LIOUVILLE-TYPE RESULTS FOR NONLINEAR ELLIPTIC EQUATIONS WITH BAOUENDI–GRUSHIN OPERATORS AND NONLINEAR GRADIENT TERMS

A. Aghajani
School of Mathematics & Computer Science, Iran University of Science and Technology, Narmak, Tehran, Iran (aghajani@iust.ac.ir)

C. Cowan
Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada (craig.cowan@umanitoba.ca)

J. Kinnunen
Department of Mathematics, Aalto University, P.O. Box 11100, FI-00076 Aalto, Finland (juha.k.kinnunen@aalto.fi)

(Submitted by: Marius Ghergu)

Abstract. We study positive supersolutions of the nonlinear elliptic equation

$$-\Delta_\gamma u + \frac{b}{u^{\beta-1}}|\nabla_\gamma u|^\beta = cu \quad \text{in } \Omega,$$

with $0 < \beta \leq 2$, where Ω is either a bounded domain or an unbounded exterior domain in $\mathbb{R}^N$, and Δ_γ is the Baouendi–Grushin operator. Extending classical Liouville-type results known for the Laplacian ($\gamma = 0$, $\beta = 1$) to the subelliptic Grushin setting, we derive conditions on the coefficient functions b and c ensuring the absence or existence of positive supersolutions, and provide explicit examples illustrating the sharpness of our results. Our approach employs generalized Hardy inequalities adapted to the Grushin structure, allowing treatment of degenerate and unbounded coefficients without restrictive boundedness assumptions.

1. Introduction

Let $N \geq 3$ be an integer and let $N = m + k$ for some positive integers m and k. Consider the space $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^k$, where a point $z \in \mathbb{R}^N$ is represented as $z = (x, y)$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$. Let $\gamma \geq 0$ be a real parameter. The Baouendi–Grushin operator Δ_γ is defined by

$$\Delta_\gamma u(z) = \Delta_x u(z) + |x|^{2\gamma} \Delta_y u(z),$$

AMS Subject Classifications: 35J60; 35H20; 35B53.

where

$$\Delta_x = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} \quad \text{and} \quad \Delta_y = \sum_{j=1}^{k} \frac{\partial^2}{\partial y_j^2}$$

denote the standard Laplace operators acting on the variables

$$x = (x_1, \dots, x_m) \quad \text{and} \quad y = (y_1, \dots, y_k),$$

respectively, and $|x|$ denotes the Euclidean norm of x in $\mathbb{R}^m$. The associated Grushin gradient is given by

$$\nabla_\gamma = (\nabla_x, |x|^\gamma \nabla_y) = \Big(\frac{\partial}{\partial x_i}, \dots, \frac{\partial}{\partial x_m}, |x|^\gamma \frac{\partial}{\partial y_1}, \dots, |x|^\gamma \frac{\partial}{\partial y_k} \Big),$$

so that $\Delta_\gamma = \nabla_\gamma \cdot \nabla_\gamma$. The vector field ∇_γ is homogeneous with respect to the anisotropic dilation

$$\delta_R(x, y) = (Rx, R^{1+\gamma} y), \quad R > 0,$$

and the Jacobian of such a transformation is

$$J(\delta_R) = R^Q, \quad Q = m + (1+\gamma)k = N + \gamma k,$$

where Q is the homogeneous dimension of $\mathbb{R}^N$. The Baouendi–Grushin operator satisfies the scaling property

$$\Delta_\gamma(u \circ \delta_R) = R^2 (\Delta_\gamma u) \circ \delta_R,$$

which reflects its intrinsic anisotropic nature. To study the geometry induced by Baouendi–Grushin operator, we introduce the Grushin distance

$$\rho = \rho(z) = \big(|x|^{2(1+\gamma)} + (1+\gamma)^2 |y|^2 \big)^{\frac{1}{2(1+\gamma)}},$$

which is homogeneous of degree one under the anistotropic dilation, that is,

$$\rho(\delta_R(x, y)) = R\rho(x, y), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^k.$$

The corresponding open metric balls centered at the origin are given by $B_r^\rho = \{z \in \mathbb{R}^N : \rho(z) < r\}$. A direct computation in [32] shows that

$$|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}.$$

Furthermore, if $u = u(z) = f(\rho(z))$ for some $f \in C^2\big((0, \infty)\big)$, then

$$\Delta_\gamma u = \frac{|x|^{2\gamma}}{\rho^{2\gamma}} \Big(f'' + (Q-1) \frac{f'}{\rho} \Big),$$

which will be a useful formula for our analysis. Throughout this paper, a domain Ω is called exterior domain, if $\{z \in \mathbb{R}^N : \rho(z) > R > 0\} \subset \Omega$ for some $R > 0$.

