Nonlinear partial differential equations
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1.1 Second order divergence type PDEs

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. We consider the Dirichlet boundary value problem

\[
\begin{cases}
Lu = f \quad \text{in} \quad \Omega, \\
u = g \quad \text{on} \quad \partial \Omega,
\end{cases}
\]

where \( u : \Omega \to \mathbb{R} \) is the unknown function. Here \( f, g : \overline{\Omega} \to \mathbb{R} \) are given functions and \( L \) denotes a second order (linear) partial differential operator of the form

\[
Lu(x) = - \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u(x)) + \sum_{i=1}^{n} b_i(x)D_i u(x) + c(x)u(x)
\]  

(1.1)

for given coefficient functions \( a_{ij}, b_i \) and \( c, i, j = 1, \ldots, n \). Here we denote the partial derivatives as

\[
D_i u(x) = \frac{\partial u}{\partial x_i}(x), \quad i = 1, \ldots, n.
\]

The operator can be written as

\[
Lu(x) = - \text{div}(A(x)Du(x)) + b(x) \cdot Du(x) + c(x)u(x)
\]

where

\[
A(x) = \begin{bmatrix}
a_{11}(x) & \cdots & a_{n1}(x) \\
a_{12}(x) & \cdots & a_{n2}(x) \\
\vdots & \ddots & \vdots \\
a_{1n}(x) & \cdots & a_{nn}(x)
\end{bmatrix}
\]
is an $n \times n$ matrix and $b(x) = (b_1(x), \ldots, b_n(x))$ is a column vector. The negative sign in front of the second order terms disappears after integration by parts and in the definition of weak solutions later. We say that (1.1) is of divergence form and we assume the symmetry condition

$$a_{ij}(x) = a_{ji}(x) \text{ for every } x \in \Omega, \ i, j = 1, \ldots, n. \quad (1.2)$$

Under this assumption the eigenvalues of the symmetric $n \times n$ matrix $A(x) = (a_{ij}(x))$ are real numbers.

**Remark 1.1.** In the constant coefficient case when every $a_{ij}$, $i, j = 1, \ldots, n$, is constant, we may always assume that $a_{ij} = a_{ji}$. To see this observe that $D_j D_i u = D_i D_j u$ and we may replace both $a_{ij}$ and $a_{ji}$ by $\frac{1}{2} (a_{ij} + a_{ji})$, which does not change the operator (exercise).

**Definition 1.2.** We say that the operator $L$ in (1.1) is uniformly elliptic, if there exists constants $0 < \lambda < \Lambda < \infty$ such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$.

**Theorem:** The uniform ellipticity condition gives uniform bounds for the speed of diffusion to each direction. In particular, the diffusion does not extinct or blow up.

**Remark 1.3.** The ellipticity means that for each point $x \in \Omega$ the symmetric matrix $A(x) = (a_{ij}(x))$ is strictly positive definite and the real eigenvalues $\lambda_i(x)$, $i = 1, \ldots, n$, of $A(x)$ satisfy

$$\lambda \leq \lambda_i(x) \leq \Lambda, \text{ for every } x \in \Omega, \ i = 1, \ldots, n.$$

**Example 1.4.** If $A(x) = I$, $b_1 = 0$, and $c = 0$, we have the Poisson equation

$$Lu(x) = -\text{div}(A(x) Du(x)) = -\text{div} Du(x) = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(x) = -\Delta u(x) = f(x).$$

**Remark 1.5.** It is rather standard in the PDE theory that the variables are not written down explicitly in functions unless there is a specific reason to do so. This makes expressions shorter and, hopefully, more readable.

### 1.2 Physical interpretation

Consider a fluid moving with velocity $b = (b_1, \ldots, b_n)$ in a domain in $\mathbb{R}^n$ and let $u = u(x, t)$ describe the concentration of a chemical in the fluid at point $x$ at
moment \(t\). Observe that the concentration changes in time. Assume that the total amount of chemical in any subdomain \(\Omega' \subset \Omega\) changes only because of inward or outward flux through the boundary \(\partial \Omega'\). This gives

\[
\frac{\partial}{\partial t} \int_{\Omega'} u \, dx = \int_{\partial \Omega'} a Du \cdot \nu \, dS - \int_{\partial \Omega'} u b \cdot \nu \, dS,
\]

(1.3)

where \(\nu = \nu(x) = (\nu_1(x), \ldots, \nu_n(x))\) is the outward pointing unit normal vector on \(\partial \Omega'\) and

\[Du(x) \cdot \nu(x) = \frac{\partial u}{\partial \nu}(x), \quad x \in \partial \Omega',\]

is the outward normal derivative of \(u\) and \(a > 0\) is the diffusion constant. The first integral on the right-hand side describes how much chemical comes in through the boundary by diffusion by assuming that the flux is proportional to the gradient, but in the opposite direction, that is, the flow is from higher concentration to lower. Note that \(Du(x) \cdot \nu(x) > 0\), \(x \in \partial \Omega\), if the concentration outside is greater than inside. The second integral on the right-hand side describes the amount of chemical that moves through the boundary by advection, that is, is transported by the flux. The negative sign is explained by the fact that \(\nu\) is an outward pointing unit normal.

By the Gauss-Green theorem

\[
\int_{\Omega'} D_i u(x) \, dx = \int_{\partial \Omega'} u(x)v_i(x) dS(x), \quad i = 1, \ldots, n,
\]

By differentiating under integral and using the Gauss-Green theorem in (1.3) we obtain

\[
\begin{align*}
\int_{\Omega'} u_t \, dx &= \int_{\partial \Omega'} a Du \cdot \nu \, dS - \int_{\partial \Omega'} u b \cdot \nu \, dS \\
&= \int_{\partial \Omega'} a \sum_{i=1}^{n} (D_i u) \nu_i \, dS - \int_{\partial \Omega'} \sum_{i=1}^{n} u b_i \nu_i \, dS \\
&= \sum_{i=1}^{n} \left( \int_{\partial \Omega'} a D_i u \nu_i \, dS - \int_{\partial \Omega'} u b_i \nu_i \, dS \right) \\
&= \sum_{i=1}^{n} \int_{\Omega'} D_i (a D_i u) \, dx - \int_{\Omega'} D_i (u b_i) \, dx \\
&= \int_{\Omega'} a \sum_{i=1}^{n} D_i (D_i u) \, dx - \int_{\Omega'} \sum_{i=1}^{n} D_i (u b_i) \, dx \\
&= \int_{\Omega'} a \Delta u \, dx - \int_{\Omega'} \text{div}(ub) \, dx \\
&= \int_{\Omega'} a \Delta u \, dx - \int_{\Omega'} \text{div}(ub) \, dx.
\end{align*}
\]

Since this holds for every \(\Omega' \subset \Omega\), we conclude that \(u\) satisfies the parabolic PDE

\[
u_t = \underbrace{a \Delta u}_{\text{diffusion}} + \underbrace{\text{div}(ub)}_{\text{advection}} = 0.
\]
The derivation of the PDE above was done in the case when \( a \) is constant, which means that the diffusion does not depend on the location of the point \( x \) in the domain \( \Omega \). If the diffusion is not uniform in the domain, that is, the coefficient \( a \) depends on the location \( x \in \Omega \), then \( a \) is a function of \( x \). If the diffusion is not isotropic in the sense that it is faster to some directions than others, then the constant diffusion matrix \( A(x) = aI \) can be replaced with a more general symmetric matrix \( A(x) = (a_{ij}(x)) \). This leads to

\[
\frac{\partial}{\partial t} \int_{\Omega} u \, dx = \sum_{i,j=1}^{n} \int_{\partial \Omega} (a_{ij}D_{j}u)v_{j} \, dS - \sum_{i=1}^{n} \int_{\partial \Omega} ub_{i}v_{i} \, dS, \quad i, j = 1, \ldots, n.
\]

and the PDE becomes

\[
u_{t} - \sum_{i,j=1}^{n} D_{j}(a_{ij}D_{i}u) + \sum_{i=1}^{n} D_{i}(b_{i}u) = 0.
\]

If the total amount of \( u \) is not conserved, then additional term \( cu \) for a creation or depletion of chemical, for example, in chemical reactions, and external source \( f \) appear. Then we have the nonhomogeneous PDE

\[
u_{t} - \sum_{i,j=1}^{n} D_{j}(a_{ij}D_{i}u) + \sum_{i=1}^{n} D_{i}(b_{i}u) - cu = f.
\]

Here \( a_{ij} = a_{ij}(x), \ b_{i} = b_{i}(x), \ c = c(x) \) and \( f = f(x) \) are functions of \( x \). This PDE can be used to model physical systems including chemical concentration, heat propagation and mass transport. If the system is in equilibrium in the sense that the solution does not depend on time, then \( u_{t} = 0 \) and we obtain the elliptic PDE

\[- \sum_{i,j=1}^{n} D_{j}(a_{ij}D_{i}u) + \sum_{i=1}^{n} D_{i}(b_{i}u) + cu = f,
\]

where \( a_{ij}, \ b_{i}, \ c \) and \( f \) are smooth enough functions for \( i, j = 1, \ldots, n \). Observe that if we apply the Leibniz rule to the advection term we obtain

\[- \sum_{i,j=1}^{n} D_{j}(a_{ij}D_{i}u) + \sum_{i=1}^{n} D_{i}(b_{i}u) + cu = f,
\]

and thus we have a PDE of type

\[
Lu = - \sum_{i,j=1}^{n} D_{j}(a_{ij}D_{i}u) + \sum_{i=1}^{n} b_{i}D_{i}u + cu = f,
\]

where \( L \) is a second order divergence type operator as in (1.1). A function \( u \in C^{2}(\Omega) \) is a classical solution of (1.4), if it satisfies the PDE at every point \( x \in \Omega \). In order to be able to show the existence of solutions for general coefficient functions \( a_{ij}, \ b_{i}, \ c \) and \( f, \ i, j = 1, \ldots, n \), we consider a weaker notion of solution.
T H E M O R A L: In order to understand the physical interpretation of a PDE it is better to consider an integrated version of a PDE instead of the pointwise version.

Remark 1.6. A nondivergence form operator

\[ Lu = - \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_{i} D_{i} u + c u \]

can be written as

\[ Lu = - \sum_{i,j=1}^{n} D_{j}(a_{ij} D_{i} u) + \sum_{i=1}^{n} \left( b_{i} + \sum_{j=1}^{n} D_{j} a_{ij} \right) D_{i} u + c u. \]

T H E M O R A L: A PDE in nondivergence form can be written in divergence form and vice versa. The main advantage of divergence form is in the arguments that are based on integration by parts.

1.3 Definition of a weak solution

It is useful to define the meaning of a PDE even if \( u \notin C^2(\Omega) \) and the coefficients \( a_{ij} \notin C^1(\Omega) \). There are two main motivations for a definition of a weak solution to a PDE.

1. Weak solutions are sometimes more accessible than classical solutions.
2. In some cases the classical solution does not exist at all. Thus weak solutions may be the only solutions to the problem.

The general strategy in existence theory for PDEs is to weaken to the notion of a solution so that a problem has a solution. Regularity theory studies whether the PDE is strong enough to give extra regularity to a weak solution. It is natural to begin with existence theory so that we know that the PDE has enough solutions.

Assumption: We consider \( L \) is as in (1.1) and make a standing assumption that \( \Omega \subset \mathbb{R}^n \) is a bounded open set,

\[ a_{ij}, b_{i}, c \in L^{\infty}(\Omega), \quad i, j = 1, \ldots, n \]

and

\[ f \in L^{2}(\Omega). \]

Moreover, we assume that symmetry condition in (1.2) and the ellipticity condition in Definition 1.2 hold true. These assumptions will not be repeated at every occasion. Sometimes we assume more smoothness on the coefficients or on the domain or set some of coefficients to zero, but these will be specified case by case.
Motivation: If $u \in C^2(\Omega)$, $a_{ij} \in C^1(\Omega)$ and $\varphi \in C^\infty_0(\Omega)$ then we can integrate by parts and $Lu = f$ gives

$$
\int_\Omega f\varphi \, dx = \int_\Omega \left( - \sum_{i,j=1}^n D_j(a_{ij}D_i u) + \sum_{i=1}^n b_i D_i u + cu \right) \varphi \, dx
$$

for every $\varphi \in C^\infty_0(\Omega)$. Observe that there are only first order derivatives of $u$ and no derivatives of the coefficients $a_{ij}$ in the integral above.

On the other hand, if

$$
\int_\Omega \left( - \sum_{i,j=1}^n D_j(a_{ij}D_i u) + \sum_{i=1}^n b_i D_i u + cu - f \right) \varphi \, dx = 0
$$

for every $\varphi \in C^\infty_0(\Omega)$ and consequently $Lu(x) = f(x)$ for every $x \in \Omega$.

**The Moral:** A function $u \in C^2(\Omega)$ is a classical solution of (1.4) if and only if it is a weak solution of (1.4) in the sense of the definition below. Observe that the negative sign in front of the second order terms disappears after integration by parts.

Next we define a weak solution to the Dirichlet problem

$$
\begin{cases}
Lu = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

so that the solution itself belongs to a Sobolev space and the boundary values are taken in the Sobolev sense.

**Definition 1.7.** A function $u \in W^{1,2}_0(\Omega)$ is a weak solution of

$$
\begin{cases}
Lu = f & \text{in } \Omega, \\
u \in W^{1,2}_0(\Omega),
\end{cases}
$$

where $L$ is as in (1.1), if

$$
\int_\Omega \left( \sum_{i,j=1}^n a_{ij}D_i u D_j \varphi + \sum_{i=1}^n b_i D_i u \varphi + cu \varphi \right) \, dx = \int_\Omega f \varphi \, dx
$$

for every $\varphi \in C^\infty_0(\Omega)$. 

THEMORAL: The definition of a weak solution is based on integration by parts. A classical solution satisfies the PDE pointwise, but a weak solution satisfies the PDE in integral sense. There are second order derivatives in the definition of a classical solution, but in the definition above is enough to assume that only first order weak derivatives exist. This is compatible with Sobolev spaces.

**Remarks 1.8:**

1. Observe that it is enough to assume that \( u \in W^{1,2}_{\text{loc}}(\Omega) \) in the definition of weak solution. This gives a local notion of solution without any boundary conditions, so that this definition applies to PDEs with Dirichlet, Neumann or other boundary conditions. This local definition is useful when we study regularity of solutions inside the domain. However, solutions are not unique without fixing the boundary values.

2. A solution \( u \in W^{1,2}_{\text{loc}}(\Omega) \) to the Dirichlet problem with nonzero boundary values \( g \in W^{1,2}_{0}(\Omega) \),

\[
\begin{cases}
    Lu = f & \text{in } \Omega, \\
    u - g \in W^{1,2}_{0}(\Omega),
\end{cases}
\]

can be obtained by considering \( w = u - g \in W^{1,2}_{0}(\Omega) \), which is a weak solution of the problem

\[
\begin{cases}
    Lw = \overline{f} & \text{in } \Omega, \\
    w \in W^{1,2}_{0}(\Omega),
\end{cases}
\]

with \( \overline{f} = f - Lg \). Both approaches lead to the same result (exercise).

**Example 1.9.** A function \( u \in W^{1,2}_{\text{loc}}(\Omega) \) is a weak solution to the Laplace equation \( \Delta u = 0 \) in \( \Omega \), if

\[
\int_{\Omega} Du \cdot D\varphi \, dx = \int_{\Omega} \sum_{i=1}^{n} D_i u D_i \varphi \, dx = 0 \quad \text{for every } \varphi \in C_0^{\infty}(\Omega). \tag{1.5}
\]

A function \( u \in W^{1,2}(\Omega) \) is a weak solution to \( \Delta u = 0 \) in \( \Omega \) with boundary values \( g \in W^{1,2}(\Omega) \), if \( u - g \in W^{1,2}_{0}(\Omega) \) and it satisfies (1.5).

**Example 1.10.** Let \( f \in L^2(\Omega) \). A function \( u \in W^{1,2}_{\text{loc}}(\Omega) \) is a weak solution to the Poisson equation \( -\Delta u = f \) in \( \Omega \), if

\[
\int_{\Omega} Du \cdot D\varphi \, dx = \int_{\Omega} f \varphi \, dx \quad \text{for every } \varphi \in C_0^{\infty}(\Omega). \tag{1.6}
\]

A function \( u \in W^{1,2}_{0}(\Omega) \) is a weak solution to \( -\Delta u = f \) in \( \Omega \) with zero boundary values, if it satisfies (1.6).

**Example 1.11.** We consider Serrin’s example of a nonsmooth weak solution, see [15]. Let \( n \geq 2 \) and \( 0 < \alpha < 1 \). The function \( u : B(0,1) \to \mathbb{R} \),

\[
u(x) = u(x_1, \ldots, x_n) = x_1 |x|^{-\alpha}
\]
is a weak solution to

$$-\sum_{i,j=1}^{n} D_j(a_{ij}D_iu) = 0 \quad \text{in} \quad B(0,1),$$

that is, \( u \in W^{1,2}(\Omega) \) and

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_iu D_j\varphi \, dx = 0$$

for every \( \varphi \in C_0^\infty(B(0,1)) \), where

$$a_{ij}(x) = \delta_{ij} + \frac{a(n-a)}{(1-a)(n-1-a)} \frac{x_ix_j}{|x|^2}, \quad i,j = 1,\ldots,n.$$ 

It is an exercise to show that the coefficients are bounded and that the uniform ellipticity condition in Definition 1.2 is satisfied with

$$\lambda = 1 \quad \text{and} \quad \Lambda = 1 + \frac{a(n-a)}{(1-a)(n-1-a)}.$$

Observe that \( \Lambda > 1 \) can be made arbitrarily close to one by choosing \( a > 0 \) small enough.

**Reason.** First we note that \( u \in W^{1,2}(\Omega) \) (exercise). Then we observe that \( u \) is a classical solution to

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x)) = 0 \quad \text{for every} \quad x \in B(0,1) \setminus \{0\}.$$

Thus

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_iu D_j\varphi \, dx = \int_{B(0,1) \setminus \{0\}} \sum_{i,j=1}^{n} a_{ij} D_iu D_j\varphi \, dx = 0$$

for every \( \varphi \in C_0^\infty(B(0,1) \setminus \{0\}) \).

Assume then that \( \varphi \in C_0^\infty(B(0,1)) \). Let \( 0 < r < \frac{1}{2} \) and let \( \eta \in C_0^\infty(B(0,2r)) \) be a cutoff function with

$$0 \leq \eta \leq 1, \quad \eta = 1 \quad \text{in} \quad B(0,r) \quad \text{and} \quad |D\eta| \leq \frac{2}{r}.$$

Then \((1-\eta)\varphi \in C_0^\infty(B(0,1) \setminus \{0\})\) and thus

$$0 = \int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_iu D_j((1-\eta)\varphi) \, dx$$

$$= \int_{B(0,1)} \sum_{i,j=1}^{n} (1-\eta)a_{ij} D_iu D_j\varphi \, dx - \int_{B(0,1)} \sum_{i,j=1}^{n} \eta a_{ij} D_iu D_j\varphi \, dx.$$
We observe that
\[
\left| \int_{B(0,1)} \sum_{i,j=1}^{n} \varphi a_{ij} D_i u D_j \eta \, dx \right| \leq \sum_{i,j=1}^{n} \int_{B(0,1)} |\varphi||a_{ij}| |D_i u||D_j \eta| \, dx
\]
\[
\leq \|\varphi\|_{L^\infty(B(0,1))} \max_{i,j} \|a_{ij}\|_{L^\infty(B(0,1))} \sum_{i,j=1}^{n} \int_{B(0,1)} |D_i u||D_j \eta| \, dx
\]
\[
\leq \|\varphi\|_{L^\infty(B(0,1))} \max_{i,j} \|a_{ij}\|_{L^\infty(B(0,1))} \frac{2}{r} \sum_{i,j=1}^{n} \int_{B(0,2r)} |D_i u| \, dx
\]
\[
\leq \|\varphi\|_{L^\infty(B(0,1))} \max_{i,j} \|a_{ij}\|_{L^\infty(B(0,1))} \frac{2}{r} \sum_{i,j=1}^{n} \left( \int_{B(0,2r)} |D_i u|^2 \, dx \right)^{\frac{1}{2}} |B(0,2r)| \frac{1}{2}
\]
\[
\leq cr^{-2/2} \left( \int_{B(0,2r)} |Du|^2 \, dx \right)^{\frac{1}{2}} \to 0 \text{ as } r \to 0.
\]

Thus
\[
0 = \lim_{r \to 0} \left( \int_{B(0,1)} \sum_{i,j=1}^{n} (1-\eta) a_{ij} D_i u D_j \varphi \, dx - \int_{B(0,1)} \sum_{i,j=1}^{n} \varphi a_{ij} D_i u D_j \eta \, dx \right)
\]
\[
= \lim_{r \to 0} \int_{B(0,1)} \sum_{i,j=1}^{n} (1-\eta) a_{ij} D_i u D_j \varphi \, dx
\]
\[
= \int_{B(0,1)} \sum_{i,j=1}^{n} \lim_{r \to 0} (1-\eta) a_{ij} D_i u D_j \varphi \, dx
\]
\[
= \int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \varphi \, dx
\]
for every \( \varphi \in C^\infty_0(B(0,1)) \). Here we used the fact that
\[
\lim_{r \to 0} (1-\eta(x)) = 1 \text{ for every } x \in B(0,1) \setminus \{0\}
\]
and the dominated convergence theorem with the integrable majorant
\[
|(1-\eta) a_{ij} D_i u D_j \varphi| \leq |a_{ij} D_i u D_j \varphi|
\]
\[
\leq \|D\varphi\|_{L^\infty(B(0,1))} \max_{i,j} \|a_{ij}\|_{L^\infty(B(0,1))}|Du| \in L^1(B(0,1)).
\]

The Moral: The weak solution above is Hölder continuous with exponent \( 1-\alpha \), that is, \( u \in C^{0,1-\alpha}(\Omega) \). This shows that a weak solution to a uniformly elliptic PDE does not need to be smoother than locally Hölder continuous.

Example 1.12. Let \( n = 1, \Omega = (0,2), b = 0 = c, a = 1 \) and
\[
f(x) = \begin{cases} 
1, & x \in (0,1), \\
2, & x \in (1,2).
\end{cases}
\]
Consider the problem
\[
\begin{aligned}
Lu(x) &= f(x), \quad x \in \Omega, \\
u(0) = u(0) = 0.
\end{aligned}
\]
with \( Lu(x) = -(au(x))' = -u''(x) \). By solving
\[
Lu(x) = -u''(x) = \begin{cases} 
1, & x \in (0, 1], \\
2, & x \in (1, 2),
\end{cases}
\]
in the subintervals \((0, 1]\) and \((1, 2)\) respectively, and requiring that the solution \( u \) belongs to \( C^1(\Omega) \), we obtain
\[
u(x) = \begin{cases} 
-\frac{1}{2}x^2 + \frac{5}{4}x, & x \in (0, 1], \\
-x^2 + \frac{9}{4}x - \frac{1}{2}, & x \in (1, 2).
\end{cases}
\]
We observe that \( u \in C^1(\Omega) \), but \( u \notin C^2(\Omega) \). In particular, \( u \) is not a classical solution to the problem above.

