The De Giorgi measure and an obstacle problem related to minimal surfaces in metric spaces

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References

Riikka Korte, Nageswari Shanmugalingam, Heli Tuominen and Juha Kinnunen, *The De Giorgi measure and an obstacle problem related to minimal surfaces in metric spaces*, J. Math. Pures Appl. (to appear).

Riikka Korte, Nageswari Shanmugalingam, Heli Tuominen and Juha Kinnunen, *Lebesgue points and capacities via boxing inequality in metric spaces, Indiana Univ. Math. J.*, **57** (2008), 401–430.

Plan of the talk

Problem: Find a set with minimal surface measure (perimeter) that separates two disjoint sets.

Context: Metric measure spaces.

Tool: Functions of bounded variation (BV).

Metric measure space

(X, d, μ) is a complete metric measure space with $\mu(X) = \infty$.

The measure is doubling, if there exists a constant $c_D \geq 1$ such that

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r))$$

for all $x \in X$ and $r > 0$.

(Coifman and Weiss, 1971)

Upper gradient

A nonnegative Borel function g on X is an upper gradient of an extended real valued function u on X if for all $x, y \in X$ and for all paths γ joining x and y in X,

$$|u(x) - u(y)| \leq \int_\gamma g\, ds$$

whenever both $u(x)$ and $u(y)$ are finite, and

$$\int_\gamma g\, ds = \infty$$

otherwise.

(Heinonen and Koskela, 1998)

Poincaré inequality

The space X supports a weak $(1,1)$-Poincaré inequality, if there exist constants $c_P > 0$ and $\tau \geq 1$ such that for all balls $B(x,r)$ of X, all locally integrable functions u on X and for all upper gradients g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq c_P r \fint_{B(x,\tau r)} g \, d\mu,$$

where

$$\begin{aligned} u_{B(x,r)} &= \fint_{B(x,r)} u \, d\mu \\ &= \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu. \end{aligned}$$

Bounded variation

For $u \in L^1_{\text{loc}}(X)$, we define the total variation as

$$\|Du\|(X) = \inf\left\{ \liminf_{i\to\infty} \int_X g_{u_i}\, d\mu : \right.$$
$$\left. u_i \in \text{Lip}_{\text{loc}}(X), u_i \to u \text{ in } L^1_{\text{loc}}(X) \right\},$$

where g_{u_i} is an upper gradient of u_i.

We say that a function $u \in L^1(X)$ is of bounded variation, $u \in BV(X)$, if

$$\|Du\|(X) < \infty.$$

(Ambrosio, 2001 and Miranda Jr., 2003)

Perimeter

A measurable set $E \subset X$ is said to have finite perimeter if

$$\|D\chi_E\|(X) < \infty.$$

We denote the perimeter measure as

$$P(E) = \|D\chi_E\|(X).$$

Observe: If $\mu(E \triangle E') = 0$, then $P(E) = P(E')$.

In particular, if $\mu(E) = 0$, then $P(E) = 0$.

The Euclidean case

In the Euclidean case with the Lebesgue measure, we have

$$\|Du\|(\mathbb{R}^n) = \sup\Big\{ \int_{\mathbb{R}^n} u \,\mathrm{div}\, f\, dx : \\ f \in C_0^1(\mathbb{R}^n;\mathbb{R}^n), \|f\| \le 1 \Big\}$$

where

$$\|f\| = \sup_{x\in\mathbb{R}^n} |f(x)|$$

and

$$f = (f_1, \dots, f_n).$$

In particular,

$$P(E) = \sup\Big\{ \int_E \mathrm{div}\, f\, dx : \\ f \in C_0^1(\mathbb{R}^n;\mathbb{R}^n), \|f\| \le 1 \Big\}.$$

If E is a set with smooth boundary, then

$$P(E) = \mathcal{H}^{n-1}(\partial E).$$

An obstacle problem

Let E and F be disjoint obstacle sets in X. Find a set G_0 such that $E \subset G_0$, $G_0 \cap F = \emptyset$ and

$$P(G_0) \leq P(G)$$

for every set G with $E \subset G$ and $G \cap F = \emptyset$.