The operator Δ_γ is a fundamental example of a degenerate elliptic operator arising in the theory of subelliptic PDEs. When $\gamma > 0$ the the Baouendi–Grushin operator exhibits anisotropic behavior, with the degeneracy localized on a lower-dimensional subspace $\{0\} \times \mathbb{R}^k$, on which the weight $|x|^{2\gamma}$ vanishes. For $\gamma = 0$, the Baouendi–Grushin operator reduces to the standard Laplacian in $\mathbb{R}^N$, recovering the classical elliptic theory. The Grushin operator was originally introduced by V.V. Grushin [34] and later studied by Baouendi [7]. It has since become a fundamental model in several fields, including sub-Riemannian geometry, harmonic analysis, and PDEs on singular spaces. Its degeneracy mimics that of more complicated structures, such as the Heisenberg group, to which it is closely related in the special case $\gamma = 1$, see [13, 14]. This connection has sparked considerable interest in the study of subelliptic operators with nontrivial geometries [1, 8, 15, 14, 22, 32, 33, 6].

In this paper, we are concerned with Liouville-type theorems for positive C^2 supersolutions of the equation

$$-\Delta_\gamma u + \frac{b}{u^{\beta-1}}|\nabla_\gamma u|^\beta = cu \quad \text{in } \Omega, \tag{1.1}$$

where $0 < \beta \leq 2$, $\Omega \subset \mathbb{R}^N$ is either a bounded domain or an unbounded exterior domain in $\mathbb{R}^N$ and the coefficient functions $b = b(z)$ and $c = c(z)$ are assumed to be continuous in Ω. Note that, for $1 \leq \beta \leq 2$, any supersolution to

$$-\Delta_\gamma u + \frac{b \cdot \nabla_\gamma u |\nabla_\gamma u|^{\beta-1}}{u^{\beta-1}} = cu \quad \text{in } \Omega, \tag{1.2}$$

where $b \in \mathbb{R}^N$ and $c \in \mathbb{R}$ are constants, is also a supersolution of (1.1) with b replaced by $|b|$. For (1.1) and (1.2), with $\gamma = 0$ and $\beta = 1$, the existence and nonexistence of supersolutions have been widely studied in the literature, particularly for $\Omega = \mathbb{R}^N$ and for exterior domains $\mathbb{R}^N$, with $N \geq 3$. A notable result of Berestycki, Hamel, and Nadirashvili [9], based on eigenvalue problem techniques, asserts that $u \equiv 0$ is the only nonnegative solution of

$$-\Delta u - b \cdot \nabla u - cu = 0 \quad \text{in } \mathbb{R}^N,$$

where $b \in \mathbb{R}^N$, $c \in \mathbb{R}$ with $4c - |b|^2 > 0$. This condition can be intuitively understood by comparing to the one-dimensional ODE

$$u'' + bu' + cu = 0 \quad \text{in } \mathbb{R},$$

where b and c are constants, which has positive exponential solutions if and only if $b^2 - 4c \geq 0$. Hence, if $4c - b^2 > 0$, the only nonnegative solution is $u \equiv 0$, even when $\mathbb{R}$ is replaced by an unbounded interval.

Berestycki, Hamel and Rossi [10] extended the results of [9] to elliptic equations with non-constant coefficients. They showed that, if the vector field $b = b(x)$ and the function $c = c(x)$ are continuous and bounded, then

$$-\Delta u + b \cdot \nabla u \geq cu \quad \text{in } \mathbb{R}^N,$$

does not have positive solutions provided that

$$\liminf_{|x|\to\infty} \Big(c(x) - \tfrac{|b(x)|^2}{4} \Big) > 0. \tag{1.3}$$

Rossi [42] further generalized these nonexistence results to the setting of fully nonlinear elliptic operators. Some refinements of these results appear in [2], which employs a generalized Hardy inequality, see also [21]. Additionally, we refer to [16], where the authors established Hadamard and Liouville-type properties for nonnegative viscosity supersolutions of fully nonlinear uniformly elliptic partial differential inequalities, both in the whole space and exterior domains. For further discussion of related problems, see [4, 3, 5, 16, 31, 11, 12].