**Claim:** \( u \) is a weak solution.

**Reason.** Since \( u \in C^1(\Omega) \) and \( u, u' \in L^\infty(\Omega) \), we have \( u \in W^{1,2}(\Omega) \). Let \( 0 < \varepsilon < 1 \) and \( \varphi \in C_0^\infty(\Omega) \). Since \( u \) is a classical solution to
\[
Lu(x) = -u''(x) = f(x)
\]
when \( x \in (0, 1-\varepsilon) \cup (1+\varepsilon, 2) \), using integration by parts, we have
\[
\int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} f(x)\varphi(x)dx = -\int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} u''(x)\varphi(x)dx = -\int_{(0,-1-\varepsilon)\cup(1+\varepsilon,2)} u'(x)\varphi'(x)dx
\]
\[
= \int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} u'(x)\varphi'(x)dx - (u'(1-\varepsilon)\varphi(1-\varepsilon) - 0 - u'(1+\varepsilon)\varphi(1+\varepsilon)).
\]

By the Lebesgue dominated convergence theorem
\[
\lim_{\varepsilon \to 0} \int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} f(x)\varphi(x)dx = \int_{(0,2)} f(x)\varphi(x)dx
\]
and
\[
\lim_{\varepsilon \to 0} \int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} u'(x)\varphi'(x)dx = \int_{(0,2)} u'(x)\varphi'(x)dx.
\]
Moreover, since \( u \in C^1(\Omega) \), we have
\[
\int_{(0,2)} u'(1-\varepsilon)\varphi(1-\varepsilon) - u'(1+\varepsilon)\varphi(1+\varepsilon) \to 0
\]
as \( \varepsilon \to 0 \). Thus
\[
\int_{(0,2)} u'(x)\varphi'(x)dx = \int_{(0,2)} f(x)\varphi(x)dx \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega).
\]
\[\blacksquare\]
THEM: Even if the coefficients are smooth and the operator is uniformly elliptic, the weak solution does not necessarily belong to $C^2(\Omega)$. In particular, the problem does not necessarily have a classical solution.

Example 1.13. Let $n = 1$, $\Omega = (0, 2)$, $f = 1$, $b = 0 = c$,

$$a(x) = \begin{cases} 1, & x \in (0, 1), \\ 2, & x \in (1, 2). \end{cases}$$

Consider the problem

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(0) = 0 = u(2), \end{cases}$$

where $Lu(x) = -(a(x)u'(x))'$. By solving the equation in the subintervals $(0, 1)$ and $(1, 2)$ respectively, as well as requiring suitable conditions at $x = 1$, we obtain

$$u(x) = \begin{cases} -\frac{1}{2}x^2 + \frac{5}{2}x, & x \in (0, 1), \\ -\frac{1}{4}x^2 + \frac{5}{12}x + \frac{1}{6}, & x \in (1, 2). \end{cases}$$

We observe that $u \notin C^1(\Omega)$. However, $u$ is a weak solution to the above problem (exercise).

THEM: If the coefficients are not smooth, the weak solution does not necessarily belong to $C^1(\Omega)$. In particular, the problem does not have a classical solution and the weak solution does not even have the first order derivatives in the classical sense.
In this chapter we discuss two methods to show that a weak solution to a PDE exists under very general conditions. The first method is a Hilbert space approach which applies to linear PDEs only. Then we consider direct methods in the calculus of variations, which is a Banach space approach and applies to nonlinear PDEs as well.

2.1 Hilbert space approach to existence

Assume that $b_i = 0$ for $i = 1, \ldots, n$. The Riesz representation theorem can be used to prove the existence of a weak solution to the Dirichlet problem

$$
\begin{cases}
- \sum_{i,j=1}^{n} a_{ij} D_i (u D_j u) + cu = f \\
u \in W^{1,2}_0(\Omega)
\end{cases}
$$

(2.1)

in any bounded open subset $\Omega$ of $\mathbb{R}^n$. More general boundary values can be considered as in Remark 1.8 (2). To this end, we define a candidate for an inner product in $W^{1,2}_0(\Omega)$ as

$$
\langle u, v \rangle = \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j v + cu v \right) dx.
$$

(2.2)

Recall that the standard inner product in $W^{1,2}_0(\Omega)$ is obtained by choosing $a_{ij} = 1$, $i, j = 1, \ldots, n$, and $c = 1$. 
Remark 2.1. By Hölder’s inequality, we have
\[
\int_{\Omega} \left| \sum_{i,j=1}^{n} a_{ij} D_i u D_j v + cuv \right| dx \\
\leq \sum_{i,j=1}^{n} \left| a_{ij} D_i u D_j v \right| dx + \int_{\Omega} |cuv| dx \\
\leq \sum_{i,j=1}^{n} \| a_{ij} \|_{L^\infty(\Omega)} \| D_i u \|_{L^2(\Omega)} \| D_j v \|_{L^2(\Omega)} + \| c \|_{L^\infty(\Omega)} \| u \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} \\
= \left( \sum_{i,j=1}^{n} \| a_{ij} \|_{L^\infty(\Omega)} + \| c \|_{L^\infty(\Omega)} \right) \| u \|_{W^{1,2}(\Omega)} \| v \|_{W^{1,2}(\Omega)} < \infty.
\]
This shows that the integrand in (2.2) is an integrable function with finite integral.

Thus \( \langle u, v \rangle \) in (2.2) is a finite number whenever \( u, v \in W^{1,2}_0(\Omega) \). Next we show that (2.2) really is an inner product under a certain condition on function \( c \).

**Lemma 2.2.** There exists a constant \( c_0 = c_0(\lambda, n) \leq 0 \) such that if \( c \geq c_0 \), then (2.2) defines an inner product in \( W^{1,2}_0(\Omega) \).

**THEOREM:** It is important to have \( c_0 \leq 0 \) so that the case \( c = 0 \) is included in the theory.

**Proof.** We show that \( \langle u, u \rangle = 0 \) implies \( u = 0 \) when \( c \geq c_0 \). To prove this, we recall the Poincaré inequality
\[
\int_{\Omega} |u|^2 dx \leq \mu \int_{\Omega} |Du|^2 dx, \quad \mu = c(\text{diam}\, \Omega)^2,
\]
which holds true for every \( u \in W^{1,2}_0(\Omega) \). By the ellipticity condition, see Definition 1.2, we have
\[
\langle u, u \rangle = \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j u + c |u|^2 \right) dx \\
\geq \lambda \int_{\Omega} |Du|^2 dx + c_0 \int_{\Omega} |u|^2 dx \\
= \frac{\lambda}{2} \int_{\Omega} |Du|^2 dx + \frac{\lambda}{2} \int_{\Omega} |Du|^2 dx + c_0 \int_{\Omega} |u|^2 dx \\
\geq \frac{\lambda}{2} \int_{\Omega} |Du|^2 dx + \left( \frac{\lambda}{2} + c_0 \right) \int_{\Omega} |u|^2 dx \\
\geq \alpha \| u \|_{W^{1,p}_0(\Omega)}^2,
\]
where
\[
\alpha = \min \left\{ \frac{\lambda}{2}, \frac{\lambda}{2} + c_0 \right\}.
\]
In particular, this shows that \( \langle u, u \rangle \geq 0 \). If \( c \geq c_0 > -\frac{1}{2\Lambda} \), then \( \alpha > 0 \) and it follows that \( \langle u, u \rangle = 0 \) implies \( \|u\|_{W^{1,p}_0(\Omega)} = 0 \) and thus \( u = 0 \). The other properties of an inner product are clear (exercise). \( \square \)

**Remark 2.3.** For the norm induced by the inner product (2.2) we have

\[
\|u\|^2 = \langle u, u \rangle = \int_{\Omega} \left( \sum_{i,j=1}^n a_{ij} D_i u D_j u + c |u|^2 \right) dx 
\leq \Lambda \int_{\Omega} |Du|^2 dx + \|c\|_{\infty} \int_{\Omega} |u|^2 dx \leq \beta \|u\|_{W^{1,2}_0(\Omega)}^2,
\]

with \( \beta = \max\{\Lambda, \|c\|_{\infty}\} \). Thus

\[
\sqrt{\alpha} \|u\|_{W^{1,2}_0(\Omega)} \leq \|u\| \leq \sqrt{\beta} \|u\|_{W^{1,2}_0(\Omega)},
\]

for every \( u \in W^{1,p}_0(\Omega) \), where \( \alpha \) is as in (2.3). This shows that \( \cdot \|_{W^{1,2}_0(\Omega)} \) and \( \cdot \| \) are equivalent norms in \( W^{1,2}_0(\Omega) \) if \( c \geq c_0 \).

**Lemma 2.4.** Let \( \widetilde{W}^{1,2}_0(\Omega) \) be \( W^{1,2}_0(\Omega) \) with the inner product given by (2.2). Then

\[
F(v) = \int_{\Omega} f v dx
\]

is a bounded linear functional on \( \widetilde{W}^{1,2}_0(\Omega) \).

**Remark 2.5.** Note that \( F(v) = \langle f, v \rangle_{L^2(\Omega)} \), where \( \langle \cdot, \cdot \rangle_{L^2(\Omega)} \) is the standard inner product in \( L^2(\Omega) \).

**Proof.** Hölder’s inequality and the proof of Lemma 2.2 imply

\[
|F(v)| = \left| \int_{\Omega} f v dx \right| \leq \left( \int_{\Omega} |f|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |v|^2 dx \right)^{\frac{1}{2}} \leq \|f\|_{L^2(\Omega)} \|v\|_{W^{1,2}_0(\Omega)} \leq \frac{1}{\sqrt{\alpha}} \|f\|_{L^2(\Omega)} \|v\|,
\]

where \( \alpha \) is given by (2.3). \( \square \)

**Theorem 2.6.** Assume that \( \Omega \) is a bounded and open subset of \( \mathbb{R}^n \) and \( f \in L^2(\Omega) \). There exists \( c_0 < 0 \) such that (2.1) has a unique weak solution for every \( c \geq c_0 \).

**The Moral:** There exists a unique solution to the Dirichlet problem with zero boundary values in the Sobolev sense in any bounded set.

**Proof.** By Definition 1.7, a function \( u \in W^{1,2}_0(\Omega) \) is a weak solution to (2.1) if

\[
\langle u, v \rangle = \int_{\Omega} \left( \sum_{i,j=1}^n a_{ij} D_i u D_j v + c u v \right) dx = \int_{\Omega} f v dx
\]
for every \( v \in C_0^\infty(\Omega) \). Here we used the inner product defined by (2.2). By Lemma 2.4
\[
F(v) = \int_\Omega f v \, dx = \langle f, v \rangle_{L^2(\Omega)}
\]
is a bounded linear functional on \( \hat{W}_0^{1,2}(\Omega) \). Note that \( \hat{W}_0^{1,2}(\Omega) \) is a Banach space, because \( \| \cdot \|_{W_0^{1,2}(\Omega)} \) and \( \| \cdot \| \) are equivalent norms in \( W_0^{1,2}(\Omega) \), see Remark 2.3. Thus \( \hat{W}_0^{1,2}(\Omega) \) is a Hilbert space when \( c \geq c_0 \) given by Lemma 2.2. By the Riesz representation theorem, there exists a unique \( u \in \hat{W}_0^{1,2}(\Omega) \) such that
\[
F(v) = \langle u, v \rangle = \int_\Omega \left( \sum_{i,j=1}^n a_{ij} D_i u D_j v + cuv \right) \, dx
\]
for every \( v \in \hat{W}_0^{1,2}(\Omega) \). By Remark 2.3, we have \( \hat{W}_0^{1,2}(\Omega) \subset W_0^{1,2}(\Omega) \). Thus \( u \in W_0^{1,2}(\Omega) \). By Remark 2.3 again, we have \( C_0^\infty(\Omega) \subset W_0^{1,2}(\Omega) \subset \hat{W}_0^{1,2}(\Omega) \) and thus
\[
\int_\Omega \left( \sum_{i,j=1}^n a_{ij} D_i u D_j v + cuv \right) \, dx = \int_\Omega f v \, dx
\]
for every \( v \in C_0^\infty(\Omega) \).

Example 2.7. Let \( \Omega \subset \mathbb{R}^n \) be any bounded open set and \( f \in L^2(\Omega) \). By Theorem 2.6 there exists a unique weak solution to the Dirichlet problem
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u \in W_0^{1,2}(\Omega),
\end{cases}
\]
that is, \( u \in W_0^{1,2}(\Omega) \) and
\[
\int_\Omega Du \cdot D\varphi \, dx = \int_\Omega f \varphi \, dx
\]
for every \( \varphi \in C_0^\infty(\Omega) \).

Example 2.8. Let \( \Omega \subset \mathbb{R}^n \) be any bounded open set, \( Lu = -\Delta u, f \in L^2(\Omega) \) and \( g \in W^{1,2}(\Omega) \). By Remark 1.8 (2) a weak solution to the Dirichlet problem
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u - g \in W_0^{1,2}(\Omega),
\end{cases}
\]
(2.4)
can be obtained by considering \( w = u - g \in W_0^{1,2}(\Omega) \) and the problem
\[
\begin{cases}
-\Delta w = \overline{f} & \text{in } \Omega, \\
w \in W_0^{1,2}(\Omega),
\end{cases}
\]
(2.5)
with \( \overline{f} = f - Lg = f + \Delta g \). By Theorem 2.6 there exists a unique weak solution \( w \in W_0^{1,2}(\Omega) \) to (2.5), that is,
\[
\int_\Omega Dw \cdot D\varphi \, dx = \int_\Omega f \varphi \, dx - \int_\Omega Dg \cdot D\varphi \, dx
\]
or equivalently
\[
\int_{\Omega} (Dw + Dg) \cdot D\phi \, dx = \int_{\Omega} f \phi \, dx
\]
for every \( \phi \in C_0^\infty(\Omega) \). This means that \( u = w + g \) is the unique solution of (2.4). Thus \( u \in W^{1,2}(\Omega) \) is a weak solution of (2.4) if and only if \(-\Delta u = f\) in weak sense and \( u - g \in W^{1,2}_0(\Omega) \).

**Example 2.9.** Let \( \Omega \subset \mathbb{R}^n \) be any bounded open set, \( f \in L^2(\Omega) \) and \( c \geq c_0 \). By Theorem 2.6 there exists a unique weak solution to the problem
\[
\begin{cases}
-\Delta u + cu = f & \text{in } \Omega, \\
u \in W^{1,2}_0(\Omega),
\end{cases}
\]
that is, \( u \in W^{1,2}_0(\Omega) \) and
\[
\int_{\Omega} Du \cdot D\phi \, dx + \int_{\Omega} cu \phi \, dx = \int_{\Omega} f \phi \, dx
\]
for every \( \phi \in C_0^\infty(\Omega) \).

**Example 2.10.** Let \( n = 1, \Omega = (0,2), c = 0 = b, f = 1 \) and
\[
a(x) = \begin{cases}
x, & x \in (0,1], \\
1, & x \in (1,2).
\end{cases}
\]
Consider the problem
\[
\begin{cases}
Lu(x) = f(x), & x \in \Omega, \\
u(0) = 0 = u(2).
\end{cases}
\]
Observe that \( L \) is not uniformly elliptic.

By solving
\[
Lu(x) = -(a(x)u'(x))' = f(x) = 1
\]
in \((0,1)\) and \((1,2)\) respectively, we obtain
\[
u(x) = \begin{cases}
-x + c_1 \ln x + c_2, & x \in (0,1], \\
-\frac{1}{2} x^2 + c_3 x + c_4, & x \in (1,2).
\end{cases}
\]
By the boundary conditions and requiring continuity at \( x = 1 \), we obtain
\[
u(x) = \begin{cases}
-x, & x \in (0,1], \\
-\frac{1}{2} x^2 + \frac{5}{2} x - 3, & x \in (1,2).
\end{cases}
\]
However, this is not a weak solution of the problem (exercise).
Example 2.11. Let $\Omega = (0, 2)$ and

$$f(x) = a(x) = \begin{cases} 
1, & x \in (0, 1), \\
0, & x \in [1, 2).
\end{cases}$$

Consider the problem

$$\begin{cases}
Lu(x) = f(x), & x \in \Omega, \\
u(0) = u(2) = 0.
\end{cases}$$

Observe that $L$ is not uniformly elliptic. Then

$$u_1(x) = \begin{cases}
-\frac{1}{2}x^2 + x, & x \in (0, 1), \\
-x^2 + \frac{5}{2}x - 1, & x \in [1, 2),
\end{cases}$$

and

$$u_2(x) = \begin{cases}
-\frac{1}{2}x^2 + x, & x \in (0, 1), \\
1 - \frac{1}{2}x, & x \in [1, 2),
\end{cases}$$

are weak solutions to $Lu = f$ (exercise).

The Moral: If the operator is not uniformly elliptic, a weak solution of a boundary value problem is not necessarily unique.

Remark 2.12. For a general operator $L$ defined by (1.1), there is a bilinear form

$$B[u, v] = \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij}D_iuD_jv + \sum_{i=1}^{n} b_iD_iuv + cuv \right) \, dx,$$

where $u, v \in W_{0}^{1,2}(\Omega)$. If the functions $b_i$, $i = 1, \ldots, n$ are not all equal to zero, then the bilinear form is not symmetric, that is, $B[u, v] \neq B[v, u]$ and the Riesz representation theorem cannot be applied as such, since $B[\cdot, \cdot]$ is not an inner product. In this case we may apply the Lax-Milgram theorem, which is a slightly extended version of the Riesz representation theorem and the Fredholm alternative, but these will not be discussed here.

Example 2.13. Let $n = 1$, $\Omega = (0, \pi)$, $a = 1$, $b = 0$, $c = -4$. The operator $L$ is uniformly elliptic, but the corresponding bilinear form

$$B[u, v] = \int_{0}^{\pi} (u'(x)v'(x) - 4u(x)v(x)) \, dx$$

is not positive definite on $W_{0}^{1,2}(\Omega)$. For example, if $u(x) = \sin x$, then

$$B[u, u] = \int_{0}^{\pi} ((\cos x)^2 - 4(\sin x)^2) \, dx = -\frac{3\pi}{2}.$$
Claim: Let \( f(x) = \sin(2x) \). Then the problem
\[
\begin{cases}
  Lu(x) = f(x), & x \in \Omega, \\
  u(0) = 0 = u(\pi),
\end{cases}
\]
does not have any solutions.

Reason. Let \( u \in W^{1,2}_0(\Omega) \). An integration by parts gives
\[
B[u, v] = \int_0^\pi \left( u'(x)v'(x) - 4u(x)v(x) \right) dx
= \int_0^\pi \left( 2u'(x)\cos(2x) - 4u(x)\sin(2x) \right) dx
= \int_0^\pi \left( 2u(x)\cos(2x) \right)' dx
= 0 \neq \int_0^\pi \left( \sin(2x) \right)^2 dx
= \int_0^\pi f(x)v(x) dx,
\]
when \( v(x) = \sin(2x) \in W^{1,2}_0(\Omega) \). Thus there does not exist a function \( u \in W^{1,2}_0(\Omega) \) for which
\[
B[u, v] = \int_\Omega f(x)v(x) dx \quad \text{for every} \quad v \in W^{1,2}_0(\Omega).
\]

Observe that the corresponding homogeneous problem
\[
\begin{cases}
  Lu(x) = 0, & x \in \Omega, \\
  u(0) = 0 = u(\pi),
\end{cases}
\]
has infinitely many solutions \( u(x) = a \sin(2x), a \in \mathbb{R} \).

### 2.2 Variational approach to existence

Existence of solutions can also be studied by direct methods in the calculus of variations. The variational integral related to the PDE
\[
- \sum_{i,j=1}^n D_j(a_{ij} D_i u) + cu = f 
\]
(2.6)
is
\[
I(v) = \frac{1}{2} \int_\Omega \left( \sum_{i,j=1}^n a_{ij} D_i v D_j v + cv^2 \right) dx - \int_\Omega f v dx
\]
(2.7)
where \( A = A(x) = (a_{ij}(x)) \) is an \( n \times n \) matrix. The PDE (2.6) is called the Euler-Lagrange equation of the variational integral (2.7).
Remark 2.14. By Hölder’s inequality, we have
\[
|I(v)| = \frac{1}{2} \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i v D_j v + cv^2 \right) dx - \int_{\Omega} f v dx
\]
\[
\leq \frac{1}{2} \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)} \int_{\Omega} |Dv|^2 dx + \frac{1}{2} \int_{\Omega} |v|^2 dx + \int_{\Omega} f v dx
\]
\[
\leq \frac{1}{2} \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)} \|Dv\|_{L^2(\Omega)}^2 + \frac{1}{2} \|c\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}
\]
\[
\leq \frac{1}{2} \left( \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \right) \|v\|_{W^{1,2}(\Omega)}^2 + \|v\|_{W^{1,2}(\Omega)} \|f\|_{L^2(\Omega)} < \infty.
\]
This shows that the integrand in (2.7) is an integrable function with finite integral for every \( v \in W^{1,2}(\Omega) \).

Example 2.15. The variational integral corresponding to the Poisson equation
\[-\Delta u = f \]
is
\[
I(v) = \frac{1}{2} \int_{\Omega} |Dv|^2 dx - \int_{\Omega} f v dx.
\]

Example 2.16. The variational integral related to Example 1.11 is
\[
I(v) = \int_{B(0,1)} |Dv(x)|^2 + a \left( \frac{x}{|x|} \cdot Dv(x) \right)^2 dx,
\]
with \( a = \frac{a(a-a)}{(1-a)(a-1-a)} > 0 \). Observe that the integrand
\[
F(x, \xi) = \left( |\xi|^2 + a \left( \frac{x}{|x|} \cdot \xi \right)^2 \right)
\]
satisfies
\[
|\xi|^2 \leq F(x, \xi) \leq (a + 1)|\xi|^2,
\]
where \( a > 0 \) can be made arbitrarily small by choosing \( a > 0 \) small enough.

Definition 2.17. A function \( u \in W^{1,2}_0(\Omega) \) is a minimizer of (2.7) with zero boundary values, if \( I(u) \leq I(v) \) for every \( v \in W^{1,2}_0(\Omega) \).

Theorem 2.2: A minimizer \( u \) minimizes the variational integral \( I(u) \) in the class of functions with zero boundary values, that is,
\[
I(u) = \inf \left\{ I(v) : v \in W^{1,2}_0(\Omega) \right\}.
\]
If there is a minimizer, then infimum can be replaced by minimum.

