Parabolic BMO with applications to PDEs

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Outline for the first part

We begin with a discussion of BMO and its connections to PDEs.

- **Goal:** To develop a theory of functions of bounded mean oscillation (BMO) related to certain nonlinear parabolic PDEs.
- **Question:** How to prove Harnack type estimates for nonlinear parabolic PDEs with non-quadratic growth?
- **Tools:** Moser iteration technique, definition of parabolic BMO that is compatible with the PDE, parabolic John–Nirenberg lemma, Calderón–Zygmund type stopping time arguments, chaining arguments.

BMO

Let

$$Q = Q(x,l) = \{y \in \mathbb{R}^n : |y_i - x_i| \le l,\ i = 1,\dots,n\}$$

be a cube with center $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and side length $l > 0$. The integral average of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ in a cube Q is denoted by

$$f_Q = \fint_Q f\,dx = \frac{1}{|Q|}\int_Q f(x)\,dx.$$

Definition

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that $f \in \mathrm{BMO}$, if for every cube Q in $\mathbb{R}^n$ there exists $c \in \mathbb{R}$, which may depend on Q, such that

$$\sup_Q \fint_Q |f - c|\,dx < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

BMO norm

$f \in \mathrm{BMO}$ if and only if

$$\|f\|_{\mathrm{BMO}} = \sup_Q \inf_{c \in \mathbb{R}} \fint_Q |f - c| \, dx < \infty.$$

Observe: This is not a norm, but it is a seminorm. We obtain a normed space by identifying functions whose difference is constant almost everywhere.

BMO with integral averages

$f \in \mathrm{BMO}$ if and only if

$$\|f\|_{\mathrm{BMO}} = \sup_Q \fint_Q |f - f_Q| \, dx < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ (exercise).

Remark. We may apply integral averages in the definition of BMO, but often it is convenient to have a more flexible theory that allows other constants as well.

John–Nirenberg lemma for BMO

The John–Nirenberg inequality gives an exponential estimate for the distribution function of oscillation of an arbitrary BMO function. The proof is based on the Calderón–Zygmund decomposition.

Theorem (John and Nirenberg 1961)

Assume that $f \in \mathrm{BMO}$. There exist positive constants $A = A(n)$ and $B = B(n)$ such that

$$|Q \cap \{|f - f_Q| > \lambda\}| \leq A \exp\left(-\frac{B\lambda}{\|f\|_{\mathrm{BMO}}}\right)|Q|$$

for every $\lambda > 0$ and every cube $Q \subset \mathbb{R}^n$.

The moral: The John–Nirenberg inequality tells that logarithmic blowup, as for $f(x) = \log|x|$, is the worst possible behavior for a BMO function.

BMO_p 1(2)

Let

$$\|f\|_{\mathrm{BMO}_p} = \sup_Q \left(\fint_Q |f - f_Q|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$, and let

$$\mathrm{BMO}_p = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{BMO}_p} < \infty \right\}.$$

The standard BMO corresponds to the case $p = 1$.

BMO_p 2(2)

Let $1 \leq p < \infty$. By Hölder's inequality

$$\fint_Q |f - f_Q| \, dx \leq \left(\fint_Q |f - f_Q|^p \, dx \right)^{\frac{1}{p}}$$

for every cube Q in $\mathbb{R}^n$. This implies

$$\|f\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}_p}$$

and thus $\mathrm{BMO}_p \subset \mathrm{BMO}$ whenever $1 \leq p < \infty$.

A consequence of the John–Nirenberg lemma

Theorem

Assume that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For every p with $1 < p < \infty$ there exists a constant $c = c(n,p)$ such that

$$\|f\|_{\mathrm{BMO}} \le \|f\|_{\mathrm{BMO}_p} \le c\|f\|_{\mathrm{BMO}}.$$

The moral: Norms $\|f\|_{\mathrm{BMO}_p}$ and $\|f\|_{\mathrm{BMO}}$ are equivalent and thus $\mathrm{BMO}_p = \mathrm{BMO}$ for every $1 \le p < \infty$. This implies that, in a certain sense, BMO functions satisfy a reverse Hölder inequality.

The first inequality in the claim follows from Hölder's inequality as before. For the second inequality, by Cavalieri's principle and the John–Nirenberg lemma, there exist constants $A = A(n)$ and $B = B(n)$ such that

$$
\begin{aligned}
\fint_Q |f - f_Q|^p \, dx &= \frac{p}{|Q|} \int_0^\infty \lambda^{p-1} |\{x \in Q : |f(x) - f_Q| > \lambda\}| \, d\lambda \\
&\leq pA \int_0^\infty \lambda^{p-1} \exp\left(-\frac{B\lambda}{\|f\|_{\mathrm{BMO}}}\right) \, d\lambda \\
&= pA \left(\frac{\|f\|_{\mathrm{BMO}}}{B}\right)^{p-1} \int_0^\infty \left(\frac{B\lambda}{\|f\|_{\mathrm{BMO}}}\right)^{p-1} \exp\left(-\frac{B\lambda}{\|f\|_{\mathrm{BMO}}}\right) \, d\lambda \\
&= pA \left(\frac{\|f\|_{\mathrm{BMO}}}{B}\right)^{p} \int_0^\infty s^{p-1} e^{-s} \, ds \\
&= pA \left(\frac{\|f\|_{\mathrm{BMO}}}{B}\right)^{p} \Gamma(p)
\end{aligned}
$$

for every cube Q in $\mathbb{R}^n$. Here Γ is the gamma function.

Median

In some cases, it is preferable to consider medians instead of integral averages. Let $0 < s < 1$. Assume that $A \subset \mathbb{R}^n$ is a measurable set with $0 < |A| < \infty$ and that $f : A \to [-\infty, \infty]$ is a measurable function. A number $a \in \mathbb{R}$ is called an s-median of f on A, if

$$|\{x \in A : f(x) > a\}| \leq s|A|$$

and

$$|\{x \in A : f(x) < a\}| \leq (1 - s)|A|.$$

Perhaps the most common parameter value is $s = \frac{1}{2}$ which gives the standard notion of median. For $0 < s < 1$, we obtain a biased notion of s-median.

Warning: In general, s-median is not unique. Let $A = [0, 1]$ and $f = \chi_{[\frac{1}{2},1]}$. Then every $a \in [0, 1]$ is $\frac{1}{2}$-median of f on A.

Maximal median 1(3)

To obtain a uniquely defined notion, we consider maximal s-median.

Definition

Let $0 < s < 1$. Assume that $A \subset \mathbb{R}^n$ is a measurable set with $0 < |A| < \infty$ and that $f : A \to [-\infty, \infty]$ is a measurable function. The maximal s-median of f over A is defined as

$$m_f^s(A) = \inf\{a \in \mathbb{R} : |\{x \in A : f(x) > a\}| < s|A|\}.$$

Note: Let $A = [0, 1]$ and $f = \chi_{[\frac{1}{2},1]}$. Then $m_f^{\frac{1}{2}}(A) = 1$.

Maximal median 2(3)

Maximal s-median is s-median (exercise), that is,

$$|\{x \in A : f(x) > m_f^s(A)\}| \leq s|A|$$

and

$$|\{x \in A : f(x) < m_f^s(A)\}| \leq (1-s)|A|.$$

Maximal median 3(3)

By Chebyshev's inequality

$$\begin{aligned} s|A| &= |A| - (1-s)|A| \\ &= |A| - |\{x \in A : |f(x)| < m^s_{|f|}(A)\}| \\ &\le |\{x \in A : |f(x)| \ge m^s_{|f|}(A)\}| \\ &\le (m^s_{|f|}(A))^{-p} \int_A |f|^p \, dx, \end{aligned}$$

which implies

$$m^s_{|f|}(A) \le \left(s^{-1} \fint_A |f|^p \, dx \right)^{\frac{1}{p}},$$

whenever $0 < p < \infty$.

Note: In this sense median is bounded by integral average. Maximal median is finite if $\int_A |f|^p \, dx < \infty$, $0 < p < \infty$. In particular, we may consider the range $0 < p < 1$.

BMO with medians 1(2)

We do not need to assume that the function is locally integrable in the definition of BMO. In particular, this gives a way to define BMO for any measurable function.

Definition

Let $0 < s \leq \frac{1}{2}$. A measurable function $f : \mathbb{R}^n \to [-\infty, \infty]$ belongs to $\mathrm{BMO}_{0,s}$ if

$$\sup_{Q} \inf_{c \in \mathbb{R}} m^s_{|f-c|}(Q) < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Note: If f is a function that takes only two values, then $\inf_{c \in \mathbb{R}} m^s_{|f-c|}(Q) = 0$ for $s > \frac{1}{2}$. This explains the range $0 < s \leq \frac{1}{2}$.

BMO with medians 2(2)

Recall that

$$\sup_{Q}\inf_{c\in\mathbb{R}} m^s_{|f-c|}(Q)$$
$$=\sup_{Q}\inf_{c\in\mathbb{R}} \inf\{a\geq 0 : |\{x\in Q : |f(x)-c|>a\}|<s|Q|\}.$$

Thus, $f\in\mathrm{BMO}_{0,s}$, if there exists $a>0$ such that for every cube Q in $\mathbb{R}^n$ there exists c (that may depend on the cube Q) such that

$$|\{x\in Q : |f(x)-c|>a\}|<s|Q|.$$

John (1965) and Strömberg (1979) and Jawerth and Torchinsky (1985) showed that

$$f \in \mathrm{BMO} \Longleftrightarrow f \in \mathrm{BMO}_{0,s}, \quad 0 < s \leq \tfrac{1}{2}.$$

See also Myyryläinen (2022) and Kinnunen and Myyryläinen (2021).

A PDE point of view

Theorem (Moser 1961, Trudinger 1967)

Assume that $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ is a nonnegative weak solution of the p-Laplace equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty.$$

Then u satisfies a scale and location invariant Harnack's inequality

$$\sup_Q u \le c(n,p) \inf_Q u$$

for every cube Q in $\mathbb{R}^n$.

Remark: This implies that a weak solution is locally Hölder continuous.

Proof.

Let $\delta > 0$. Then

$$\begin{aligned}\sup_Q u &\le c\left(\fint_{2Q} u^{\delta}\,dx\right)^{\frac{1}{\delta}}\\ &\le c\left(\fint_{2Q} u^{-\delta}\,dx\right)^{-\frac{1}{\delta}}\\ &\le c\inf_Q u\end{aligned}$$

for every cube Q in $\mathbb{R}^n$. The first and last inequalities follow from a Moser-type iteration argument. The second inequality follows from the fact that $\log u \in \mathrm{BMO}$ and the John–Nirenberg lemma. $\square$

Question: For which nonlinear parabolic PDEs is it possible to develop a similar theory? What is the appropriate definition of the parabolic BMO?

