Supercaloric functions for the parabolic p-Laplace equation

Juha Kinnunen, Aalto University, Finland
juha.k.kinnunen@aalto.fi
http://math.aalto.fi/∼jkkinnun/

June 1, 2022

References 1(2)

- R. Giri, J. Kinnunen and K. Moring, *Supercaloric functions for the parabolic p-Laplace equation in the fast diffusion case*, NoDEA Nonlinear Differential Equations Appl. 28 (2021), no. 3, 33.
- J. Kinnunen, P. Lehtelä, P. Lindqvist and M. Parviainen, *Supercaloric functions for the porous medium equation*, J. Evol. Equ. 19 (2019), 249–270.
- T. Kuusi, P. Lindqvist, and M. Parviainen: *Shadows of infinities*, Ann. Mat. Pura Appl. (4) 195 (2016), no. 4, 1185–1206.
- J. Kinnunen, P. Lindqvist and T. Lukkari, *Perron's method for the porous medium equation*, J. Eur. Math. Soc. 18 (2016), 2953–2969.

References 2(2)

- J. Kinnunen and P. Lindqvist, *Unbounded supersolutions of some quasilinear parabolic equations: a dichotomy*, Nonlinear Anal. 131 (2016), 229–242, Nonlinear Anal. 131 (2016), 289–299.
- J. Kinnunen and P. Lindqvist, *Definition and properties of supersolutions to the porous medium equation*, J. reine angew. Math. 618 (2008), 135–168.
- J. Kinnunen and P. Lindqvist, *Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation*, Ann. Mat. Pura Appl. (4) 185 (2006), no. 3, 411–435.
- J. Kinnunen and P. Lindqvist, *Summability of semicontinuous supersolutions to a quasilinear parabolic equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), no. 1, 59–78.

Outline of the talk 1(3)

We discuss supersolutions to the parabolic p-Laplace equation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty.$$

The three cases $1 < p < 2$ (fast diffusion), $p = 2$ (the heat equation) and $p > 2$ (slow diffusion) are different, for example, when the "diffusion coefficient" $|\nabla u| \to 0$.

Motivation: Supersolutions arise in obstacle problems, problems with measure data, Perron-Wiener-Brelot method, boundary regularity, polar sets, removable sets and capacity.

Outline of the talk 2(3)

Classes of supersolutions:

- Weak supersolutions (test functions under the integral)
- Supercaloric functions (defined through a comparison principle)
- Solutions to a measure data problem (solutions obtained as limit of approximations, entropy solutions)
- Viscosity supersolutions (test functions evaluated at contact points)

Outline of the talk 3(3)

- **Goal**
  - To discuss a nonlinear theory of supercaloric functions for the parabolic p-Laplace equation.
- **Questions**
  - Connections of supercaloric functions to supersolutions
  - Sobolev space properties of supercaloric functions
  - Infinity sets of supercaloric functions
- **Toolbox**
  - Energy estimates
  - Harnack inequalities
  - Regularity results
  - Obstacle problems

Space-time cylinders

- Let Ω be an open subset of $\mathbb{R}^n$ and let $0 \leq t_1 < t_2 \leq T$.
- We denote space-time cylinders as

$$\Omega_T = \Omega \times (0, T) \quad \text{and} \quad \Omega'_{t_1,t_2} = \Omega' \times (t_1, t_2),$$

  where $\Omega' \subset \Omega$ is an open set.
- The parabolic boundary of Ω'_{t_1,t_2} is

$$\partial_p \Omega'_{t_1,t_2} = (\overline{\Omega'} \times \{t_1\}) \cup (\partial\Omega' \times (t_1, t_2]),$$

  that is, only the initial and lateral boundaries are taken into account.

Sobolev spaces

- $W^{1,p}(\Omega)$ denotes the Sobolev space that consists of functions $u \in L^p(\Omega)$ such that the weak gradient $\nabla u \in L^p(\Omega)$.
- The parabolic Sobolev space $L^p(0,T;W^{1,p}(\Omega))$ consists of measurable functions $u : \Omega_T \to [-\infty, \infty]$ such that $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$ for almost every $t \in (0,T)$ and
$$\iint_{\Omega_T} \left(|u(x,t)|^p + |\nabla u(x,t)|^p \right) dx\, dt < \infty.$$
- $u \in L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ if u belongs to the parabolic Sobolev space for every $\Omega'_{t_1,t_2} = \Omega' \times (t_1,t_2) \Subset \Omega_T$.

Weak solutions

Let $1 < p < \infty$ and let Ω be an open set in $\mathbb{R}^n$. A function $u \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega))$ is a weak solution to

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0,$$

if

$$\iint_{\Omega_T} \left(-u\partial_t\varphi + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right)\, dx\, dt = 0$$

for every $\varphi \in C_0^\infty(\Omega_T)$. Furthermore, u is a weak supersolution if the integral above is nonnegative for all nonnegative test functions $\varphi \in C_0^\infty(\Omega_T)$.

It is possible to consider more general equations if this type, but we focus on the prototype equation. We may also consider solutions defined, for example, in $\Omega \times (-\infty, \infty)$ or $\mathbb{R}^{n+1}$.

Structural properties 1(2)

- Constants can be added to solutions.
- The equation is nonlinear: A sum of two solutions is not a solution, in general.
- A minimum of two supersolutions is a supersolution. In particular, the truncations

$$\min\{u, k\}, \quad k = 1, 2, \dots,$$

are supersolutions. Thus we may always consider bounded supersolutions and estimates which are independent of the level of truncation.

Structural properties 2(2)

- Solutions cannot be scaled for $p \neq 2$, since

$$\begin{aligned}
&\partial_t(\lambda u) - \operatorname{div}(|\nabla(\lambda u)|^{p-2}\nabla(\lambda u)) = 0 \\
&\implies \lambda\partial_t u - \lambda^{p-1}\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad \lambda > 0.
\end{aligned}$$

- Estimates respect an intrinsic geometry that depends on the solution itself. Roughly speaking, if $|\nabla u| \approx \lambda$, then

$$\begin{aligned}
&\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in} \quad Q(x,R)\times(0,R^2) \\
&\implies \partial_t u - \lambda^{p-2}\operatorname{div}(\nabla u) = 0 \quad \text{in} \quad Q(x,R)\times(0,R^2) \\
&\implies \partial_t u - \Delta u = 0 \quad \text{in} \quad Q(x,R)\times(0,\lambda^{2-p}R^2).
\end{aligned}$$

The statement of the following existence result can be found in Björn et al. (Theorem 2.3). For the proof, we refer to Ivert (Theorem 3.2), which applies results of Fontes. For related existence results, see Bögelein et al. (Theorem 1.2).