We also note that Liouville-type theorems for the Grushin operator have attracted significant interest. Capuzzo-Dolcetta and Cutrì [17] investigated the problem

$$-\Delta_\gamma u \geq u^q, \quad u \geq 0 \quad \text{in } \mathbb{R}^N$$

and employing techniques for sublaplacians on stratified Lie groups, they proved that, for a fixed integer $\gamma > 1$, this problems does not have nontrivial positive solutions in the range $1 < q \leq \frac{Q}{Q-2}$. D'Ambrosio and Lucente [26] explored necessary conditions for the solvability of

$$L(x, y, D_x, D_y)u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \quad \text{in } \mathbb{R}^m \times \mathbb{R}^k,$$

where the operator L encompasses Grushin-type operators as special cases. Liouville theorems for nonnegative solutions of the equation

$$-\Delta_\gamma u = u^p \quad \text{in } \mathbb{R}^N \tag{1.4}$$

have also been studied. Monticelli [41] established a Liouville theorem for nonnegative classical solutions in the entire space and by Yu [45] for nonnegative weak solutions to (1.4) with the optimal exponent $p < \frac{Q+2}{Q-2}$. Both studies applied the Kelvin transform combined with the moving planes method. Moreover, Monti and Morbidelli [40] studied classification results for (1.4) in the critical exponent case $p = \frac{Q+2}{Q-2}$ using the moving spheres technique, a variant of the moving planes method widely applied in elliptic equations. For other Liouville-type results related to the Grushin operator, we refer the reader to [24, 26, 8, 13, 14, 20, 40, 41, 16, 29, 30, 37] and references therein.

In this paper, we consider positive supersolutions to a more general problem (1.1) in bounded or exterior domains $\Omega \subset \mathbb{R}^N$, where the coefficient functions $b, c \in C(\Omega)$ satisfy the condition

$$D(z) = c(z) - \left(\tfrac{\beta}{2}\right)^{\frac{2}{2-\beta}} \left(\tfrac{2}{\beta} - 1\right) b(z)^{\frac{2}{2-\beta}} \geq 0 \quad \text{in } \Omega,$$

if $0 < \beta < 2$, and $c(z) > 0$, $\|b\|_{L^\infty(\Omega)} < 1$ in Ω, if $\beta = 2$. If Ω is an exterior domain, we assume these conditions hold eventually, that is, for sufficiently large $|z|$. By employing a generalized version of the Hardy inequality, we obtain new Liouville-type results that appear to be essentially sharp. In particular, in the case of an exterior domain we include the degenerate case

$$\limsup_{|z|\to\infty} D(z) = 0$$

and allow unbounded coefficient functions b and c.

2. Hardy inequalities for the Grushin operator

Our main results are derived using certain generalized Hardy-type inequalities for the Grushin operator, proved in this section. The proof applies the divergence theorem to selected vector fields, following the method of Mitidieri [39], see also [28]. These inequalities are well adapted to the analysis of equation (1.1). For more extensive treatments of Hardy-type inequalities related to Grushin-type operators, see [18, 1, 32, 27, 23, 25, 36, 43, 35].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain and assume that $u, v \in C^2(\Omega)$ are positive functions.*

(i) *For every $\phi \in C_c^\infty(\Omega)$ we have*

$$\int_\Omega \frac{-\Delta_\gamma u}{u} \phi^2 \, dz \leq \int_\Omega |\nabla_\gamma \phi|^2 \, dz. \tag{2.1}$$

As a consequence, we have

$$\left(\tfrac{Q-2}{2}\right)^2 \int_\Omega \frac{|x|^{2\gamma}}{\rho^{2+2\gamma}} \phi^2 \, dz \leq \int_\Omega |\nabla_\gamma \phi|^2 \, dz. \tag{2.2}$$

In particular, when $\gamma = 0$, so that $Q = N$, we obtain

$$\left(\tfrac{N-2}{2}\right)^2 \int_\Omega \frac{\phi^2}{|z|^2} \, dz \leq \int_\Omega |\nabla \phi|^2 \, dz.$$