Remark 2.18. For nonzero boundary values \( g \in W^{1,2}(\Omega) \), we may consider
\[
I(u) = \inf \left\{ I(v) : v \in W^{1,2}(\Omega), v - g \in W^{1,2}_0(\Omega) \right\}.
\]
Thus a function \( u \in W^{1,2}(\Omega) \) is a minimizer of (2.7) with boundary values \( g \in W^{1,2}(\Omega) \), if \( I(u) \leq I(v) \) for every \( v \in W^{1,2}(\Omega) \) with \( v - g \in W^{1,2}_0(\Omega) \). We consider zero boundary values case in the argument below, but the methods apply to nonzero boundary values as well (exercise).
Theorem 2.19. If \( u \in W^{1,2}_0(\Omega) \) is a minimizer of (2.7), then it is a weak solution to (2.7).

The Moral: A minimizer of a variational integral with given boundary values in the Sobolev sense is a weak solution to the Dirichlet problem for the corresponding Euler-Lagrange equation.

Proof. Let \( \varphi \in C_0^\infty(\Omega) \) and \( \varepsilon \in \mathbb{R} \). Then

\[
I(u) - I(u + \varepsilon \varphi) = \frac{1}{2} \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i(u + \varepsilon \varphi) D_j(u + \varepsilon \varphi) + c(u + \varepsilon \varphi)^2 \right) \, dx - \int_{\Omega} f(u + \varepsilon \varphi) \, dx
\]

Since \( u \) is a minimizer, \( i(\varepsilon) \) has minimum at \( \varepsilon = 0 \), which implies that \( i'(0) = 0 \). A direct computation shows that

\[
i'(\varepsilon) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} \left( D_i u D_j \varphi + D_i \varphi D_j u + 2 \varepsilon D_i \varphi D_j \varphi \right) \, dx
\]

Thus

\[
i'(0) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} \left( D_i u D_j \varphi + D_i \varphi D_j u + 2 \varepsilon D_i \varphi D_j \varphi \right) \, dx + \int_{\Omega} c(u \varphi + \varepsilon \varphi^2) \, dx - \int_{\Omega} f \varphi \, dx
\]

and we obtain

\[
i'(0) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} \left( D_i u D_j \varphi + D_i \varphi D_j u + 2 \varepsilon D_i \varphi D_j \varphi \right) \, dx + \int_{\Omega} c u \varphi \, dx - \int_{\Omega} f \varphi \, dx
\]

As \( a_{ij} = a_{ji} \) and \( i'(0) = 0 \), we obtain

\[
i'(0) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \varphi \, dx + \int_{\Omega} c u \varphi \, dx - \int_{\Omega} f \varphi \, dx
\]

for every \( \varphi \in C_0^\infty(\Omega) \). This shows that \( u \) is a weak solution to (2.6).

\[\Box\]

Lemma 2.20. Assume that \( f \in L^2(\Omega) \). The variational integral (2.7) is bounded from below in \( W^{1,2}_0(\Omega) \) provided \( c \geq c_0 \), where \( c_0 \) is as in the proof of Lemma 2.2.
THEMOMRAL: We already know that $|I(v)| < \infty$ for every $W^{1,2}_0(\Omega)$. The lemma asserts that there is a constant $m$ such that $I(v) \geq m$ for every $W^{1,2}_0(\Omega)$, that is,

$$\inf\{I(v) : v \in W^{1,2}_0(\Omega)\} > -\infty.$$ 

This excludes the case that the infimum is $-\infty$.

Proof. By the ellipticity condition, see Definition 1.2, we have

$$I(v) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^n a_{ij} D_i v D_j v \, dx + \frac{1}{2} \int_{\Omega} c v^2 \, dx - \int_{\Omega} f v \, dx$$

$$\geq \frac{\lambda}{2} \int_{\Omega} |Dv|^2 \, dx + \frac{1}{2} \int_{\Omega} c v^2 \, dx - \int_{\Omega} f v \, dx$$

$$\geq \frac{\lambda}{2} \int_{\Omega} |Dv|^2 \, dx + \frac{c_0}{2} \int_{\Omega} v^2 \, dx - \frac{c}{2} \int_{\Omega} f^2 \, dx$$

$$\left(0 \leq \left(\sqrt{\lambda} v - \frac{1}{\sqrt{c}} f\right)^2 = \lambda v^2 - 2vf + \frac{1}{c} f^2, \text{see Corollary 3.3}\right)$$

$$\geq \frac{\lambda}{2} \left(\frac{\lambda}{\mu} + c_0 - \epsilon \right) \int_{\Omega} v^2 \, dx - \frac{c}{2\epsilon} \int_{\Omega} f^2 \, dx$$

(Poincaré inequality)

$$\geq - \frac{1}{2\epsilon} \int_{\Omega} f^2 \, dx \left(\frac{\lambda}{\mu} + c_0 - \epsilon > 0\right)$$

for every $v \in W^{1,2}_0(\Omega)$, when $\epsilon > 0$ is chosen so small that $\frac{\lambda}{\mu} + c_0 - \epsilon > 0$. This is possible, since in the proof of Lemma 2.2 we have $c_0 > -\frac{\lambda}{\mu}$, or equivalently, $\frac{\lambda}{\mu} + c_0 > 0$. \qed

Remark 2.21. From the proof we see that

$$I(v) \geq \frac{\lambda}{2} \int_{\Omega} |Dv|^2 \, dx + \frac{c_0 - \epsilon}{2} \int_{\Omega} v^2 \, dx - \frac{c}{2\epsilon} \int_{\Omega} f^2 \, dx$$

which implies that

$$\|v\|^2_{W^{1,2}_0(\Omega)} = \int_{\Omega} |v|^2 \, dx + \int_{\Omega} |Dv|^2 \, dx \leq c_1 \|f\|^2_{L^2(\Omega)} + c_2 I(v)$$

for every $v \in W^{1,2}_0(\Omega)$. Here $c_1$ and $c_2$ are independent of $v$. In particular, this shows that

$$\|v\|_{W^{1,2}_0(\Omega)} \to \infty \implies I(v) \to \infty.$$ 

This property is called coercivity.

Theorem 2.22. There exists a constant $c_0$ such that the variational integral (2.7) has a minimizer $u \in W^{1,2}_0(\Omega)$ for every $f \in L^2(\Omega)$ when $c \geq c_0$. 
By Theorem 2.19, every minimizer is a solution to the Euler-Lagrange equation and Theorem 2.22 gives a variational proof of the existence of a solution to the Dirichlet problem. This approach does not use the Hilbert space structure and, as we shall see, it generalizes to nonlinear PDEs as well.

**Proof.**

[1] By Lemma 2.20, the variational integral \( I(v) \) is bounded from below in \( W^{1,2}_0(\Omega) \) and hence

\[
\inf_{v \in W^{1,2}_0(\Omega)} I(v)
\]

is a finite number. The definition of infimum implies that there exists a minimizing sequence \( u_k \in W^{1,2}_0(\Omega), \ k = 1, 2, \ldots \), such that

\[
\lim_{k \to \infty} I(u_k) = \inf_{v \in W^{1,2}_0(\Omega)} I(v).
\]

The existence of the limit \( \lim_{k \to \infty} I(u_k) \) implies the sequence \( (I(u_k)) \) is bounded, that is,

\[
|I(u_k)| \leq M, \ k = 1, 2, \ldots,
\]

for some constant \( M < \infty \). By Remark 2.21, we see that

\[
\|u_k\|_{W^{1,2}_0(\Omega)}^2 \leq c_1 \|f\|_{L^2(\Omega)}^2 + c_2 M, \ \ k = 1, 2, \ldots,
\]

which shows that \( (u_k) \) is a bounded sequence in \( W^{1,2}_0(\Omega) \).

[2] By the sequential weak compactness of \( W^{1,2}(\Omega) \) there exists a subsequence \( (u_{k_l}) \) and a function \( u \) in \( W^{1,2}_0(\Omega) \) such that \( u_{k_l} \rightharpoonup u \) and \( Du_{k_l} \rightharpoonup Du \) weakly in \( L^2(\Omega) \) as \( l \to \infty \). This implies that

\[
\lim_{l \to \infty} \int_{\Omega} f u_{k_l} \, dx = \int_{\Omega} f u \, dx.
\]

By the ellipticity condition, see Definition 1.2, we have

\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i (u_{k_l} - u) D_j (u_{k_l} - u) + c (u_{k_l} - u)^2 \right) \, dx
\]

\[
\geq \lambda \int_{\Omega} |D(u_{k_l} - u)|^2 \, dx + \int_{\Omega} c (u_{k_l} - u)^2 \, dx \geq 0
\]

from which we conclude that

\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_{k_l} D_j u_{k_l} + cu_{k_l}^2 \right) \, dx
\]

\[
\geq 2 \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_{k_l} D_j u + cu_{k_l} u \right) \, dx - \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j u + cu^2 \right) \, dx.
\]
Since \(D_i u_k \to D_i u\) weakly in \(L^2(\Omega)\), \(i = 1, \ldots, n\), and \(a_{ij} D_j u \in L^2(\Omega)\), we obtain
\[
\liminf_{l \to \infty} \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_k D_j u_k + cu_k^2 \right) \, dx \\
\geq 2 \liminf_{l \to \infty} \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_k D_j u + cu_k^2 \right) \, dx - \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j u + cu^2 \right) \, dx \\
= \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j u + cu^2 \right) \, dx.
\]
Thus
\[
I(u) = \frac{1}{2} \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j u + cu^2 \right) \, dx - \int_{\Omega} f u \, dx \\
\leq \frac{1}{2} \liminf_{l \to \infty} \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_k D_j u_k + cu_k^2 - f u_k \right) \, dx \\
= \liminf_{l \to \infty} I(u_k) \\
= \lim_{k \to \infty} I(u_k),
\]
and finally
\[
\inf_{v \in W_0^{1,2}(\Omega)} I(v) \leq I(u) \leq \lim_{k \to \infty} I(u_k) = \inf_{v \in W_0^{1,2}(\Omega)} I(v)
\]
from which we conclude that
\[
I(u) = \inf_{v \in W_0^{1,2}(\Omega)} I(v).
\]

**Remark 2.23.** The proof above is based on the following steps:

1. Choose a minimizing sequence.
2. Use coercivity, see Remark 2.21 to show that the minimizing sequence is bounded in the Sobolev space.
3. Use reflexivity to show that there is a weakly converging subsequence.
4. Use lower semicontinuity of the variational integral to show that the limit is a minimizer.
5. Use strict convexity of the variational integral to show uniqueness.

Next we discuss an abstract version of the existence result. Let \(X\) be a Banach space. We begin with recalling some definitions.

**Definition 2.24.**

1. We say that \(x_k \in X\), \(k = 1, 2, \ldots\), converges weakly to \(x \in X\) if \(x^*(x_k) \to x^*(x)\) as \(k \to \infty\) for every \(x^* \in X^*\). Here \(X^*\) denotes the dual of \(X\).
2. By the Eberlein-Shmulyan theorem a Banach space is reflexive if and only if every bounded sequence has a weakly converging subsequence.
(3) A function \( I : X \to \mathbb{R} \) is sequentially weakly lower semicontinuous, if
\[
I(u) \leq \liminf_{k \to \infty} I(u_k)
\]
whenever \( u_k \rightharpoonup u \) weakly in \( X \).

(4) A function \( I : X \to \mathbb{R} \) is coercive, if
\[
\|u_k\|_X \to \infty \implies I(u_k) \to \infty
\]
as \( k \to \infty \).

(5) A function \( I : X \to \mathbb{R} \) is convex, if
\[
I((1-t)x + ty) \leq (1-t)I(x) + tI(y)
\]
for every \( x, y \in X \) and \( t \in [0,1] \). \( I \) is strictly convex if
\[
I((1-t)x + ty) < (1-t)I(x) + tI(y)
\]
for every \( x, y \in X \), \( x \neq y \) and \( t \in (0,1) \).

**Theorem 2.25.** Assume that \( I : X \to \mathbb{R} \) is a coercive, sequentially weakly lower semicontinuous and strictly convex variational integral on a reflexive Banach space \( X \). Then there exists a unique \( u \in X \) such that
\[
I(u) = \inf_{v \in X} I(v).
\]

**Proof.** [1] We show that \( m = \inf_{v \in X} I(v) \) is finite. Assume, for a contradiction, that it is not, in which case \( m = -\infty \). By the definition of infimum, there exists a sequence \( (u_k) \) such that \( I(u_k) \to -\infty \) as \( k \to \infty \). If \( (u_k) \) is a bounded sequence in \( X \), by reflexivity, it has a weakly converging subsequence such that \( u_{k_l} \rightharpoonup u \) weakly as \( l \to \infty \) for some \( u \in X \). Since \( I \) is sequentially weakly lower semicontinuous, we have
\[
I(u) \leq \liminf_{l \to \infty} I(u_{k_l}) = -\infty
\]
and thus \( I(u) = -\infty \), which is a contradiction with the fact that \( |I(u)| < \infty \). Thus \( (u_k) \) is an unbounded sequence in \( X \) and there exists a subsequence \( (u_{k_l}) \) such that \( \|u_{k_l}\| \to \infty \) as \( l \to \infty \). By coercivity, \( I(u_{k_l}) \to \infty \) as \( l \to \infty \). This is a contradiction with \( I(u_k) \to -\infty \) as \( k \to \infty \). Thus
\[
m = \inf_{v \in X} I(v) > -\infty.
\]

[2] Let \( (u_k) \) be a minimizing sequence such that \( I(u_k) \to m \) as \( k \to \infty \). As a converging sequence of real numbers \( (I(u_k)) \) is bounded. We show that \( (u_k) \) is a bounded sequence in \( X \). Assume, for a contradiction, that it is unbounded. Then there exists a subsequence \( (u_{k_l}) \) such that \( \|u_{k_l}\| \to \infty \) as \( l \to \infty \). By coercivity, \( I(u_{k_l}) \to \infty \) as \( l \to \infty \). This is a contradiction with \( I(u_k) \to m < \infty \) as \( k \to \infty \).
Since \((u_k)\) is a bounded sequence in \(X\), by reflexivity, it has a weakly converging subsequence \((u_{k_l})\) such that \(\|u_{k_l}\| \to u\) as \(l \to \infty\) for some \(u \in X\). Since \(I\) is sequentially weakly lower semicontinuous, we have
\[
I(u) \leq \liminf_{k \to \infty} I(u_{k_l}) \leq \lim_{k \to \infty} I(u_k) = m
\]
and
\[
m \leq I(u) \leq \lim_{k \to \infty} I(u_k) = m.
\]
This shows that \(I(u) = m\) and that \(u\) is a minimizer.

To show that the minimizer is unique assume, for a contradiction, that \(u_1 \in X\) and \(u_2 \in X\) are minimizers with \(u_1 \neq u_2\). We consider
\[
u = \frac{1}{2}u_1 + \frac{1}{2}u_2.
\]
Since \(u_1 \neq u_2\), by strict convexity
\[
I(u) = I\left(\frac{1}{2}u_1 + \frac{1}{2}u_2\right) < \frac{1}{2}I(u_1) + \frac{1}{2}I(u_2) = m.
\]
Thus \(I(u) < m\) and this is a contradiction with the fact that \(u_1\) is a minimizer. A similar argument applies for \(u_2\) as well. Thus \(u_1 = u_2\) and the minimizer is unique.

\[\square\]

THE MORAL: The variational approach is based on Banach space techniques. This applies to nonlinear variational integrals as well.

Example 2.26. Let \(n = 1\) and \(\Omega = (0,1)\) and
\[
I(u) = \int_{0}^{1} \left(\frac{1}{2}u(x)^2 + (1-u'(x))^2\right) dx, \quad u \in W^{1,4}_0(\Omega).
\]

Claim: This variational problem does not have a solution, that is, there does not exist a function \(u \in W^{1,4}_0(\Omega)\) such that
\[
I(u) = \inf \{I(v) : v \in W^{1,4}_0(\Omega)\}.
\]

Reason. Let \(u \in W^{1,4}_0(\Omega)\). By the Sobolev embedding we may assume that \(u\) is continuous. Since the integrand is nonnegative, we have \(I(u) \geq 0\) for every \(u \in W^{1,4}_0(\Omega)\). We show that \(I(u) > 0\) for every \(u \in W^{1,4}_0(\Omega)\). To see this, we note that if \(u(x) = 0\) for every \(x \in \Omega\), then \(I(u) = 1 > 0\). If \(u\) is not identically zero, then there exists \(k \in \mathbb{N}\) such that
\[
\left|\{x \in \Omega : |u(x)| > \frac{1}{k}\}\right| > 0.
\]
Thus
\[
I(u) = \int_{0}^{1} \left(\frac{1}{2}u(x)^2 + (1-u'(x))^2\right) dx \geq \int_{0}^{1} \frac{1}{2}u(x)^2 dx
\]
\[
\geq \frac{1}{2} \left(\frac{1}{k}\right)^2 \left|\{x \in \Omega : |u(x)| > \frac{1}{k}\}\right| > 0.
\]
Consider a sequence of sawtooth functions 
\[ u_j \in W^{1,4}_0(\Omega), \quad j = 1, 2, \ldots, \]
such that
\[ |u_j(x)| \leq \frac{1}{2^j}, \quad j = 1, 2, \ldots, \quad \text{and} \quad |u_j'(x)| = 1 \quad \text{for almost every} \quad x \in \Omega. \]

Then
\[ I(u_j) = \int_0^1 \left( \frac{1}{2} u_j(x)^2 + (1 - u_j'(x))^2 \right) dx = \int_0^1 \frac{1}{2} u_j(x)^2 dx \leq \int_0^1 \frac{1}{2} \left( \frac{1}{2^j} \right)^2 dx \to 0 \quad \text{as} \quad j \to \infty. \]

This implies that
\[ m = \inf_{v \in W^{1,4}_0(\Omega)} I(v) = 0. \]

Since \( I(u) > 0 \) for every \( u \in W^{1,4}_0(\Omega) \), there does not exist a function \( u \in W^{1,4}_0(\Omega) \) such that \( I(u) = 0 = m \).

**THE MORAL:** A minimizer may not exist, if the variational integral is not sequentially weakly lower semicontinuous.

**Example 2.27.** Let \( n = 1, \Omega = (-1, 1), g : \Omega \to \mathbb{R}, g(x) = x, \) and
\[ I(u) = \int_{-1}^1 u'(x)^2 x^4 \, dx, \]
where \( u \in W^{1,2}(\Omega) \) such that \( u - g \in W^{1,2}_0(\Omega) \). Again, we may assume that \( u \) is continuous and \( u(-1) = -1 \) and \( u(1) = 1 \).

**Claim:** This variational problem does not have a solution, that is, there does not exist a function \( u \in W^{1,2}(\Omega) \) with \( u - g \in W^{1,2}_0(\Omega) \) such that
\[ I(u) = \inf \left\{ I(v) : v \in W^{1,2}(\Omega), v - g \in W^{1,2}_0(\Omega) \right\}. \]

**Reason.** Let \( 0 < \varepsilon < 1 \) and \( u_{\varepsilon} : \Omega \to \mathbb{R}, \)
\[ u_{\varepsilon}(x) = \begin{cases} -1, & x \in [-1, -\varepsilon], \\ \varepsilon, & x \in (-\varepsilon, \varepsilon), \\ 1, & x \in (\varepsilon, 1]. \end{cases} \]

Then \( u_{\varepsilon} \in W^{1,2}(\Omega) \) with \( u_{\varepsilon} - g \in W^{1,2}_0(\Omega) \)
\[ 0 \leq I(u_{\varepsilon}) = \int_{-\varepsilon}^{\varepsilon} \left( \frac{1}{\varepsilon^2} \right)^2 x^4 \, dx = \frac{1}{\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} x^4 \, dx = \frac{2\varepsilon^5}{5} = \frac{2\varepsilon^3}{5} \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

Since \( I(v) \geq 0 \) for every \( v \in W^{1,2}(\Omega) \), we conclude that
\[ m = \inf \left\{ I(v) : v \in W^{1,2}(\Omega), v - g \in W^{1,2}_0(\Omega) \right\} = 0. \]
Thus the infimum of the variational integral is zero. Let \( u \in W^{1,2}(\Omega) \) with \( u - g \in W^{1,2}_0(\Omega) \). Let \( \varphi_j \in C^\infty(\Omega) \), \( j = 1, 2, \ldots \), such that \( \varphi_j \to u \) in \( W^{1,2}(\Omega) \) and
\[
\lim_{j \to \infty} \varphi_j(x) = u(x) \quad \text{for every} \quad x \in \Omega.
\]

Since \( u(1) - u(-1) = 1 - (-1) = 2 \) and \( u \) is continuous, there exist \( x, y \in \Omega \) such that \( u(x) - u(y) \geq 1 \). Thus
\[
1 \leq u(x) - u(y) = \lim_{j \to \infty} (\varphi_j(x) - \varphi_j(y))
\]
\[
= \lim_{j \to \infty} \int_y^x (\varphi_j)'(t) \, dt \leq \lim_{j \to \infty} \int_y^x |(\varphi_j)'(t)| \, dt \leq \int_{-1}^1 |u'(t)| \, dt.
\]
This implies that there exists \( k \in \mathbb{N} \) such that
\[
\left| \left\{ x \in \Omega : |u'(x)| > \frac{1}{k} \right\} \right| > 0
\]
and consequently
\[
I(u) = \int_{-1}^1 u'(x)^2 \, dx \geq \int_{\{x : |u'(x)| > \frac{1}{k}\}} u'(x)^2 \, dx
\]
\[
= \sum_{j=1}^\infty \int_{\{x : |u'(x)| > \frac{1}{k}\}} u'(x)^2 \, dx
\]
\[
\geq \sum_{j=1}^\infty \frac{1}{k^2} (2^{-j})^4 \left| \left\{ x \in \Omega : |u'(x)| > \frac{1}{k} \right\} \right| \cap \left\{ x \in \Omega : 2^{-j} \leq |x| < 2^{-j+1} \right\} > 0,
\]
since at least one of the terms in the sum is positive. Since \( I(u) > 0 \) for every \( u \in W^{1,2}(\Omega) \), there does not exist a function \( u \in W^{1,2}(\Omega) \) such that \( I(u) = 0 = m \).

We observe that \( I \) is not coercive, since
\[
\|u_\varepsilon\|_{W^{1,2}(\Omega)} \geq \|(u_\varepsilon)'\|_{L^2(\Omega)} = \left( \int_{-\varepsilon}^{\varepsilon} \left( \frac{1}{\varepsilon} \right)^2 \, dx \right)^{\frac{1}{2}}
\]
\[
= \left( \frac{2\varepsilon}{\varepsilon^2} \right)^{\frac{1}{2}} = \left( \frac{2}{\varepsilon} \right)^{\frac{1}{2}} \to \infty \quad \text{as} \quad \varepsilon \to 0,
\]
but \( I(u_\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Note that the integrand \( F(\xi) = \xi^2 x^4 \) is convex.