Theorem

Let $1 < p < \infty$ and let Ω be a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary and $g \in C(\partial_p \Omega_T)$. There exists a unique weak solution $u \in C(\overline{\Omega_T})$ to the parabolic p-Laplace equation with $u = g$ on $\partial_p \Omega_T$.

Existence 2(2)

- A. Björn, J. Björn, U. Gianazza and M. Parviainen, *Boundary regularity for degenerate and singular parabolic equations*, Calc. Var. Partial Differential Equations 52 (2015), no. 3-4, 797–827.
- V. Bögelein, F. Duzaar and P. Marcellini, *Existence of evolutionary variational solutions via the calculus of variations*, J. Differential Equations 256 (2014), no. 12, 3912–3942.
- M. Fontes, *Initial-boundary value problems for parabolic equations*, Ann. Acad. Sci. Fenn. Math. 34 (2009), no. 2, 583–605.
- P.-A. Ivert, *On the boundary value problem for p-parabolic equations in methods of spectral analysis in mathematical physics*, Oper. Theory Adv. Appl. 186 (2009), 229–239.

Continuity properties

Let $\frac{2n}{n+2} < p < \infty$.

- A weak solution is continuous after a possible redefinition on a set of measure zero (DiBenedetto). A weak solution to the p-parabolic equation belongs to $C^{1,\alpha}_{\mathrm{loc}}(\Omega_T)$ and this is optimal. The gradient of a solution is not necessarily continuous for more general equations with p-structure.
- A weak supersolution is lower semicontinuous after a possible redefinition on a set of measure zero. See T. Kuusi, *Lower semicontinuity of weak supersolutions to nonlinear parabolic equations*, Differential Integral Equations 22 (2009), no. 11-12, 1211–1222.

The moral: A weak solution has a continuous representative and a weak supersolution has a lower semicontinuous representative.

Remark

No regularity in time is assumed, in particular, for weak supersolutions. For example,

$$u(x,t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

is a weak supersolution.

Standard references

- E. DiBenedetto, *Degenerate parabolic equations*, Springer 1993.
- E. DiBenedetto, U. Gianazza and V. Vespri, *Harnack's inequality for degenerate and singular parabolic equations*, Springer 2012.

Main ingredients in the regularity theory

The regularity theory is based on

- energy estimates (Caccioppoli estimates) and
- Sobolev inequalities.

Caccioppoli inequality

Lemma

Let $1 < p < \infty$, $\varepsilon > 0$. Assume that u is a nonnegative weak supersolution in Ω_T. There exists a constant $c = c(p, \varepsilon)$ such that

$$\iint_{\Omega_T} |\nabla u|^p u^{-\varepsilon-1} \varphi^p \, dx \, dt + \operatorname*{ess\,sup}_{0<t<T} \int_{\Omega} u^{1-\varepsilon} \varphi^p \, dx \leq c \iint_{\Omega_T} u^{p-1-\varepsilon} |\nabla \varphi|^p \, dx \, dt + c \iint_{\Omega_T} u^{1-\varepsilon} |\partial_t(\varphi^p)| \, dx \, dt$$

for every nonnegative test function $\varphi \in C_0^\infty(\Omega_T)$.

Proof.

Apply $u^{-\varepsilon}\varphi^p$ as a test function. □

Remark

- The lack of scaling property of the parabolic p-Laplace equation is reflected in the Caccioppoli inequality: The powers on the integrals corresponding to the elliptic equation add up to $p - \varepsilon - 1 + p$, but the powers on the time part add up to $1 - \varepsilon + p$.
- The leading term on the right-hand side is defferent in the cases $1 < p < 2$ and $p > 2$, since

$$p - 1 - \varepsilon > 1 - \varepsilon \iff p > 2.$$

Parabolic Sobolev inequality

Lemma

Let $0 < m < \infty$ and $1 \le p < \infty$. Assume that $u \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega))$ and $\varphi \in C_0^\infty(\Omega_T)$. There exists a constant $c = c(p, m, n)$ such that

$$
\begin{aligned}
&\iint_{\Omega_T} |\varphi u|^{p^*} \, dx \, dt \\
&\le c \iint_{\Omega_T} |\nabla(\varphi u)|^p \, dx \, dt \left(\operatorname*{ess\,sup}_{0<t<T} \int_{\Omega} |\varphi u|^m \, dx \right)^{\frac{p}{n}},
\end{aligned}
$$

where $p^* = p + \frac{pm}{n}$.

Proof.

Apply the standard Sobolev inequality on time slices and integrate over the time variable. $\square$

Comparison principle

Theorem (Kilpeläinen-Lindqvist 1996)

Let $1 < p < \infty$ and let Ω be a bounded open set in $\mathbb{R}^n$. Assume that v is a weak supersolution and u is a weak subsolution in Ω_T. If v and $-u$ are lower semicontinuous in $\overline{\Omega}_T$ and $u \leq v$ on $\partial_p \Omega_T$, then $u \leq v$ almost everywhere in Ω_T.

Note: Comparison principle implies uniqueness.

The Barenblatt solution in the slow diffusion case

Example

The Barenblatt solution for $p > 2$ is

$$u(x,t) = \begin{cases} (\lambda t)^{-\frac{n}{\lambda}} \left(C - \frac{p-2}{p} (\lambda t)^{-\frac{p}{\lambda(p-1)}} |x|^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}}, & t > 0, \\ 0, \quad t \le 0, \end{cases}$$

where $\lambda = n(p-2) + p$ and the constant C is usually chosen so that

$$\int_\Omega u(x,t)\, dx = 1 \quad \text{for every} \quad t > 0.$$

Observe: For $p > 2$, the Barenblatt solution is compactly supported for every $t > 0$. There is a moving boundary and disturbances propagate with finite speed.

The Barenblatt solution in the supercritical case

Example

The Barenblatt solution for $\frac{2n}{n+1} < p < 2$ is

$$u(x,t) = \begin{cases} (\lambda t)^{-\frac{n}{\lambda}} \left(C + \frac{2-p}{p} (\lambda t)^{-\frac{p}{\lambda(p-1)}} |x|^{\frac{p}{p-1}} \right)^{-\frac{p-1}{2-p}}, & t > 0, \\ 0, \quad t \leq 0, \end{cases}$$

where λ and C are as in the slow diffusion case. Note that $\lambda > 0$ for $p > \frac{2n}{n+1}$.