(ii) *Let $0 \le \alpha \le 1$. Then*

$$\begin{aligned}\alpha \int_\Omega \tfrac{-\Delta_\gamma u}{u}\phi^2\,dz + (1-\alpha)\int_\Omega \tfrac{-\Delta_\gamma v}{v}\phi^2\,dz \\ + \alpha(1-\alpha)\int_\Omega \left|\nabla_\gamma \log \tfrac{u}{v}\right|^2 \phi^2\,dz \le \int_\Omega |\nabla_\gamma \phi|^2\,dz\end{aligned} \tag{2.3}$$

and

$$\alpha \int_\Omega \tfrac{-\Delta_\gamma u}{u}\phi^2\,dz + \alpha(1-\alpha)\int_\Omega \tfrac{|\nabla_\gamma u|^2}{u^2}\phi^2\,dz \le \int_\Omega |\nabla_\gamma \phi|^2\,dz, \tag{2.4}$$

for every $\phi \in C_c^\infty(\Omega)$.

The following simple lemma will be used in the proof of Proposition 2.1. We include a proof for the sake of completeness.

Lemma 2.2. *Let $u \in C^2(\Omega)$ and let $F \in C^2$ be a real-valued function on the range of u. Then*

$$\Delta_\gamma(F(u)) = F''(u)|\nabla_\gamma u|^2 + F'(u)\Delta_\gamma u.$$

Proof. By the definition of Δ_γ and the chain rule, we have

$$\begin{aligned}\Delta_\gamma(F(u)) &= \Delta_x F(u) + |x|^{2\gamma}\Delta_y F(u) \\ &= F''(u)|\nabla_x u|^2 + F'(u)\Delta_x u + |x|^{2\gamma}\left(F''(u)|\nabla_y u|^2 + F'(u)\Delta_y u\right) \\ &= F''(u)\left(|\nabla_x u|^2 + |x|^{2\gamma}|\nabla_y u|^2\right) + F'(u)(\Delta_x u + |x|^{2\gamma}\Delta_y u) \\ &= F''(u)|\nabla_\gamma u|^2 + F'(u)\Delta_\gamma u.\end{aligned}$$

□

Proof of Proposition 2.1. First we prove (2.1). Let $W \in C^2(\Omega)$ and $\phi \in C_c^\infty(\Omega)$. By integration by parts we have

$$\begin{aligned}-\int_\Omega (\Delta_\gamma W)\,\phi^2\,dz = -\int_\Omega \nabla_\gamma\cdot\nabla_\gamma W\,\phi^2\,dz = 2\int_\Omega \nabla_\gamma W\cdot\phi\nabla_\gamma\phi\,dz \\ \le \int_\Omega |\nabla_\gamma W|^2\phi^2\,dz + \int_\Omega |\nabla_\gamma \phi|^2\,dz,\end{aligned}$$

which after regrouping terms gives

$$-\int_\Omega \left(\Delta_\gamma W - |\nabla_\gamma W|^2\right)\phi^2\,dz \le \int_\Omega |\nabla_\gamma \phi|^2\,dz. \tag{2.5}$$

Let $W = \log E$, where $E \in C^2(\Omega)$, $E > 0$. Lemma 2.2 implies that

$$\Delta_\gamma W = \Delta_\gamma \log E = \tfrac{|\nabla_\gamma u|^2}{E^2} + \tfrac{\Delta_\gamma E}{E} \quad \text{and} \quad \nabla_\gamma W = \tfrac{\nabla_\gamma E}{E}.$$

By substituting these equalities into (2.5) we arrive at

$$\int_\Omega \frac{\Delta_\gamma E}{E}\phi^2\,dz \le \int_\Omega |\nabla_\gamma \phi|^2\,dz.$$

This proves (2.1).

Let $E = E(z) = \rho(z)^\theta$. A direct computation gives

$$\frac{-\Delta_\gamma E}{E} = -\theta(\theta+Q-2)\frac{|x|^{2\gamma}}{\rho^{2+2\gamma}},$$

and by (2.1) we obtain

$$-\theta(\theta+Q-2)\int_\Omega \frac{|x|^{2\gamma}}{\rho^{2+2\gamma}}\phi^2\,dz \le \int_\Omega |\nabla_\gamma\phi|^2\,dz.$$

By setting $\theta = \frac{Q-2}{2}$ we arrive at (2.2).

