Q_\infty = \lim_{j \to \infty} Q_j.$$

Let $1 < p < \infty$ and assume that the measure is doubling and supports a weak $(1,p)$-Poincaré inequality. Let $u > 0$ be a continuous weak solution of the doubly nonlinear equation. Then there exists a constant $C = C(p, D_0, P_0) > 0$ such that

$$\operatorname*{ess\,sup}_{Q_\infty} |\nabla u|^2 \le C \Big(\fint_{Q_0} |Du|^p \, d\mu \, dt + 1 \Big).$$

Open problems

(1) More direct proofs that are based only on the definitions in the correct geometry.

(2) Necessary and sufficient Wiener type criterion for the solution of the Dirichlet problem.

(3) Theory for equations with measure data.

(4) Reverse Hölder inequalities for the gradient.

Summary

Literature for the doubly nonlinear equation is very small.

There are unexpected difficulties in dealing with the doubly nonlinear equation. Indeed, methods for the evolutionary p-Laplace equation are used extensively in the arguments.

Despite of difficulties it is possible to develop regularity theory for the doubly nonlinear equation.