Observe: The only difference to the slow diffusion case is that the positive part is missing. The Barenblatt solution is positive everywhere for every $t > 0$ and disturbances propagate with infinite speed.

The Barenblatt solution in the subcritical case

Barenblatt solution does not exist in the subcritical case $1 < p \leq \frac{2n}{n+1}$ and, to our knowledge, the theory is not yet well understood.

Properties of the Barenblatt solution

Let u be the Barenblatt solution with $\frac{2n}{n+1} < p < \infty$.

- u has a point singularity at $(0,0)$, that is,

$$\lim_{\substack{(y,t)\to(0,0)\\ t>0}} u(y,t) = \infty.$$

- u is a weak solution in the upper half-space $\mathbb{R}^n \times (0,\infty)$.
- $u \in L^q_{\mathrm{loc}}(\mathbb{R}^{n+1})$ for every $0 < q < p - 1 + \frac{p}{n}$.
- The weak gradient exists and $\nabla u \in L^q_{\mathrm{loc}}(\mathbb{R}^{n+1})$ for every $0 < q < p - 1 + \frac{1}{n+1}$.
- u satisfies the condition in the definition of weak supersolution in $\mathbb{R}^{n+1}$, but

$$\int_{-1}^{1} \int_{|x|<1} |\nabla u(x,t)|^p \, dx \, dt = \infty,$$

and thus $\nabla u \notin L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Thus u is not a weak supersolution in $\mathbb{R}^{n+1}$.

Measure data problem

The Barenblatt solution u solves the measure data problem

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = M\delta$$

in the weak sense, where $M > 0$ and δ is Dirac's delta, that is,

$$\iint_{\mathbb{R}^{n+1}} \left(-u\partial_t\varphi + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right)\,dx\,dt = M\varphi(0)$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

Observe: It is also possible to consider Dirac's delta as an initial value in the upper half-space.

p-supercaloric functions

Definition (Kilpeläinen-Lindqvist 1996)

Let $1 < p < \infty$. A function $u : \Omega_T \to [0, \infty]$ is p-supercaloric, if

1. u is lower semicontinuous,
2. u is finite in a dense subset of Ω_T and
3. u satisfies the following comparison principle in every interior cylinder $\Omega'_{t_1,t_2} \Subset \Omega_T$: If $h \in C(\overline{\Omega'_{t_1,t_2}})$ is a weak solution of the parabolic p-Laplace equation in Ω'_{t_1,t_2} and $u \geq h$ on $\partial_p \Omega'_{t_1,t_2}$, then $u \geq h$ in Ω'_{t_1,t_2}.

For $p = 2$ we have supercaloric functions (supertemperatures) for the heat equation. See N.A. Watson, *Introduction to heat potential theory, American Mathematical Society*, 2012.

Remarks 1(2)

- The Barenblatt solution is a p-supercaloric function in $\mathbb{R}^{n+1}$.
- By the Schwarz alternating method is enough to compare in boxes instead of all cylindrical subdomains (Korte, Kuusi and Parviainen 2010).
- The minimum of two p-supercaloric functions is p-supercaloric.
- The class of p-supercaloric functions is closed under increasing convergence if the limit function is finite on a dense set.
- A p-supercaloric function does not, a priori, belong to a Sobolev space. The only connection to the equation is through the comparison principle.
- A lower semicontinous function is locally bounded from below. By adding a constant, we may assume that the function is nonnegative in case we are interested in local properties.

Remarks 2(2)

- A lower semicontinuous representative of a weak supersolution is p-supercaloric.
  Idea of the proof: Weak supersolutions satisfy the comparison principle.
- A locally bounded p-supercaloric function is a weak supersolution. In particular, the truncations

$$\min\{u, k\}, \quad k = 1, 2, \dots,$$

  are weak supersolutions (K.-Lindqvist 2006).
  Idea of the proof: Approximate a given p-supercaloric function pointwise by an increasing sequence of weak supersolutions, constructed through successive obstacle problems. By the boundedness assumption, the limit function is a weak supersolution.

Takeaways

- There are no other **bounded** p-supercaloric functions than weak supersolutions, once the question of lower semicontinuity is taken into account. Thus if we are only interested in bounded functions these classes coincide.
- The Barenblatt solution is a p-supercaloric functions, but not a weak supersolution. As we shall see, there are several ways to construct **unbounded** p-supercaloric functions, that are not weak supersolutions. Thus, in general, these are different classes of functions.
- We can **approximate** any p-supercaloric function u by its truncations $\min\{u, k\}$, $k = 1, 2, \dots$, which are weak supersolutions. For weak supersolutions useful estimates are available, in particular, Sobolev, Caccioppoli and Harnack inequalities.

Unbounded p-supercaloric functions 1(2)

Example

Assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded open set and let $p > 2$. The friendly giant is

$$u(x,t) = \begin{cases} t^{-\frac{1}{p-2}} v(x), & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $v \in W_0^{1,p}(\Omega)$ is the unique positive weak solution to the nonlinear elliptic eigenvalue problem

$$\operatorname{div}(|\nabla v|^{p-2} \nabla v) + \frac{1}{p-2} v = 0 \quad \text{in} \quad \Omega.$$

The friendly giant is a p-supercaloric function in $\Omega \times \mathbb{R}$.

Remarks

- Singularity occurs on the entire time slice $\Omega \times \{0\}$, that is,

$$\lim_{\substack{(y,t)\to(x,0)\\ t>0}} u(y,t) = \infty \quad \text{for every} \quad x \in \Omega.$$

  This cannot happen for the heat equation when $p = 2$.
- u is a weak solution in $\Omega \times (0,\infty)$.
- $u \notin L^{p-2}_{\mathrm{loc}}(\Omega \times \mathbb{R})$. In particular, $u \notin L^{p}_{\mathrm{loc}}(\Omega \times \mathbb{R})$ and thus u is not a weak supersolution in $\Omega \times \mathbb{R}$.
- The friendly giant can be modified so that the obtained p-supercaloric function is not locally integrable to any positive power.