In other words, find a set with a minimal surface measure that separates the sets E and F.

Our work is a generalization of Euclidean results by De Giorgi, Colombini and Piccinini, *Frontiere orientate di misura minima e questioni collegate*, Scuola Normale Superiore, Pisa, 1972.

An existence result

Theorem. (KKST, 2009) Let E and F be disjoint sets in X. Then there exists a set G_0 with $E \subset G_0$ and $G_0 \cap F = \emptyset$ such that

$$P(G_0) \leq P(G)$$

for every set G with $E \subset G$ and $G \cap F = \emptyset$.

This means that a minimizing set exists.

Proof of the existence

Strategy: The direct methods in the calculus of variations.

Step 1: Denote

$$\lambda = \inf\{P(G) : E \subset G,\, G \cap F = \emptyset\}.$$

Take a minimizing sequence of sets G_i, with

$$E \subset G_i \quad \text{and} \quad G_i \cap F = \emptyset$$

for every $i = 1, 2, \ldots$, such that

$$\lim_{i\to\infty} P(G_i) = \lambda.$$

Step 2: Use a compactness result (Miranda, 2003) to obtain a subsequence G_{i_j}, $j = 1, 2, \ldots$, and a Borel set G_0 such that

$$\chi_{G_{i_j}} \to \chi_{G_0} \quad \text{in} \quad L^1_{\text{loc}}(X)$$

as $j \to \infty$.

Step 3: By passing to a subsequence, if necessary, we can also assume that

$$\chi_{G_{i_j}} \to \chi_{G_0} \quad \text{almost everywhere}$$

as $j \to \infty$. Hence, by changing G_0 on a set of measure zero, we may assume that $E \subset G_0$ and $G_0 \cap F = \emptyset$. This implies $\lambda \leq P(G_0)$.

Step 4: A lower semicontinuity result (Miranda, 2003) implies that

$$P(G_0) \leq \liminf_{j \to \infty} P(G_{i_j}) = \lambda.$$

This shows that $\lambda = P(G_0)$ and hence G_0 is a minimizing set.

An example

Let $X = \mathbb{R}^2$ with the Euclidean distance and Lebesgue measure,

$$E = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1,\ x_2 = 0\}$$

and

$$F = \{x \in \mathbb{R}^2 : |x| \geq 4\}.$$

Then E itself will do as a minimizing set and $P(E) = 0$ since the Lebesgue measure of E is zero. Hence

$$\inf P(G) = 0,$$

where the infimum is taken over all sets G with $E \subset G$ and $G \cap F = \emptyset$.

Problems

(1) The perimeter does not see sets of measure zero.

(2) There are too many admissible test sets.

We can try to restrict ourselves to a smaller class of test sets.

A class of good test sets

We denote by $\mathcal{G}$ the collection of all $E \subset X$ such that E is μ-measurable,

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0$$

for every $x \in E$, and

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap (X\setminus E))}{\mu(B(x,r))} > 0$$

for every $x \in X \setminus E$.

In other words, $E \in \mathcal{G}$ if both E and $X \setminus E$ are thick in the sense that the upper density of the set is positive at all points belonging to the set.

Properties of good sets

It is possible to associate to every set $E \subset X$ a set $E' \in \mathcal{G}$ so that $\mu(E \triangle E') = 0$. Indeed, there is a set in this class that differs from the original set as little as possible.

Clearly $E \in \mathcal{G}$ if and only if $X \setminus E \in \mathcal{G}$.

A reformulated problem

Let E and F be disjoint sets in X. Find a set $G_0 \in \mathcal{G}$ with $E \subset G_0$ and $G_0 \cap F = \emptyset$ such that

$$P(G_0) \leq P(G)$$

for every set $G \in \mathcal{G}$ with $E \subset G$ and $G \cap F = \emptyset$.

Problem: The example shows that there may be no minimizing set in this class.