**The Moral:** A minimizer may not exist, if the variational integral is not coercive.

**Remark 2.28.** We consider the Dirichlet problem for the Laplace equation in the unit disc in the two-dimensional case. Let \( \Omega = B(0, 1) \) be the unit disc in \( \mathbb{R}^2 \) and assume that \( g \in C(\partial \Omega) \) is a continuous function on the boundary. The problem is to find \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) such that
\[
\begin{cases}
\Delta u = 0 & \text{in} \quad \Omega, \\
u = g & \text{on} \quad \partial \Omega.
\end{cases}
\]
This problem can be solved by separation of variables with Fourier series in polar coordinates. Recall that any point in the plane can be uniquely determined by its distance from the origin $r$ and the angle $\theta$ that the line segment from the origin to the point forms with the $x_1$-axis, that is,

$$(x_1, x_2) = (r \cos \theta, r \sin \theta),\quad (x_1, x_2) \in \mathbb{R}^2,\quad 0 < r < \infty,\quad -\pi < \theta < \pi,$$

where $r^2 = x_1^2 + x_2^2$ and $\tan \theta = \frac{x_2}{x_1}$. In polar coordinates, we have

$$\Omega = \{(r, \theta) : 0 < r < 1,\quad -\pi < \theta < \pi\} \quad \text{and} \quad \partial \Omega = \{(1, \theta) : -\pi < \theta < \pi\}.$$  

The two-dimensional Laplace operator in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2},\quad 0 < r < \infty,\quad -\pi < \theta < \pi.$$  

By separation of variables, we obtain

$$u(r, \theta) = a_0 + \sum_{j=1}^{\infty} r^j \left( a_j \cos(j \theta) + b_j \sin(j \theta) \right),$$

where $a_j$ and $b_j$ are the Fourier cosine and sine coefficients of $g$, respectively. If $\sum_{j=1}^{\infty} (|a_j| + |b_j|) < \infty$, the series converges uniformly in $\bar{\Omega}$ and its derivatives converge uniformly on compact subsets of $\Omega$. Thus $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $u = g$ on $\partial \Omega$. This shows that $u$ is a classical solution to the Dirichlet problem in the unit disc.

$$\int_{B(0,\rho)} |Du|^2 \, dx = \int_0^{2\pi} \int_0^{\rho} \left( |u_r|^2 + \frac{1}{r^2} |u_\theta|^2 \right) r \, dr \, d\theta$$

$$= \pi \sum_{j=1}^{\infty} j \rho^{2j} (a_j^2 + b_j^2).$$

If we choose

$$u(r, \theta) = \sum_{j=1}^{\infty} \frac{j^2}{j^{2j}} \sin(j \theta),$$

then the boundary function is

$$g(\theta) = u(1, \theta) = \sum_{j=1}^{\infty} \frac{1}{j^2} \sin(j \theta).$$

In this case

$$\int_{B(0,1)} |Du|^2 \, dx = \pi \sum_{j=1}^{\infty} \frac{j}{j^2} = \infty$$

and thus $u \in W^{1,2}(\Omega)$.

\textbf{The Moral:} The classical solution of the Dirichlet problem with continuous boundary values may fail to belong to $W^{1,2}(\Omega)$. 

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Remark 2.29. Let \( n = 2 \) and \( \Omega = B(0, 1) \setminus \{0\} \). Consider the Dirichlet problem
\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]
where \( g(x) = 1 - |x| \). Note that \( g \in W^{1,2}(\Omega) \cap C(\overline{\Omega}) \). Then \( u : \Omega \to \mathbb{R}, u(x) = 0 \) is the weak solution with boundary values \( g \), that is, \( u - g \in W^{1,2}(\Omega) \). Observe that
\[
0 = \lim_{x \to 0^-} u(x) \neq \lim_{x \to 0^+} g(x) = 1.
\]

**The Moral:** The boundary values of a weak solution to a Dirichlet problem are not necessarily attained in the classical sense.

2.3 Uniqueness

Let us briefly discuss the uniqueness question. To this end, we need a useful lemma.

**Lemma 2.30.** If \( u \in W^{1,2}_0(\Omega) \) is a weak solution of (2.1), then
\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j v + cuv \right) dx = \int_{\Omega} f v dx
\]
for every \( v \in W^{1,2}_0(\Omega) \).

**The Moral:** The advantage of this result is that we may use \( W^{1,2}_0(\Omega) \) functions as test functions in the definition of a weak solution instead of \( C_0^\infty(\Omega) \) functions, see Definition 1.7. Especially, we can use a weak solution itself as a test function. The result holds also under the assumption \( u \in W^{1,2}(\Omega) \).

**Proof.** Let \( \varphi_k \in C_0^\infty(\Omega), k = 1, 2, \ldots \), such that \( \varphi_k \to v \) in \( W^{1,2}(\Omega) \) as \( k \to \infty \). Then
\[
\left| \int_{\Omega} a_{ij} D_i u D_j \varphi_k dx - \int_{\Omega} a_{ij} D_i u D_j v dx \right| \\
= \left| \int_{\Omega} a_{ij} D_i u (D_j \varphi_k - D_j v) dx \right| \\
\leq \|a_{ij}\|_\infty \|D_i u\|_{L^2(\Omega)} \|D_j \varphi_k - D_j v\|_{L^2(\Omega)} \to 0,
\]
as \( k \to \infty \), \( i, j = 1, \ldots, n \). Thus
\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u D_j v dx = \lim_{k \to \infty} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \varphi_k dx.
\]
Similar arguments show that
\[
\int_{\Omega} cuv dx = \lim_{k \to \infty} \int_{\Omega} cu \varphi_k dx
\]
and 
\[ \int_{\Omega} f v \, dx = \lim_{k \to \infty} \int_{\Omega} f \varphi_k \, dx. \]

By the definition of a weak solution, see Definition 1.7, we have 
\[ \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j v + cu v \right) \, dx = \int_{\Omega} f \varphi_k \, dx \]
for every \( k = 1, 2, \ldots \), since \( \varphi_k \in C_0^\infty(\Omega) \). This implies that 
\[ \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_1 D_j v + cu_1 v \right) \, dx = \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_2 D_j v + cu_2 v \right) \, dx \]
\[ = \lim_{k \to \infty} \int_{\Omega} f \varphi_k \, dx = \int_{\Omega} f v \, dx. \]

**Theorem 2.31.** The solution of (2.1) is unique, provided \( c \geq c_0 \), where \( c_0 \) is as in the proof of Lemma 2.2.

**Proof.** Let \( u_1, u_2 \in W^{1,2}_0(\Omega) \) be weak solutions. By Lemma 2.30
\[ \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_1 D_j v + cu_1 v \right) \, dx = \int_{\Omega} f v \, dx \]
and 
\[ \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u_2 D_j v + cu_2 v \right) \, dx = \int_{\Omega} f v \, dx \]
for every \( v \in W^{1,2}_0(\Omega) \). By subtracting the equations from each other and choosing \( v = u_1 - u_2 \in W^{1,2}_0(\Omega) \), we have
\[ \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij}(D_i u_1 - D_i u_2)(D_j u_1 - D_j u_2) + c(u_1 - u_2)(u_1 - u_2) \right) \, dx = 0 \]

With the ellipticity property, see Definition 1.2, this implies that 
\[ \lambda \int_{\Omega} |D u_1 - D u_2|^2 \, dx + \int_{\Omega} c(u_1 - u_2)^2 \, dx \leq 0. \]

By using the fact that \( c \geq -\frac{\lambda}{2\mu} \) and the Poincaré inequality, as in the proof of Lemma 2.2, we have 
\[ \int_{\Omega} c(u_1 - u_2)^2 \, dx \geq -\frac{\lambda}{2\mu} \int_{\Omega} (u_1 - u_2)^2 \, dx \geq -\frac{\lambda}{2} \int_{\Omega} |D u_1 - D u_2|^2 \, dx. \]

By combining these estimates, we conclude that 
\[ -\frac{\lambda}{2} \int_{\Omega} |D u_1 - D u_2|^2 \, dx \leq \int_{\Omega} c(u_1 - u_2)^2 \, dx \leq -\lambda \int_{\Omega} |D u_1 - D u_2|^2 \, dx. \]

Thus 
\[ \int_{\Omega} |D u_1 - D u_2|^2 \, dx = 0 \quad \text{and} \quad \int_{\Omega} c(u_1 - u_2)^2 \, dx = 0. \]

This implies \( u_1 = u_2 \) almost everywhere in \( \Omega \). \( \square \)
2.4 Comparison and maximum principles

In this section we show that the same technique as in the proof of uniqueness gives certain versions of comparison and maximum principles.

**Theorem 2.32 (Comparison principle).** Assume that \( u, w \in W^{1,2}(\Omega) \) are weak solutions of (2.1) and \( c \geq c_0 \). If \( (u - w)_+ \in W^{1,2}_0(\Omega) \), then \( u \leq w \) in \( \Omega \).

**The Moral:** The assumption \( (u - w)_+ \in W^{1,2}_0(\Omega) \) means that \( u \leq w \) on \( \partial \Omega \) in Sobolev space sense. Thus the comparison principle asserts that if a solution is above another on the boundary, then it is above also inside the domain.

**Proof.** The idea is the same as in the proof of the uniqueness. By Lemma 2.30

\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j v + c u v \right) \, dx = \int_{\Omega} f v \, dx
\]

and

\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_i w D_j v + c w v \right) \, dx = \int_{\Omega} f v \, dx
\]

for every \( v \in W^{1,2}_0(\Omega) \). By subtracting the equations from each other we have

\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_j (u - w) D_i v + c (u - w) v \, dx = 0.
\]

We choose \( v = (u - w)_+ \in W^{1,2}_0(\Omega) \) and obtain

\[
0 = \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} D_j (u - w) D_i (u - w)_+ + c (u - w)_+^2 \right) \, dx
\]

\[
\geq \int_{\Omega} \lambda |D(u - w)_+|^2 + c (u - w)_+^2 \, dx.
\]

Since \( c \geq \frac{\lambda}{2\mu} \) and by the Poincaré inequality we have

\[
0 \geq \int_{\Omega} \lambda |D(u - w)_+|^2 + c (u - w)_+^2 \, dx
\]

\[
\geq \int_{\Omega} \lambda |D(u - w)_+|^2 - \frac{\lambda}{2\mu} (u - w)_+^2 \, dx
\]

\[
\geq \int_{\Omega} \lambda |D(u - w)_+|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} |D(u - w)_+|^2 \, dx
\]

\[
= \frac{\lambda}{2} \int_{\Omega} |D(u - w)_+|^2 \, dx.
\]

By the Poincaré inequality, we have

\[
0 \leq \int_{\Omega} |(u - w)_+|^2 \, dx \leq \mu \int_{\Omega} |D(u - w)_+|^2 \, dx \leq 0.
\]

This implies that \( (u - w)_+ = 0 \) almost everywhere in \( \Omega \), that is, \( u \leq w \) almost everywhere in \( \Omega \). \( \square \)
Remark 2.33. The proof above shows that if \( u, w \in W^{1,2}(\Omega) \) are sub- and supersolutions respectively, that is, 
\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}D_i u D_j v + c u v \, dx \leq \int_{\Omega} f v \, dx
\]
and 
\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}D_i w D_j v + c w v \, dx \geq \int_{\Omega} f v \, dx
\]
for every \( v \in W^{1,2}_0(\Omega) \) with \( v \geq 0 \), and \( (u - w)_+ \in W^{1,2}_0(\Omega) \), then \( u \leq w \) in \( \Omega \).

**Theorem 2.34 (Weak maximum principle).** Let \( u \in W^{1,2}(\Omega) \) be a weak solution of (2.1) with \( f = 0 \) and \( c \geq 0 \). Then
\[
\text{esssup}_{\Omega} u \leq \sup_{\partial \Omega} u_+.
\]

**The moral:** The maximum principle asserts, roughly speaking, that a solution attains its maximum on the boundary of the domain. More precisely, a solution cannot attain a strict maximum inside the domain.

**Proof.** Set \( M = \sup_{\partial \Omega} u_+ \geq 0 \). Then \( (u - M)_+ \in W^{1,2}_0(\Omega) \). To see this, choose a decreasing sequence \( l_i \to M \) so that \( (u - l_i)_+ = (u_+ - l_i)_+ \in W^{1,2}_0(\Omega) \). Since \( \Omega \) is bounded, it follows that \( u - l_i \to u - M \) in \( W^{1,2}(\Omega) \). This implies
\[
(u - l_i)_+ \to (u - M)_+ \text{ in } W^{1,2}(\Omega)
\]
and thus \( (u - M)_+ \in W^{1,2}_0(\Omega) \).

We use \( v = (u - M)_+ \) as a test function and obtain
\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij}D_i u D_j v + c u v \right) \, dx = 0
\]
and the constant function \( M \) is a weak supersolution, that is,
\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij}D_i M D_j v + c M v \right) \, dx = \int_{\Omega} c M v \, dx \geq 0.
\]
Here we used \( M, c, v \geq 0 \). We subtract these from each other and conclude that
\[
\lambda \int_{\Omega} |D(u - M)_+|^2 + c(u - M)_+^2 \, dx \leq 0.
\]
From this it follows that \( u \leq M \) almost everywhere in \( \Omega \). \( \square \)
Higher order regularity

In the previous chapter, we proved existence of a solution by weakening the definition of a solution. Now we study the regularity of weak solutions: Are weak solutions of the PDE

$$Lu = f \quad \text{in } \Omega,$$

where $L$ is as in (1.1), smoother than $W^{1,2}_{\text{loc}}(\Omega)$ under suitable assumptions on the coefficients? Are they classical solutions to the problem? Example 1.11 shows that this is not true under the $L^\infty$-assumption on the coefficients, so that additional assumptions have to be imposed. Our main result shows that if the coefficients are smooth then the solution is smooth.

3.1 Motivation

Consider the Poisson equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

A formal computation using the integration by parts shows that

$$\int_{\mathbb{R}^n} f^2 \, dx = \int_{\mathbb{R}^n} (\Delta u)^2 \, dx = \int_{\mathbb{R}^n} \left( \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \right)^2 \, dx$$

$$= \int_{\mathbb{R}^n} \left( \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \right) \left( \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} \right) \, dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 u}{\partial x_j^2} \, dx$$

$$= -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 u}{\partial x_j^2} \, dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \, dx$$

$$= \int_{\mathbb{R}^n} |D^2 u|^2 \, dx,$$
where
\[
D^2 u = \begin{bmatrix}
\frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\
\frac{\partial^2 u}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n \partial x_n}
\end{bmatrix}
\]
is the matrix of the second derivatives and
\[
|D^2 u|^2 = \sum_{i,j=1}^{n} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2.
\]
Next we use this recursively. This is called a bootstrap argument.

[Step 1] By the previous computation, the $L^2$-norm of the second derivatives of $u$ can be estimated by the $L^2$-norm of $f$.

[Step 2] By differentiating the PDE, we have
\[
-\Delta \left( \frac{\partial u}{\partial x_k} \right) = -\frac{\partial}{\partial x_k} (\Delta u) = \frac{\partial f}{\partial x_k}, \quad k = 1, \ldots, n,
\]
that is,
\[
-\Delta \bar{u} = \bar{f},
\]
where
\[
\bar{u} = \frac{\partial u}{\partial x_k} \quad \text{and} \quad \bar{f} = \frac{\partial f}{\partial x_k}, \quad k = 1, \ldots, n.
\]
Thus the partial derivatives satisfy a similar PDE. By the same method as in Step 1, we can estimate the $L^2$-norm of the third derivatives of $u$ by the first derivatives of $f$.

[Step 3] Continuing this way, we see that the $L^2$-norm of the $(m + 2)^{nd}$ derivatives of $u$ can be controlled by the $L^2$-norm of the $m^{th}$ derivatives of $f$ for $m = 0, 1, 2, \ldots$.

**THE MORAL:** This formal argument suggests that $u$ has two more derivatives in $L^2$ than $f$. In particular, if $f \in C^\infty_0(\mathbb{R}^n)$, then $u \in W^{m,2}(\mathbb{R}^n)$ for every $m = 1, 2, \ldots$, and thus $u \in C^\infty(\mathbb{R}^n)$.

Observe, however, that we assumed that $u$ is smooth in the iterative process above, and thus it is not really a proof for smoothness. Next we want to make this heuristic idea more precise.

### 3.2 Young’s inequality

Before stating the main results of this chapter, we recall two useful versions of Young’s inequality.
Lemma 3.1 (Young's inequality). Let $1 < p < \infty$ and $a, b \geq 0$, then
\[ ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \]
where $\frac{1}{p} + \frac{1}{p'} = 1$ or equivalently $p' = \frac{p}{p-1}$.

Remark 3.2. Young’s inequality for $p = 2$ follows immediately from
\[ (a-b)^2 \geq 0 \iff a^2 - 2ab + b^2 \geq 0 \iff \frac{a^2}{2} + \frac{b^2}{2} \geq ab. \]

Lemma 3.3 (Young's inequality with $\varepsilon$). Let $1 < p < \infty$, $a, b \geq 0$ and $\varepsilon > 0$. Then
\[ ab \leq \varepsilon a^p + cb^{p'}, \]
where
\[ c = c(\varepsilon, p) = (\varepsilon p)^{-\frac{1}{p}} \frac{p-1}{p} \]

Proof. We apply Young's inequality to $\widetilde{a} = (\varepsilon p)^{\frac{1}{p}} a$ and $\widetilde{b} = (\varepsilon p)^{-\frac{1}{p}} b$. This gives
\[ ab = (\varepsilon p)^{\frac{1}{p}} a \cdot (\varepsilon p)^{-\frac{1}{p}} b \]
\[ \leq \frac{p\varepsilon a^p}{p} + (\varepsilon p)^{-\frac{1}{p}} \frac{b^{p'}}{p'} \]
\[ = \varepsilon a^p + (\varepsilon p)^{-\frac{1}{p}} \frac{p-1}{p} b^{p'}. \]

\[ \square \]

Remark 3.4. For $p = 2$, $a, b \geq 0$ and $\varepsilon > 0$, we have
\[ ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2. \]

Remark 3.5. It is essential that $\varepsilon$ can be chosen as small as we please. We shall use the inequality in the following context. Suppose that $f \in L^p(A)$ and $g \in L^p(A)$ and that
\[ \int_A |f|^p \, dx \leq c \int_A |f|^{p-1} |g| \, dx \]
for some constant $c > 0$.

Then by applying Young’s inequality with $\varepsilon$ we obtain
\[ \int_A |f|^p \, dx \leq c \int_A |f|^{p-1} |g| \, dx \]
\[ \leq c \varepsilon \int_A |f|^{(p-1)\frac{p}{p-1}} \, dx + c(\varepsilon, p) \int_A |g|^p \, dx. \]

Now we can move the $L^p$-integral of $f$ to the left-hand-side and obtain
\[ (1 - c\varepsilon) \int_A |f|^p \, dx \leq c(\varepsilon, p) \int_A |g|^p \, dx. \]

If $1 - c\varepsilon > 0$ or equivalently $\varepsilon < \frac{1}{c}$, then the estimate above implies that
\[ \int_A |f|^p \, dx \leq \frac{c(\varepsilon, p)}{1 - c\varepsilon} \int_A |g|^p \, dx. \]
3.3 Difference quotients

The proof of the main result of this section uses difference quotients and thus this approach is called the difference quotient method. We recall the definition and basic properties of difference quotients.

**Definition 3.6.** Let $f \in L^1_{\text{loc}}(\Omega)$ and $\Omega' \subseteq \Omega$. The $k^{th}$ difference quotient is

$$D^h_k f(x) = \frac{f(x + he_k) - f(x)}{h}, \quad k = 1, \ldots, n,$$

for $x \in \Omega'$ and $h \in \mathbb{R}$ such that $0 < |h| < \text{dist}(\Omega', \partial \Omega)$. We denote

$$D^h f = (D^h_1 f, \ldots, D^h_n f).$$

**The Moral:** Note that the definition of the difference quotient makes sense at every $x \in \Omega$ whenever $0 < |h| < \text{dist}(x, \partial \Omega)$. If $\Omega = \mathbb{R}^n$, then the definition makes sense for every $h \neq 0$.

The following properties of the difference quotients follow directly from the definition.

**Lemma 3.7.**

1. If $f, g \in L^2(\mathbb{R}^n)$ are compactly supported functions, then

$$\int_{\mathbb{R}^n} f(x) D^h_k g(x) dx = - \int_{\mathbb{R}^n} g(x) D^{-h}_k f(x) dx, \quad k = 1, \ldots, n.$$

2. If $f$ has weak partial derivatives $D_i f$, $i = 1, \ldots, n$, then

$$D_i D^h_k f = D^h_k D_i f, \quad i, k = 1, 2, \ldots, n.$$

3. If $f, g \in L^2(\mathbb{R}^n)$, then

$$D^h_k (fg) = g^h(x) D^h_k f(x) + f(x) D^h_k g(x),$$

where $g^h(x) = g(x + he_k)$.

**Proof:**

$$\int_{\mathbb{R}^n} f(x) D^h_k g(x) dx = \int_{\mathbb{R}^n} f(x) \frac{g(x + he_k) - g(x)}{h} dx$$

$$= \int_{\mathbb{R}^n} \frac{g(x + he_k) f(x)}{h} dx - \int_{\mathbb{R}^n} \frac{g(x) f(x)}{h} dx$$

$$= \int_{\mathbb{R}^n} \frac{g(x) f(x - he_k)}{h} dx - \int_{\mathbb{R}^n} \frac{g(x) f(x)}{h} dx$$

$$= - \int_{\mathbb{R}^n} g(x) \frac{f(x - he_k) - f(x)}{(-h)} dx.$$
Let $\varphi \in C^\infty_0(\mathbb{R}^n)$. Then
\[
\int_{\mathbb{R}^n} D_h^k f \frac{\partial \varphi}{\partial x_i} dx = \int_{\mathbb{R}^n} \frac{f(x + he_k) - f(x)}{h} \frac{\partial \varphi}{\partial x_i}(x) dx
\]
\[
= \frac{1}{h} \left( \int_{\mathbb{R}^n} f(x + he_k) \frac{\partial \varphi}{\partial x_i}(x) dx - \int_{\mathbb{R}^n} f(x) \frac{\partial \varphi}{\partial x_i}(x) dx \right)
\]
\[
= -\frac{1}{h} \left( \int_{\mathbb{R}^n} D_i f(x + he_k) \varphi(x) dx - \int_{\mathbb{R}^n} D_i f(x) \varphi(x) dx \right)
\]
\[
= -\int_{\mathbb{R}^n} D_h^k D_i f \varphi(x) dx.
\]

We recall a characterization of the Sobolev spaces by integrated difference quotients.

**Theorem 3.8.**

(1) Assume $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. Then for every $\Omega' \subset \subset \Omega$, we have
\[
\|D_h^k u\|_{L^p(\Omega')} \leq c \|Du\|_{L^p(\Omega)}
\]
for some constant $c = c(n, p)$ and all $0 < |h| < \text{dist}(\Omega', \partial \Omega)$.

(2) If $u \in L^p(\Omega')$, $1 < p < \infty$, and there is a constant $c$ such that
\[
\|D_h^k u\|_{L^p(\Omega')} \leq c
\]
for all $0 < |h| < \text{dist}(\Omega', \partial \Omega)$, then $u \in W^{1,p}(\Omega')$ and
\[
\|Du\|_{L^p(\Omega')} \leq c.
\]

**Proof.** See Sobolev spaces.

3.4 A difference quotient method

We assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and we consider a PDE of the type
\[
Lu = -\sum_{i,j=1}^n D_j(a_{ij}D_i u) + \sum_{i=1}^n b_i D_i u + cu = f,
\]
see (1.1). We continue to require the uniform ellipticity condition, see Definition 1.2 and we will make additional assumptions about the smoothness of the coefficients $a_{ij}, b_i$ and $c$.