Comparison with the friendly giant

The friendly giant plays an important role as a minorant for p-supercaloric functions which blow up at time $t_0 = 0$. Assume that u is a nonnegative p-supercaloric function in Ω_T with the property that

$$\lim_{\substack{(y,t)\to(x,0)\\ t>0}} u(y,t) = \infty \quad \text{for every} \quad x \in \Omega.$$

Let $\sigma > 0$. The comparison principle gives

$$u(x,t) \ge v(x)(t+\sigma)^{-\frac{1}{p-2}} \quad \text{for every} \quad (x,t) \in \Omega_{T-\sigma},$$

where v is a solution to the elliptic eigenvalue problem as in the construction of the friendly giant. By letting $\sigma \to 0$, we have

$$u(x,t) \ge v(x)t^{-\frac{1}{p-2}} \quad \text{for every} \quad (x,t) \in \Omega_T.$$

In particular,

$$\liminf_{\substack{(y,t)\to(x,0)\\ t>0}} u(y,t)t^{\frac{1}{p-2}} > 0 \quad \text{for every} \quad x \in \Omega.$$

Unbounded p-supercaloric functions 2(2)

Example

Let $\frac{2n}{n+1} < p < 2$. If $C = 0$ in the Barenblatt solution, we obtain the infinite point source solution (IPSS)

$$u(x,t) = \begin{cases} \left(\dfrac{ct}{|x|^p}\right)^{\frac{1}{2-p}}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

where

$$c = (2-p)\left(\frac{p}{2-p}\right)^{p-1}\left(\frac{p}{2-p} - n\right).$$

The IPSS is a p-supercaloric function in $\mathbb{R}^{n+1}$.

Observe: The IPSS has a standing singularity at $x = 0$ for every $t > 0$.

Properties of the IPSS

Let u be the infinite point source solution with $\frac{2n}{n+1} < p < 2$.

- u has a standing singularity at $x = 0$, that is,

$$\lim_{\substack{(y,t)\to(x,s)\\ t>s}} u(y,t) = \infty$$

  for every $s > 0$.

- u is a weak solution in $(\mathbb{R}^n \setminus \{0\}) \times (0,\infty)$.
- $u \notin L^{\frac{n}{p}(2-p)}_{\mathrm{loc}}(\mathbb{R}^{n+1})$. In particular, $u \notin L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and thus u is not a weak supersolution in $\mathbb{R}^{n+1}$.
- The IPPS can be modified so that the obtained p-supercaloric function is not locally integrable to any positive power.

The stationary case

If $u(x,t) = u(x)$ is independent of t, the p-parabolic equation becomes the elliptic p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

in Ω. In this case

$$u(x) = |x|^{\frac{p-n}{p-1}}$$

is a solution in $\Omega \setminus \{0\}$. This gives an example of an unbounded p-supercaloric function for $1 < p < n$.

Dichotomy of unbounded supercaloric functions

Question: What kind of Sobolev space properties do unbounded p-supercaloric functions have?

Answer: There exist two mutually exclusive classes of p-supercaloric functions:

- **The Barenblatt class ("good")**, modeled by the Barenblatt solution.
- **The complementary class ("bad")**, modeled by the friendly giant ($p > 2$) or the infinite point source solution ($\frac{2n}{n+1} < p < 2$).

Moral: A p-supercaloric function behaves like the Barenblatt solution or blows up at least with the rate given by the friendly giant and the IPSS. There is no third alternative.

A rest stop

We shall discuss the slow diffusion case ($p > 2$) and the fast diffusion case ($\frac{2n}{n+1} < p < 2$) separately. Similar techniques apply in both cases, but the behavior of supercaloric functions is different in both cases as shown by the examples.

We begin with the the slow diffusion case.

Theorem (Kuusi, Lindqvist and Parviainen 2016)

Let $p > 2$. For a p-supercaloric function u in Ω_T, the following statements are equivalent:

1. $u \in L^{p-2}_{\mathrm{loc}}(\Omega_T)$.
2. $u \in L^{q}_{\mathrm{loc}}(\Omega_T)$ *for any* $q < p - 1 + \frac{p}{n}$.
3. $\operatorname*{ess\,sup}_{\delta<t<T-\delta} \int_{\Omega'} u(x,t)\,dx < \infty$ *for every* $\Omega' \Subset \Omega$ *and* $\delta \in (0, \frac{T}{2})$.

   Thus $u \in L^{\infty}_{\mathrm{loc}}(0,T; L^{1}_{\mathrm{loc}}(\Omega))$.

Note: The Barenblatt solution shows that the upper bound for the exponent are sharp.

Dichotomy in the slow diffusion case

Assume that u is a p-supercaloric function with $p > 2$. Then either

$$u \in L^q_{\mathrm{loc}}(\Omega_T) \quad \text{for every} \quad q < p - 1 + \frac{p}{n}$$

or

$$u \notin L^{p-2}_{\mathrm{loc}}(\Omega_T).$$

Thus the local integrability of a p-supercaloric function is either up to $q < p - 1 + \frac{p}{n}$ or worse than $p - 2$. There is a gap

$$\left[p - 2, p - 1 + \frac{p}{n}\right).$$

The lower bound for the gap is given by the friendly giant and the upper bound by the Barenblatt solution.

The gap of integrability in the slow diffusion case

Lemma

Let $p > 2$ and assume that u is a p-supercaloric function in Ω_T. If $u \in L^{p-2+s}_{\mathrm{loc}}(\Omega_T)$ for some $s > 0$, then $u \in L^{p-1+\frac{p}{n}-\sigma}_{\mathrm{loc}}(\Omega_T)$ for any $\sigma > 0$.

Remark: The endpoint $p - 2$ is obtained by the complementary class.

Proof

Preliminary reductions:

- Since u is locally bounded from below, by adding a constant, we may assume that $u \geq 1$.
- The truncations $u_k = \min\{u, k\}$, $k = 2, 3, \dots,$ are weak supersolutions.
- We may assume that $s > 0$ is as small as we want, since since the local integrability assumption holds for any $0 < s < q - p$. In particular, we may assume that $0 < s < 1$.
- We apply a Moser type iteration scheme, which is based on the Sobolev inequality and the Caccioppoli estimate.

Proof

Let $\varphi \in C_0^\infty(\Omega_T)$, $0 \le \varphi \le 1$ and $\varphi = 1$ in a compact subset of Ω_T. By the parabolic Sobolev inequality, we have

$$\iint_{\Omega_T} \varphi^{p^*} u_k^{p-2+s+\frac{sp}{n}}\,dx\,dt = \iint_{\Omega_T} \left|\varphi u_k^{\frac{p-2+s}{p}}\right|^{p^*} dx\,dt$$

$$\le c \iint_{\Omega_T} \left|\nabla\left(\varphi u_k^{\frac{p-2+s}{p}}\right)\right|^p dx\,dt \left(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega \varphi^m u_k^s\,dx\right)^{\frac{p}{n}},$$

with $m = \frac{ps}{p-2+s}$ and $p^* = p + \frac{pm}{n}$. We estimate the integrals on the right-hand side separately by applying the Caccioppoli inequality.