Let $u, v \in C^2(\Omega)$, $u, v > 0$, and set $W = u^\alpha v^{1-\alpha}$, $0 \le \alpha \le 1$. We claim that

$$\frac{-\Delta_\gamma W}{W} = \alpha\frac{-\Delta_\gamma u}{u} + (1-\alpha)\frac{-\Delta_\gamma v}{v} + \alpha(1-\alpha)\left|\nabla_\gamma \log\frac{u}{v}\right|^2. \tag{2.6}$$

To prove the above formula first we note that for two functions w_1 and w_2 we have

$$\begin{aligned}\nabla_\gamma(w_1w_2) &= (\nabla_x(w_1w_2), |x|^\gamma\nabla_y(w_1w_2))\\ &= (w_1\nabla_x w_2 + w_2\nabla_x w_1, |x|^\gamma w_1\nabla_y w_2 + |x|^\gamma w_2\nabla_y w_1)\\ &= w_1\nabla_\gamma w_2 + w_2\nabla_\gamma w_1.\end{aligned}$$

Morover, for a function w and a vector field V, we have

$$\begin{aligned}\nabla_\gamma\cdot(wV) &= \nabla_x\cdot(wV) + |x|^\gamma\nabla_y\cdot(wV)\\ &= \nabla_x w\cdot V + w\nabla_x\cdot V + |x|^\gamma\nabla_y w\cdot V + |x|^\gamma w\nabla_y\cdot V\\ &= \nabla_\gamma w\cdot V + w\nabla_\gamma\cdot V.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta_\gamma(w_1w_2) &= \nabla_\gamma\cdot\nabla_\gamma(w_1w_2) = \nabla_\gamma\cdot\big(w_1\nabla_\gamma w_2 + w_2\nabla_\gamma w_1\big)\\ &= w_1\Delta_\gamma w_2 + 2\nabla_\gamma w_1\cdot\nabla_\gamma w_2 + w_2\Delta_\gamma w_1.\end{aligned}$$

Using the equality above for $W = u^\alpha v^{1-\alpha}$, we obtain

$$\begin{aligned}\frac{\Delta_\gamma W}{W} &= \alpha(\alpha-1)\frac{|\nabla_\gamma u|^2}{u^2} + \alpha\frac{\Delta_\gamma u}{u} - \alpha(1-\alpha)\frac{|\nabla_\gamma v|^2}{v^2} + (1-\alpha)\frac{\Delta_\gamma v}{v}\\ &\quad + 2\alpha(1-\alpha)\frac{\nabla_\gamma u\cdot\nabla_\gamma v}{uv}\\ &= \alpha\frac{\Delta_\gamma u}{u} + (1-\alpha)\frac{\Delta_\gamma v}{v} - \alpha(1-\alpha)\left(\frac{|\nabla_\gamma u|^2}{u^2} + \frac{|\nabla_\gamma v|^2}{v^2} - 2\frac{\nabla_\gamma u\cdot\nabla_\gamma v}{uv}\right)\\ &= \alpha\frac{\Delta_\gamma u}{u} + (1-\alpha)\frac{\Delta_\gamma v}{v} - \alpha(1-\alpha)|\nabla_\gamma\log u - \nabla_\gamma\log v|^2\end{aligned}$$

$$= \alpha \tfrac{\Delta_\gamma u}{u} + (1-\alpha)\tfrac{\Delta_\gamma v}{v} - \alpha(1-\alpha)\left|\nabla_\gamma \log \tfrac{u}{v}\right|^2.$$

This proves (2.6). Multiplying (2.6) by ϕ^2 and applying (2.1), we obtain (2.3). Note that (2.3) also holds for nonnegative functions by replacing u and v with $u+\varepsilon$ and $v+\varepsilon$ and passing to the limit as $\varepsilon \to 0$.