**Theorem 3.9 (Second order interior estimate).** Assume that

$$a_{ij} \in C^1(\Omega), \quad b_i, c \in L^\infty(\Omega), \quad i, j = 1, \ldots, n, \quad \text{and} \quad f \in L^2(\Omega).$$

Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in $\Omega$, where $L$ is as in (1.1). Then $u \in W^{2,2}_{\text{loc}}(\Omega)$ and for every $\Omega' \subseteq \Omega$, we have

$$\|u\|_{W^{2,2}(\Omega')} \leq c \left( \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right),$$

where the constant $c$ depends only on $\Omega'$, $\Omega$ and the coefficients of $L$.

**THEOREM:** This regularity result asserts that the weak solution that assumed to belong to $W^{1,2}(\Omega)$ is more regular and belongs to $W^{2,2}_{\text{loc}}(\Omega)$, if the coefficients $a_{ij} \in C^1(\Omega)$. In addition, this result comes with an estimate. Example 1.11 shows that this cannot hold under the assumption $a_{ij} \in L^\infty(\Omega)$. Note that no boundary conditions are assumed, so that this regularity result applies to PDEs with Dirichlet, Neumann or other boundary conditions.

**Remarks 3.10:**

1. Note that we do not require $u \in W^{1,2}_0(\Omega)$, that is, we are not assuming that $u = 0$ on $\partial \Omega$ in the Sobolev sense.
2. The claim $u \in W^{2,2}_{\text{loc}}(\Omega)$ implies that $u$ actually solves the PDE almost everywhere in $\Omega$, that is,

$$Lu(x) = f(x) \quad \text{for almost every } x \in \Omega.$$

**Reason.** By the definition of the second order weak derivative gives

$$\int_\Omega f \varphi dx = \int_\Omega \left( \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi + \sum_{i=1}^n b_i D_i u \varphi + c u \varphi \right) dx$$

$$= \int_\Omega \left( - \sum_{i,j=1}^n D_j (a_{ij} D_i u) \varphi + \sum_{i=1}^n b_i D_i u \varphi + c u \varphi \right) dx$$

and consequently

$$\int_\Omega \left( - \sum_{i,j=1}^n D_j (a_{ij} D_i u) + \sum_{i=1}^n b_i D_i u + c u - f \right) \varphi dx = \int_\Omega (Lu - f) \varphi dx = 0$$

for every $\varphi \in C^\infty_0(\Omega)$. This implies that $Lu - f = 0$ almost everywhere in $\Omega$. $\blacksquare$
Proof. \([1]\) Choose \(\Omega''\) such that \(\Omega' \subset \Omega'' \subset \Omega\). Let \(\eta \in C_0^\infty(\Omega'')\) be a cutoff function such that \(\eta = 1\) in \(\Omega'\) and \(0 \leq \eta \leq 1\).

\([2]\) Let \(u\) be a weak solution of \(Lu = f\) in \(\Omega\). Then by Lemma 2.30,

\[
\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_i u D_j v \, dx = \int_{\Omega} \bar{f} v \, dx \tag{3.1}
\]

for every \(v \in W_0^{1,2}(\Omega)\), where

\[\bar{f} = f - \sum_{i=1}^{n} b_i D_i u - cu.\]

We point out that Lemma 2.30 holds also without assumption that \(b_i = 0, \ i = 1, \ldots, n\) (exercise).

\([3]\) Use

\[v = -D_k^{-h}(\eta^2 D_k^h u), \quad k = 1, \ldots, n,\]

as a test function in (3.1), where

\[D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}\]

is a difference quotient with \(|h| > 0\) small enough. Observe that \(v \in W_0^{1,2}(\Omega)\) for small enough \(|h| > 0\). We write the resulting expression as \(A = B\) for

\[A = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_i u D_j v \, dx \quad \text{and} \quad B = \int_{\Omega} \bar{f} v \, dx.\]

\([4]\) For \(A\) we have

\[
A = -\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_i u D_j \left[ D_k^{-h}(\eta^2 D_k^h u) \right] \, dx \quad \text{(Lemma 3.7 (2))}
\]

\[= \sum_{i,j=1}^{n} \int_{\Omega} D_k^h (a_{ij} D_i u) D_j (\eta^2 D_k^h u) \, dx \quad \text{(Lemma 3.7 (1))}
\]

\[= \sum_{i,j=1}^{n} \int_{\Omega} \left[ a_{ij} D_k^h (D_i u) D_j (\eta^2 D_k^h u) \right. \]

\[\left. + (D_k^h a_{ij}) D_i u D_j (\eta^2 D_k^h u) \right] \, dx \quad \text{(Lemma 3.7 (3))}
\]

\[= \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_k^h (D_i u) D_k^h (D_j u) \eta^2 \, dx \]

\[+ \sum_{i,j=1}^{n} \int_{\Omega} \left( a_{ij} D_k^h (D_i u) D_k^h (D_j u) \eta^2 \right) \]

\[+ (D_k^h a_{ij}) D_i u D_k^h (D_j u) \eta^2 + (D_k^h a_{ij}) D_i u D_k^h (D_j u) \eta^2 \] \(\eta \text{D} \eta\)

\[dx = A_1 + A_2.\]
Recall that $a_{ij}(x) = a_{ij}(x + h e_k)$. The uniform ellipticity (see Definition 1.2) implies that

$$A_1 = \sum_{i,j=1}^{n} \int_{\hat{\Omega}} a_{ij}^k D_i^h(D_i u) D_j^h(D_j u) \eta^2 \, dx \geq \lambda \int_{\hat{\Omega}} \eta^2 |D^h u|^2 \, dx.$$ 

On the other hand, by using the properties $a_{ij} \in L^\infty(\Omega)$, $D^h_i a_{ij} \in L^\infty(\Omega)$, $\eta^2 \leq \eta$ in $\Omega$ and Young’s inequality with $\epsilon$, see Corollary 3.3, with $p = 2$, we have

$$|A_2| \leq c \int_{\Omega} \left( |\eta| D_h^k D u |D^h u| + |\eta| D_h^k D u |D u| + |\eta| D_h^k u |D u| \right) \, dx$$

$$\leq c c \int_{\Omega} |\eta^2| D_h^k D u|^2 \, dx + c(e) \int_{\Omega} (|D^h u|^2 + |D u|^2) \, dx$$

$$\leq c c \int_{\Omega} |\eta^2| D_h^k D u|^2 \, dx + c(e) \int_{\Omega} |D u|^2 \, dx.$$ 

In the last inequality we used the fact that

$$\|D_h^k u\|_{L^2(\Omega')} \leq c \|D u\|_{L^2(\Omega)}, \quad k = 1, \ldots, n,$$

for some constant $c = c(n, p)$ and all $0 < |h| < \text{dist}(\Omega'^c, \partial\Omega)$, see Theorem 3.8 (1). By choosing $\epsilon > 0$ so that $c c = \frac{\lambda}{2}$, we have

$$|A_2| \leq \frac{\lambda}{2} \int_{\Omega} \eta^2 |D^h u|^2 \, dx + c \int_{\Omega} |D u|^2 \, dx.$$ 

This gives the lower bound

$$A \geq A_1 - |A_2|$$

$$\geq \lambda \int_{\Omega} \eta^2 |D_h^k u|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} \eta^2 |D_h^k D u|^2 \, dx - c \int_{\Omega} |D u|^2 \, dx$$

$$= \frac{\lambda}{2} \int_{\Omega} \eta^2 |D_h^k u|^2 \, dx - c \int_{\Omega} |D u|^2 \, dx.$$ 

We estimate $B$ by using Young’s inequality with $\epsilon$, see Corollary 3.3, and obtain

$$|B| \leq \int_{\Omega} |\eta| |v| \, dx = \int_{\Omega} |f - \sum_{i=1}^{n} b_i D_i u - c u| |v| \, dx$$

$$\leq c \int_{\Omega} (|f| + |D u| + |u|) |v| \, dx$$

$$\leq c c \int_{\Omega} |v|^2 \, dx + c(e) \int_{\Omega} (|f|^2 + |D u| + |u|^2) \, dx$$

$$\leq c c \int_{\Omega} |v|^2 \, dx + c(e) \int_{\Omega} (|f|^2 + |u|^2 + |D u|^2) \, dx,$$
A combination of estimates from (4) and (5) gives

\[
\int_{\Omega} |v|^2 \, dx = \int_{\Omega} |v|^2 \, dx = \int_{\Omega} |D_k^{-h}(\eta^2 D_k^h u)|^2 \, dx \\
\leq c \int_{\Omega} |D(\eta^2 D_k^h u)|^2 \, dx = c \int_{\Omega} |D(\eta^2 D_k^h u)|^2 \, dx \\
\leq c \int_{\Omega} \left(2|\eta|D\eta|D_k^h u| + \eta^2 |D(D_k^h u)|\right)^2 \, dx \quad \text{(Leibniz)} \\
\leq c \int_{\Omega} \eta^2 |D\eta|D_k^h u|^2 \, dx + c \int_{\Omega} \eta^4 |D_k^h D u|^2 \, dx \\
\leq c \int_{\Omega} |D u|^2 \, dx + c \int_{\Omega} \eta^2 |D_k^h D u|^2 \, dx. \quad (\eta^4 \leq \eta^2)
\]

Thus

\[
|B| \leq c \int_{\Omega} \eta^2 |D_k^h D u|^2 \, dx + c \int_{\Omega} (|f|^2 + |u|^2 + |D u|^2) \, dx.
\]

By choosing \( \varepsilon > 0 \) so that \( c \varepsilon = \frac{\lambda}{4} \), we obtain

\[
|B| \leq \frac{\lambda}{4} \int_{\Omega} \eta^2 |D_k^h D u|^2 \, dx + c \int_{\Omega} (|f|^2 + |u|^2 + |D u|^2) \, dx.
\]

\[\boxed{6}\] A combination of estimates from (4) and (5) gives

\[
\frac{\lambda}{2} \int_{\Omega} \eta^2 |D_k^h D u|^2 \, dx - c \int_{\Omega} |D u|^2 \, dx \leq A \\
= B \leq \frac{\lambda}{4} \int_{\Omega} \eta^2 |D_k^h D u|^2 \, dx + c \int_{\Omega} (|f|^2 + |u|^2 + |D u|^2) \, dx.
\]

Thus

\[
\int_{\Omega} |D_k^h D u|^2 \, dx \leq \int_{\Omega} \eta^2 |D_k^h D u|^2 \, dx \leq c \int_{\Omega} (|f|^2 + |u|^2 + |D u|^2) \, dx
\]

for \( k = 1, \ldots, n \) and all sufficiently small \( |h| \neq 0 \). The characterization of Sobolev spaces by integrated difference quotients, see Theorem 3.8 (2), implies \( D u \in W^{1,2}(\Omega') \) and thus \( u \in W^{2,2}(\Omega') \) with the estimate

\[
\|u\|_{W^{2,2}(\Omega')} \leq c \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right).
\]

This is almost what we want except that there is the Sobolev norm \( \|u\|_{W^{1,2}(\Omega)} \) instead of \( \|u\|_{L^2(\Omega)} \) on the right-hand side.

\[\boxed{7}\] To complete the proof, choose a cutoff function \( \eta \in C^\infty_0(\Omega) \) such that \( \eta = 1 \) on \( \Omega' \) and \( 0 \leq \eta \leq 1 \). By Lemma 2.30 we may apply

\[
v = \eta^2 u \in W^{1,2}_0(\Omega)
\]

as a test function in (3.1), that is,

\[
\sum_{i,j=1}^n \int_{\Omega} a_{ij} D_i u D_j (\eta^2 u) \, dx = \int_{\Omega} \mathcal{F} \eta^2 u \, dx.
\]
Now
\[
\sum_{i,j=1}^{n} a_{ij} D_i u D_j (\eta^2 u) dx = \sum_{i,j=1}^{n} a_{ij} D_i u (\eta^2 D_j u + 2u \eta D_j \eta) dx \\
\geq \lambda \int_{\Omega} \eta^2 |D u|^2 dx - \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_i u 2u \eta D_j \eta dx,
\]
where
\[
\left| \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_i u 2u \eta D_j \eta dx \right| \leq c \sum_{i,j=1}^{n} \int_{\Omega} \eta |D_j u||u| dx \leq c \int_{\Omega} \eta |D u||u| dx.
\]
Thus
\[
\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_i u D_j (\eta^2 u) dx \geq \lambda \int_{\Omega} \eta^2 |D u|^2 dx - c \int_{\Omega} \eta |D u||u| dx.
\]
On the other hand, we can use Young’s inequality with \( \varepsilon \) to obtain
\[
\int_{\Omega} \eta^2 u dx \leq c \int_{\Omega} (|f| + |D u| + |u|) \eta^2 u dx \\
\leq c \varepsilon \int_{\Omega} \eta^2 |D u|^2 dx + c \varepsilon \int_{\Omega} |u|^2 dx + c \int_{\Omega} |f|^2 dx.
\]
Choosing \( \varepsilon > 0 \) such that \( c \varepsilon = \frac{1}{2} \) and combining the previous estimates, we have
\[
\int_{\Omega} \eta^2 |D u|^2 dx \leq c \int_{\Omega} (|f|^2 + |u|^2) dx + c \int_{\Omega} |u||D u| dx,
\]
where the last term can again be estimated by Young’s inequality as
\[
c \int_{\Omega} |u||D u| dx \leq c \varepsilon \int_{\Omega} \eta^2 |D u|^2 dx + c(\varepsilon) \int_{\Omega} |u|^2 dx.
\]
By choosing \( \varepsilon > 0 \) such that \( c \varepsilon = \frac{1}{2} \), we finally have
\[
\int_{\Omega} \eta^2 |D u|^2 dx \leq c \int_{\Omega} (|f|^2 + |u|^2) dx + \frac{1}{2} \int_{\Omega} \eta^2 |D u|^2 dx + c \int_{\Omega} |u|^2 dx,
\]
which implies
\[
\int_{\Omega} |D u|^2 dx \leq \int_{\Omega} \eta^2 |D u|^2 dx \leq c \int_{\Omega} (|f|^2 + |u|^2) dx. \tag{3.2}
\]

The argument in (6), with \( \Omega \) replaced by \( \Omega'' \), combined with (7) gives
\[
\|u\|_{W^{2,2}(\Omega')} \leq c (\|f\|_{L^2(\Omega')} + \|u\|_{W^{1,2}(\Omega')}).
\]
\[
\|u\|_{W^{2,2}(\Omega')} \leq c (\|f\|_{L^2(\Omega')} + \|u\|_{L^2(\Omega')}).
\]

This completes the proof. \( \square \)

**The Moral:** The proof is based on choosing appropriate test functions. In step (2) we use \( v = -D_h^{2h}(\eta^2 D_h^k u) \), \( k = 1, \ldots, n \), and in step (7) we use \( v = \eta^2 u \) as a test function in (3.1). These are the only points in the proof where we use the PDE.
Remark 3.11. The proof of the previous theorem gives an extremely useful energy estimate. Assume that \( \Omega \subset \mathbb{R}^n \) is a bounded open set, \( a_{ij}, b_i, c \in L^\infty(\Omega) \), \( i, j = 1, \ldots, n \) and \( f \in L^2(\Omega) \). Let \( u \in W^{1,2}(\Omega) \) be a weak solution of \( Lu = f \) in \( \Omega \), where \( L \) is as in (1.1). Then by (3.2), there exists a constant \( c = c(\lambda) \) such that

\[
\int_{\Omega'} |Du|^2 \, dx \leq c \int_{\Omega} (|f|^2 + |u|^2) \, dx,
\]

whenever \( \Omega' \Subset \Omega \). Observe, that Poincaré inequality states that

\[
\int_{\Omega} |u|^2 \, dx \leq c(diam \Omega)^2 \int_{\Omega} |Du|^2 \, dx
\]

for every \( u \in W^{1,2}_0(\Omega) \). Thus the energy estimate above is a reverse Poincaré inequality.

**Moral:** The use of the difference quotients in the proof means that we, roughly speaking, differentiate the equation and obtain a PDE for the derivatives of a weak solution.

### 3.5 A bootstrap argument

**Motivation:** Our goal is to use Theorem 3.9 recursively provided the coefficients and the right-hand side of the PDE are smooth enough. To this end, we would like to show that weak derivatives of a weak solution are solutions to certain PDE as well. Assume that \( a_{ij} \) are constants, \( b_i = 0, i, j = 1, \ldots, n \), \( c = 0 \) and \( f = 0 \). Then we have

\[
Lu = -\sum_{i,j=1}^n D_j(a_{ij}D_iu) = -\sum_{i,j=1}^n a_{ij}D_jD_iu = 0.
\]

Let \( \psi \in C_0^\infty(\Omega) \) and choose

\[
\varphi = D_k \psi \in C_0^\infty(\Omega), \quad k = 1, \ldots, n,
\]

as a test function in the definition of a weak solution. This gives

\[
\int_{\Omega} \sum_{i,j=1}^n a_{ij}D_iuD_j\varphi \, dx = 0.
\]

Recall that by Theorem 3.9 we have \( u \in W^{2,2}_{0\text{loc}}(\Omega) \) and thus

\[
D_k u \in W^{1,2}_{0\text{loc}}(\Omega), \quad k = 1, \ldots, n.
\]
By the definition of the weak derivative, that is, integration by parts, this gives

\[ \sum_{i,j=1}^{n} \int_{\Omega} (a_{ij} D_i u)(D_j \psi) \, dx = \sum_{i,j=1}^{n} \int_{\Omega} (a_{ij} D_i u)(D_k D_j \psi) \, dx \]

\[ = - \sum_{i,j=1}^{n} \int_{\Omega} D_k (a_{ij} D_i u) D_j \psi \, dx \]

\[ = - \sum_{i,j=1}^{n} \int_{\Omega} (a_{ij} D_k D_i u) D_j \psi \, dx \]

\[ = - \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i (D_k u) D_j \psi \, dx = 0. \]

\[ \text{THE MORAL:} \] This means that \( D_k u, k = 1, \ldots, n, \) is a weak solution to the same PDE as \( u. \) This procedure can be iterated and used to show smoothness of weak solutions.

Next we extend this argument to more general PDEs.

**Theorem 3.12 (Higher order interior estimate).** Let \( m \) be a nonnegative integer. Assume that

\[ a_{ij}, b_i, c \in C^{m+1}(\Omega), \quad i, j = 1, \ldots, n, \quad \text{and} \quad f \in W^{m,2}(\Omega). \]

Let \( u \in W^{1,2}(\Omega) \) be a weak solution of \( Lu = f \) in \( \Omega, \) where \( L \) is as in (1.1). Then \( u \in W^{m+2,2}_{\text{loc}}(\Omega) \) and for every \( \Omega' \subset \Omega, \) we have

\[ \| u \|_{W^{m+2,2}(\Omega')} \leq c \left( \| f \|_{W^{m,2}(\Omega)} + \| u \|_{L^2(\Omega)} \right), \]

where the constant \( c \) depends only on \( \Omega', \Omega' \) and the coefficients of \( L. \)

\[ \text{THE MORAL:} \] This regularity result asserts that a weak solution belongs locally to a higher order Sobolev space, if the coefficients and the right-hand side of the PDE are smooth enough. In addition, this result comes with an estimate. In this sense \( u \) has two more derivatives than \( f. \) Thus the degree of regularity can be increased stepwise provided the data is smooth.

**Proof.** \(^{[1]}\) We prove the claim by induction on \( m. \) The case \( m = 0 \) follows from Theorem 3.9.

\(^{[2]}\) Let \( u \in W^{1,2}(\Omega) \) be a weak solution of \( Lu = f \) in \( \Omega. \) Assume that for some nonnegative integer \( m, \) we have \( u \in W^{m+2,2}_{\text{loc}}(\Omega) \) and

\[ \| u \|_{W^{m+2,2}(\Omega')} \leq c \left( \| f \|_{W^{m,2}(\Omega)} + \| u \|_{L^2(\Omega)} \right) \] (3.3)

for every \( \Omega' \subset \Omega, \) where the constant \( c \) depends only on \( \Omega', \Omega \) and the coefficients of \( L. \) We shall show that the claim holds for \( m + 1. \) To this end, assume that

\[ a_{ij}, b_i, c \in C^{m+2}(\Omega), \quad i, j = 1, \ldots, n, \quad \text{and} \quad f \in W^{m+1,2}(\Omega). \] (3.4)
Assume that Theorem 3.9 implies where \( \bar{\psi} \). After a number of integrations by parts, we obtain

\[
\int \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j \psi + \sum_{i=1}^{n} b_i D_i u \psi + cu \psi \right) dx = \int f \psi dx.
\]

This gives

\[
\sum_{i,j=1}^{n}(-1)^{|\alpha|} \int a_{ij} D_i u D_j (D^\alpha \phi) dx + \sum_{i=1}^{n}(-1)^{|\alpha|} \int b_i D_i u D^\alpha \phi dx
\]

\[+ (-1)^{|\alpha|} \int c u D^\alpha \phi dx = (-1)^{|\alpha|} \int f D^\alpha \phi dx.
\]

After a number of integrations by parts, we obtain

\[
\int \left( \sum_{i,j=1}^{n} a_{ij} D_i \bar{u} D_j \phi + \sum_{i=1}^{n} b_i D_i \bar{u} \phi + c \bar{u} \phi \right) dx = \int \bar{f} \phi dx,
\]

where \( \bar{u} = D^\alpha u \in W^{1,2}(\Omega') \) and

\[
\bar{f} = D^\alpha f - \sum_{\beta < \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} \left[ - \sum_{i,j=1}^{n} D_j \left( D^{\alpha - \beta} a_{ij} D^\beta D_i u \right)
\right.
\]

\[+ \sum_{i=1}^{n} D^{\alpha - \beta} b_i D^\beta D_i u + \left. D^{\alpha - \beta} c D^\beta u \right], \tag{3.5}
\]

where \( \binom{\alpha}{\beta} = \frac{\alpha!}{\beta! (\alpha - \beta)!} \). This shows that \( \bar{u} \) is a weak solution to

\[
L \bar{u} = \bar{f} \quad \text{in} \quad \Omega'.
\]

By (3.5), (3.3) and (3.4) we conclude that \( \bar{f} \in L^2(\Omega') \) with

\[
\| \bar{f} \|_{L^2(\Omega')} \leq c \left( \| f \|_{W^{m+2,2}(\Omega)} + \| u \|_{L^2(\Omega)} \right).
\]

Theorem 3.9 implies \( \bar{u} \in W^{2,2}(\Omega') \) with the estimate

\[
\| \bar{u} \|_{W^{2,2}(\Omega')} \leq c \left( \| \bar{f} \|_{L^2(\Omega')} + \| \bar{u} \|_{L^2(\Omega')} \right)
\]

\[\leq c \left( \| f \|_{W^{m+2,2}(\Omega)} + \| u \|_{L^2(\Omega)} \right).
\]

This holds true for every multi-index \( \alpha \) with \( |\alpha| = m + 1 \) and \( \bar{u} = D^\alpha u \). This implies \( u \in W^{m+3,2}(\Omega') \) and

\[
\| u \|_{W^{m+3,2}(\Omega')} \leq \| \bar{u} \|_{W^{m+2,2}(\Omega')} + \| D^\alpha u \|_{W^{2,2}(\Omega')}
\]

\[\leq c \left( \| f \|_{W^{m+2,2}(\Omega)} + \| u \|_{L^2(\Omega)} \right) + c \left( \| f \|_{W^{m+2,2}(\Omega)} + \| u \|_{L^2(\Omega)} \right)
\]

\[\leq c \left( \| f \|_{W^{m+2,2}(\Omega)} + \| u \|_{L^2(\Omega)} \right).
\]

This completes the proof. \( \square \)
Remark 3.13. By the higher order Sobolev embedding, we obtain that \( u \in C^1(\Omega) \) when \( 2(m+2) > n \) and \( u \in C^2(\Omega) \) when \( 2m > n \). Thus
\[
\bigcap_{m=0}^{\infty} W^{m+2,2}_{\text{loc}}(\Omega) = C^\infty(\Omega).
\]

Theorem 3.12 can be applied recursively with \( m = 0, 1, 2, \ldots \) to conclude smoothness of a weak solution if the data is smooth.