Proof

Since $u_k \geq 1$ and $p > 2$, we have $u_k^s \leq u_k^{p-2+s}$. For the first step of iteration, let $0 < \varepsilon < 1$ in the Caccioppoli estimate so that $1 - \varepsilon = s$. For first term on the right-hand side we have

$$
\begin{aligned}
&\iint_{\Omega_T} \left|\nabla\left(\varphi u_k^{\frac{p-2+s}{p}}\right)\right|^p dx\,dt \\
&\leq c\Big(\iint_{\Omega_T} u_k^{p-2+s}|\nabla\varphi|^p\,dx\,dt + \iint_{\Omega_T} u_k^{s-2}|\nabla u_k|^p\varphi^p\,dx\,dt\Big) \\
&\leq c\Big(\iint_{\Omega_T} u_k^{p-2+s}|\nabla\varphi|^p\,dx\,dt + \iint_{\Omega_T} u_k^{s}\,|\partial_t(\varphi^p)|\,dx\,dt\Big) \\
&\leq c\iint_{\Omega_T} u_k^{p-2+s}\big(|\nabla\varphi|^p + \left|\partial_t(\varphi^p)\right|\big)\,dx\,dt \\
&\leq c\iint_{\Omega_T} u^{p-2+s}\,dx\,dt < \infty
\end{aligned}
$$

for every $k = 2, 3, \dots$.

Proof

We apply the Caccioppoli inequality for second term on the right-hand side and obtain

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<T} \int_\Omega \varphi^m u_k^s \, dx \\
&\le c\Big(\iint_{\Omega_T} u_k^{p-2+s} |\nabla(\varphi^{\frac{m}{p}})|^p \, dx\, dt + \iint_{\Omega_T} u_k^s \,|\partial_t(\varphi^m)| \, dx\, dt \Big) \\
&\le c \iint_{\Omega_T} u_k^{p-2+s} \big(|\nabla(\varphi^{\frac{m}{p}})|^p + \big|\partial_t(\varphi^m)\big|\big) \, dx\, dt \\
&\le c \iint_{\Omega_T} u^{p-2+s} \, dx\, dt < \infty
\end{aligned}
$$

for every $k = 2, 3, \dots$.

Proof

By combining the estimates and applying the monotone convergence theorem we conclude

$$\begin{aligned}&\iint_{\Omega_T} \varphi^q u^{p-2+s(1+\frac{p}{n})}\,dx\,dt\\&=\lim_{k\to\infty}\iint_{\Omega_T} \varphi^q u_k^{p-2+s(1+\frac{p}{n})}\,dx\,dt<\infty.\end{aligned}$$

Since $p>2$, it follows that $p-2+s(1+\frac{p}{n})>s$. Thus

$$u\in L^{p-2+s(1+\frac{p}{n})}_{\mathrm{loc}}(\Omega_T).$$

Note: If we can obtain this estimate for any $s<1$, then we reach the upper bound $p-2+(1+\frac{p}{n})=p-1+\frac{p}{n}$.

Proof

We repeat this process with $s(1+\frac{p}{n})<1$ in place of s and we obtain a better exponent

$$p-2+s\Big(1+\frac{p}{n}\Big)\Big(1+\frac{p}{n}\Big)=p-2+s\Big(1+\frac{p}{n}\Big)^2.$$

After $j\in\mathbb{N}$ steps we reach the exponent $p-2+s(1+\frac{p}{n})^j$. The index $j\in\mathbb{N}$ is chosen so that

$$s\Big(1+\frac{p}{n}\Big)^j=1-\frac{\sigma}{1+\frac{p}{n}}.$$

Since the local integrability assumption holds for any $0<s<q-p$, we can always accomplish this.

Proof

In order to pass from $p-2+s\big(1+\frac{p}{n}\big)^j$ to $p-1+\frac{p}{n}-\sigma$ we apply a similar argument once more to obtain the exponent

$$\begin{aligned} &p-2+s\Big(1+\frac{p}{n}\Big)^j\Big(1+\frac{p}{n}\Big)\\ &=p-2+\Big(1-\frac{\sigma}{1+\frac{p}{n}}\Big)\Big(1+\frac{p}{n}\Big)\\ &=p-1+\frac{p}{n}-\sigma. \end{aligned}$$

This concludes the proof, but we remark that an explicit estimate can be worked out, which we omit, since only a finite number of iterations was needed.

Gradient estimate

Theorem

Let $p > 2$ and assume that u is a p-supercaloric function in Ω_T. If $u \in L^{p-2+s}_{\mathrm{loc}}(\Omega_T)$ for some $s > 0$, then $\nabla u \in L^q_{\mathrm{loc}}(\Omega_T)$ whenever $0 < q < p - 1 + \frac{1}{n+1}$. By the previous theorem this holds true with a weaker assumption $u \in L^{p-2}_{\mathrm{loc}}(\Omega_T)$.

Note: The Barenblatt solution shows that the bound for the exponent above are sharp. The exponent is strictly small than p, which means that the gradient does not belong to the natural Sobolev space. This is a serious problem in energy estimates, comparison principles and uniqueness results.

Proof

Since u is locally bounded from below, by adding a constant, we may assume that $u \geq 1$. Let $0 < t_1 < t_2 < T$, $\Omega' \Subset \Omega$ and $0 < \varepsilon < 1$. We consider the truncations $u_k = \min\{u, k\}$, $k = 1, 2, \ldots$. By Hölder's inequality, we have

$$
\begin{aligned}
\int_{t_1}^{t_2} \int_{\Omega'} |\nabla u_k|^q \, dx \, dt &= \int_{t_1}^{t_2} \int_{\Omega'} \left(u_k^{-\frac{1+\varepsilon}{p}} |\nabla u_k| \right)^q u_k^{q\frac{1+\varepsilon}{p}} \, dx \, dt \\
&\leq \left(\int_{t_1}^{t_2} \int_{\Omega'} u_k^{-1-\varepsilon} |\nabla u_k|^p \, dx \, dt \right)^{\frac{q}{p}} \left(\int_{t_1}^{t_2} \int_{\Omega'} u_k^{q\frac{1+\varepsilon}{p-q}} \, dx \, dt \right)^{1-\frac{q}{p}} \\
&= \left(\frac{p}{p-1-\varepsilon} \right)^q \left(\int_{t_1}^{t_2} \int_{\Omega'} \left| \nabla \left(u_k^{\frac{p-1-\varepsilon}{p}} \right) \right|^p dx \, dt \right)^{\frac{q}{p}} \\
&\quad \cdot \left(\int_{t_1}^{t_2} \int_{\Omega'} u_k^{q\frac{1+\varepsilon}{p-q}} \, dx \, dt \right)^{1-\frac{q}{p}}.
\end{aligned}
$$

Proof

The first integral on the right-hand side is uniformly bounded with respect to k as in the proof of the previous theorem. The second integral on the right-hand side is finite whenever

$$q\frac{1+\varepsilon}{p-q} < p-1+\frac{p}{n}$$

by the previous theorem. From this we can conclude that ∇u_k is uniformly bounded in $L^q_{\mathrm{loc}}(\Omega_T)$ for any $0 < q < p-1+\frac{1}{n+1}$.