The parabolic Sobolev space

- Ω is an open subset of $\mathbb{R}^n$ and $0 \leq t_1 < t_2 \leq T$.
- The space-time cylinders are

$$\Omega_T = \Omega \times (0, T) \quad \text{and} \quad \Omega'_{t_1,t_2} = \Omega' \times (t_1, t_2),$$

  where $\Omega' \subset \Omega$ is an open set.
- The parabolic Sobolev space $L^p(0, T; W^{1,p}(\Omega))$ consists of measurable functions $u : \Omega_T \to [-\infty, \infty]$ such that $x \mapsto u(x, t)$ belongs to $W^{1,p}(\Omega)$ for almost all $t \in (0, T)$, and

$$\iint_{\Omega_T} (|u|^p + |Du|^p) \, dx \, dt < \infty.$$

- $u \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega))$, if u belongs to the parabolic Sobolev space for every $\Omega'_{t_1,t_2} \Subset \Omega_T$.

The doubly nonlinear equation

A weak solution to the doubly nonlinear equation

$$\frac{\partial}{\partial t}(|u|^{p-2}u) - \operatorname{div}(|Du|^{p-2}Du) = 0, \qquad 1 < p < \infty,$$

is a function $u = u(x,t) \in L^p_{\mathrm{loc}}(\mathbb{R}; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n))$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(|Du|^{p-2}Du \cdot D\varphi - |u|^{p-2}u \frac{\partial \varphi}{\partial t} \right) dx\, dt = 0$$

for every $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$.

It is possible to consider more general equations of this type, but we only discuss the prototype equation here. From now on, the parameter $p > 1$ will be fixed. When $p = 2$, we have the heat equation. We focus on the case $\Omega = \mathbb{R}^n$ and $-\infty < t < \infty$.

The Barenblatt solution

Example

The function

$$u(x,t) = t^{\frac{-n}{p(p-1)}} e^{-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}}, \; x \in \mathbb{R}^n, \; t > 0,$$

is a solution of the doubly nonlinear equation in the upper half space $\mathbb{R}^{n+1}_+$.

Observe: $u(x,t) > 0$ for every $x \in \mathbb{R}^n$ and $t > 0$. This indicates infinite speed of propagation of disturbances. When $p = 2$ we have the heat kernel.

Structural properties when $p \neq 2$

- Solutions can be scaled.
- Constants cannot be added to a solution.
- The sum of two solutions is not a solution.

Some known results

- Existence (Ivanov–Mkrtychyan–Jäger 1997, Sturm 2017)
- Asymptotic behavior (Manfredi–Vespri 1994, Savaré–Vespri 1993, Tedeev–Vespri 2015)
- Nonnegative weak solutions satisfy a scale and location invariant parabolic Harnack inequality in the space-time cylinders $B(x,r) \times (t - r^p, t + r^p)$. (Fornaro–Sosio–Vespri 2015, Gianazza–Vespri 2006, Kinnunen–Kuusi 2007, Trudinger 1968)
- Nonnegative weak solutions and are locally Hölder continuous. (Ivanov 1994, Kuusi–Lageoglu–Siljander–Urbano 2012, Porzio–Vespri 1993, Vespri 1992)
- Sign-changing weak solutions are locally Hölder continuous. (Bögelein–Duzaar–Liao 2021)
- The gradient of a weak solution satisfies a reverse Hölder inequality. (Bögelein–Duzaar–Kinnunen–Scheven 2020)

Unexpected features

One might expect that the doubly nonlinear equation would have a similar behaviour as the heat equation, but this is not clear.

- The parabolic Harnack inequality does not immediately give local Hölder continuity, because constants cannot be added to solutions.
- The role of the intrinsic geometry is not clear. For example, the intrinsic geometry is not needed in Harnack estimates, but it is used in regularity theory.
- The question about uniqueness for the Dirichlet boundary value problem seems to be unsettled without additional assumptions. (Ivanov 1997, Lindgren–Lindqvist 2019)
- Does a weak solution to the doubly nonlinear equation belong to $C^{1,\alpha}_{\mathrm{loc}}(\Omega_T)$? This is optimal by the elliptic theory. The gradient of a solution is not necessarily continuous for more general equations with p-structure.

Parabolic geometry

- If $u(x, t)$ is a solution, so does $u(\lambda x, \lambda^p t)$ with $\lambda > 0$.
- This suggests that in the natural geometry for the doubly nonlinear equation the time variable scales as the modulus of the space variable raised to power p.
- Consequently, the Euclidean cubes have to be replaced by parabolic rectangles respecting this scaling in all estimates.

Parabolic rectangles

Definition

Let $1 < p < \infty$, $0 < \gamma < 1$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $l > 0$. Denote

$$\begin{aligned} R = R(x,t,l) &= Q(x,l) \times (t - l^p, t + l^p), \\ R^+(\gamma) &= Q(x,l) \times (t + \gamma l^p, t + l^p) \quad \text{and} \\ R^-(\gamma) &= Q(x,l) \times (t - l^p, t - \gamma l^p). \end{aligned}$$

We say that R is a parabolic rectangle with center at (x,t) and sidelength l. $R^\pm(\gamma)$ are the upper and lower parts of R. Parameter γ is the time lag.

Harnack's inequality

Theorem (Trudinger 1968, K.–Kuusi 2007)

Assume that $u > 0$ is a weak solution of the doubly nonlinear equation and let $0 < \gamma < 1$. Then u satisfies scale and location invariant Harnack's inequality

$$\sup_{R^-(\gamma)} u \le c(n,p,\gamma) \inf_{R^+(\gamma)} u$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$.

Remark. The parabolic rectangles respect the natural geometry of the doubly nonlinear equation.

Time lag

Harnack's inequality gives a bound on a positive solution at a given time in terms of its values at later times. The time lag $\gamma > 0$ is an unavoidable feature of the theory rather than a mere technicality. The result is not true with $\gamma = 0$. This can be seen from the Barenblatt solution and the heat kernel already when $p = 2$, since Harnack's inequality does not hold on a given time slice.

Proof.

Let $\delta > 0$. Then

$$\begin{aligned}
\sup_{R^-(\gamma)} u &\le c \left(\fint_{2R^-(\gamma)} u^\delta \, dx \, dt \right)^{\frac{1}{\delta}} \\
&\le c \left(\fint_{2R^+(\gamma)} u^{-\delta} \, dx \, dt \right)^{-\frac{1}{\delta}} \\
&\le c \inf_{R^+(\gamma)} u
\end{aligned}$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$. Here

$$2R^-(\gamma) = R^-(x, t, 2l)(\gamma) = Q(x, 2l) \times (t - (2l)^p, t - \gamma(2l)^p)$$

and

$$2R^+(\gamma) = R^+(x, t, 2l)(\gamma) = Q(x, 2l) \times (t + \gamma(2l)^p, t + (2l)^p).$$

Outline of the proof

- **Step 1:** Energy estimates for weak solutions of the doubly nonlinear equation.
- **Step 2:** Parabolic Sobolev estimates.
- **Step 3:** The Moser iteration argument which is is based on a successive use of Sobolev's inequality and energy estimates. This gives the first and last inequalities on the previous slide.
- **Step 4:** Logarithm of a positive weak solution belongs to parabolic BMO.
- **Step 5:** The remaining inequality follows from a logarithmic energy estimate and a parabolic John–Nirenberg lemma.

Energy estimate for negative exponents

Lemma (K.–Kuusi 2007)

Assume that $u > 0$ is a weak solution of the doubly nonlinear equation and let $\varepsilon > 0$ with $\varepsilon \neq p-1$. There exists a constant $c = c(p,\varepsilon)$ such that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |Du|^p u^{-\varepsilon-1}\varphi^p \,dx\,dt + \operatorname*{ess\,sup}_{t\in\mathbb{R}} \int_{\mathbb{R}^n} u^{p-1-\varepsilon}\varphi^p\,dx$$
$$\leq c\int_{\mathbb{R}}\int_{\mathbb{R}^n} u^{p-1-\varepsilon}|D\varphi|^p\,dx\,dt + c\int_{\mathbb{R}}\int_{\mathbb{R}^n} u^{p-1-\varepsilon}\varphi^{p-1}\left|\frac{\partial\varphi}{\partial t}\right|\,dx\,dt$$

for every $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ with $\varphi \geq 0$.

Proof.

Choose $u^{-\varepsilon}\varphi^p$ as a test function. □

Energy estimate for positive exponents

Lemma (K.–Kuusi 2007)

Assume that $u > 0$ is a weak solution of the doubly nonlinear equation in $\mathbb{R}^{n+1}$ and let $\varepsilon > 0$. There exists a constant $c = c(p, \varepsilon)$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |Du|^p u^{\varepsilon-1} \varphi^p \, dx \, dt + \operatorname*{ess\,sup}_{t \in \mathbb{R}} \int_{\mathbb{R}^n} u^{p-1+\varepsilon} \varphi^p \, dx$$
$$\leq c \int_{\mathbb{R}} \int_{\mathbb{R}^n} u^{p-1+\varepsilon} |D\varphi|^p \, dx \, dt + c \int_{\mathbb{R}} \int_{\mathbb{R}^n} u^{p-1+\varepsilon} \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| dx \, dt$$

for every $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ with $\varphi \geq 0$.

Proof.

Choose $u^\varepsilon \varphi^p$ as a test function. $\square$

Logarithmic energy estimate

Lemma (K.–Kuusi 2007)

Assume that $u > 0$ is a weak solution of the doubly nonlinear equation and let $\varepsilon > 0$. There exists a constant $c = c(p)$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |D(\log u)|^p \varphi^p \, dx \, dt + \operatorname*{ess\,sup}_{t \in \mathbb{R}} \left| \int_{\mathbb{R}^n} \log u \varphi^p \, dx \right|$$
$$\leq c \int_{\mathbb{R}} \int_{\mathbb{R}^n} |D\varphi|^p \, dx \, dt + c \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\log u| \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| \, dx \, dt$$

for every $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ with $\varphi \geq 0$.

Proof.

Choose $u^{1-p}\varphi^p$ as a test function. □

Cutoff function 1(2)

Let

$$R = R(x,t,l) = Q(x,l) \times (t - l^p, t + l^p)$$

be a parabolic rectangle in $\mathbb{R}^{n+1}$ and denote

$$2R = R(x,t,2l) = Q(x,2l) \times (t - (2l)^p, t + (2l)^p).$$

Consider a cutoff function $\varphi \in C_0^\infty(2R)$, $0 \le \varphi \le 1$, $\varphi = 1$ in R with

$$|D\varphi| \le \frac{c}{l} \quad \text{and} \quad \left|\frac{\partial \varphi}{\partial t}\right| \le \frac{c}{l^p}.$$

Note: Cutoff functions are used to localize estimates.