Theorem 3.14 (Smoothness). Assume that
\[
a_{ij}, b_i, c \in C^\infty(\Omega), \quad i, j = 1, \ldots, n, \quad \text{and} \quad f \in C^\infty(\Omega).
\]

Let \( u \in W^{1,2}(\Omega) \) be a weak solution of \( Lu = f \) in \( \Omega \), where \( L \) is as in (1.1). Then \( u \in C^\infty(\Omega) \).

The Moral: A weak solution is smooth if the data is smooth. Note that no boundary conditions are assumed, so that this regularity result applies to PDEs with Dirichlet, Neumann or other boundary conditions. Moreover, it shows that possible singularities on the boundary do not propagate inside the domain. Observe that these regularity results are based on estimates that are proved from structural ellipticity properties of the PDE. Thus the result holds for a whole class of PDEs instead of a particular PDE.

Remarks 3.15:

1. For the corresponding estimates up to the boundary, we refer to [2], p. 336–345.
2. The corresponding estimates in the spaces of Hölder continuous functions are called the Schauder estimates, but this topic will not be discussed here.
3. The corresponding estimates in \( L^p \) with \( 1 < p < \infty \) are called the Calderón-Zygmund estimates, but this topic will not be discussed here.
In the previous chapter we discussed regularity of weak solutions under smoothness assumptions on the coefficients, but this chapter focuses regularity of weak solutions under the assumption that the coefficients are only bounded and measurable functions. We give a treatment of a remarkable De Giorgi-Nash-Moser result that weak solutions of the equation

$$- \text{div}(ADu) = - \sum_{i,j=1}^{n} D_j(a_{ij}D_i u) = 0 \quad \text{in} \quad \Omega$$

(4.1)

are locally Hölder continuous under the ellipticity assumption

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad 0 < \lambda \leq \Lambda,$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. See Definitions 1.2 and 1.7 for precise definitions. This result was proved by De Giorgi and Nash independently in the 1950’s and it is one of the major results in PDEs. We shall consider Moser’s proof of this result. Throughout we assume that $a_{ij} \in L^{\infty}(\Omega), i,j = 1, \ldots, n$, that is, the coefficients are only bounded and measurable functions. Instead of the general equation (1.1), we only consider the case $b_i = 1, i = 1, \ldots, n, c = 0$ and $f = 0$ in this chapter. Essential features and challenges of the theory are already visible in this case.

### 4.1 Serrin’s example

We begin with reconsidering Serrin’s example of a pathological weak solution, see [15] and Example 1.11. This example shows that under the assumption that $a_{ij} \in L^{\infty}(\Omega), i,j = 1, \ldots, n$, the best result we can hope for is that weak solutions are locally Hölder continuous. Next we modify the example to justify the assumption that $u \in W^{1,2}_{\text{loc}}(\Omega)$ from the point of view of regularity theory.
Example 4.1. Let $n \geq 2$ and $0 < \epsilon < 1$. The function $u : B(0, 1) \to \mathbb{R}$,

$$u(x) = u(x_1, \ldots, x_n) = x_1 |x|^{1-n-\epsilon}$$

is a classical solution to

$$- \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u(x)) = 0 \quad \text{for every} \quad x \in B(0, 1) \setminus \{0\},$$

where

$$a_{ij}(x) = \delta_{ij} + (a - 1) \frac{x_i x_j}{|x|^2}, \quad i, j = 1, \ldots, n,$$

and

$$a = \frac{n-1}{\epsilon(\epsilon + n-2)}.$$ 

It is an exercise to show that the coefficients are bounded and that the uniform ellipticity condition in Definition 1.2 is satisfied with $\lambda = 1$ and $\Lambda = a$.

We have

$$u \in W^{1,p}(\Omega), \quad \text{if} \quad p < \frac{n}{n + \epsilon - 1}. $$

(exercise). Observe that $p < 2$, when $n \geq 2$, and thus

$$u \notin W^{1,2}(\Omega) \quad \text{for every} \quad 0 < \epsilon < 1.$$ 

However, as in Example 1.11, we see that

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \varphi \, dx = 0$$

for every $\varphi \in C^\infty_0(B(0,1))$ (exercise). In this sense $u$ is a weak solution to

$$- \sum_{i,j=1}^{n} D_j(a_{ij} D_i u) = 0 \quad \text{in} \quad B(0, 1),$$

but $u \notin W^{1,2}(\Omega)$ for every $0 < \epsilon < 1$.

Clearly the function $u$ is neither locally bounded nor has a local maximum principle. Moreover, local boundedness fails to hold even in the class of solutions in $W^{1,p}(\Omega)$, where $p < 2$ is any fixed number when $n = 2$.

It can be used, moreover, to show that the Dirichlet problem need not have a unique solution. In fact, let $v \in W^{1,2}(\Omega)$ be the unique weak solution with the same boundary values on $\partial \Omega$ as $u$. Then $u - v = 0$ on $\partial \Omega$, but $u - v$ is not identically zero in $\Omega$. This shows that the identically zero function and $v - u$ are weak solutions to the Dirichlet problem with zero boundary values. Thus the problem has two solutions corresponding to the same data, provided we give up the requirement that these solutions belong to $W^{1,2}(\Omega)$.

The Moral: Local boundedness, uniqueness and maximum principle do not hold without the assumption that a weak solution belongs to $W^{1,2}_{\text{loc}}(\Omega)$. Thus the usual requirement that a weak solution belongs to $W^{1,2}_{\text{loc}}(\Omega)$ is an essential part of the theory.
4.2 Super- and subsolutions

**Motivation:** Assume that \( u \in C^2(\Omega) \), \( a_{ij} \in C^1(\Omega) \) satisfies

\[
- \sum_{i,j=1}^{n} D_j(a_{ij}D_iu) \geq 0 \quad \text{in} \quad \Omega.
\]

Let \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi \geq 0 \). Then we can integrate by parts and obtain

\[
0 \leq \int_{\Omega} - \sum_{i,j=1}^{n} D_j(a_{ij}D_iu)\varphi \, dx = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}D_iuD_j\varphi \, dx
\]

for every \( \varphi \in C_0^\infty(\Omega) \).

On the other hand, if

\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}D_iuD_j\varphi \, dx \geq 0
\]

for every \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi \geq 0 \), then

\[
\int_{\Omega} - \sum_{i,j=1}^{n} D_j(a_{ij}D_iu)\varphi \, dx \geq 0
\]

for every \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi \geq 0 \) and consequently

\[
- \sum_{i,j=1}^{n} D_j(a_{ij}D_iu) \geq 0 \quad \text{in} \quad \Omega.
\]

**The Moral:** A function \( u \in C^2(\Omega) \) is a classical supersolution of \((4.1)\) if and only if it is a weak supersolution of \((4.1)\) in the sense of the definition below. Observe that the negative sign in front of the second order terms disappears after integration by parts.

**Definition 4.2.** \( u \in W^{1,2}_{\text{loc}}(\Omega) \) is a weak supersolution of \((4.1)\), if

\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x)D_iuD_j\varphi \, dx \geq 0
\]

for every \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi \geq 0 \). For a subsolution, we require

\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x)D_iuD_j\varphi \, dx \leq 0
\]

for all such test functions.

**The Moral:** Every weak solution is a weak super- and subsolution. The advantage is that the properties of super- and subsolutions can be considered separately.
CHAPTER 4. LOCAL HÖLDER CONTINUITY

Remarks 4.3:
(1) By Lemma 2.30, a function \( u \in W^{1,2}_{\text{loc}}(\Omega) \) is a weak supersolution (subsolution and solution, respectively) in \( \Omega \) if and only if
\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u D_j v \, dx \geq 0
\]
for every \( v \in W^{1,2}_0(\Omega) \) with \( v > 0 \) almost everywhere in \( \Omega \) (exercise).

(2) \( u \in W^{1,2}_{\text{loc}}(\Omega) \) is a weak solution if and only if it is both super- and subsolution in \( \Omega \) (exercise).

(3) \( u \) is a weak supersolution if and only if \( -u \) is a weak subsolution (exercise).

Lemma 4.4. If \( u \in W^{1,2}_{\text{loc}}(\Omega) \) is a weak subsolution of (4.1), then \( u^+ = \max\{u, 0\} \) is a weak subsolution in \( \Omega \).

THEOREM: The class of weak subsolutions is closed with respect to truncation from below. The class of weak solutions does not have the corresponding property.

Proof. By properties of Sobolev spaces, we have \( u^+ \in W^{1,2}_{\text{loc}}(\Omega) \). Let \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi \geq 0 \). Denote
\[
v_k = \min\{ku^+, 1\}, \quad k = 1, 2, \ldots
\]
Then \( (v_k) \) is an increasing sequence, \( 0 \leq v_k \leq 1 \), \( k = 1, 2, \ldots \),
\[
\lim_{k \to \infty} v_k(x) = \chi_{\{x \in \Omega : u(x) > 0\}}(x), \quad x \in \Omega,
\]
and we choose \( v_k \varphi \in W^{1,2}_0(\Omega) \) as a test function. Notice that \( v_k \varphi \geq 0 \) and that
\[
D_j v_k = \begin{cases} kD_j u & \text{almost everywhere in } \{x \in \Omega : 0 < ku(x) < 1\}, \\ 0 & \text{almost everywhere in } \{x \in \Omega : k u(x) \geq 1\} \cup \{x \in \Omega : u(x) \leq 0\}. \end{cases}
\]
The Leibniz rule gives
\[
0 \geq \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u D_j (v_k \varphi) \, dx
= \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u (\varphi D_j v_k + v_k D_j \varphi) \, dx
= k \int_{\{x \in \Omega : 0 < ku(x) < 1\}} \sum_{i,j=1}^{n} a_{ij} \varphi D_i u D_j u \, dx + \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} v_k D_i u D_j \varphi \, dx.
\]
The previous estimate together with the ellipticity implies that
\[
\int_{\Omega} \sum_{i,j=1}^{n} a_{ij} v_k D_i u D_j \varphi \, dx \leq -k \int_{\{x \in \Omega : 0 < ku(x) < 1\}} \varphi \sum_{i,j=1}^{n} a_{ij} D_i u D_j u \, dx
\leq -k \lambda \int_{\{x \in \Omega : 0 < ku(x) < 1\}} \varphi |Du|^2 \, dx \leq 0.
\]
Since
\[ \left| \sum_{i,j=1}^{n} a_{ij} v_k D_i u D_j \varphi \right| \leq \sum_{i,j=1}^{n} |a_{ij}| |v_k| |D_i u| |D_j \varphi| \]
\[ \leq \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)} \|D_j \varphi\|_{L^\infty(\Omega)} |D_i u| \]
\[ \leq c \sum_{i,j=1}^{n} |D_i u| \in L^1(\Omega), \]
we may use the Lebesgue dominated convergence theorem to conclude that
\[ \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} v_k D_i u D_j \varphi \, dx = \lim_{k \to \infty} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} v_k D_i u D_j \varphi \, dx \]
\[ = \lim_{k \to \infty} \sum_{i,j=1}^{n} a_{ij} v_k D_i u D_j \varphi \, dx \leq 0 \]
for every \( \varphi \in C^\infty(\Omega) \) with \( \varphi \geq 0 \).

**THE MORAL:** The proof is based on a clever choice of a test function.

**Remark 4.5.** The following versions of the previous result are left as exercises.

1. If \( u \) is a weak subsolution, then \( \max\{u, k\}, k \in \mathbb{Z} \), is a weak subsolution.
2. If \( u, v \) are weak subsolutions, then \( \max\{u, v\} \) is a weak subsolution.
3. If \( u \) is a weak supersolution, then \( \min\{u, k\}, k \in \mathbb{Z} \), is a weak supersolution.
4. If \( u, v \) are weak supersolutions, then \( \min\{u, v\} \) is a weak supersolution.
5. If \( u \) is a weak subsolution and \( f \in C^2(\mathbb{R}) \) with \( f(0) = 0 \), \( f'' \geq 0 \) (\( f \) is convex) and \( f' \geq 0 \), then \( f \circ u \) is a weak subsolution.
6. If \( u \) is a weak supersolution and \( f \in C^2(\mathbb{R}) \) with \( f(0) = 0 \), \( f'' \leq 0 \) (\( f \) is concave) and \( f' \geq 0 \), then \( f \circ u \) is a weak supersolution.
7. If \( u \) is a weak solution and \( f \in C^2(\mathbb{R}) \) is convex, then \( f \circ u \) is a weak subsolution.

In properties (5)–(7) we assume \( f \in C^2(\mathbb{R}) \) is such that the chain rule holds for \( f \circ u \).

**THE MORAL:** The classes of super- and subsolutions are more flexible than solutions. In particular, super- and subsolutions can be modified as above. The corresponding modifications are not possible in the class of weak solutions.

### 4.3 Caccioppoli estimates

Next we prove a Caccioppoli type energy estimate. The purpose of Caccioppoli type estimates is to provide estimates for the gradient of the solutions with respect
to the function itself. A combination of a Caccioppoli type estimate and Sobolev embedding provides us reverse H"older inequalities. In many cases the PDE is used only to prove Caccioppoli estimates and the rest of the argument applies to all functions that satisfy the corresponding estimate. This is a powerful method, since it applies to a whole class of PDEs simultaneously.

**Theorem 4.6 (Caccioppoli estimate for subsolutions).** Assume that $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak subsolution of (4.1) in $\Omega$ and let $\alpha > 0$. Then there exists $c = c(\lambda, \Lambda)$ such that

$$\int_{\{x \in \Omega : u(x) > 0\}} u^{\alpha-1} |Du|^2 \varphi^2 \, dx \leq c \int_{\{x \in \Omega : u(x) > 0\}} u^{\alpha+1} |D\varphi|^2 \, dx$$

for every $\varphi \in C^0_\infty(\Omega)$ with $\varphi \geq 0$.

**Remark 4.7.** By choosing $\alpha = 1$, we have

$$\int_{\Omega} |Du|^2 \varphi^2 \, dx \leq c \int_{\Omega} |u|^2 |D\varphi|^2 \, dx.$$  

Take a cutoff function $\varphi \in C^\infty_0(B(x,2r))$ with $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B(x,r)$ and $|D\varphi| \leq \frac{c}{r}$. This gives

$$\int_{B(x,r)} |Du|^2 \, dy = \int_{B(x,2r)} |Du|^2 \varphi^2 \, dy \leq \int_{B(x,2r)} |Du|^2 \varphi^2 \, dy \leq c \int_{B(x,2r)} |u|^2 |D\varphi|^2 \, dy \leq \frac{c}{r^2} \int_{B(x,2r)} |u|^2 \, dy.$$  

If $u$ is a solution of (4.1), then also $u - u_{B(x,2r)}$ is a solution, and thus

$$\int_{B(x,r)} |Du|^2 \, dy \leq \frac{c}{r^2} \int_{B(x,2r)} |u - u_{B(x,2r)}|^2 \, dy.$$  

On the other hand, by the Poincaré inequality

$$\int_{B(x,2r)} |u - u_{B(x,2r)}|^2 \, dy \leq c r^2 \int_{B(x,2r)} |Du|^2 \, dy$$

for every $u \in W^{1,2}_{\text{loc}}(\Omega)$. Thus the Caccioppoli estimate can be seen as a reverse Poincaré inequality. Compare to Remark 3.11.

**Proof.** By Lemma 4.4 we may assume that $u = u^+$. We would like to apply $u^\alpha \varphi^2$ as a test function, but it is not clear that this function belongs to $W^{1,2}_0(\Omega)$. Thus we modify the test function in the following manner. Let

$$\psi_k = \varphi^2 \min\{u^+, ku\} \quad k = 1, 2, \ldots,$$

Observe that $\psi_k \in W^{1,2}_0(\Omega)$ and $\psi_k \geq 0$, $k = 1, 2, \ldots$. Moreover, $(\psi_k)$ is an increasing sequence,

$$\lim_{k \to \infty} \psi_k(x) = u(x)^2, \quad x \in \Omega,$$
\[ D_j \psi_k = 2 \psi(D_j \varphi) \min\{u^a, ku\} + (D_j \min\{u^a, ku\}) \psi^2, \quad j = 1, \ldots, n. \]

Since \( u \) is a weak subsolution, we have
\[
0 \geq \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \psi_k \, dx
= \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u (D_j \min\{u^a, ku\}) \psi^2 \, dx + 2 \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} D_i u (D_j \varphi) \psi \min\{u^a, ku\} \, dx.
\]

Denote
\[
\Omega_k = \{ x \in \Omega : 0 < u^a(x) \leq ku(x), \quad k = 1, 2, \ldots \}
\]

Notice that \( D_j u = 0 \) almost everywhere in the set where \( u = 0 \). Therefore we have
\[
D_j \min\{u^a, ku\} = \begin{cases} a u^{a-1} D_j u & \text{almost everywhere in } \Omega_k, \\ ku & \text{almost everywhere in } \Omega \setminus \Omega_k. \end{cases}
\]

The previous inequality implies that
\[
a \int_{\Omega_k} \sum_{i,j=1}^{n} a_{ij} D_i u D_j u u^{a-1} \psi^2 \, dx + k \int_{\Omega \setminus \Omega_k} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \psi \psi^2 \, dx
\leq 2 \int_{\Omega} \psi \min\{u^a, ku\} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \varphi \psi^2 \, dx
\leq 2c \int_{\Omega_k} \psi u^a \sum_{i,j=1}^{n} |D_i u||D_j \varphi| \, dx + 2kc \int_{\Omega \setminus \Omega_k} \psi u \sum_{i,j=1}^{n} |D_i u||D_j \varphi| \, dx
\leq c \int_{\Omega_k} \psi u^a |D u||D \varphi| \, dx + kc \int_{\Omega \setminus \Omega_k} \psi u |D u||D \varphi| \, dx.
\]

Next we first apply the uniform ellipticity condition to the previous estimate, and then we use Young’s inequality with epsilon (exercise) to have
\[
a \lambda \int_{\Omega_h} u^{a-1} |D u|^2 \psi^2 \, dx + k \lambda \int_{\Omega_h} |D u|^2 \psi^2 \, dx
\leq c \int_{\Omega_k} \psi u^a |D u||D \varphi| \, dx + kc \int_{\Omega \setminus \Omega_k} \psi u |D u||D \varphi| \, dx
\leq \frac{a \lambda}{2} \int_{\Omega_h} u^{a-1} |D u|^2 \psi^2 \, dx + \frac{\lambda k}{2} \int_{\Omega_h} |D u|^2 \psi^2 \, dx
+ \frac{c}{a} \int_{\Omega_k} u^{a+1} |D \varphi|^2 \, dx + ck \int_{\Omega \setminus \Omega_k} u^2 |D \varphi|^2 \, dx.
\]

Since \( u^a \leq ku \) in \( \Omega_k \) and \( u \in W^{1,2}_{\text{loc}}(\Omega) \), we have
\[
\int_{\Omega_k} u^{a-1} |D u|^2 \psi^2 \, dx \leq k \int_{\Omega_k} |D u|^2 \psi^2 \, dx < \infty
\]

and
\[
\int_{\Omega \setminus \Omega_k} |D u|^2 \psi^2 \, dx \leq \int_{\Omega} |D u|^2 \psi^2 \, dx < \infty.
\]
so that these terms can be absorbed into the left-hand side. This gives

\[ \alpha \int_{\Omega_k} u^{\alpha-1}|Du|^2 \phi^2 \, dx + k \int_{\Omega \setminus \Omega_k} |Du|^2 \phi^2 \, dx \]

\[ \leq \frac{c}{\alpha} \int_{\Omega_k} u^{\alpha+1} |D\phi|^2 \, dx + ck \int_{\Omega \setminus \Omega_k} u^2 |D\phi|^2 \, dx, \]

where

\[ k \int_{\Omega \setminus \Omega_k} u^2 |D\phi|^2 \, dx \leq \int_{\Omega \setminus \Omega_k} u^{\alpha+1} |D\phi|^2 \, dx \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty, \]

since \( ku \leq u^\alpha \) in \( \Omega \setminus \Omega_k \). Here we may assume that \( u^{\alpha+1} |D\phi|^2 \in L^1(\Omega) \), since otherwise the claim is clear. Consequently

\[ \int_{\Omega} u^{\alpha-1} |Du|^2 \phi^2 \, dx = \lim_{k \to \infty} \int_{\Omega_k} u^{\alpha-1} |Du|^2 \phi^2 \, dx \]

\[ \leq \lim_{k \to \infty} \left( \frac{c}{a^2} \int_{\Omega_k} u^{\alpha+1} |D\phi|^2 \, dx + \frac{ck}{\alpha} \int_{\Omega \setminus \Omega_k} u^2 |D\phi|^2 \, dx \right) \]

\[ = \frac{c}{a^2} \int_{\Omega} u^{\alpha+1} |D\phi|^2 \, dx. \]

The last equality follows from the Lebesgue dominated convergence theorem. □

**Theorem 4.8 (Caccioppoli estimate for supersolutions).** Assume that \( u \in W^{1,2}_{\text{loc}}(\Omega), \ u \geq 0 \), is a weak supersolution of (4.1) in \( \Omega \) and let \( \alpha < 0 \). Then there exists \( c = c(\lambda, \Lambda) \) such that

\[ \int_{\{x \in \Omega: u(x) > 0\}} u^{\alpha-1} |Du|^2 \phi^2 \, dx \leq \frac{c}{|a|^2} \int_{\{x \in \Omega: u(x) > 0\}} u^{\alpha+1} |D\phi|^2 \, dx \]

for every \( \phi \in C_0^\infty(\Omega) \) with \( \phi \geq 0 \).