A measure data problem

The gradient estimate can be applied in a measure data problem.

Theorem

Let$p > 2$. Assume that u is a p-supercaloric function in Ω_T and that u belongs the Barenblatt class. There exists a Radon measure μ on $\mathbb{R}^{n+1}$, such that u is a weak solution to the measure data problem

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu.$$

Moral: Every good supercaloric function is a solution to a measure data problem.

Proof

By the discussion above,

$$u \in L^1_{\mathrm{loc}}(\Omega_T) \quad \text{and} \quad \nabla u \in L^{p-1}_{\mathrm{loc}}(\Omega_T).$$

Since $u_k = \min\{u, k\}$, $k = 1, 2, \dots$, are weak supersolutions, we have

$$\begin{aligned}\iint_{\Omega_T} &\left(-u\partial_t\varphi + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right)\, dx\, dt \\ &= \lim_{k\to\infty} \iint_{\Omega_T} \left(-u_k\partial_t\varphi + |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla\varphi\right)\, dx\, dt \geq 0\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$. By the Riesz representation theorem there exists a nonnegative Radon measure μ on $\mathbb{R}^{n+1}$ such that

$$\iint_{\Omega_T} \left(-u\partial_t\varphi + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right)\, dx\, dt = \iint_{\Omega_T} \varphi\, d\mu$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

Existence results for the smeasure data problem

A converse result holds as well: For every finite Radon measure there exists a p-supercaloric function that satisfies the corresponding measure data problem.

- L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. 87 (1989), 149–169.
- J. Kinnunen, T. Lukkari and M. Parviainen, *An existence result for superparabolic functions*, J. Funct. Anal. 258 (2010), 713–728.

Uniqueness of the solution of the measure data problem with given boundary values is a challenging problem, since we cannot use the very weak solution as a test function.

Complementary class in the slow diffusion case

Theorem (Kuusi, Lindqvist and Parviainen 2016)

Let $p > 2$. For a p-supercaloric function u in Ω_T, the following statements are equivalent:

1. $u \notin L^{p-2}_{\mathrm{loc}}(\Omega_T)$.
2. *There exists* $\Omega' \Subset \Omega$ *and* $\delta \in (0, \frac{T}{2})$ *such that*

$$\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_{\Omega'} u(x,t)\,dx = \infty.$$

   Thus $u \notin L^{\infty}_{\mathrm{loc}}(0,T; L^1_{\mathrm{loc}}(\Omega))$.

3. *There exists a point* $(x_0, t_0) \in \Omega_T$, *such that*

$$\liminf_{\substack{(x,t)\to(x_0,t_0)\\ t>t_0}} u(x,t)(t-t_0)^{\frac{1}{p-2}} > 0.$$

Moral

Functions in the complementary class with $p > 2$ blow up at least with the rate given by the friendly giant. In this case there is no hope for Sobolev space estimates.

Main ingredient of the proof for the complementary class

For the complementary class, we apply a weak Harnack inequality for weak supersolutions to get a lower bound.

- T. Kuusi, *Harnack estimates for weak supersolutions to nonlinear degenerate parabolic equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (2008), no. 4, 673–716.

A weak Harnack inequality in the slow diffusion case

Lemma

Let $p > 2$. Assume that u is a nonnegative weak supersolution to the parabolic p-Laplace equation in Ω_T and let $B(x_0, 4r) \times (0, T) \subset \Omega_T$. Then there exist positive constants $c_1 = c_1(n, p)$ and $c_2 = c_2(n, p)$ such that for almost every $t \in (0, T)$, we have

$$\fint_{B(x_0,r)} u(x,t)\,dx \le \frac{1}{2}\left(\frac{c_1 r^p}{T-t}\right)^{\frac{1}{p-2}} + c_2 \operatorname*{ess\,inf}_{Q} u,$$

where $Q = B(x_0, 2r) \times (t + \frac{\theta}{2}, t + \theta)$ with

$$\theta = \min\left\{T - t, c_1 r^p \left(\fint_{B(x_0,r)} u(x,t)\,dx\right)^{2-p}\right\}.$$

A key ingredient in the argument

Let $B(x_0,4r)\subset\Omega$. Let $0<t_j<T$, $j=1,2,\dots$, with $t_j\to t_0$ as $j\to\infty$. If

$$\lim_{j\to\infty}\int_{B(x_0,r)} u(x,t_j)\,dx=\infty,$$

then the Harnack inequality (after some arrangements) gives

$$u(x,t)\geq c(n,p)\frac{r^{\frac{p}{p-2}}}{|t-t_0|^{\frac{1}{p-2}}}>0$$

for every $x\in B(x_0,2r)$ and $t_0<t<T$.

Barenblatt class in the fast diffusion case

Theorem (Giri, K. and Moring 2021)

Let $\frac{2n}{n+1} < p < 2$. For a p-supercaloric function u in Ω_T, the following statements are equivalent:

1. $u \in L_{\mathrm{loc}}^{\frac{n}{p}(2-p)}(\Omega_T)$.
2. $u \in L_{\mathrm{loc}}^{q}(\Omega_T)$ *for any* $q < p - 1 + \frac{p}{n}$.
3. $\displaystyle \operatorname*{ess\,sup}_{\delta<t<T-\delta} \int_{\Omega'} u(x,t)\,dx < \infty$ *for every* $\Omega' \Subset \Omega$ *and* $\delta \in (0, \frac{T}{2})$.

   Thus $u \in L_{\mathrm{loc}}^{\infty}(0,T; L_{\mathrm{loc}}^{1}(\Omega))$.

Note: Since $\frac{2n}{n+1} < p < 2$, we have $0 < \frac{n}{p}(2-p) < 1$ and $p - 1 + \frac{p}{n} > 1$.