Finally, setting $v \equiv C$ (a constant) in (2.3), for any $u > 0$ and $0 \le \alpha \le 1$, we have

$$\alpha \int_\Omega \tfrac{-\Delta_\gamma u}{u}\phi^2\, dz + \alpha(1-\alpha)\int_\Omega \tfrac{|\nabla_\gamma u|^2}{u^2}\phi^2\, dz \le \int_\Omega |\nabla_\gamma \phi|^2\, dz.$$

□

Proposition 2.3. *Let $\Omega = \{z \in \mathbb{R}^N : \rho(z) \ge R_0\}$ be an exterior domain in $\mathbb{R}^N$ and suppose that $w = w(z)$ satisfies*

$$\int_\Omega w\phi^2\, dz \le \int_\Omega |\nabla_\gamma \phi|^2\, dz, \tag{2.7}$$

for every $\phi \in C_c^\infty(\Omega)$. Then

$$\liminf_{\rho(z)\to\infty} \tfrac{\rho(z)^{2+2\gamma}}{|x|^{2\gamma}} w(z) \le \left(\tfrac{Q-2}{2}\right)^2.$$

Proof. Let $d > 1$, $R > 2R_0$, and let $\psi = \psi_R(\rho) \in C^\infty\big([0,\infty)\big)$ be a cutoff function such that $0 \le \psi \le 1$, $\psi \equiv 0$ on $[R_0, \frac{R}{2}] \cup [2dR, \infty)$, $\psi \equiv 1$ on $[R, dR]$, $|\psi'(\rho)| \le \frac{4}{R}$, $\frac{R}{2} < \rho < R$, and $|\psi'(\rho)| \le \frac{4}{dR}$, $dR < \rho < 2dR$. Consider the test function

$$\phi = \phi(z) = \rho(z)^{-\delta}\psi(\rho(z)),$$

where $\delta = \frac{Q-2}{2}$. A direct computation gives

$$\begin{aligned}
\int_\Omega |\nabla_\gamma \phi|^2\, dz &= \int_{\frac{R}{2}<\rho<2dR} |\nabla \phi|^2\, dz \\
&= \int_{\frac{R}{2}<\rho<R} |\nabla_\gamma \phi|^2\, dz + \int_{R<\rho<dR} |\nabla_\gamma \phi|^2\, dz + \int_{dR<\rho<2dR} |\nabla_\gamma \phi|^2\, dz \\
&= I_1(R) + I_2(R) + I_3(R).
\end{aligned}$$

Since $|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}$, we have

$$\begin{aligned}
|\nabla_\gamma \phi|^2 &= \left|-\delta\rho^{-\delta-1}\psi(\rho) + \rho^{-\delta}\psi'(\rho)\right|^2 |\nabla_\gamma \rho|^2 \\
&\le \big(\delta + \rho|\psi'(\rho)|\big)^2 |x|^{2\gamma}\rho^{-Q-2\gamma}.
\end{aligned}$$

The bounds on ψ' imply that

$$|\nabla_\gamma \phi|^2 \le (\delta+4)^2 |x|^{2\gamma}\rho^{-Q-2\gamma} \quad \text{in } \left\{\tfrac{R}{2} < \rho < R\right\} \cup \{dR < \rho < 2dR\},$$

and since $\psi \equiv 1$ in $R < \rho < dR$ we have

$$|\nabla_\gamma \phi|^2 \le \delta^2 |x|^{2\gamma} \rho^{-Q-2\gamma} \quad \text{in } \{R < \rho < dR\}.$$

A standard computation shows that

$$\int_{B^\rho_{r_2} \setminus B^\rho_{r_1}} |x|^{2\gamma} \rho^{-Q-2\gamma}\, dz = C_{m,\gamma,k} \ln \tfrac{r_2}{r_1}.$$

It follows that

$$I_1(R), I_3(R) \le C_{m,\gamma,k} (\delta + 4)^2 \ln 2, \quad I_2(R) \le C_{m,\gamma,k} \delta^2 \ln d,$$

and thus

$$\int_\Omega |\nabla_\gamma \phi|^2\, dz \le C_{m,\gamma,k} \big(2(\delta+4)^2 \ln 2 + \delta^2 \ln d\big). \tag{2.8}$$

For the left-hand side of (2.7), we have

$$\begin{aligned}\int_\Omega w\phi^2\, dz &\ge \int_{R<\rho<dR} w\rho^{-2\delta}\, dz = \int_{R<\rho<dR} \big(\tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} w\big) |x|^{2\gamma} \rho^{-Q-2\gamma}\, dz \\ &\ge \inf_{R<\rho<dR} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} w \int_{R<\rho<dR} |x|^{2\gamma} \rho^{-Q-2\gamma}\, dz \\ &= C_{m,\gamma,k} \ln d \inf_{R<\rho<dR} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} w.\end{aligned}$$

Combining (2.7) and (2.8), we obtain

$$\inf_{R<\rho<dR} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} w - \delta^2 \le \tfrac{2 \ln 2}{\ln d} (\delta + 4)^2.$$

Letting $d \to \infty$ and then $R \to \infty$, we conclude that

$$\liminf_{\rho\to\infty} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} w \le \delta^2 = \big(\tfrac{Q-2}{2}\big)^2,$$

as claimed. □

3. Preliminary results

Using the Hardy-type inequalities in the previous section, we obtain general results, which assert that, if (1.1) has a positive supersolution u, then the coefficient functions $c = c(z)$ and $b = b(z)$ satisfy certain inequalities that are independent of u.