Cutoff function 2(2)

Assume that $u > 0$ is a weak solution of the doubly nonlinear equation in $\mathbb{R}^{n+1}$ and let $R = Q \times [t - l^p, t + l^p]$ be a parabolic rectangle. The energy estimate for positive exponents with $\varepsilon = 1$ gives

$$\begin{aligned}
&\int_R |Du|^p \, dx \, dt + \operatorname*{ess\,sup}_{t \in \mathbb{R}} \int_Q u^p \, dx \\
&\le \int_{\mathbb{R}} \int_{\mathbb{R}^n} |Du|^p \varphi^p \, dx \, dt + \operatorname*{ess\,sup}_{t \in \mathbb{R}} \int_{\mathbb{R}^n} u^p \varphi^p \, dx \\
&\le c \int_{\mathbb{R}} \int_{\mathbb{R}^n} u^p |D\varphi|^p \, dx \, dt + c \int_{\mathbb{R}} \int_{\mathbb{R}^n} u^p \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| dx \, dt \\
&\le \frac{c}{l^p} \int_{2R} u^p \, dx \, dt.
\end{aligned}$$

Note: The cutoff function above is compatible with the energy estimates.

Parabolic Sobolev inequality

Lemma

Assume that $u = u(x,t) \in L^p(\mathbb{R}; W^{1,p}(\mathbb{R}^n))$ *and let* $\kappa > 1$. *Then there exists a constant* $c = c(n,p,\kappa)$ *such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |u|^{\kappa p}\, dx\, dt \le c \int_{\mathbb{R}} \int_{\mathbb{R}^n} |Du|^p\, dx\, dt \cdot \left(\operatorname*{ess\,sup}_{t \in \mathbb{R}} \int_{\mathbb{R}^n} |u|^{(\kappa-1)n}\, dx \right)^{\frac{p}{n}}.$$

Proof.

Apply the standard Sobolev inequality on time slices and integrate over the time variable. □

Assume that $u > 0$ is a weak solution of the doubly nonlinear equation in $\mathbb{R}^{n+1}$. By choosing $\kappa = 1 + \frac{p}{n}$ we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} u^{\kappa p}\,dx\,dt \le c \int_{\mathbb{R}} \int_{\mathbb{R}^n} |Du|^p\,dx\,dt \left(\operatorname*{ess\,sup}_{t\in\mathbb{R}} \int_{\mathbb{R}^n} u^p\,dx \right)^{\frac{p}{n}}.$$

Both terms on the right-hand side can be estimated by the energy estimates after applying a cutoff function. This gives an estimate of the form

$$\fint u^{\kappa p}\,dx\,dt \lesssim \left(\fint u^p\,dx\,dt \right)^{1+\frac{p}{n}},$$

where the integral average on the right-hand side is taken over a larger parabolic rectangle as on the left-hand side.

Note: There is a gain of $\kappa > 1$ in the estimate above.

This implies a reverse Hölder inequality

$$\left(\fint u^{\kappa p}\,dx\,dt\right)^{\frac{1}{\kappa p}} \lesssim \left(\fint u^{p}\,dx\,dt\right)^{\frac{1}{p}}$$

with $\kappa > 1$. The main idea in the Moser iteration technique is to apply this inequality recursively. Observe that at every step of iteration the integrability exponent is increased by a factor $\kappa > 1$. At the limit we obtain an estimate for the essential supremum, since

$$\sup u = \lim_{k\to\infty}\left(\fint u^{\kappa^k p}\,dx\,dt\right)^{\frac{1}{\kappa^k p}}.$$

Logarithmic estimate 1(4)

The following version of the logarithmic energy estimate will be convenient for us.

Lemma (K.–Kuusi 2007)

Let u be a positive weak solution of the doubly nonlinear equation and let $0 < \gamma < 1$. Then there exist constants $c = c(p, n, \gamma)$, $c' = c'(p, n, \gamma)$ and $a \in \mathbb{R}$, that depends on the parabolic rectangle R, such that

$$|R^+(\gamma) \cap \{\log u < -\lambda + a - c'\}| \leq \frac{c}{\lambda^{p-1}} |R^+(\gamma)|$$

and

$$|R^-(\gamma) \cap \{\log u > \lambda + a + c'\}| \leq \frac{c}{\lambda^{p-1}} |R^-(\gamma)|$$

for every $\lambda > 0$.

Proof. Follows from the logarithmic energy estimate by applying a test function that is independent of time.

Logarithmic estimate 2(4)

Let $q = \frac{p-1}{2}$ and denote $f = -\log u$. By Cavalieri's principle

$$
\begin{aligned}
&\int_{R^+} (f+a)_+^q \, dx\, dt \le q \int_0^\infty \lambda^{q-1} |\{R^+ : f + a > \lambda\}| \, d\lambda \\
&= q \int_0^\infty \lambda^{q-1} |\{R^+ : -\log u + a > (\lambda - c') + c'\}| \, d\lambda \\
&= q \int_{-c'}^\infty (\lambda + c')^{q-1} |\{R^+ : -\log u + a > \lambda + c'\}| \, d\lambda \\
&\le |R^+|(1+c')^q + q \int_1^\infty (\lambda + c')^{q-1} |\{R^+ : \log u < -\lambda + a - c'\}| \, d\lambda \\
&\le |R^+|(1+c')^q + cq|R^+| \int_1^\infty (\lambda + c')^{q-p} \, d\lambda \\
&= c(n,p,\gamma)|R^+|.
\end{aligned}
$$

In the last inequality we applied the logarithmic estimate.

Similarly

$$\begin{aligned}
&\int_{R^-} (f+a)_-^q \, dx\, dt \le q \int_0^\infty \lambda^{q-1} |\{R^- : -f-a>\lambda\}| \, d\lambda \\
&= q \int_0^\infty \lambda^{q-1} |\{R^- : \log u - a > (\lambda - c') + c'\}| \, d\lambda \\
&\le |R^-|(1+c')^q + q \int_1^\infty (\lambda + c')^{q-1} |\{R^- : \log u > \lambda + a + c'\}| \, d\lambda \\
&\le |R^-|(1+c')^q + cq|R^-| \int_1^\infty (\lambda + c')^{q-p} \, d\lambda \\
&= c(n,p,\gamma)|R^-|.
\end{aligned}$$

In the last inequality we applied the logarithmic estimate.

Logarithmic estimate 4(4)

Lemma (K.–Saari 2016)

Assume that $u > 0$ is a weak solution of the doubly nonlinear equation, let $0 < \gamma < 1$ and denote $f = -\log u$. Then there exists $c \in \mathbb{R}$, which depends on the parabolic rectangle R, such that

$$\sup_{R} \left(\fint_{R^+} (f-c)_+^q \, dx\, dt + \fint_{R^-} (f-c)_-^q \, dx\, dt \right)^{\frac{1}{q}} < \infty,$$

where the supremum is taken over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$ and $q = \frac{p-1}{2}$.

Observe: $0 < q < 1$ for $p < 3$. For example, $q = \frac{1}{2}$ for $p = 2$, see Moser 1964 and Trudinger 1968. By Hölder's inequality, we may take $q = 1$ for $p \geq 3$.

We omit the differentials $dx\,dt$ in the notation.

Definition

Let $0 < q < \infty$, $0 < \gamma < 1$ and assume that $f \in L^q_{\mathrm{loc}}(\mathbb{R}^{n+1})$. We say that $f \in \mathrm{PBMO}^+_{q,\gamma} = \mathrm{PBMO}^+$ if for every parabolic rectangle R there exists $c \in \mathbb{R}$, which may depend on R, such that

$$\sup_R \left(\fint_{R^+(\gamma)} (f-c)_+^q + \fint_{R^-(\gamma)} (f-c)_-^q \right)^{\frac{1}{q}} < \infty,$$

where the supremum is taken over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$. If the condition above is satisfied with the direction of the time axis reversed, we denote $f \in \mathrm{PBMO}^-$.

Remark

$f \in L^q_{\mathrm{loc}}(\mathbb{R}^{n+1})$ belongs to PBMO^+ if and only if for every parabolic rectangle R there exists a constant $c \in \mathbb{R}$, that may depend on R, with

$$\fint_{R^+(\gamma)} (f-c)_+^q \le M \quad \text{and} \quad \fint_{R^-(\gamma)} (f-c)_-^q \le M,$$

where $M \in \mathbb{R}$ is a constant that is independent of R.

Note: There are two oscillation conditions in the definition of the parabolic BMO. Note that c is the same constant for $R^+(\gamma)$ and $R^-(\gamma)$ in a parabolic rectangle R. This gives a coupling between the conditions.

Remark

- PBMO is interesting already for functions $f = f(t)$ that depend only on the time variable. In this case the condition is $(\gamma = 0,\ q = 1,\ p = 1)$

$$\sup_{x,h} \left(\frac{1}{h} \int_x^{x+h} (f(t) - c)_+ \, dt + \frac{1}{h} \int_{x-h}^{x} (f(t) - c)_- \, dt \right) < \infty.$$

  This has been studied extensively.

- For time independent functions $f = f(x,t) = f(x)$ we obtain the standard BMO $(q = 1)$, since

$$\begin{aligned} &\fint_{R^+(\gamma)} (f(x) - c)_+ \, dx \, dt + \fint_{R^-(\gamma)} (f(x) - c)_- \, dx \, dt \\ &= \fint_Q (f(x) - c)_+ \, dx + \fint_Q (f(x) - c)_+ \, dx \\ &= \fint_Q |f(x) - c| \, dx. \end{aligned}$$

Example

Let $1 < p < \infty$ and $0 < \gamma < 1$. The function

$$u(x,t) = t^{\frac{-n}{p(p-1)}} e^{-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}}, \; x \in \mathbb{R}^n, \; t > 0,$$

is a solution of the doubly nonlinear equation in the upper half space $\mathbb{R}^{n+1}_+$. By the logarithmic estimate above, we conclude that the function

$$\begin{aligned} f(x,t) = -\log u(x,t) &= -\log t^{\frac{-n}{p(p-1)}} - \log e^{-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}} \\ &= \frac{n}{p(p-1)} \log t + \frac{p-1}{p} \left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}, \; x \in \mathbb{R}^n, \; t > 0, \end{aligned}$$

belongs to $\mathrm{PBMO}^+_{\gamma,q}(\mathbb{R}^{n+1}_+)$ with $q = \frac{p-1}{2}$. Later we see that f belongs to $\mathrm{PBMO}^+_{\gamma,q}(\mathbb{R}^{n+1}_+)$ for every $0 < q < \infty$.