Dichotomy in the fast diffusion case

Let $\frac{2n}{n+1} < p < 2$. Either

$$u \in L^q_{\mathrm{loc}}(\Omega_T) \quad \text{for every} \quad q < p - 1 + \frac{p}{n}$$

or

$$u \notin L^{\frac{n}{p}(2-p)}_{\mathrm{loc}}(\Omega_T).$$

Observe that $\frac{n}{p}(2-p) < 1 < p - 1 + \frac{p}{n}$. Thus the local integrability of a p-supercaloric function is either up to $q < p - 1 + \frac{p}{n}$ or worse than $\frac{n}{p}(2-p)$. There is a gap

$$\left[\frac{n}{p}(2-p), p - 1 + \frac{p}{n}\right).$$

The lower bound for the gap is given by the infinite point source solution and the upper bound by the Barenblatt solution. As $p \searrow \frac{2n}{n+1}$ we have $\frac{n}{p}(2-p) \nearrow 1$ and $p - 1 + \frac{p}{n} \searrow 1$. The gap shrinks to a point as p tends to the critical value $\frac{2n}{n+1}$.

Lemma

Let $\frac{2n}{n+1} < p < 2$ and assume that u is a p-supercaloric function in Ω_T. If $u \in L^s_{\mathrm{loc}}(\Omega_T)$ for some $s > \frac{n}{p}(2-p)$, then $u \in L^{p-1+\frac{p}{n}-\sigma}_{\mathrm{loc}}(\Omega_T)$ for any $\sigma > 0$.

Remark: The endpoint $\frac{n}{p}(2-p)$ is obtained by the complementary class.

Proof

Preliminary reductions:

- Since u is locally bounded from below, by adding a constant, we may assume that $u \geq 1$.
- The truncations $u_k = \min\{u, k\}$, $k = 2, 3, \ldots$, are weak supersolutions.
- We may assume $0 < s < 1$, otherwise we have $u \in L^1_{\mathrm{loc}}(\Omega_T)$ and we can directly proceed to the last step of the proof.
- The proof is similar to the case $p > 2$, but the leading term in the Caccioppoli inequality changes.

Proof

Let $\varphi \in C_0^\infty(\Omega_T)$, $0 \le \varphi \le 1$ and $\varphi = 1$ in a compact subset of Ω_T. By the parabolic Sobolev inequality, we have

$$\iint_{\Omega_T} \varphi^q u_k^{s-(2-p)+\frac{sp}{n}}\,dx\,dt = \iint_{\Omega_T} \left|\varphi u_k^{\frac{s-(2-p)}{p}}\right|^q dx\,dt$$

$$\le c \iint_{\Omega_T} \left|\nabla\left(\varphi u_k^{\frac{s-(2-p)}{p}}\right)\right|^p dx\,dt \left(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega \varphi^m u_k^s\,dx\right)^{\frac{p}{n}},$$

with $m = \frac{ps}{s-(2-p)}$ and $q = p + \frac{pm}{n}$. We estimate the integrals on the right-hand side separately by applying the Caccioppoli inequality

Proof

Since $u_k \geq 1$ and $\frac{2n}{n+1} < p < 2$, we have $u_k^{s-(2-p)} \leq u_k^s$. For the first step of iteration, let $0 < \varepsilon < 1$ in the Caccioppoli estimate so that $1 - \varepsilon = s$. For first term on the right-hand side we have

$$\begin{aligned}
&\iint_{\Omega_T} \left|\nabla\left(\varphi u_k^{\frac{s-(2-p)}{p}}\right)\right|^p dx\,dt \\
&\leq c\Big(\iint_{\Omega_T} u_k^{s-(2-p)}|\nabla\varphi|^p\,dx\,dt + \iint_{\Omega_T} u_k^{s-2}|\nabla u_k|^p\varphi^p\,dx\,dt\Big) \\
&\leq c\Big(\iint_{\Omega_T} u_k^{s-(2-p)}|\nabla\varphi|^p\,dx\,dt + \iint_{\Omega_T} u_k^s\,|\partial_t(\varphi^p)|\;dx\,dt\Big) \\
&\leq c\iint_{\Omega_T} u_k^s\big(|\nabla\varphi|^p + \big|\partial_t(\varphi^p)\big|\big)\,dx\,dt \\
&\leq c\iint_{\Omega_T} u^s\,dx\,dt < \infty,
\end{aligned}$$

for every $k = 2, 3, \dots$.

Proof

We apply the Caccioppoli inequality for second term on the right-hand side and obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<T} \int_\Omega \varphi^m u_k^s \, dx \\
&\le c\Big(\iint_{\Omega_T} u_k^{s-(2-p)} |\nabla(\varphi^{\frac{m}{p}})|^p \, dx\, dt + \iint_{\Omega_T} u_k^s \, |\partial_t(\varphi^m)| \, dx\, dt \Big) \\
&\le c \iint_{\Omega_T} u_k^s \big(|\nabla(\varphi^{\frac{m}{p}})|^p + \big|\partial_t(\varphi^m)\big| \big) \, dx\, dt \\
&\le c \iint_{\Omega_T} u^s \, dx\, dt < \infty
\end{aligned}$$

for every $k = 2, 3, \dots$.

Proof

By combining the estimates and applying the monotone convergence theorem we conclude

$$\begin{aligned}\iint_{\Omega_T} &\varphi^q u^{s-(2-p)+\frac{sp}{n}}\,dx\,dt\\ &= \lim_{k\to\infty}\iint_{\Omega_T}\varphi^q u_k^{s-(2-p)+\frac{sp}{n}}\,dx\,dt<\infty.\end{aligned}$$

Since $s>\frac{n}{p}(2-p)$, it follows that $s-(2-p)+\frac{sp}{n}>s$. Thus $u\in L^{s(1+\frac{p}{n})-(2-p)}_{\mathrm{loc}}(\Omega_T)$.

Note: If we can obtain this estimate for any $s<1$, then we reach the upper bound $(1+\frac{p}{n})-(2-p)=p-1+\frac{p}{n}$. This can be done as in the slow diffusion case.

Gradient estimate

Theorem

Let $\frac{2n}{n+1} < p < 2$ and let Ω be an open set in $\mathbb{R}^n$. Assume that u is a p-supercaloric function in Ω_T with $u \in L^s_{\mathrm{loc}}(\Omega_T)$ for some $s > \frac{n}{p}(2-p)$, Then $\nabla u \in L^q_{\mathrm{loc}}(\Omega_T)$ whenever $0 < q < p - 1 + \frac{1}{n+1}$.

The proof is the same as in the slow diffusion case.