**The Moral:** This is the same estimate as in Theorem 4.6 for negative values of \( \alpha \).

**Proof.** Let \( u_k = u + \frac{1}{k}, \ k = 1, 2, \ldots, \) and apply \( u_k^\alpha \phi^2 \in W^{1,2}_0(\Omega) \) as a test function. Then

\[ D_j(u_k^\alpha \phi^2) = 2\phi(D_j \phi)u_k^\alpha + \alpha u_k^{\alpha-1}(D_j u_k)\phi^2, \quad j = 1, \ldots, n. \]

Since \( u \) is a weak supersolution, we have

\[ 0 \leq \int_{\Omega} \sum_{i,j=1}^n a_{ij} D_i u D_j(u_k^\alpha \phi^2) \, dx \]

\[ = 2 \int_{\Omega} \phi u_k^\alpha \sum_{i,j=1}^n a_{ij} D_i u D_j \phi \, dx + \alpha \int_{\Omega} \phi^2 u_k^{\alpha-1} \sum_{i,j=1}^n a_{ij} D_i u D_j u \, dx. \]
By using the previous equation and ellipticity, we obtain the estimate

\[
\int_\Omega \varphi \frac{u_k^{a-1} |Du|^2}{u} \, dx \leq \frac{1}{\lambda} \int_\Omega \varphi \frac{u_k^{a-1}}{u} \sum_{i,j=1}^n a_{ij} D_i u D_j u \, dx
\]

\[
\leq - \frac{2}{\lambda} \int_\Omega \varphi u_k^{a} \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi \, dx
\]

\[
\leq \frac{c}{|a|} \int_\Omega \varphi \frac{u_k^{a}}{|D\varphi|} \, dx \quad (a_{ij} \in L^\infty(\Omega))
\]

\[
= \frac{c}{|a|} \int_\Omega \varphi u_k^{a} \frac{u_k^{a+1}}{|D\varphi|} \, dx
\]

\[
\leq \frac{1}{2} \int_\Omega \varphi \frac{u_k^{a-1}}{u^2} |Du|^2 \, dx + \frac{c}{|a|^2} \int_\Omega u_k^{a+1} |D\varphi|^2 \, dx. \quad \text{(Young with } \varepsilon)\]

Thus

\[
\int_\Omega \varphi \frac{u_k^{a-1}}{u^2} |Du|^2 \, dx \leq \frac{c}{|a|^2} \int_\Omega u_k^{a+1} |D\varphi|^2 \, dx.
\]

By the monotone and dominated convergence theorem, we conclude that

\[
\int_\Omega \varphi \frac{u_k^{a-1}}{u^2} |Du|^2 \, dx = \int_\Omega \lim_{k \to \infty} \varphi \frac{u_k^{a-1}}{u} |Du|^2 \, dx
\]

\[
= \lim_{k \to \infty} \int_\Omega \varphi \frac{u_k^{a-1}}{u} |Du|^2 \, dx
\]

\[
\leq \lim_{k \to \infty} \frac{c}{|a|^2} \int_\Omega u_k^{a+1} |D\varphi|^2 \, dx
\]

\[
= \frac{c}{|a|^2} \int_\Omega \lim_{k \to \infty} u_k^{a+1} |D\varphi|^2 \, dx
\]

\[
\leq \frac{c}{|a|^2} \int_\Omega u_k^{a+1} |D\varphi|^2 \, dx.
\]

Observe that if \(a + 1 < 0\), we may use the monotone convergence theorem in taking the limit inside the integral. If \(-1 \leq a < 0\), then \(u_k^{a+1} \leq (u + 1)^{a+1}\) and we may apply the dominated convergence theorem. \(\square\)

**Theorem 4.9 (Logarithmic Caccioppoli inequality).** If \(u > 0\) is a weak supersolution of (4.1) in \(\Omega\), then there exists \(c = c(\lambda, \Lambda)\) such that

\[
\int_\Omega \varphi^2 |D \log u|^2 \, dx \leq c \int_\Omega |D\varphi|^2 \, dx
\]

for every \(\varphi \in C^\infty_0(\Omega), \varphi \geq 0\).

**Remark 4.10.** The right hand side is independent of \(u\).

**Proof.** Theorem 4.8 with \(a = -1\) gives

\[
\int_\Omega \varphi^2 |D \log u|^2 \, dx = \int_\Omega \varphi^2 \frac{|Du|^2}{u^2} \, dx \leq c \int_\Omega |D\varphi|^2 \, dx. \quad \square
\]
4.4 Essential supremum and infimum

Our goal is to obtain estimates for the maximum and the minimum of a solution to a PDE. Since functions in Sobolev spaces are defined only up to a set of measure zero, we recall the definition of essential supremum and infimum.

**Definition 4.11.** Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set and $f : A \to [-\infty, \infty]$ a Lebesgue measurable function. The essential supremum of $f$ is

$$\text{ess sup } f(x) = \inf\{M : f(x) \leq M \text{ for almost every } x \in A\}$$

and the essential infimum of $f$ is

$$\text{ess inf } f(x) = \sup\{m : f(x) \geq m \text{ for almost every } x \in A\}.$$

**The Moral:** Essential supremum is supremum outside sets of measure zero.

**Remark 4.12.** Observe that for the standard supremum we have

$$\sup_{x \in A} f(x) = \inf\{M : \{x \in A : f(x) > M\} = \emptyset\}.$$

Analogously, essential infimum is infimum outside sets of measure zero.

$$\inf_{x \in A} f(x) = \sup\{m : \{x \in A : f(x) < m\} = \emptyset\}.$$

Moreover,

$$f(x) \leq \text{ess sup } f(x) \text{ for almost every } x \in A$$

and

$$f(x) \geq \text{ess inf } f(x) \text{ for almost every } x \in A.$$

The integral average of $f$ in $A$, $0 < |A| < \infty$, is denoted by

$$\frac{1}{|A|} \int_A f \, dx.$$

Let $-\infty < p < q < \infty$, $p \neq 0$, $q \neq 0$ and assume that $0 < |A| < \infty$. By Hölder’s, or Jensen’s, inequality

$$\text{ess inf } |f| \leq \left(\int_A |f|^p \, dx\right)^{\frac{1}{p}} \leq \left(\int_A |f|^q \, dx\right)^{\frac{1}{q}} \leq \text{ess sup } |f|.$$

Thus the integral average is an increasing function of the power.
Theorem 4.13. Let \( f : A \rightarrow [-\infty, \infty] \) be a Lebesgue measurable function and \( 0 < |A| < \infty \). Then

1. \( \lim_{p \to \infty} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}} = \text{esssup}_{A} |f| \) and
2. \( \lim_{p \to \infty} \left( \int_A |f|^{-p} \, dx \right)^{-\frac{1}{p}} = \text{essinf}_{A} |f| \).

**The Moral:** This gives a method to derive estimates for supremum and infimum by uniform estimates for integral averages with powers. The Moser iteration technique is based on this observation.

**Remarks 4.14:**

1. The integral average can be replaced with the integral.
2. We leave as an exercise to prove that
   \[ \lim_{p \to 0} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}} = e^{\int_A \log |f| \, dx}. \]

**Proof.** (1) Assume that \( \text{esssup}_A |f| < \infty \). Then
   \[ \int_A |f|^p \, dx \leq \text{esssup}_A |f|^p \int_A 1 \, dx = |A| \text{esssup}_A |f|^p, \]
   which implies that for every \( p, 1 \leq p < \infty \),
   \[ \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}} \leq \text{esssup}_A |f| \]
   and, in particular, that
   \[ \limsup_{p \to \infty} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}} \leq \text{esssup}_A |f|. \]

This clearly holds true also in the case \( \text{esssup}_A |f| = \infty \).

Denote \( E_\lambda = \{ x \in A : |f(x)| > \lambda \} \). For every \( \lambda \) with \( 0 < \lambda < \text{esssup}_A |f| \), we have \( |E_\lambda| > 0 \). Since \( |f|^p \geq \lambda^p \) in \( E_\lambda \), we obtain
   \[ \lambda^p |E_\lambda| \leq \int_{E_\lambda} |f|^p \, dx \leq \int_A |f|^p \, dx. \]

By taking the \( p \)-th root we have
   \[ \lambda |E_\lambda|^{\frac{1}{p}} \leq \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}}. \]

Observe that for any \( E_\lambda \) with \( 0 < |E_\lambda| < \infty \), we have \( |E_\lambda|^{\frac{1}{p}} \to 1 \) as \( p \to \infty \). Thus
   \[ \lambda \leq \liminf_{p \to \infty} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}}. \]
As $0 < |A| < \infty$, we have also that $|A|^{-\frac{1}{p}} \rightarrow 1$ as $p \rightarrow \infty$ and thus

$$
\lambda \leq \liminf_{p \rightarrow \infty} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}}.
$$

By letting $\lambda \rightarrow \text{esssup}_A |f|$, we obtain

$$
\text{esssup}_A |f| \leq \liminf_{p \rightarrow \infty} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}}.
$$

All together we have now proved that

$$
\text{esssup}_A |f| \leq \liminf_{p \rightarrow \infty} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}} \leq \limsup_{p \rightarrow \infty} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}} \leq \text{esssup}_A |f|,
$$

which implies that the limit exists and

$$
\text{esssup}_A |f| = \lim_{p \rightarrow \infty} \left( \int_A |f|^p \, dx \right)^{\frac{1}{p}}.
$$

(2) Clearly

$$
\int_A |f|^{-p} \, dx \leq \left( \int_A |f|^p \, dx \right)^{-\frac{1}{p}} \leq \text{essinf}|f|,
$$

and thus

$$
\left( \int_A |f|^{-p} \, dx \right)^{-\frac{1}{p}} \geq \text{essinf}|f|.
$$

By letting $p \rightarrow \infty$ we see that

$$
\lim_{p \rightarrow \infty} \left( \int_A |f|^{-p} \, dx \right)^{\frac{1}{p}} \geq \text{essinf}|f|.
$$

Let $F_\lambda = \{ x \in A : |f(x)| < \lambda \}$. For every $\lambda > \text{essinf}_A |f|$, we have $|F_\lambda| > 0$ and by using the fact that $|f|^{-p} \geq \lambda^{-p}$ in $F_\lambda$, we obtain

$$
\lambda^{-p} |F_\lambda| \leq \int_{F_\lambda} |f|^{-p} \, dx \leq \int_A |f|^{-p} \, dx.
$$

This is equivalent to

$$
\lambda |F_\lambda|^{-\frac{1}{p}} \geq \left( \int_A |f|^{-p} \, dx \right)^{-\frac{1}{p}}.
$$

As $|F_\lambda|^{-\frac{1}{p}} \rightarrow 1$ as $p \rightarrow \infty$, we conclude

$$
\lambda \geq \limsup_{p \rightarrow \infty} \left( \int_A |f|^{-p} \, dx \right)^{-\frac{1}{p}} = \limsup_{p \rightarrow \infty} \left( \int_A |f|^{-p} \, dx \right)^{-\frac{1}{p}}.
$$

Since this holds for every $\lambda > \text{essinf}_A |f|$, we obtain

$$
\text{essinf}_A |f| \geq \limsup_{p \rightarrow \infty} \left( \int_A |f|^{-p} \, dx \right)^{-\frac{1}{p}}.
$$

\( \square \)

**Remark 4.15.** Part (2) of the theorem above could be also proved by applying the part (1) to the function $\frac{1}{|f|}$. 
4.5 Estimates from above

The next result shows that weak subsolutions to a PDE are locally bounded from above. The proof is based on the Moser iteration technique together with a Caccioppoli inequality and a Sobolev inequality.

**Theorem 4.16 (Local boundedness from above).** Assume that \( u \in W^{1,2}_{\text{loc}}(\Omega) \) is a weak subsolution of (4.1) in \( \Omega \). Then there exists \( c = c(n, \lambda, \Lambda, \beta) \) such that

\[
\text{esssup}_{B(x,r)} u^+ \leq c \left( \frac{1}{(R-r)^n} \int_{B(x,R)} (u^+)^\beta \, dy \right)^{\frac{1}{\beta}},
\]

whenever \( \beta > 1 \) and \( B(x,R) \subset \Omega, \, 0 < r < R \).

**The Moral:** By choosing \( R = 2r \) we have

\[
\text{esssup}_{B(x,r)} u \leq \text{esssup}_{B(x,r)} u^+ \leq c \left( \int_{B(x,2r)} (u^+)^\beta \, dy \right)^{\frac{1}{\beta}} \leq c \left( \int_{B(x,2r)} |u|^{\beta} \, dy \right)^{\frac{1}{\beta}}.
\]

In particular, weak subsolutions are locally bounded from above. Sometimes this result is called the weak maximum principle, since it gives an estimate of the supremum in terms of positive powers of integral averages. This is a counterpart of the mean value property of harmonic functions for more general PDEs.

**Proof.** Assume that \( u \in L^\beta_{\text{loc}}(\Omega) \). Observe that this holds, in particular, when \( \beta = 2 \). By Lemma 4.4, we may assume that \( u = u^+ \). Choose a cutoff function \( \varphi \in C_0^\infty(B(x,R)) \) such that \( \varphi = 1 \) in \( B(x,r) \), \( 0 \leq \varphi \leq 1 \) and \( |D\varphi| \leq \frac{c}{R^2} \). By the Caccioppoli estimate, Theorem 4.6, we obtain

\[
\int_{\Omega} |D(\varphi u^\frac{\beta}{2})|^2 \, dy = \int_{\Omega} \left| \frac{\beta}{2} u^{\frac{\beta}{2}-1} Du \right|^2 \varphi^2 \, dy
\]

\[
= \beta^2 \int_{\Omega} u^{\beta-2} |Du|^2 \varphi^2 \, dy
\]

\[
\leq c \left( \frac{\beta}{\beta - 1} \right)^2 \int_{\Omega} u^\beta |D\varphi|^2 \, dy.
\]

Since

\[
|D(\varphi u^\frac{\beta}{2})| \leq |\varphi D(u^\frac{\beta}{2})| + |u^\frac{\beta}{2} D\varphi|,
\]

we obtain

\[
\int_{\Omega} |D(\varphi u^\frac{\beta}{2})|^2 \, dy \leq 2 \left( \int_{\Omega} |\varphi D(u^\frac{\beta}{2})|^2 \, dy + \int_{\Omega} |u^\frac{\beta}{2} D\varphi|^2 \, dy \right)
\]

\[
\leq c \left( \left( \frac{\beta}{\beta - 1} \right)^2 + 1 \right) \int_{\Omega} |u^\frac{\beta}{2} D\varphi|^2 \, dy.
\]
Notice that
\[ \left( \frac{\beta}{\beta - 1} \right)^2 + 1 = \frac{\beta^2 + (\beta - 1)^2}{(\beta - 1)^2} \leq \frac{4\beta^2 + 4\beta + 1}{(\beta - 1)^2} = \left( \frac{2\beta + 1}{\beta - 1} \right)^2. \]

By the Gagliardo-Nirenberg-Sobolev inequality, we have
\[ \left( \int_{\Omega} |\varphi u_\beta^2|^{2x} \, dy \right)^{\frac{1}{x}} \leq c \int_{\Omega} |D(\varphi u_\beta^2)|^2 \, dy, \quad \kappa = \frac{n}{n-2} > 1. \]

By combining the previous estimates and using the properties of the cutoff function, we obtain the estimate
\[ \left( \int_{\mathcal{B}(x,r)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}} \leq \left( \frac{2\beta + 1}{\beta - 1} \right)^2 \left( \int_{\mathcal{B}(x,R)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}}. \quad (4.2) \]

This is a reverse Hölder inequality. Observe that from \( u \in L^\beta_{\text{loc}}(\Omega) \), we may conclude that \( u \in L^{\beta_0}_{\text{loc}}(\Omega) \) with \( \kappa > 1 \). This gives us a bootstrap method to increase the level of local integrability stepwise. In particular, starting from \( \beta = 2 \), we may iterate \((4.2)\) and conclude that \( u \in L^\beta_{\text{loc}}(\Omega) \) for every \( 1 < \beta < \infty \). Thus all integrals in this proof are finite. Note that if \( \beta_0 > 1 \) and \( \beta \geq \beta_0 \), then
\[ \frac{2\beta + 1}{\beta - 1} \leq \frac{2\beta_0 + 1}{\beta_0 - 1} = B \]

and
\[ \left( \int_{\mathcal{B}(x,r)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}} \leq (cB)^{\frac{1}{\beta}} \left( \frac{2}{R-r} \right)^{\frac{2\beta}{\beta_0}} \left( \int_{\mathcal{B}(x,R)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}}. \quad (4.3) \]

for every \( \beta > \beta_0 \).

**Step 1** Let \( r_0 = R, r_1 = r + \frac{R-r}{2^2} \). By \((4.3)\) we have
\[ \left( \int_{\mathcal{B}(x,r_1)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}} \leq (cB)^{\frac{1}{\beta}} \left( \frac{2}{R-r} \right)^{\frac{2\beta}{\beta_0}} \left( \int_{\mathcal{B}(x,R)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}}. \]

**Step 2** Let \( r_2 = r + \frac{R-r}{2^3} \). By applying \((4.3)\) twice we have
\[ \left( \int_{\mathcal{B}(x,r_2)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}} \leq (cB)^{\frac{1}{\beta}} \left( \frac{2}{R-r} \right)^{\frac{2\beta}{\beta_0}} \left( \int_{\mathcal{B}(x,r_1)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}} \leq (cB)^{\frac{1}{\beta}} \left( \frac{2}{R-r} \right)^{\frac{2\beta}{\beta_0}} \left( \int_{\mathcal{B}(x,R)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}}. \]

**Step k** Let \( r_k = r + \frac{R-r}{2^k}, \ k = 1, 2, \ldots \) By applying \((4.3)\) recursively we have
\[ \left( \int_{\mathcal{B}(x,r_k)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}} \leq (cB)^{\frac{1}{\beta}} \left( \frac{2}{R-r} \right)^{\frac{2\beta}{\beta_0}} \left( \int_{\mathcal{B}(x,R)} u^{\beta_0} \, dy \right)^{\frac{1}{\beta_0}}. \quad (4.4) \]
Let us calculate the sums that appear in the estimate. First, we have
\[ \sum_{i=1}^{k} \frac{1}{k^{i-1}} \leq \sum_{i=1}^{\infty} \frac{1}{k^{i-1}} = \frac{1}{1 - \frac{1}{k}} = \frac{1 - n^{-2}}{n} = \frac{n}{2}. \]

In the second sum, we recognize a derivative of a geometric series and obtain
\[ \sum_{i=1}^{k} \frac{i}{k^{i-1}} \leq \sum_{i=1}^{\infty} \frac{i}{k^{i-1}} = \frac{1}{(1 - \frac{1}{k})^{2}} = \left(\frac{n}{2}\right)^{2}. \]

The left-hand side of (4.4) converges to the $L^\infty$-norm of $u$ as $k \to \infty$ and thus we conclude that
\[ \text{ess sup}_{B(x,r)} u = \lim_{k \to \infty} \left( \int_{B(x,r)} u^{x^{0}} \, dy \right)^{\frac{1}{x^{0}}} \leq c(R - r)^{\frac{\beta}{2}} \left( \int_{B(x,R)} u^{\beta} \, dy \right)^{\frac{1}{\beta}}. \]

The claim follows from this, since we denoted $u = u^+$. \hfill \Box

**Corollary 4.17 (Local boundedness).** Assume that $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution of (4.1) in $\Omega$. Then there exists $c = c(n, \lambda, \Lambda, \beta)$ such that
\[ \text{ess sup}_{B(x,r)}|u| \leq c \left( \frac{1}{(R - r)^{\beta}} \int_{B(x,R)}|u|^\beta \, dy \right)^{\frac{1}{\beta}} \]
whenever $\beta > 1$ and $B(x,R) \subseteq \Omega$, $0 < r < R$.

**THE MORAL:** By choosing $R = 2r$ we have
\[ \text{ess sup}_{B(x,r)}|u| \leq c \left( \int_{B(x,2r)}|u|^\beta \, dy \right)^{\frac{1}{\beta}}. \]

In particular, every weak solution is locally bounded.

**Proof.** By Lemma 4.4, $u^+$ is a weak subsolution and thus by Theorem 4.16 we have $u^+ \in L^\infty_{\text{loc}}(\Omega)$ with
\[ \text{ess sup}_{B(x,r)} u^+ \leq c \left( \frac{1}{(R - r)^{\beta}} \int_{B(x,R)}(u^+)^\beta \, dy \right)^{\frac{1}{\beta}}. \]

On the other hand, since $u$ is a weak solution $-u$ is a weak solution as well. Again by Lemma 4.4, $(-u)^+ = u^-$ is a weak subsolution, and by Theorem 4.16 we have $u^- \in L^\infty_{\text{loc}}(\Omega)$ with
\[ \text{ess sup}_{B(x,r)} u^- \leq c \left( \frac{1}{(R - r)^{\beta}} \int_{B(x,R)}(u^-)^\beta \, dy \right)^{\frac{1}{\beta}}. \]
This shows that \( u = u^+ - u^- \in L^\infty_{\text{loc}}(\Omega) \). Moreover,
\[
\text{ess sup}_{B(x,r)} |u| = \text{ess sup}_{B(x,r)}(u^+ + u^-) \\
\leq c \left( \frac{1}{(R - r)^n} \int_{B(x,R)} (u^+)^\beta \, dy \right)^{\frac{1}{\beta}} + c \left( \frac{1}{(R - r)^n} \int_{B(x,R)} (u^-)^\beta \, dy \right)^{\frac{1}{\beta}} \\
\leq c \left( \frac{1}{(R - r)^n} \int_{B(x,R)} |u|^\beta \, dy \right)^{\frac{1}{\beta}}
\]
whenever \( \beta > 1 \) and \( B(x,R) \subset \Omega \), \( 0 < r < R \).

We will next present a technical lemma, which will be used in proving that Theorem 4.16 actually holds for all \( \beta > 0 \).

**Lemma 4.18.** Let \( \psi : [0,T] \to \mathbb{R} \) be a nonnegative bounded function. If there exists \( A > 0, \alpha > 0 \) and \( 0 < \varepsilon < 1 \) such that
\[
\Psi(r) \leq A(1-r)^{-\alpha} + \varepsilon \Psi(R)
\]
for every \( 0 \leq r < R \leq T \), then there exists \( c = c(\alpha,\varepsilon) \) such that
\[
\Psi(r) \leq cA(1-r)^{-\alpha}
\]
for every \( 0 < r < R \leq T \).