Parabolic BMO "norm"

$f \in \mathrm{PBMO}^+$ if and only if

$$\|f\|_{\mathrm{PBMO}^+} = \sup_{R} \inf_{c \in \mathbb{R}} \left(\fint_{R^+(\gamma)} (f-c)_+^q + \fint_{R^-(\gamma)} (f-c)_-^q \right)^{\frac{1}{q}} < \infty,$$

where the supremum is taken over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$.

Remark. This is not a norm, since every constant function has zero norm.

Fact: The infimum over all $c \in \mathbb{R}$ is attained in every parabolic rectangle R.

Lemma

- $\|f+a\|_{\mathrm{PBMO}^+} = \|f\|_{\mathrm{PBMO}^+}$ *for every* $a \in \mathbb{R}$.
- $\|f+g\|_{\mathrm{PBMO}^+} \le \max\{2^{\frac{1}{q}-1}, 2^{1-\frac{1}{q}}\}(\|f\|_{\mathrm{PBMO}^+} + \|g\|_{\mathrm{PBMO}^+})$.
- $\|af\|_{\mathrm{PBMO}^+} = \begin{cases} a\|f\|_{\mathrm{PBMO}^+}, & a \ge 0, \\ -a\|f\|_{\mathrm{PBMO}^-}, & a < 0. \end{cases}$
- $\|\max\{f,g\}\|_{\mathrm{PBMO}^+} \le \max\{1, 2^{\frac{1}{q}-1}\}(\|f\|_{\mathrm{PBMO}^+} + \|g\|_{\mathrm{PBMO}^+})$.
- $\|\min\{f,g\}\|_{\mathrm{PBMO}^+} \le \max\{1, 2^{\frac{1}{q}-1}\}(\|f\|_{\mathrm{PBMO}^+} + \|g\|_{\mathrm{PBMO}^+})$.

Remark. The inconvenient constants for $q \neq 1$ can be avoided by considering other definition of the parabolic BMO norm, but the current approach is convenient in the proof of the John–Nirenberg lemma.

Takeaways

- PBMO is closed under addition and multiplication by positive constants.
- Multiplication by negative constants reverses the time direction.
- PBMO is closed under maximum and minimum.
- PBMO is not a vector space and $\|\cdot\|_{\mathrm{PBMO}^+}$ is not a norm.

Truncation

Every function $f \in \mathrm{PBMO}^+$ can be approximated pointwise by a sequence of bounded PBMO^+ functions, since truncations

$$f_k(x) = \min\{\max\{f(x), -k\}, k\},$$

$k = 1, 2, \dots,$ belong to PBMO^+ with

$$\|f_k\|_{\mathrm{PBMO}^+} \le \max\{1, 2^{\frac{2}{q}-2}\}\|f\|_{\mathrm{PBMO}^+},$$

and it holds that $f_k \to f$ pointwise and in $L^q_{\mathrm{loc}}(\mathbb{R}^{n+1})$ as $k \to \infty$.

Remark. We may assume that functions in PBMO are bounded in arguments, if the estimates are independent of the level of truncation.

Parabolic John–Nirenberg lemma

Theorem (Kinnunen–Myyryläinen–Yang 2022)

Let $0 < \gamma < 1$, $0 < q \le 1$ *and* $0 < \rho \le \gamma$. *Assume that* $f \in \mathrm{PBMO}^{+}_{\gamma,q}$. *There exist positive constants* $A = A(n, p, q, \gamma, \rho)$, $B = B(n, p, q, \gamma, \rho)$ *and* $c \in \mathbb{R}$, *which depends on the parabolic rectangle* R, *such that*

$$|R^{+}(\rho) \cap \{(f - c)_{+} > \lambda\}| \le A \exp\left(-\left(\frac{B\lambda}{\|f\|_{\mathrm{PBMO}^{+}_{\gamma,q}}}\right)^{q}\right) |R^{+}(\rho)|$$

and

$$|R^{-}(\rho) \cap \{(f - c)_{-} > \lambda\}| \le A \exp\left(-\left(\frac{B\lambda}{\|f\|_{\mathrm{PBMO}^{+}_{\gamma,q}}}\right)^{q}\right) |R^{-}(\rho)|$$

for every $\lambda > 0$ *and every parabolic rectangle* $R \subset \mathbb{R}^{n+1}$

History

- Moser 1964, 1967 and Fabes–Garofalo 1985 discussed the case $p = 2$ with $q = \frac{1}{2}$ and $q = 1$. The proof by Fabes–Garofalo is based on a proof of the elliptic John–Nirenberg lemma by Calderón. We extend the argument by Fabes–Garofalo to the full range $1 < p < \infty$.
- Aimar 1988 developed a very general theory on metric measure spaces. This theory can be applied with the parabolic metric $d((x,t),(y,s)) = \max\{\|x-y\|_\infty, |t-s|^{\frac{1}{p}}\}$.
- Berkovits 2011, 2012 studied the case $p = 1$.
- Kinnunen–Saari 2016 and Saari 2018 studied connections to the doubly nonlinear equation by using results of Aimar.

Parabolic John–Nirenberg lemma

Proof.

- It is enough to prove only the first inequality, since the second inequality follows by applying the first inequality it to the function $-f(x,-t)$.
- A "dyadic" partition of a parabolic rectangle that respects the parabolic geometry.
- A stopping time and covering arguments, as in the proof of the Calderón–Zygmund lemma, for parabolic rectangles.
- A decay estimate for the distribution set and an iteration argument.
- A chaining argument to recover $0 < \rho \leq \gamma$.

$\square$

Remarks

- The John–Nirenberg lemma cannot hold with $\gamma = 0$, because this would imply a parabolic Harnack inequality without a time lag.
- The proof of the next theorem shows that the John–Nirenberg lemma gives a characterization of PBMO^+.

Theorem

Let $0 < \rho \le \gamma < 1$ *and* $0 < q \le r < \infty$. *Then there exist constants* $c_1 = c_1(q, r, \gamma, \rho)$ *and* $c_2 = c_2(q, r, \gamma, \rho)$ *such that*

$$c_1 \|f\|_{\mathrm{PBMO}^+_{\gamma,q}} \le \|f\|_{\mathrm{PBMO}^+_{\rho,r}} \le c_2 \|f\|_{\mathrm{PBMO}^+_{\gamma,q}}.$$

Moral:

- If $f \in \mathrm{PBMO}_{q,\gamma}$ for some $0 < \gamma < 1$, then $f \in \mathrm{PBMO}_{q,\gamma}$ for every $0 < \gamma < 1$. Thus the length of the time lag does not matter in the definition of PBMO.
- If $f \in \mathrm{PBMO}_{q,\gamma}$ for some $0 < q < \infty$, then $f \in \mathrm{PBMO}_{q,\gamma}$ for every $0 < q < \infty$. Thus the exponent does not matter in the definition of PBMO. In particular, $f \in \mathrm{PBMO}$ is locally integrable to any positive power.

Proof 1(5)

We begin with the first inequality. Let R be a parabolic rectangle in $\mathbb{R}^{n+1}$. By Hölder's inequality, we obtain

$$
\begin{aligned}
&\left(\fint_{R^+(\gamma)}(f-c)_+^q+\fint_{R^-(\gamma)}(f-c)_-^q\right)^{\frac{1}{q}}\\
&\le c(q)\left(\left(\fint_{R^+(\gamma)}(f-c)_+^q\right)^{\frac{1}{q}}+\left(\fint_{R^-(\gamma)}(f-c)_-^q\right)^{\frac{1}{q}}\right)\\
&\le c(q)\left(\left(\fint_{R^+(\gamma)}(f-c)_+^r\right)^{\frac{1}{r}}+\left(\fint_{R^-(\gamma)}(f-c)_-^r\right)^{\frac{1}{r}}\right)\\
&\le c(q,r)\left(\fint_{R^+(\gamma)}(f-c)_+^r+\fint_{R^-(\gamma)}(f-c)_-^r\right)^{\frac{1}{r}}
\end{aligned}
$$

with $c(q,r)=\max\{1,2^{\frac{1}{q}-1}\}\max\{1,2^{1-\frac{1}{r}}\}$.

Proof 2(5)

Since $R^{\pm}(\gamma) \subset R^{\pm}(\rho)$ and $|R^{\pm}(\gamma)| = \frac{1-\gamma}{1-\rho}|R^{\pm}(\rho)|$, we have

$$\begin{aligned}
&\left(\fint_{R^+(\gamma)} (f-c)_+^r + \fint_{R^-(\gamma)} (f-c)_-^r \right)^{\frac{1}{r}} \\
&\leq \left(\frac{1-\rho}{1-\gamma} \right)^{\frac{1}{r}} \left(\fint_{R^+(\rho)} (f-c)_+^r + \fint_{R^-(\rho)} (f-c)_-^r \right)^{\frac{1}{r}} .
\end{aligned}$$

By taking supremum over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$, we arrive at

$$\|f\|_{\mathrm{PBMO}^+_{\gamma,q}} \leq c\|f\|_{\mathrm{PBMO}^+_{\rho,r}},$$

with

$$c = c(q,r,\gamma,\rho) = \left(\frac{1-\rho}{1-\gamma} \right)^{\frac{1}{r}} \max\{1, 2^{\frac{1}{q}-1}\} \max\{1, 2^{1-\frac{1}{r}}\}.$$

Proof 3(5)

To show the second inequality, we assume that $0 < q \leq 1$. This is not an issue since after establishing the second inequality for $0 < q \leq 1$ we can use the first inequality to obtain the corresponding inequality with $0 < q \leq r$.

Proof 4(5)

Cavalieri's principle and the John–Nirenberg lemma imply that

$$
\begin{aligned}
&\fint_{R^+(\rho)} (f-c)_+^r \\
&= \frac{r}{|R^+(\rho)|} \int_0^\infty \lambda^{r-1} |R^+(\rho) \cap \{(f-c)_+ > \lambda\}| \, d\lambda \\
&\le Ar \int_0^\infty \lambda^{r-1} \exp\left(-\left(\frac{B\lambda}{\|f\|_{\mathrm{PBMO}^+_{\gamma,q}}}\right)^q\right) d\lambda \\
&= Ar \left(\frac{\|f\|_{\mathrm{PBMO}^+_{\gamma,q}}}{B^{\frac{1}{q}}}\right)^{r-1} \frac{\|f\|_{\mathrm{PBMO}^+_{\gamma,q}}}{B^{\frac{1}{q}} q} \int_0^\infty s^{\frac{r}{q}-1} e^{-s} \, ds \\
&= \frac{Ar}{B^{\frac{r}{q}} q} \Gamma\left(\tfrac{r}{q}\right) \|f\|^r_{\mathrm{PBMO}^+_{\gamma,q}},
\end{aligned}
$$

where we made a change of variables $s = \frac{B}{\|f\|^q_{\mathrm{PBMO}^+_{\gamma,q}}} \lambda^q$. Here $\Gamma\left(\frac{r}{q}\right)$ is the gamma function.