Proposition 3.1. *Assume that* (1.1) *has a supersolution* $u > 0$ *in a domain* $\Omega \subset \mathbb{R}^N$.

(i) *If* $0 < \beta < 2$*, then the coefficient functions* $b = b(z)$ *and* $c = c(z)$ *satisfy*

$$\sup_{0<t<1}\left(t\int_\Omega c\phi^2\,dz - t(1-t)^{\frac{\beta}{\beta-2}}\int_\Omega \big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta b s}{2}\big)^{\frac{2}{2-\beta}}\phi^2\,dz\right) \leq \int_\Omega |\nabla_\gamma \phi|^2\,dz, \tag{3.1}$$

for every $\phi \in C_c^\infty(\Omega)$*. In particular, if* $\beta = 1$*, the inequality above simplifies to*

$$\sqrt{\int_\Omega c\phi^2\,dz} - \sqrt{\int_\Omega \tfrac{b^2}{4}\phi^2\,dz} \leq \sqrt{\int_\Omega |\nabla_\gamma\phi|^2\,dz}.$$

(ii) *If* $\beta = 2$ *and* $b_\infty = \|b\|_{L^\infty(\Omega)} < 1$*, then*

$$(1-b_\infty)\int_\Omega c\phi^2\,dz \leq \int_\Omega |\nabla_\gamma\phi|^2\,dz, \tag{3.2}$$

for every $\phi \in C_c^\infty(\Omega)$*.*

Proof. (i) Let $0 < \beta < 2$ and u be a positive supersolution of (1.1). Let $v = \frac{u^t}{t}$, $0 < t < 1$. Applying (2.2) with $F(s) = \frac{s^t}{t}$, we obtain

$$-\Delta_\gamma v = (1-t)u^{t-2}|\nabla_\gamma u|^2 - u^{t-1}\Delta_\gamma u,$$

or equivalently,

$$\tfrac{1}{t}\tfrac{-\Delta_\gamma v}{v} = (1-t)\tfrac{|\nabla_\gamma u|^2}{u^2} - \tfrac{\Delta_\gamma u}{u}.$$

By (1.1) we have

$$\tfrac{1}{t}\tfrac{-\Delta_\gamma v}{v} \geq (1-t)\tfrac{|\nabla_\gamma u|^2}{u^2} + c - b\tfrac{|\nabla_\gamma u|^\beta}{u^\beta}. \tag{3.3}$$

Let $T = \frac{|\nabla_\gamma u|}{u}$. An elementary calculation shows that

$$\begin{aligned}(1-t)\tfrac{|\nabla_\gamma u|^2}{u^2} - b\tfrac{|\nabla_\gamma u|^\beta}{u^\beta} &= (1-t)T^2 - bT^\beta \\ &\geq -\big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}}\big(\tfrac{2}{\beta}-1\big)(1-t)^{\frac{\beta}{\beta-3}}b^{\frac{2}{2-\beta}},\end{aligned}$$

for $T \geq 0$. Substituting this into (3.3), we obtain

$$\tfrac{1}{t}\tfrac{-\Delta_\gamma v}{v} \geq c - \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}}\big(\tfrac{2}{\beta}-1\big)(1-t)^{\frac{\beta}{\beta-3}}b^{\frac{2}{2-\beta}}.$$

By the Hardy-type inequality in (2.1), we have

$$
\begin{aligned}
t\int_\Omega c\phi^2\,dz - t(1-t)^{\frac{\beta}{\beta-2}}\int_\Omega \left(\tfrac{2}{\beta}-1\right)\left(\tfrac{\beta b}{2}\right)^{\frac{2}{2-\beta}}\phi^2\,dz \\
\le \int_\Omega \frac{-\Delta_\gamma v}{v}\,dz \le \int_\Omega |\nabla_\gamma \phi|^2\,dz.
\end{aligned}
\tag{3.4}
$$

We obtain (3.1) by taking the supremum over $t\in(0,1)$.