In order to be able to obtain the existence of a minimizing set $G_0 \in \mathcal{G}$, we need to relax the conditions $E \subset G$ and $F \cap G = \emptyset$.

We could try to minimize the functional

$$I(G) = 2H^{n-1}(E \setminus G) + 2H^{n-1}(G \cap F)$$

where $G \in \mathcal{G}$.

Problem: No lower semicontinuity result.

We apply the following equivalent measure proposed by De Giorgi that has better semicontiuity properties.

The De Giorgi measure

Let $\varepsilon > 0$. For $E \subset X$, let

$$\sigma_\varepsilon(E) = \inf \left\{ P(G) + \frac{\mu(G)}{\varepsilon} : G \in \mathcal{G},\, E \subset G \right\}.$$

The De Giorgi measure of E is

$$\sigma(E) = \sup_{\varepsilon > 0} \sigma_\varepsilon(E) = \lim_{\varepsilon \to 0} \sigma_\varepsilon(E).$$

Theorem. (KKST, 2009) The De Giorgi measure is a Borel regular outer measure. In particular,

$$\sigma(E) = \text{inf}\{\sigma(B) : B \text{ is a Borel set, } E \subset B\}$$

for every $E \subset X$.

A relaxed problem

Let E and F be disjoint sets in X. Find a set $G_0 \in \mathcal{G}$ such that

$$I(G_0) \leq I(G),$$

for every $G \in \mathcal{G}$, where

$$I(G) = P(G) + \sigma(E \setminus G) + \sigma(F \cap G).$$

Observe: The obstacles are described in the penalty term.

The example revisited

Let $X = \mathbb{R}^2$ with the Euclidean distance and Lebesgue measure,

$$E = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1,\, x_2 = 0\}$$

and

$$F = \{x \in \mathbb{R}^2 : |x| \geq 4\}.$$

Then the minimizing set $G_0 = \emptyset$ and

$$\inf_{G \in \mathcal{G}} I(G_0) = 4.$$

An existence result for the relaxed problem

Theorem. (KKST, 2009) Let E and F be disjoint sets in X. Then there exists $G_0 \in \mathcal{G}$ such that

$$I(G_0) \leq I(G)$$

for every $G \in \mathcal{G}$.

Proof: The direct methods in the calculus of variations: Compactness and lower semicontinuity.

A semicontinuity result

Theorem. (KKST, 2009) Let E and F be disjoint sets in X. Suppose that $G \in \mathcal{G}$ satisfies

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap G)}{\mu(B(x,r))} = 0$$

for every $x \in E \setminus G$ and

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap (X \setminus G))}{\mu(B(x,r))} = 0$$

for every $x \in G \cap F$. If $G_i \in \mathcal{G}$, $i = 1, 2, \ldots$, are such that

$$\chi_{G_i} \to \chi_G$$

in $L^1_{\mathrm{loc}}(X)$ as $i \to \infty$, then

$$I(G) \leq \liminf_{i \to \infty} I(G_i).$$

Density assumptions

In the collection of sets $G' \in \mathcal{G}$ with

$$\mu(G' \triangle G) = 0$$

the set G satisfying

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap G)}{\mu(B(x,r))} = 0$$

for every $x \in E \setminus G$ has the largest possible intersection with E.

Similarly, the set G satisfying

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap (X \setminus G))}{\mu(B(x,r))} = 0$$

for every $x \in G \cap F$ has the smallest possible intersection with F.