Proof 5(5)

Similarly

$$\fint_{R^-(\rho)} (f-c)_-^r \le \frac{Ar}{B^{\frac{r}{q}} q} \Gamma\left(\tfrac{r}{q}\right) \|f\|^r_{\mathrm{PBMO}^+_{\gamma,q}}$$

By summing two estimates above and taking supremum over all parabolic rectangles R, we conclude that

$$\|f\|_{\mathrm{PBMO}^+_{\rho,r}} \le c \|f\|_{\mathrm{PBMO}^+_{\gamma,q}}$$

with

$$c = c(q,r,\gamma,\rho) = \left(2 \frac{Ar}{B^{\frac{r}{q}} q} \Gamma\left(\tfrac{r}{q}\right) \right)^{\frac{1}{r}}.$$

Lemma

Assume that $f \in \mathrm{PBMO}^{+}_{\gamma,q}$*. There exist positive constants* $A = A(n,p,\gamma)$, $B = B(n,p,\gamma)$ *and* $c \in \mathbb{R}$*, which depends on the parabolic rectangle* R*, such that*

$$\fint_{R^+(\gamma)} e^{\delta(f-c)_+} \le 1 + \frac{A\delta \|f\|_{\mathrm{PBMO}^{+}_{\gamma,q}}}{B - \delta \|f\|_{\mathrm{PBMO}^{+}_{\gamma,q}}} < \infty$$

and

$$\fint_{R^-(\gamma)} e^{\delta(f-c)_-} \le 1 + \frac{A\delta \|f\|_{\mathrm{PBMO}^{+}_{\gamma,q}}}{B - \delta \|f\|_{\mathrm{PBMO}^{+}_{\gamma,q}}} < \infty,$$

whenever $0 < \delta < \frac{B}{\|f\|_{\mathrm{PBMO}^{+}_{\gamma,q}}}$*. Here* A*,* B *and* c *are the same constants as in the John–Nirenberg lemma.*

Proof.

Cavalieri's principle and the John–Nirenberg inequality with $q = 1$ and $\rho = \gamma$ imply

$$
\begin{aligned}
\int_{R^+(\gamma)} e^{\delta(f-c)_+} &= \int_0^\infty |R^+(\gamma) \cap \{e^{\delta(f-c)_+} > \lambda\}|\, d\lambda \\
&= \int_0^1 \dots \, d\lambda + \int_1^\infty \dots \, d\lambda \\
&\le |R^+(\gamma)| + \int_0^\infty e^s |R^+(\gamma) \cap \{(f-c)_+ > \tfrac{s}{\delta}\}|\, ds \\
&\le |R^+(\gamma)| + A|R^+(\gamma)| \int_0^\infty \exp\left(s \left(1 - \frac{B}{\delta \|f\|_{\mathrm{PBMO}^+_{q,\gamma}}} \right) \right) ds \\
&= |R^+(\gamma)| + A|R^+(\gamma)| \frac{\delta \|f\|_{\mathrm{PBMO}^+_{q,\gamma}}}{B - \delta \|f\|_{\mathrm{PBMO}^+_{q,\gamma}}} < \infty.
\end{aligned}
$$

Here we made a change of variables $\lambda = e^s$. The second inequality follows in a similar way. □

Harnack's inequality revisited 1(4)

Assume that $u > 0$ is a weak solution of the doubly nonlinear equation in $\mathbb{R}^{n+1}$ and let $0 < \gamma < 1$. By the logarithmic energy estimate we have

$$f = -\log u \in \mathrm{PBMO}^+_{q,\gamma},$$

and

$$\|f\|_{\mathrm{PBMO}^+_{q,\gamma}} \leq c(n,p,\gamma) < \infty.$$

with $q = \frac{p-1}{2}$.

Note: The bound for the PBMO norm is independent of the solution.

By the exponential estimate, there exist positive constants $A = A(n,p,\gamma)$, $B = B(n,p,\gamma)$ and $c \in \mathbb{R}$, which depends on the parabolic rectangle R, such that

$$\begin{aligned}\fint_{R^+(\gamma)} u^{-\delta} &= \fint_{R^+(\gamma)} e^{\delta f} = e^{\delta c} \fint_{R^+(\gamma)} e^{\delta(f-c)}\\ &\le e^{\delta c} \fint_{R^+(\gamma)} e^{\delta(f-c)_+}\\ &\le e^{\delta c}\left(1+\frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B-\delta\|f\|_{\mathrm{PBMO}^+}}\right) < \infty,\end{aligned}$$

whenever $0 < \delta < \frac{B}{\|f\|_{\mathrm{PBMO}^+}}$. Similarly, we have

$$\fint_{R^-(\gamma)} u^{\delta} \le e^{-\delta c}\left(1+\frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B-\delta\|f\|_{\mathrm{PBMO}^+}}\right).$$

Harnack's inequality revisited 3(4)

Thus

$$
\begin{aligned}
&\fint_{R^-(\gamma)} u^\delta \fint_{R^+(\gamma)} u^{-\delta} \\
&\le e^{\delta c}\left(1+\frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B-\delta\|f\|_{\mathrm{PBMO}^+}}\right) e^{-\delta c}\left(1+\frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B-\delta\|f\|_{\mathrm{PBMO}^+}}\right) \\
&= \left(1+\frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B-\delta\|f\|_{\mathrm{PBMO}^+}}\right)^2 < \infty
\end{aligned}
$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$. By choosing $\delta = \frac{B}{2\|f\|_{\mathrm{PBMO}^+}}$, we have

$$
\left(1+\frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B-\delta\|f\|_{\mathrm{PBMO}^+}}\right)^2 = A^2 = A(n,p,\gamma)^2.
$$

This shows that there exists $\delta = \delta(n, p, \gamma) > 0$ and $c = c(n, p, \gamma)$ such that

$$\left(\fint_{R^-(\gamma)} u^\delta \right)^{\frac{1}{\delta}} \le c \left(\fint_{R^+(\gamma)} u^{-\delta} \right)^{-\frac{1}{\delta}}$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$. This is the missing estimate in the proof of Harnack's inequality.

An alternative approach

There exists another proof of the parabolic Harnack inequality by Moser, which applies an abstract lemma of Bombieri instead of the parabolic John–Nirenberg lemma.

- Enrico Bombieri, *Theory of minimal surfaces and a counterexample to the Bernstein conjecture in high dimension*, Mimeographed Notes of Lectures Held at Courant Institute, New York University (1970)
- Enrico Bombieri and Enrico Giusti, *Harnack's inequality for elliptic differential equations on minimal surfaces*, Invent. Math. 15 (1972), 24–46.
- Juha Kinnunen and Tuomo Kuusi, *Local behaviour of solutions to doubly nonlinear parabolic equations*, Math. Ann. 337 (2007), 705–728.
- Jürgen Moser, *On a pointwise estimate for parabolic equations*, Comm. Pure Appl. Math. 24 (1971), 727–740.

An approach with medians

It is possible to develop the corresponding theory for the parabolic BMO with medians. The main advantage is that, instead of assuming local integrability, it is enough to assume that a function in measurable in the definition of the parabolic BMO. The parabolic John–Nirenberg lemma with medians implies that the parabolic BMO with medians coincides with the original definition of the parabolic BMO. In particular, all results for the parabolic BMO also hold for the parabolic BMO with medians and vice versa.

A local to global property

- If $f \in \mathrm{BMO}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain satisfying a suitable chaining condition, then John–Nirenberg inequality holds not only locally over cubes but also globally over whole Ω. (Reimann–Rychener, Smith-Stegenga, Staples)
- Olli Saari has obtained similar parabolic local to global results in space-time cylinders, see O. Saari, *Parabolic* BMO *and global integrability of supersolutions to doubly nonlinear parabolic equations*, Rev. Mat. Iberoamericana 32 (2016), 1001–1018.
- The proofs are based on delicate chaining arguments.

A global integrability result

Theorem (Saari 2016)

Let u be a positive weak solution to the doubly nonlinear equation on $\Omega \times (0, T)$, where Ω is a nice domain (satisfying a quasihyperbolic boundary condition). Then there exists $\epsilon > 0$ such that

$$u^\epsilon \in L^1(\Omega \times (0, T - \epsilon)).$$

Proof.

Follows from a global exponential integrability as in the proof of Harnack's inequality. □

Remark: This result seems to be new even for the heat equation.

Takeaways

- It is possible to develop theory for parabolic BMO related to the doubly nonlinear parabolic PDE.
- The time lag is plays an essential role in the theory.
- There exists a rather complete one-dimensional theory without the time lag.
- The proofs are based on delicate Calderón–Zygmund type covering arguments with parabolic rectangles.
- The results and methods can be applied in nonlinear PDEs.

Outline for the second part

Next we discuss parabolic Muckenhoupt weights and their connection to PDEs.

- **Goal:** To develop a higher dimensional theory for Muckenhoupt weights and functions of bounded mean oscillation (BMO) related to certain nonlinear parabolic PDEs. This extends the existing one-dimensional theory to higher dimensions.
- **Questions:** Characterization of the weighted norm inequalities for parabolic maximal functions through Muckenhoupt weights, Coifman-Rochberg type characterization of the parabolic BMO, Jones-Rubio de Francia type factorization of the parabolic Muckenhoupt weights, applications to PDEs.
- **Tools:** Definitions that are compatible with the PDEs, Calderón-Zygmund type covering arguments, harmonic analysis techniques related to the weighted norm inequalities.

Elliptic concepts 1(2)

- Let

$$Q = Q(x, l) = \{y \in \mathbb{R}^n : |y_i - x_i| \le l,\ i = 1, \dots, n\}$$

be a cube with center $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and side length $l > 0$.

- The integral average of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ in a cube Q is denoted by

$$f_Q = \fint_Q f\,dx = \frac{1}{|Q|}\int_Q f(x)\,dx.$$

- The Hardy–Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f|,$$

where the supremum is over all cubes $Q \subset \mathbb{R}^n$ containing x.