**Proof:** Let \( 0 < \tau < 1 \), \( t_0 = r \) and
\[
t_{i+1} = t_i + (1 - \tau)r^i(R - r), \quad i = 0, 1, 2, \ldots.
\]
Then
\[
\Psi(t_0) \leq \varepsilon \Psi(t_1) + A(1-t_0)^{-\alpha} \quad \text{(assumption)}
\]
\[
= \varepsilon \Psi(t_1) + A(1 - \tau)^{-\alpha}(R - r)^{-\alpha} \quad \text{(definition of } t_i)\]
\[
\leq \varepsilon \Psi(t_2) + A(t_1 - t_2)^{-\alpha} + A(1 - \tau)^{-\alpha}(R - r)^{-\alpha} \quad \text{(assumption)}
\]
\[
= \varepsilon^2 \Psi(t_2) + A(1 - \tau)^{-\alpha}(R - r)^{-\alpha} + A(1 - \tau)^{\alpha}(R - r)^{-\alpha} \quad \text{(definition of } t_i)\]
\[
= \varepsilon^2 \Psi(t_2) + A(1 - \tau)^{-\alpha}(R - r)^{-\alpha}(\varepsilon \tau^{\alpha} + 1)
\]
\[
\leq \varepsilon^k \Psi(t_k) + A(1 - \tau)^{-\alpha}(1 - \tau)^{-\alpha} \sum_{i=0}^{k-1} \varepsilon^i \tau^{-\alpha}.
\]
Choose \( \tau \) such that \( \frac{\varepsilon}{\tau^{\alpha}} < 1 \). Then
\[
\Psi(r) \leq \lim_{k \to \infty} \left( \varepsilon^k \Psi(t_k) + A(1 - \tau)^{-\alpha}(1 - \tau)^{-\alpha} \sum_{i=0}^{k-1} \varepsilon^i \tau^{-\alpha} \right)
\]
\[
= c(\alpha,\varepsilon)A(1-r)^{-\alpha}.
\]
Here the first term converges to zero because \( \Psi \) is bounded.
**Lemma 4.19.** Theorem 4.16 and Corollary 4.17 hold for every $\beta > 0$.

**Proof.**

\[
\text{esssup}_{B(x, r)} u^+ \leq c(R - r)^{-\frac{n}{2}} \left( \int_{B(x, r)} (u^+)^2 \, dy \right)^{\frac{1}{2}} \quad \text{(Theorem 4.16 with } \beta = 2) 
\]

\[
\leq c(R - r)^{-\frac{n}{2}} \left( \int_{B(x, r)} (u^+)^{\beta} \left( \text{esssup}_{B(x, r)} u^+ \right)^{2-\beta} \, dy \right)^{\frac{1}{2}} \quad (\beta < 2) 
\]

\[
= c(R - r)^{-\frac{n}{2}} \left( \int_{B(x, r)} (u^+)^{\beta} \, dy \right)^{\frac{1}{2}} \left( \text{esssup}_{B(x, r)} u^+ \right)^{1-\frac{n}{2}} 
\]

\[
\leq c \text{esssup}_{B(x, r)} u^+ + \frac{c(\epsilon)}{(R - r)^{\beta}} \left( \int_{B(x, r)} (u^+) \, dy \right)^{\frac{1}{2}} \quad \text{(Young with } \epsilon, \beta < 2) 
\]

The claim follows from Lemma 4.18 by choosing $T = R$ and

\[\Psi(r) = \text{esssup}_{B(x, r)} u^+ \].

\[\square\]

### 4.6 Estimates from below

The following property of super- and subsolutions gives us a tool to apply Theorem 4.16 to obtain a lower bound for the infimum of supersolutions in terms of negative powers of integral averages.

**Lemma 4.20.** If $u \geq c > 0$ is a weak supersolution of (4.1), then $v = \frac{1}{u}$ is a weak subsolution.

**Proof.** Let $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$. If $\psi = u^{-2}\varphi$, then $\psi \in W^{1,2}_0(\Omega)$ and

\[D_j \psi = -2u^{-3}D_j u \varphi + u^{-2}D_j \varphi, \quad j = 1, \ldots, n,\]

and

\[0 \leq \int_{\Omega} \sum_{i,j=1}^n a_{ij} D_i u D_j \psi \, dx\]

\[= -2 \int_{\Omega} u^{-3} \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi \, dx + \sum_{i,j=1}^n \int_{\Omega} a_{ij} u^{-2} D_i u D_j \varphi \, dx\]

\[\leq - \sum_{i,j=1}^n \int_{\Omega} a_{ij} D_i v D_j \varphi \, dx\]

for every $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$. Here we used the facts that

\[\int_{\Omega} u^{-3} \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi \, dx \geq \lambda \int_{\Omega} u^{-3} |D u|^2 \varphi \, dx \geq 0\]

and $D_i v = -\frac{D_i u}{u^2}$. \[\square\]
Lemma 4.21. Let \( u \in W^{1,2}_{\text{loc}}(\Omega), u \geq 0 \), be a weak supersolution of (4.1). Then there exists \( c = c(n, \lambda, \Lambda, \beta) \) such that

\[
\left( \frac{1}{(R-r)^n} \int_{B(x,R)} u^{-\beta} \, dy \right)^{-\frac{1}{\beta}} \leq c \text{ess inf}_B u
\]

for every \( \beta > 0, B(x,R) \subset \Omega, 0 < r < R \).

The Moral: By choosing \( R = 2r \) we have

\[
\left( \int_{B(x,2r)} u^{-\beta} \, dy \right)^{-\frac{1}{\beta}} \leq c \text{ess inf}_B u.
\]

Proof. Since we can add constants to weak supersolutions, the function \( u_k = u + \frac{1}{k}, k = 1, 2, \ldots \), is a weak supersolution. By Lemma 4.20, \( \frac{1}{u_k} \) is a weak subsolution. Now by Theorem 4.16 and Lemma 4.19, we have

\[
\text{ess sup}_{B(x,r)} \frac{1}{u_k} \leq c \left( \frac{1}{(R-r)^n} \int_{B(x,R)} \left( \frac{1}{u_k} \right)^{\beta} \, dy \right)^{\frac{1}{\beta}}
\]

or equivalently

\[
\left( \int_{B(x,R)} u_k^{-\beta} \, dy \right)^{-\frac{1}{\beta}} \leq c(R-r)^{-\frac{1}{\beta}} \left( \text{ess sup}_{B(x,r)} \frac{1}{u_k} \right)^{-1} \leq c(R-r)^{-\frac{1}{\beta}} \text{ess inf}_{B(x,r)} u_k.
\]

The claim follows by letting \( k \to \infty \).

Remark 4.22. Another way to prove Lemma 4.21 is to run the Moser iteration technique as in the proof of Theorem 4.16 using Theorem 4.8 for weak supersolutions. This approach completely avoids Lemma 4.20 (exercise).

### 4.7 Harnack’s inequality

Recall that Harnack’s inequality for nonnegative solutions of the Laplace equation can be proved by the mean value property. If \( u \geq 0 \) is a weak solution to (4.1), then by Theorem 4.16 we have

\[
\text{ess sup}_{B(x,r)} u \leq c \left( \int_{B(x,2r)} u^{\beta} \, dy \right)^{\frac{1}{\beta}}
\]

and by Lemma 4.21 we have

\[
\left( \int_{B(x,2r)} u^{-\beta} \, dy \right)^{\frac{1}{\beta}} \leq c \text{ess inf}_{B(x,r)} u
\]

Next we prove the missing inequality

\[
\left( \int_{B(x,2r)} u^{\beta} \, dy \right)^{\frac{1}{\beta}} \leq c \left( \int_{B(x,2r)} u^{-\beta} \, dy \right)^{-\frac{1}{\beta}}
\]
in order to obtain
\[
\text{esssup } u - \text{essinf } u \leq c_{B(x,r)} u \leq c_{B(x,r)} v,
\]
whenever \(B(x,2r) \subset \Omega\). This is Harnack’s inequality for nonnegative weak solutions. Harnack’s inequality states that locally the supremum of a positive solution is bounded by a constant times the infimum of the solution. The only missing piece is the passage over zero. We shall use the theory of BMO functions, in particular, the John-Nirenberg lemma, to overcome this problem. For the theory of BMO functions we refer to the Harmonic Analysis course.

In the theory of BMO it is more convenient to use cubes instead of balls. This is just a technical point and we could work either with cubes or balls throughout. A closed cube is a bounded interval in \(\mathbb{R}^n\), whose sides are parallel to the coordinate axes and equally long, that is,
\[
Q = [a_1,b_1] \times \cdots \times [a_n,b_n]
\]
with \(b_1 - a_1 = \cdots = b_n - a_n\). The side length of a cube \(Q\) is denoted by \(l(Q)\). In case we want to specify the center, we write
\[
Q(x,l) = \{ y \in \mathbb{R}^n : |y_i - x_i| \leq \frac{l}{2}, i = 1, \ldots, n \}
\]
for a cube with center at \(x \in \mathbb{R}^n\) and side length \(l > 0\).

Assume then that \(u \geq 0\) is a weak solution to (4.1). Denote
\[
v_k = \log \left( u + \frac{1}{k} \right), \quad k = 1, 2, \ldots
\]
Then \(u + \frac{1}{k} > 0\), \(k = 1, 2, \ldots\), is a solution to (4.1). Let \(Q(x,2l) \subset \Omega\) and take a cutoff function \(\psi \in C_0^\infty(Q(x,2l)), 0 \leq \psi \leq 1\) such that \(\psi = 1\) on \(Q(x,l)\) and \(|D\psi| \leq \frac{c}{l}\). By the logarithmic Caccioppoli estimate, Theorem 4.9, we have
\[
\int_{Q(x,l)} |Dv_k|^2 \, dy = \int_{Q(x,l)} \left| D \log \left( u + \frac{1}{k} \right) \right|^2 \, dy
\leq \int_{Q(x,l)} \psi^2 \left| D \log \left( u + \frac{1}{k} \right) \right|^2 \, dy
\leq c \int_{Q(x,2l)} |D\psi|^2 \, dy
\leq \frac{c}{l^2} \int_{Q(x,2l)} 1 \, dy = cl^{n-2}, \quad k = 1, 2, \ldots
\]
In particular, this implies that \(|Dv_k| \in L_{\text{loc}}^2(\Omega)\). On the other hand, since \(v_k\) is bounded from below and \(u \in L^2(\Omega)\), we conclude that \(v_k \in L_{\text{loc}}^2(\Omega)\). This implies that \(v_k \in W_{\text{loc}}^{1,2}(\Omega)\), \(k = 1, 2, \ldots\). By the Poincaré inequality we have
\[
\int_{Q(x,l)} |v_k - (v_k)_{Q(x,l)}|^2 \, dy \leq c l^2 \int_{Q(x,l)} |Dv_k|^2 \, dy \leq c, \quad k = 1, 2, \ldots
\]
for every cube $Q(x,l)$ such that $Q(x,2l) \subseteq \Omega$. By Hölder’s, or Jensen’s, inequality
\[
\int_{Q(x,l)} |v_k - (v_k)_{Q(x,l)}| \, dy \leq \left( \int_{Q(x,l)} |v_k - (v_k)_{Q(x,l)}|^2 \, dy \right)^{\frac{1}{2}} \leq c < \infty, \quad k = 1, 2, \ldots,
\]
for every cube $Q(x,l)$ such that $Q(x,2l) \subseteq \Omega$ with $c$ depending only on $n$, $\lambda$ and $\Lambda$.

Observe, that $c$ is independent of the solution $u$ and $k$. This shows that $v_k$ is of bounded mean oscillation (BMO) over such cubes. By the exponential integrability result for BMO-functions, there are $\gamma > 0$ and $c < \infty$, depending only on $n$, $\lambda$ and $\Lambda$, such that
\[
\int_{Q(x,l)} e^{\gamma(v_k - (v_k)_{Q(x,l)})} \, dy \leq c
\]
for every cube $Q(x,l)$ such that $Q(x,2l) \subseteq \Omega$. This implies that
\[
\int_{B(x,r)} e^{\gamma k} \, dy \int_{B(x,r)} e^{-\gamma k} \, dy \leq c \int_{Q(x,2r)} e^{\gamma k} \, dy \int_{Q(x,2r)} e^{-\gamma k} \, dy
= c \int_{Q(x,2r)} e^{\gamma(v_k - (v_k)_{Q(x,2r)})} \, dy \int_{Q(x,2r)} e^{-\gamma(v_k - (v_k)_{Q(x,2r)})} \, dy
\leq c \left( \int_{Q(x,2r)} e^{\gamma(v_k - (v_k)_{Q(x,2r)})} \, dy \right)^2 \leq c,
\]
whenever $Q(x,4r) \subseteq \Omega$. From this we conclude that
\[
\left( \int_{B(x,r)} u^\gamma \, dy \right)^{\frac{1}{\gamma}} = \lim_{k \to \infty} \left( \int_{B(x,r)} \left( u + \frac{1}{k} \right)^\gamma \, dy \right)^{\frac{1}{\gamma}}
= \lim_{k \to \infty} \left( \int_{B(x,r)} e^{\gamma k} \, dy \right)^{\frac{1}{\gamma}}
\leq \lim_{k \to \infty} c \left( \int_{B(x,r)} e^{-\gamma k} \, dy \right)^{-\frac{1}{\gamma}}
= c \lim_{k \to \infty} \left( \int_{B(x,r)} (u + \frac{1}{k})^{-\gamma} \, dy \right)^{-\frac{1}{\gamma}}
= c \left( \int_{B(x,r)} u^{-\gamma} \, dy \right)^{-\frac{1}{\gamma}} \quad (4.5)
\]
for every ball $B(x,r)$ such that $Q(x,4r) \subset B(x,2\sqrt{\pi}r) \subseteq \Omega$.

**The Moral:** This is a reverse Hölder inequality, since by Hölder’s, or Jensen’s, inequality we always have
\[
\left( \int_{B(x,r)} u^{-\gamma} \, dy \right)^{\frac{1}{\gamma}} \leq \left( \int_{B(x,r)} u^\gamma \, dy \right)^{\frac{1}{\gamma}}.
\]
Reverse Hölder inequalities are very powerful tools in harmonic analysis and PDEs.

Harnack’s inequality can be seen as a quantitative version of the maximum principle.
CHAPTER 4. LOCAL HÖLDER CONTINUITY

**Theorem 4.23 (Harnack’s inequality).** Assume that $u \geq 0$ is a weak solution of (4.1) in $\Omega$. Then there exists a constant $c$ such that

$$\text{esssup}_{B(x,r)} u \leq c \text{essinf}_{B(x,r)} u$$

for every ball $B(x,r)$ such that $Q(x,4r) \subset B(x,2\sqrt{2}r) \subseteq \Omega$.

**Theorem:** By a chaining argument Harnack’s inequality gives the pointwise estimate

$$\frac{1}{c} u(y) \leq u(x) \leq cu(y)$$

for almost every points $x, y \in \Omega' \subseteq \Omega$. This means that the values of nonnegative weak solution in $\Omega'$ are comparable. Thus if $u$ is small (or large) somewhere in $\Omega'$ it is small (or large) everywhere in $\Omega'$. In particular, if $u(y) = 0$ for some $y \in \Omega$, then $u(x) = 0$ for every $x \in \Omega$. The assumption that $u \geq 0$ is essential in the result.

**Proof.** Theorem 4.16, Lemma 4.19, Lemma 4.21 and (4.5) imply

$$\text{esssup}_{B(x,r)} u \leq c \left( \int_{B(x,2r)} u^\gamma \, dy \right)^{1/\gamma} \leq c \left( \int_{B(x,2r)} u^{-\gamma} \, dy \right)^{-1/\gamma} \leq c \text{essinf}_{B(x,r)} u.$$

□

### 4.8 Local Hölder continuity

Next we shall prove that Harnack’s inequality implies that weak solutions of (4.1) are locally Hölder continuous after a possible redefinition on a set of measure zero. Observe that a weak solution belongs to $W^{1,2}_{\text{loc}}(\Omega)$ and is defined only up to a set of measure zero. This is the main reason that we have essential supremum and infimum in Harnack’s inequality, see Theorem 4.23. For a continuous function, we can apply the standard supremum and infimum instead. Denote

$$m(r) = \text{essinf}_{B(x,r)} u,$$

$$M(r) = \text{esssup}_{B(x,r)} u,$$

and the oscillation of $u$ by

$$\text{osc}_{B(x,r)} u = M(r) - m(r).$$

By Corollary 4.17, we have $-\infty < m(r) \leq M(r) < \infty$. Since constants can be added to weak solutions, we see that the functions $u - m(2r)$ and $M(2r) - u$ are weak solutions of (4.1) as well. Notice that $u - m(2r) \geq 0$ and $M(2r) - u \geq 0$ in $B(x,2r)$.

By Harnack’s inequality, Theorem 4.23, for small enough $r > 0$ we have

$$M(r) - m(2r) = \text{esssup}_{B(x,r)}(u - m(2r))$$

$$\leq c \text{essinf}_{B(x,r)}(u - m(2r))$$

$$= c(m(r) - m(2r)).$$
A similar argument gives

\[ M(2r) - m(r) = \text{esssup}_{B(x,r)} (M(2r) - u) \]
\[ \leq c \text{essinf}_{B(x,r)} (M(2r) - u) \]
\[ = c(M(2r) - M(r)), \]

everywhere \( r > 0 \) is small enough. By combining these estimates we have

\[ M(r) - m(2r) + M(2r) - m(r) \leq c(m(r) - m(2r) + M(2r) - M(r)), \]

which is implies

\[ M(r) - m(r) \leq \frac{c - 1}{c + 1} (M(2r) - m(2r)), \]

that is,

\[ \text{osc}_{B(x,r)} u \leq \gamma \text{osc}_{B(x,2r)} u, \quad 0 < \gamma < 1. \] (4.6)

Let \( 0 < r < R \) and choose \( i \) such that

\[ R \left( \frac{1}{2^{i+1}} \right) < r < \frac{R}{2^i}. \]

Then by iterating (4.6), we obtain

\[ \text{osc}_{B(x,r)} u \leq \gamma^i \text{osc}_{B(x,2r)} u \leq c \left( \frac{R}{r} \right)^{\alpha} \text{osc}_{B(x,2r)} u, \]

where \( \alpha = -\frac{\log \gamma}{\log 2} \). The last inequality follows, because

\[ \frac{r}{R} > \frac{1}{2^{i+1}} = \left( \frac{1}{2} \right)^{i+1} \]

implies that

\[ i > \frac{\log R}{\log \frac{r}{R}} - 1, \]

and as \( \gamma < 1 \), we have

\[ \gamma^i = e^{i \log \gamma} \leq e^{i \log (\frac{\log R}{\log \frac{r}{R}} - 1)} \log R = \frac{1}{\gamma} \left( \frac{r}{R} \right)^{\alpha}. \]

Let \( x, y \in B(x_0, r) \). Then \( R = 2r > |x - y| \) and

\[ |u(x) - u(y)| \leq \text{osc}_{B(x, \frac{|x - y|}{2r})} u \leq c \left( \frac{|x - y|}{R} \right)^{\alpha} \text{osc}_{B(x, 2r)} u \]
\[ \leq c \left( \frac{|x - y|}{r} \right)^{\alpha} \text{osc}_{B(x_0, 3r)} u \leq c \left( \frac{|x - y|}{r} \right)^{\alpha} \text{esssup}_{B(x_0, 3r)} |u| \]

for almost every \( x, y \in B(x_0, r) \). Notice that

\[ \text{esssup}_{B(x_0, 3r)} |u| < \infty \]

by Corollary 4.17. This implies that \( u \) is locally Hölder continuous by redefining it on a set of measure zero. The argument to show that there exists a locally Hölder continuous representative is similar as in the proof of Morrey's inequality.
Theorem 4.24 (Local Hölder continuity). Every weak solution of (4.1) is locally Hölder continuous.

Remark 4.25. Let \( \Omega = B(0, r) \) and \( y \in \partial B(0, r) \). The Poisson kernel for the ball \( B(0, r) \) gives the function

\[
    u(x) = \frac{1}{n\alpha(n)r} \frac{r^2 - |x|^2}{|x-y|^n}, \quad x \in B(0, r).
\]

Then \( \Delta u(x) = 0 \) for every \( x \in \Omega \), but is not Hölder continuous in \( \Omega \) for any \( 0 < \alpha \leq 1 \).

Reason. If \( u \) is Hölder continuous in \( \Omega \) for some \( 0 < \alpha \leq 1 \), then it is Hölder continuous in \( \overline{\Omega} \) with the same \( \alpha \). This implies that \( u \in L^\infty(\Omega) \). This is not possible, since \( u \notin L^\infty(\Omega) \). However, \( u \) is locally Hölder continuous in \( \Omega \). \[ \blacksquare \]

The Moral: Weak solutions are locally Hölder continuous, but not in general Hölder continuous in the whole domain.

Finally we show that Harnack's inequality implies that weak solutions of (4.1) satisfy the strong maximum principle.

Theorem 4.26 (Strong maximum principle). If a weak solution of (4.1) attains its maximum in a connected open set \( \Omega \), then it is a constant function.

Proof. If there exists \( x_0 \in \Omega \) such that

\[
    u(x_0) = \max_{x \in \Omega} u(x),
\]

then \( u(x_0) - u(x) \) is a nonnegative weak solution in \( \Omega \). Therefore by Harnack's inequality, Theorem 4.23, we have

\[
    \sup_{x \in B(x_0, r)} (u(x_0) - u(x)) \leq c \min_{x \in B(x_0, r)} (u(x_0) - u(x)) = 0.
\]

Thus \( u(x_0) - u(x) = 0 \) for every \( x \in B(x_0, r) \). Since \( \Omega \) is connected, every point \( x \in \Omega \) can be connected to the point \( x_0 \) with a finite chain of balls \( B(x_i, r_i) \), \( i = 0, \ldots, k \), such that \( x_k = x \) and

\[
    B(x_i, r_i) \cap B(x_{i+1}, r_{i+1}) \neq \emptyset, \quad i = 0, 1, \ldots, k - 1.
\]

By using Harnack's inequality in every ball, we have \( u(x) = u(x_0) \). \[ \square \]

Remarks 4.27:

1. An analogous argument gives a strong minimum principle as well.
2. The strong maximum principle implies the standard maximum principle:
   if \( u \in C(\overline{\Omega}) \) is a weak solution in a bounded open set \( \Omega \), then

\[
    \max_{\Omega} u = \max_{\partial \Omega} u.
\]
**Theorem 4.28 (Comparison principle).** Let \( u \) and \( v \) be weak solutions in \( \Omega \). By Theorem 4.24 we may assume that they both are continuous functions in \( \Omega \). If \( \Omega' \subset \Omega \) and \( u \leq v \) on \( \partial \Omega' \) then \( u \leq v \) in \( \Omega' \).

**Proof.** As \( u \leq v \) on \( \partial \Omega' \), \( u - v \leq 0 \) on \( \partial \Omega' \). The partial differential equation in (4.1) is linear and therefore \( u - v \) is a weak solution in \( \Omega \). The maximum principle, Theorem 4.26, implies that

\[
\max_{\Omega}(u - v) = \max_{\partial \Omega'}(u - v) \leq 0.
\]

Therefore \( u - v \leq 0 \) in \( \Omega' \) and thus \( u \leq v \) in \( \Omega' \). \( \square \)

**Remark 4.29.** This argument uses the linearity, but the result holds true also for certain nonlinear partial differential equations.
Bibliography