A connection to the original problem

Theorem. (KKST, 2009) Let E and F be disjoint sets in X and denote

$$\lambda = \min\{P(G) : E \subset G, G \cap F = \emptyset\},$$
$$\gamma = \min\{I(G) : G \in \mathcal{G}\} \text{ and}$$
$$\nu = \inf\{P(G) : G \in \mathcal{G}, E \subset G, G \cap F = \emptyset\}.$$

Then

$$\lambda \le \gamma \le \nu.$$

If

$$\limsup_{r\to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} > 0$$

for every $x \in E$ and

$$\limsup_{r\to 0} \frac{\mu(B(x,r) \cap F)}{\mu(B(x,r))} > 0$$

for every $x \in F$, then

$$\lambda = \gamma = \nu.$$

Hausdorff measure of codimension one

Let $E \subset X$ and $R > 0$. We define

$$\mathcal{H}_R(E) = \inf \Big\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : r_i \le R, \\ E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \Big\}$$

and

$$\mathcal{H}(E) = \lim_{R \to 0} \mathcal{H}_R(E).$$

The number $\mathcal{H}(E)$, which is possibly infinite, is called the Hausdorff measure of codimension one of E.

De Giorgi $\approx$ Hausdorff

Theorem. (KKST, 2009) There exist positive constants c_1 and c_2, depending only on the constants in the doubling condition and the Poincaré inequality, such that

$$c_1\mathcal{H}(E) \le \sigma(E) \le c_2\mathcal{H}(E)$$

for every set $E \subset X$

Remark. In the Euclidean case, if E is a C^1-surface of codimension one, then

$$\sigma(E) = 2\mathcal{H}^{n-1}(E).$$

(De Giorgi, Colombini and Piccinini, 1972)

A counterexample by Hutchinson (Boll. U.M.I. (5) 18-B (1981), 619-628) shows that the equality does not hold in general.

Two essential tools

(1) The total variation measure is concentrated on the measure theoretic boundary by

L. Ambrosio, *Fine properties of sets of finite perimeter in doubling metric measure spaces*, Set-Valued Anal. **10** (2-3) (2002), 111–128.

(2) The boxing inequality proved in the metric setting in

J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen, *Lebesgue points and capacities via boxing inequality in metric spaces*, Indiana Univ. Math. J. **57** (1) (2008), 401–430,

and independently in

T. Mäkäläinen, *Adams inequality on metric spaces*, Rev. Mat. Iberoamericana (to appear).

Measure theoretic boundary

The measure theoretic boundary of $E \subset X$, denoted by $\partial^* E$, is the set of points $x \in X$, where both E and its complement have positive density, that is,

$$\limsup_{r \to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} > 0$$

and

$$\limsup_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} > 0.$$

Theorem. (Ambrosio, 2002) Assume that $E \subset X$ is a set of finite perimeter. Then

$$\frac{1}{c}P(E) \leq \mathcal{H}(\partial^* E) \leq cP(E),$$

where c depends only on the doubling constant and the Poincaré inequality.

Open question: Note that the theorem above does not imply that the Hausdorff measure of $\partial^* E$ would be infinite whenever the perimeter measure of E is infinite.

The boxing inequality

Theorem. (KKST, 2008) Let $E \subset X$ be an open set of finite perimeter with $\mu(E) < \infty$, τ the dilation constant in the weak $(1,1)$-Poincaré inequality. Then there exists a collection of disjoint balls $B(x_i, \tau r_i)$, $i = 1, 2, \dots$, such that

$$E \subset \bigcup_{i=1}^{\infty} B(x_i, 5\tau r_i),$$

$$\frac{1}{2c_D} < \frac{\mu(E \cap B(x_i, r_i))}{\mu(B(x_i, r_i))} \leq \frac{1}{2}$$

for $i = 1, 2, \dots$, and

$$\sum_{i=1}^{\infty} \frac{\mu(B(x_i, 5\tau r_i))}{5\tau r_i} \leq c\, P(E).$$

The constant c depends only on the doubling constant c_D, the constants in the weak $(1,1)$-Poincaré inequality.

Summary

We study the existence of a set with minimal perimeter that separates two disjoint sets in a metric measure space equipped with a doubling measure and supporting a Poincaré inequality.

A measure constructed by De Giorgi is used to state a relaxed problem, whose solution coincides with the solution to the original problem for measure theoretically thick sets.

The De Giorgi measure on metric is comparable to the Hausdorff measure of codimension one, but has better lower semicontinuity properties.

The theory of functions of bounded variation on metric spaces is used extensively in the arguments.