Elliptic concepts 2(2)

- Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $w \geq 0$, be a weight. The Muckenhoupt A_p condition with $1 < p < \infty$ is

$$\sup_Q \fint_Q w \left(\fint_Q w^{\frac{1}{1-p}} \right)^{p-1} < \infty.$$

- The Muckenhoupt A_1 condition is

$$\sup_Q \fint_Q w \left(\inf_Q w \right)^{-1} < \infty.$$

- $A_\infty = \bigcup_{p \geq 1} A_p$.
- Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. $f \in \mathrm{BMO}$, if

$$\|f\|_{\mathrm{BMO}} = \sup_Q \fint_Q |f - f_Q| < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Classical results in Harmonic Analysis

The following statements are equivalent:

- $M : L^p(w) \to L^p(w)$, $p > 1$, is bounded, (maximal function theorem)
- $w \in A_p$, (Muckenhoupt's theorem)
- $w = uv^{1-p}$ with $u, v \in A_1$. (Jones–Rubio de Francia factorization)

In addition:

- $\mathrm{BMO} = \{\lambda \log w : w \in A_p, \lambda > 0\}$, (John–Nirenberg lemma)
- $f \in \mathrm{BMO} \iff f = \alpha \log M\mu - \beta \log M\nu + b$ with μ, ν positive Borel measures with almost everywhere finite maximal functions, $b \in L^\infty(\mathbb{R}^n)$ and $\alpha, \beta \geq 0$ (Coifman–Rochberg characterization).

Coifman-Rochberg lemma

Theorem (Coifman–Rochberg 1980)

- *Assume that* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *is such that* $Mf(x) < \infty$ *for almost every* $x \in \mathbb{R}^n$ *and let* $0 < \delta < 1$. *Then* $w(x) = (Mf(x))^\delta$ *is an* A_1 *weight whose* A_1*-constant depends only on dimension* n *and* δ.
- *Conversely, if* $w \in A_1$, *then there exist* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $0 < \delta < 1$ *and* b, *with* $b, \frac{1}{b} \in L^\infty(\mathbb{R}^n)$, *such that* $w(x) = b(x)(Mf(x))^\delta$ *for almost every* $x \in \mathbb{R}^n$.

Note: This result asserts that every A_1 weight is essentially of the form $(Mf(x))^\delta$ for some $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $0 < \delta < 1$. Moreover, this gives a useful method to construct A_1 weights, see R. Coifman and R. Rochberg, Another characterization of BMO, Proc. Amer. Math. Soc. 79 (1980), 249–254.

Maximal functions and BMO

Theorem

Let $f \in \mathrm{BMO}(\mathbb{R}^n)$ *and assume that* $Mf < \infty$ *for almost every* $x \in \mathbb{R}^n$. *Then* $Mf \in \mathrm{BMO}(\mathbb{R}^n)$ *and there exists a constant* $c = c(n)$ *such that*

$$\|Mf\|_{\mathrm{BMO}} \le c\|f\|_{\mathrm{BMO}}.$$

References

- C. Bennett, R.A. DeVore and R. Sharpley, Weak L^∞ and BMO, Ann. Math.113 (1981), 601–611.
- F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. Appl. 7(1987), 273–279.
- O. Saari, Poincaré inequalities for the maximal function, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19 (2019), 1065–1083.

A PDE point of view

A nonnegative weak solution $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ to the elliptic p-Laplace equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0, \qquad p \in (1,\infty),$$

satisfies the following properties:

- $\log u \in \mathrm{BMO}$, (logarithmic Caccioppoli's estimate)
- $u \in A_1 \subset A_p$ for every $p > 1$, (weak Harnack's inequality)
- $\sup_Q u \leq c(n,p) \inf_Q u$. (Harnack's inequality)

These are the key points in Moser's and Trudinger's regularity theory in the 1960s.

Question: For which nonlinear parabolic PDEs is it possible to develop a similar theory? What are the correct definitions of the parabolic Muckenhoupt classes and what are the connections to the parabolic BMO?

Parabolic Muckenhoupt condition

Definition

Let $0 < \gamma < 1$ and $q > 1$. $w \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, $w \geq 0$, is in the parabolic Muckenhoupt class $A^+_q(\gamma)$, if

$$[w]_{A^+_q(\gamma)} = \sup_R \left(\fint_{R^-(\gamma)} w \right) \left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}} \right)^{q-1} < \infty,$$

where the supremum is over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$. If the condition above is satisfied with the direction of the time axis reversed, we denote $w \in A^-_q(\gamma)$.

Observe: The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ between the upper and lower parts $R^\pm(\gamma)$ is essential for us.

Remarks

- Classical A_q weights with a trivial extension in time belong to the parabolic $A_q^+(\gamma)$ class.
- If $w \in A_q^+(\gamma)$, then $e^t w \in A_q^+(\gamma)$, for example. Parabolic Muckenhoupt weights can grow arbitrarily fast in time, since we may replace e^t with any positive increasing function.

Example

Consider $w : \mathbb{R} \to (0, \infty)$, $w(x) = e^x$. Then $w \in A_2^+ \setminus A_2$ (with $\gamma = 0$), since

$$\begin{aligned}[w]_{A_2^+} &= \sup_{x,h} \left(\frac{1}{h} \int_{x-h}^{x} w \right) \left(\frac{1}{h} \int_{x}^{x+h} w^{-1} \right) \\ &= \sup_{h} \frac{2e^{-h} - e^{-2h} - 1}{h^2} < \infty\end{aligned}$$

and

$$\begin{aligned}[w]_{A_2} &= \sup_{x,h} \left(\frac{1}{h} \int_{x}^{x+h} w \right) \left(\frac{1}{h} \int_{x}^{x+h} w^{-1} \right) \\ &= \sup_{h} \frac{e^{h} + e^{-h} - 2}{h^2} = \infty.\end{aligned}$$

PDE connection

Let $1 < p < \infty$, $1 < q < \infty$ and $0 < \gamma < 1$. Assume that $u > 0$ is a weak solution to the doubly nonlinear equation.

$$
\begin{aligned}
[u]_{A_q^+(\gamma)} &= \sup_R \left(\fint_{R^-(\gamma)} u \right) \left(\fint_{R^+(\gamma)} u^{\frac{1}{1-q}} \right)^{q-1} \\
&\le \sup_R \Big(\sup_{R^-(\gamma)} u \Big) \Big(\inf_{R^+(\gamma)} u \Big)^{\frac{1}{1-q}(q-1)} \\
&= \sup_R \Big(\sup_{R^-(\gamma)} u \Big) \Big(\inf_{R^+(\gamma)} u \Big)^{-1} \le c(n,p,\gamma) < \infty,
\end{aligned}
$$

where the supremum is taken over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$. The last bound follows from parabolic Harnack's inequality.

Note: This shows that positive solutions to the doubly nonlinear equation belong to $A_q^+(\gamma)$ for every $0 < q < 1$ and $0 < \gamma < 1$.

Barenblatt solution

Example

Let $1 < p < \infty$. The function

$$u(x,t) = t^{\frac{-n}{p(p-1)}} e^{-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}}, \; x \in \mathbb{R}^n, \; t > 0,$$

is the Barenblatt solution to the doubly nonlinear equation

$$\frac{\partial}{\partial t}(|u|^{p-2}u) - \operatorname{div}(|Du|^{p-2}Du) = 0$$

in the upper half-space $\mathbb{R}^{n+1}_+$. Thus $u \in A_q^+(\gamma)$ in the upper half-space $\mathbb{R}^{n+1}_+$ for every $0 < \gamma < 1$.

Nestedness

Lemma

For $1 < q < r < \infty$, *we have* $A_q^+(\gamma) \subset A_r^+(\gamma)$.

Note: The parabolic Muckenhoupt condition becomes stronger as q decreases.

Proof

By Hölder's inequality

$$\left(\fint_{R^+(\gamma)} w^{\frac{1}{1-r}}\right)^{r-1} \le \left(\fint_{R^+(\gamma)} (w^{\frac{1}{1-r}})^{\frac{r-1}{q-1}}\right)^{(r-1)\frac{q-1}{r-1}} = \left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}}\right)^{q-1}.$$

This implies

$$\left(\fint_{R^-(\gamma)} w\right)\left(\fint_{R^+(\gamma)} w^{\frac{1}{1-r}}\right)^{r-1} \le \left(\fint_{R^-(\gamma)} w\right)\left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}}\right)^{q-1} \le [w]_{A_q^+(\gamma)}.$$

Duality

Lemma

Let $1 < q < \infty$. Then $w \in A_q^+(\gamma)$ if and only if $w^{1-q'} \in A_{q'}^-(\gamma)$ with $q' = \frac{q}{q-1}$.

Note: $w \in A_q^+(\gamma) \iff w^{\frac{1}{1-q}} \in A^-_{\frac{q}{q-1}}(\gamma)$.

Proof 1(2)

If $w \in A_q^+(\gamma)$, then

$$\begin{aligned}
&\left(\fint_{R^+(\gamma)} w^{1-q'}\right)\left(\fint_{R^-(\gamma)} (w^{1-q'})^{\frac{1}{1-q'}}\right)^{q'-1}\\
&= \left(\fint_{R^+(\gamma)} w^{1-q'}\right)\left(\fint_{R^-(\gamma)} w\right)^{q'-1}\\
&= \left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}}\right)\left(\fint_{R^-(\gamma)} w\right)^{\frac{1}{q-1}} \le [w]_{A_q^+(\gamma)}^{\frac{1}{q-1}}.
\end{aligned}$$

This implies $w^{1-q'} \in A_{q'}^-(\gamma)$.

Proof 2(2)

Conversely, if $w^{1-q'} \in A^-_{q'}(\gamma)$, then

$$\begin{aligned}
&\left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}}\right)\left(\fint_{R^-(\gamma)} w\right)^{\frac{1}{q-1}}\\
&= \left(\fint_{R^+(\gamma)} w^{1-q'}\right)\left(\fint_{R^-(\gamma)} w\right)^{q'-1}\\
&= \left(\fint_{R^+(\gamma)} w^{1-q'}\right)\left(\fint_{R^-(\gamma)} (w^{1-q'})^{\frac{1}{1-q'}}\right)^{q'-1} \le [w^{1-q'}]_{A^-_{q'}(\gamma)}.
\end{aligned}$$

This implies $w \in A^+_q(\gamma)$.

Theorem

Assume that $v, w \in A_q^+(\gamma)$. *Then* $\min\{v, w\} \in A_q^+(\gamma)$ *and* $\max\{v, w\} \in A_q^+(\gamma)$.

Note: The class of parabolic Muckenhoupt weights is closed under taking minimum and maximum.

Proof

$$
\begin{aligned}
\fint_{R^-(\gamma)} \max\{v,w\} &\le \fint_{R^-(\gamma)} v + \fint_{R^-(\gamma)} w \\
&\le [v]_{A_q^+(\gamma)} \left(\fint_{R^+(\gamma)} v^{\frac{1}{1-q}} \right)^{1-q} + [w]_{A_q^+(\gamma)} \left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}} \right)^{1-q} \\
&\le \left([v]_{A_q^+(\gamma)} + [w]_{A_q^+(\gamma)}\right) \left(\fint_{R^+(\gamma)} \max\{v,w\}^{\frac{1}{1-q}} \right)^{1-q}.
\end{aligned}
$$

By the duality property $v^{\frac{1}{1-q}}, w^{\frac{1}{1-q}} \in A^-_{\frac{q}{q-1}}(\gamma)$. The first part of the proof implies

$$
(\min\{v,w\})^{\frac{1}{1-q}} = \max\{v^{\frac{1}{1-q}}, w^{\frac{1}{1-q}}\} \in A^-_{\frac{q}{q-1}}(\gamma).
$$

By the duality property, we have $\min\{v,w\} \in A_q^+(\gamma)$.