Remark

- If $p > \frac{2n+1}{n+1}$, then $p - 1 + \frac{1}{n+1} > 1$ and it follows that ∇u is the Sobolev gradient.
- If $\frac{2n}{n+1} < p \le \frac{2n+1}{n+1}$, then $p - 1 + \frac{1}{n+1} \le 1$. In this case we consider the truncations $u_k = \min\{u, k\}$, $k = 1, 2, \dots$ and define the weak gradient by

$$\nabla u = \lim_{k\to\infty} \nabla u_k,$$

which is a measurable function but does not necessarily belong to $L^1_{\mathrm{loc}}(\Omega_T)$.

Complementary class in the fast diffusion case

Theorem (Giri, K. and Moring 2021)

Let $\frac{2n}{n+1} < p < 2$. For a p-supercaloric function u in Ω_T, the following statements are equivalent:

1. $u \notin L^{\frac{n}{p}(2-p)}_{\mathrm{loc}}(\Omega_T)$,
2. *There exists $\Omega' \Subset \Omega$ and $\delta \in (0, \frac{T}{2})$ such that*

$$\sup_{\delta<t<T-\delta} \int_{\Omega'} u(x,t)\,dx = \infty.$$

   Thus $u \notin L^\infty_{\mathrm{loc}}(0,T; L^1_{\mathrm{loc}}(\Omega))$.

3. *There exists a point $(x_0, t_0) \in \Omega_T$, such that*

$$\liminf_{x\to x_0} u(x,t)|x-x_0|^{\frac{p}{2-p}} > 0$$

   for every $t > t_0$.

Moral

Functions in the complementary class with $\frac{2n}{n+1} < p < 2$ blow up at least with the rate given by the infinite point source solution. In this case there is no hope for Sobolev space estimates.

Main ingredients of the proof for the complementary class

For the complementary class, we apply Harnack type inequalities.

- An L^1-Harnack inequality for weak solutions (DiBenedetto, Gianazza and Vespri, Proposition A.1.1).
- A weak Harnack inequality for weak supersolutions (Gianazza, Lukkari and Liao 2018).

L^1-Harnack inequality

Lemma

Let $1 < p < 2$ and let Ω be an open set in $\mathbb{R}^n$. Assume that h is a nonnegative continuous weak solution in Ω_T. There exists a constant $c = c(p, n)$ such that

$$
\begin{aligned}
&\sup_{s<\tau<t} \fint_{B(x_0,r)} h(x,\tau)\,dx \\
&\le c \inf_{s<\tau<t} \fint_{B(x_0,2r)} h(x,\tau)\,dx + c\left(\frac{t-s}{r^p}\right)^{\frac{1}{2-p}}
\end{aligned}
$$

whenever $B(x_0, 2r) \times [s, t] \subset \Omega_T$.

A weak Harnack inequality

Lemma

Let $\frac{2n}{n+1} < p < 2$. Assume that u is a nonnegative weak supersolution to the parabolic p-Laplace equation in Ω_T and let $B(x_0, 4r) \times (0, T) \subset \Omega_T$. Then there are positive constants $c_1 = c_1(n, p) \in (0, 1)$ and $c_2 = c_2(n, p) \in (0, 1)$ such that, for almost every $s \in (0, T)$, we have

$$\inf_{B(x_0,2r)} u(\cdot, t) \geq c_1 \fint_{B(x_0,2r)} u(x, s)\, dx$$

for almost every $t \in [s + \frac{3}{4}\theta r^p, s + \theta r^p]$ with

$$\theta = c_2 \Big(\fint_{B(x_0,2r)} u(x, s)\, dx \Big)^{2-p}$$

whenever $B(x_0, 16r) \times [s, s + \theta r^p] \subset \Omega_T$.

Comparison of the slow and fast diffusion cases

- In slow diffusion case the decisive integrability exponent is $p-2$ and in the fast diffusion case it is $\frac{n}{p}(2-p)$.
- Space and time change roles in the complementary classes.
- In the slow diffusion case infinities may occupy a whole time slice at some moment of time $t_0 \in (0, T)$, that is,

$$\lim_{\substack{(x,t)\to(x_0,t_0)\\ t>t_0}} u(x,t) = \infty \quad \text{for every} \quad x_0 \in \Omega.$$

- In the singular case this cannot happen. In this case, there exists $(x_0, t_0) \in \Omega_T$ such that

$$\lim_{\substack{x\to x_0\\ t>t_0}} u(x,t) = \infty \quad \text{for every} \quad t \in (t_0, T).$$

Past and future

It is essential that the limit in the definition of the infinity set is determined only by the future times $t > t_0$, while the past and present times $t \leq t_0$ are totally excluded.

This is in striking contrast to the pointwise value of the function, which can always be determined only by the past. An extension of Brelot's classical theorem for p-supercaloric functionswith $p > 2$ (K.-Lindqvist 2006) states that

$$u(x,t) = \operatorname*{ess\,lim\,inf}_{\substack{(y,s)\to(x,t)\\ s<t}} u(y,s)$$

Here the notion of the essential limes inferior means that any set of $(n+1)$-dimensional Lebesgue measure zero can be neglected in the calculation of the lower limit. This implies that if two p-supercaloric funcitions coincide almost everywhere, they coincide everywhere.

Takeaways

- Locally bounded p-supercaloric functions are weak supersolutions.
- An unbounded nonnegative p-supercaloric function has a Barenblatt type behavior or it blows up at least with the rate given by the friendly giant ($2 < p < \infty$) or the infinite point source solution ($\frac{2n}{n+1} < p < 2$).
- Functions in the Barenblatt class have a natural Sobolev space properties. There exists a measure data problem and the associated Riesz measure.
- Functions in the complementary class are lacking several properties, such as local integrability. Thus these functions are not easily tractable.

Open problems

- The corresponding theory in the critical ($p = \frac{2n}{n+1}$) and subcritical ($1 < p < \frac{2n}{n+1}$) cases?
- Is it possible to develop capacity theory for $1 < p < 2$?
  - $p \geq 2$: K., Korte, Kuusi and Parviainen 2013.
- Are polar sets for p-supercaloric functions sets of capacity zero for $1 < p < 2$?
  - $p \geq 2$: K., Korte, Kuusi and Parviainen 2013.
- Are sets of capacity zero removable for bounded p-supercaloric functions for $1 < p < 2$?
  - $p \geq 2$: Avelin and Saari 2017.
- The fast diffusion case is open for the PME as well (PME).
  - $m \geq 1$: K., Lehtelä, Lindqvist and Parviainen 2019.