Forward in time doubling

Notation: We write

$$w(E) = \int_E w = \int_E w(x,t)\,dx\,dt$$

throughout.

Lemma

Assume that $w \in A_q^+(\gamma)$. Then

$$\frac{w(R^-(\gamma))}{w(E)} \le [w]_{A^+(\gamma)} \left(\frac{|R^-(\gamma)|}{|E|} \right)^q$$

for every $E \subset R^+(\gamma)$.

Proof

Let $E \subset R^+(\gamma)$. By Hölder's inequality we have

$$\begin{aligned}
\left(\frac{|E|}{|R^+(\gamma)|}\right)^q w(R^-(\gamma)) &= \left(\fint_{R^+(\gamma)} \chi_E\right)^q w(R^-(\gamma)) \\
&= \left(\fint_{R^+(\gamma)} \chi_E w^{\frac{1}{q}} w^{-\frac{1}{q}}\right)^q w(R^-(\gamma)) \\
&\le \left(\fint_{R^+(\gamma)} \chi_E w\right)\left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}}\right)^{q-1} w(R^-(\gamma)) \\
&= \left(\fint_{R^-(\gamma)} w\right)\left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}}\right)^{q-1} \int_E w \\
&\le [w]_{A^+(\gamma)} w(E).
\end{aligned}$$

Remark

Parabolic $A_q^+(\gamma)$ weights are not necessarily doubling in the sense that

$$w(2R) \le cw(R)$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$ because they can grow arbitrarily fast in time. However, $w \in A_q^+(\gamma)$ is doubling in the sense that

$$\begin{aligned}
\frac{w(2R^-(\gamma))}{w(R^+(\gamma))} &\le [w]_{A^+(\gamma)} \left(\frac{|2R^-(\gamma)|}{|R^+(\gamma))|} \right)^q \\
&= [w]_{A^+(\gamma)} \left(\frac{|2R^-(\gamma)|}{|R^-(\gamma))|} \right)^q \\
&= (2^{n+p})^q [w]_{A^+(\gamma)}
\end{aligned}$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$. Here

$$2R^-(\gamma) = R^-(x,t,2l)(\gamma) = Q(x,2l) \times (t-(2l)^p, t-\gamma(2l)^p).$$

Forward and backward in time estimates

Lemma

- *Assume that* $w \in A_q^+(\gamma)$ *and let* $a, b > 0$. *There exits a constant* $c = c(n, p, [w]_{A_q^+(\gamma)}, \gamma, a)$ *such that*

$$\fint_{R^-(\gamma)} w \le c \fint_{a+R^-(\gamma)} w.$$

  Analogously, there exits a constant $c = c(n, p, [w]_{A_q^+(\gamma)}, \gamma, b)$ *such that*

$$\fint_{R^+(\gamma)} w^{\frac{1}{1-q}} \le c \fint_{R^+(\gamma)-b} w^{\frac{1}{1-q}}.$$

Note: We can move $R^-(\gamma)$ backward in time and $R^+(\gamma)$ forward in time with the estimates above.

Proof

If $a + R^-(\gamma) \subset R$, there exist a finite number of congruent, possibly overlapping, parabolic rectangles $\widehat{R}$ such that the lower parts $\widehat{R}^-(\gamma) \subset R^-(\gamma)$ cover $R^-(\gamma)$ and the upper parts $\widehat{R}^+(\gamma) \subset a + R^-(\gamma)$ cover $a + R^-(\gamma)$. If $a + R^-(\gamma) = R^+(\gamma)$, we may choose $\widehat{R} = R$. For every such a rectangle $\widehat{R}$, by Hölder's inequality, we have

$$\fint_{\widehat{R}^-(\gamma)} w \le [w]_{A_q^+(\gamma)} \left(\fint_{\widehat{R}^+(\gamma)} w^{\frac{1}{1-q}} \right)^{1-q} \le [w]_{A_q^+(\gamma)} \fint_{\widehat{R}^+(\gamma)} w.$$

The claim follows by summing up and using the bounded overlap. For general $a > 0$, the result follows by applying this recursively. The proof of the other inequality is analogous.

Remark

Assume that $w \in A_q^+(\gamma)$. By the previous result

$$\left(\fint_{R^-(\gamma)-a} w\right)\left(\fint_{R^+(\gamma)+b} w^{\frac{1}{1-q}}\right)^{q-1}$$
$$\leq c\left(\fint_{R^-(\gamma)} w\right)\left(\fint_{R^+(\gamma)} w^{\frac{1}{1-q}}\right)^{q-1} \leq c[w]_{A_q^+(\gamma)},$$

with $c = c(n, p, [w]_{A_q^+(\gamma)}, \gamma, a, b)$.

Role of the time lag

It follows from the definition that $A_q^+(\gamma) \subset A_q^+(\gamma')$ for $0 < \gamma < \gamma' < 1$. The next result asserts that the conditions $A_q^+(\gamma)$ are equivalent for every $0 < \gamma < 1$. Thus the size of the time lag γ does not play any role in the theory of parabolic Muckenhoupt weights.

Lemma (Equivalence)

If $w \in A_q^+(\gamma)$ for some $0 < \gamma < 1$, then $w \in A_q^+(\gamma')$ for every $0 < \gamma' < 1$.

Proof.

The proof follows from duality, forward in time doubling condition and a subdivision argument using the fact that the parabolic rectangles become flat at small scales for $p > 1$. $\square$

The parabolic maximal operator

Definition

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $0 < \gamma < 1$. The parabolic forward in time maximal function is defined as

$$M^{\gamma+} f(x,t) = \sup \fint_{R^+(\gamma)} |f|,$$

where the supremum is over all parabolic rectangles $R = R(x,t)$ centered at (x,t). The parabolic backward in time operator $M^{\gamma-}$ is defined analogously.

Observe: By the Lebesgue differentiation theorem, we have

$$|f(x,t)| \leq M^{\gamma+} f(x,t)$$

for almost every $(x,t) \in \mathbb{R}^{n+1}$.

Characterization of parabolic Muckenhoupt weights

Theorem (K.–Saari 2016)

Let $1 < q < \infty$ and $0 < \gamma < 1$. The following claims are equivalent:

- $w \in A_q^+(\gamma)$,
- *(Weak type estimate)* $M^{\gamma+} : L^q(w) \to L^{q,\infty}(w)$ *with*

$$w(\{M^{\gamma+} f > \lambda\}) \leq \frac{c}{\lambda^q} \int_{\mathbb{R}^{n+1}} |f|^q w$$

  for every $\lambda > 0$,
- *(Strong type estimate)* $M^{\gamma+} : L^q(w) \to L^q(w)$ *with*

$$\int_{\mathbb{R}^{n+1}} |M^{\gamma+} f|^q w \leq c \int_{\mathbb{R}^{n+1}} |f|^q w.$$

The constants in each of the conditions only depend on n, p, q, γ, $[w]_{A_q^+(\gamma)}$ *and each other.*

Proof.

- First we prove that $w \in A_q^+(\gamma)$ if and only if the weak type estimate holds.
- The necessity follows as in the classical case by applying the weak type estimate with a characteristic function of a parabolic rectangle.
- The sufficiency part of the weak type estimate uses a modification of a covering argument by L. Forzani, F.J. Martín-Reyes and S. Ombrosi, Weighted inequalities for the two-dimensional one-sided Hardy-Littlewood maximal function, Trans. Amer. Math. Soc. 363 (2011), 1699–1719.
- The strong type estimate follows from a reverse Hölder type inequality and the Marcinkiewicz interpolation theorem.

$\square$

One-dimensional theory

For the one-sided maximal operator ($\gamma = 0$, that is, without a lag)

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|$$

and the corresponding one-sided Muckenhoupt weights $w \in A_p^+$,

$$\sup_{x,h} \frac{1}{h} \int_{x-h}^{x} w \left(\frac{1}{h} \int_x^{x+h} w^{1-p'} \right)^{p-1} < \infty,$$

it is known that $M^+ : L^p(w) \to L^p(w) \iff w \in A_p^+$.
(Sawyer 1986)

Takeaways

- There is a complete one-dimensional theory including A_∞^+, one-sided reverse Hölder inequality and one-sided BMO . (Cruz–Uribe, Martín–Reyes, Neugebauer, Olesen, Pick, de la Torre,...)
- The time lag disappears in the one-dimensional case.
- Higher dimensional case has turned out to be more challenging. Some partial results are known. (Berkovits, Forzani, Lerner, Martín–Reyes, Ombrosi 2010–2011)
- Our approach gives a complete characterization in the higher dimensional case with a time lag.

Reverse Hölder inequality

Lemma (K.–Saari 2016)

Assume that $w \in A_q^+(\gamma)$ *with* $0 < \gamma < 1$. *Then there exist* $\varepsilon > 0$ *and a constant* $c = c(n, p, q, \gamma, [w]_{A_q^+(\gamma)})$ *such that*

$$\left(\fint_{R^-(\gamma)} w^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \leq c \fint_{R^+(\gamma)} w$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$.

Observe: This is weaker than the standard RHI, because there is a time lag between the rectangles R^- and R^+.

Proof.

- First we prove a distribution set estimate

$$w(R^- \cap \{w > \lambda\}) \leq c\lambda|R \cap \{w > \beta\lambda\}|.$$

- We apply a delicate parabolic Calderón-Zygmund type decomposition.
- Once the distribution set estimate is done, the claim follows from Cavalieri's principle, covering arguments and iteration.

$\square$

Self-improvement

Theorem

Assume that $w \in A_q^+(\gamma)$ with $1 < q < \infty$. There exists a constant $\varepsilon = \varepsilon(n, p, q, \gamma, [w]_{A_q}) > 0$ such that $w \in A_{q-\varepsilon}^+(\gamma)$.

Note: Assume that $w \in A_q^+(\gamma)$. By the nestedness property $w \in A_{q+\varepsilon}^+(\gamma)$ for every $\varepsilon > 0$, but there also exists a small $\varepsilon > 0$ for which $w \in A_{q-\varepsilon}^+(\gamma)$.

Proof 1(2)

The duality property implies

$$w^{\frac{1}{1-q}} \in A^-_{\frac{q}{q-1}}(\gamma).$$

Let $R \subset \mathbb{R}^{n+1}$ be a parabolic rectangle. By the reverse Hölder inequality, there exist c and $\widehat{q} > 1$ such that

$$\left(\fint_{R^+(\gamma)} w^{\frac{\widehat{q}}{1-q}} \right)^{\frac{1}{\widehat{q}}} \le c \fint_{R^-(\gamma)} w^{\frac{1}{1-q}}.$$

Proof 2(2)

Let $\widehat{R} \subset \mathbb{R}^{n+1}$ be a parabolic rectangle such that $\widehat{R}^{+}(\gamma) = R^{-}(\gamma)$. Let $0 < \varepsilon < q - 1$ be determined by $\frac{\widehat{q}}{q-1} = \frac{1}{q-\varepsilon-1}$. This implies

$$\begin{aligned}\left(\fint_{R^{+}(\gamma)} w^{\frac{1}{1-(q-\varepsilon)}} \right)^{q-\varepsilon-1} &= \left(\fint_{R^{+}(\gamma)} w^{\frac{\widehat{q}}{1-q}} \right)^{\frac{q-1}{\widehat{q}}} \\ &\leq c \left(\fint_{R^{-}(\gamma)} w^{\frac{1}{1-q}} \right)^{q-1} \\ &\leq c \left(\fint_{\widehat{R}^{-}(\gamma)} w \right)^{-1}.\end{aligned}$$

This shows that $w \in A^{+}_{q-\varepsilon}(\gamma)$, since $\widehat{R}^{-}(\gamma) = R^{-}(\gamma) - a$ for some $a > 0$.

A characterization of parabolic BMO

Lemma

Let $1 < q < \infty$ *and* $0 < \gamma < 1$. *Then*

$$\mathrm{PBMO}^{+} = \{-\lambda \log w : w \in A_q^{+}(\gamma), \lambda > 0\}.$$

Proof 1(5)

Assume that $f \in \mathrm{PBMO}^+$. By the John-Nirenberg lemma there exist positive constants $A = A(n,p,\gamma)$, $B = B(n,p,\gamma)$ and $c \in \mathbb{R}$, which depends on the parabolic rectangle R, such that

$$\fint_{R^-(\gamma)} e^{\delta(f-c)_-} \le 1 + \frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B - \delta\|f\|_{\mathrm{PBMO}^+}} < \infty,$$

and

$$\fint_{R^+(\gamma)} e^{\delta(f-c)_+} \le 1 + \frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B - \delta\|f\|_{\mathrm{PBMO}^+}} < \infty$$

whenever $0 < \delta < \frac{B}{\|f\|_{\mathrm{PBMO}^+}}$ This implies

$$\begin{aligned}\fint_{R^-(\gamma)} e^{-\delta f} &= e^{-c\delta} \fint_{R^-(\gamma)} e^{\delta(c-f)} \le e^{-c\delta} \fint_{R^-(\gamma)} e^{\delta(f-c)_-} \\ &\le e^{-c\delta}\left(1 + \frac{A\delta\|f\|_{\mathrm{PBMO}^+}}{B - \delta\|f\|_{\mathrm{PBMO}^+}}\right).\end{aligned}$$

Proof 2(5)

and

$$
\begin{aligned}
\fint_{R^+(\gamma)} e^{\frac{\delta f}{q-1}} &= e^{\frac{c\delta}{q-1}} \fint_{R^+(\gamma)} e^{\frac{\delta(f-c)}{q-1}} \\
&\le e^{\frac{c\delta}{q-1}} \fint_{R^+(\gamma)} e^{\frac{\delta(f-c)_+}{q-1}} \\
&\le e^{\frac{c\delta}{q-1}} \left(1 + \frac{A\frac{\delta}{q-1}\|f\|_{\mathrm{PBMO}^+}}{B - \frac{\delta}{q-1}\|f\|_{\mathrm{PBMO}^+}} \right)
\end{aligned}
$$

for some $1 < q < \infty$. This shows that

$$
w = e^{-\delta f} \in A_q^+(\gamma)
$$

and $f = -\delta^{-1} \log w$.

Proof 3(5)

To prove the other direction, assume that $w \in A_q^+(\gamma)$ with $q \le 2$. Let

$$c = \log \fint_{R^-(\gamma)} w.$$

By Jensen's inequality and the parabolic Muckenhoupt condition, we have

$$\begin{aligned}
\exp \fint_{R^+(\gamma)} (c - \log w)_+ &\le \fint_{R^+(\gamma)} \exp(c - \log w)_+ \\
&\le 1 + \fint_{R^+(\gamma)} \exp\left(c - \frac{1}{1-q'} \log w^{1-q'}\right) \\
&\le 1 + e^c \left(\fint_{R^+(\gamma)} w^{1-q'}\right)^{q-1} \\
&= 1 + \left(\fint_{R^-(\gamma)} w\right)\left(\fint_{R^+(\gamma)} w^{1-q'}\right)^{q-1} \le 1 + [w]_{A_q^+(\gamma)}.
\end{aligned}$$

Proof 4(5)

On the other hand, again by Jensen's inequality,

$$\begin{aligned}
\exp \fint_{R^-(\gamma)} (c - \log w)_- &= \exp \fint_{R^-(\gamma)} (\log w - c)_+ \\
&\le \fint_{R^-(\gamma)} \exp(\log w - c)_+ \le 1 + \fint_{R^-(\gamma)} \exp(\log w - c) \\
&\le 1 + e^{-c} \fint_{R^-(\gamma)} w \\
&= 1 + \left(\fint_{R^-(\gamma)} w \right)^{-1} \fint_{R^-(\gamma)} w \le 2.
\end{aligned}$$

Proof 5(5)

This implies that

$$\log(2(1+[w]_{A_q^+})) \geq \fint_{R^+(\gamma)} (-\log w - (-c))_+ + \fint_{R^-(\gamma)} (-\log w - (-c))_-$$

and consequently

$$u = -\log w \in \mathrm{PBMO}^+(\gamma).$$

Applying the same argument for $A_{q'}^-(\gamma)$ with $q > 2$ shows that $-\log w^{1-q'} \in \mathrm{PBMO}^-(\gamma)$ and consequently

$$-(q'-1)\log w \in \mathrm{PBMO}^+(\gamma).$$

Coifman–Rochberg theorem

Theorem (K.–Saari 2016)

Let $f \in \mathrm{PBMO}^+$ and $0 < \gamma < 1$. Then there are positive Borel measures μ, ν satisfying

$$M^{\gamma+}\mu < \infty \quad \text{and} \quad M^{\gamma-}\nu < \infty$$

almost everywhere in $\mathbb{R}^{n+1}$, a bounded function b and constants $\alpha, \beta \geq 0$ such that

$$f = -\alpha \log M^{\gamma-}\mu + \beta \log M^{\gamma+}\nu + b.$$

Conversely, if the above holds with $\gamma = 0$, then $f \in \mathrm{PBMO}^+$.

Observe: This gives a method to produce examples of parabolic BMO functions.

Proof.

- $\mathrm{PBMO}^+ = \{-\lambda \log w : w \in A_q^+(\gamma), \lambda > 0\}$. (The John-Nirenberg lemma)
- Let $0 < \delta < 1$ and $0 < \gamma < \delta 2^{1-p}$. Then

$$w \in A_q^+(\delta) \iff w = uv^{1-p},$$

where $u \in A_1^+(\gamma)$ and $v \in A_1^-(\gamma)$. A weight w belongs to the parabolic Muckenhoupt $A_1^+(\gamma)$ class, if

$$M^{\gamma-} w \leq Cw$$

almost everywhere in $\mathbb{R}^{n+1}$. The class $A_1^-(\gamma)$ is defined by reversing the direction of time. (Jones factorization)
- The rest follows from a similar reasoning as in the classical case. (Coifman–Rochberg, Coifman–Jones–Rubio de Francia)

$\square$

Takeaways

- It is possible to develop theory for parabolic Muckenhoupt weights related to the doubly nonlinear parabolic PDE. The results are new even for the heat equation.
- A complete Muckenhoupt type characterization of the weighted norm inequalities can be obtained with connections to the parabolic BMO.
- The time lag is both a challenge and an opportunity.
- There is a rather complete one-dimensional theory without the lag.
- The proofs are based on delicate Calderón–Zygmund type covering arguments.

References 1(3)

- H. Aimar, *Elliptic and parabolic BMO and Harnack's inequality*, Trans. Amer. Math. Soc. 306 (1988), 265–276.
- L. Berkovits, *Parabolic John–Nirenberg spaces*, J. Funct. Spaces Appl. 2012, Art. ID 901917, 9 pp.
- L. Berkovits, *Parabolic Muckenhoupt weights in the Euclidean space*, J. Math. Anal. Appl. 379 (2011), 524–537.
- E. B. Fabes and N. Garofalo, *Parabolic BMO and Harnack's inequality*, Proc. Amer. Math. Soc. 95 (1985), 63?69.
- L. Forzani, F. J. Martín-Reyes and S. Ombrosi, S., *Weighted inequalities for the two-dimensional one-sided Hardy-Littlewood maximal function*, Trans. Amer. Math. Soc. 363 (2011), 1699–1719.
- J. Kinnunen and T. Kuusi, *Local behaviour of solutions to doubly nonlinear parabolic equations*, Math. Ann. 337 (2007), 705–728.
- J. Kinnunen, K. Myyryläinen and D. Yang, *John–Nirenberg inequalities for parabolic* BMO (2022) (submitted).

- J. Kinnunen and O. Saari, *On weights satisfying parabolic Muckenhoupt conditions*, Nonlinear Anal. 131 (2016), 289–299.
- A. Lerner and S. Ombrosi, *A boundedness criterion for general maximal operators*, Publ. Mat. 54 (2010), 53–71.
- F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. 117 (1993), 691–698.
- F. J. Martín-Reyes, P. Ortega Salvador and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. 319 (1990), 517–534.
- J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17 (1964), 101–134.
- J. Moser, *Correction to: A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. 20 (1967), 231–236.

References 3(3)

- K. Myyryläinen, *Median-type John–Nirenberg space in metric measure spaces*, J. Geom. Anal. 32 (2022), no. 4, 131.
- S. Ombrosi, *Weak weighted inequalities for a dyadic one-sided maximal function in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. 133 (2005), 769–1775.
- O. Saari, *Parabolic* BMO *and global integrability of supersolutions to doubly nonlinear parabolic equations*, Rev. Mat. Iberoamericana 32 (2016), 1001–1018.
- O. Saari, *Parabolic* BMO *and the forward-in-time maximal operator*, Ann. Mat. Pura Appl. (4) 197 (2018), 1477–1497.
- E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. 297 (1986), 53–61.
- N. S. Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math. 21 (1968), 205–226.