(ii) Let $\beta = 2$, $b_\infty = \|b\|_{L^\infty(\Omega)} < 1$ and assume that u is a positive supersolution of (1.1). Then

$$
\frac{-\Delta_\gamma u}{u} + b_\infty \frac{|\nabla_\gamma u|^2}{u^2} \ge c \quad \text{in } \Omega.
$$

By multiplying both sides by $(1-b_\infty)\phi^2$, $\phi \in C_c^\infty(\Omega)$, and integrating over Ω, we obtain

$$
(1-b_\infty)\int_\Omega \frac{-\Delta_\gamma u}{u}\phi^2\,dz + b_\infty(1-b_\infty)\int_\Omega \frac{|\nabla_\gamma u|^2}{u^2}\phi^2\,dz \ge (1-b_\infty)\int_\Omega c\phi^2\,dz.
$$

By applying (2.4) with $\alpha = 1-b_\infty$, we arrive at

$$
(1-b_\infty)\int_\Omega c\phi^2\,dz \le \int_\Omega |\nabla_\gamma\phi|^2 dz,
$$

which proves (3.2).

□

If $\beta \neq 1$, the supremum in (3.1) cannot be computed explicitly, in general. However, through certain estimates, we obtain the following more explicit result.

Corollary 3.2. *Assume that* (1.1) *has a positive supersolution $u > 0$ in a domain $\Omega \subset \mathbb{R}^N$. If $0 < \beta < 2$, then the coefficient functions $b = b(z)$ and $c = c(z)$ satisfy*

$$
\begin{aligned}
\int_\Omega \left(c - \left(\tfrac{\beta}{2}\right)^{\frac{2}{2-\beta}}\left(\tfrac{2}{\beta}-1\right) b^{\frac{2}{2-\beta}}\right)\phi^2\,dz \\
\le \tfrac{2\beta}{2-\beta}\left(\int_\Omega c\phi^2\,dz\right)^{\frac12}\left(\int_\Omega |\nabla_\gamma\phi|^2\,dz\right)^{\frac12},
\end{aligned}
\tag{3.5}
$$

for every $\phi \in C_c^\infty(\Omega)$*. On the other hand, if* $0<\beta\le 1$*, then the following refined inequality holds*

$$\begin{aligned}\tfrac{\beta}{2-\beta}\Bigg(\Big(\int_\Omega c\phi^2\,dz\Big)^{\frac{2-\beta}{2}}-\Big(\int_\Omega \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}}\big(\tfrac{2}{\beta}-1\big)b^{\frac{2}{2-\beta}}\phi^2\,dz\Big)^{\frac{2-\beta}{2}}\Bigg)^2\\ \le\Big(\int_\Omega c\phi^2\,dz\Big)^{1-\beta}\int_\Omega|\nabla_\gamma\phi|^2\,dz,\end{aligned}\tag{3.6}$$

for every $\phi\in C_c^\infty(\Omega)$*. As a consequence, if* $\Omega\subset\mathbb{R}^N$ *is an exterior domain and* $0<\beta<2$ *then* (1.1) *does not have positive supersolutions, if*

$$\sup_{R>2R_0}\left(\frac{\inf_{R<\rho(z)<dR}\frac{\rho(z)^{2+2\gamma}}{|x|^{2\gamma}}b(z))}{\sup_{\frac{R}{2}<\rho(z)<2dR}\frac{\rho(z)^{1+\gamma}}{|x|^{\gamma}}\sqrt{c(z)}}\right)=\infty.\tag{3.7}$$

Proof. We write (3.4) in the form

$$At-Bt(1-t)^{\frac{\beta}{\beta-2}}\le\int_\Omega|\nabla_\gamma\phi|^2\,dz,$$

where

$$A=\int_\Omega c\phi^2\,dz\quad\text{and}\quad B=\int_\Omega\big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta b}{2}\big)^{\frac{2}{2-\beta}}\phi^2\,dz.$$

Assume that $A>B$ for a given $\phi\in C_c^\infty(\Omega)$ and let

$$g(t)=At-Bt(1-t)^{\frac{\beta}{\beta-2}},\quad 0<t<1.$$

An elementary calculation shows that

$$\max_{0<t<1}g(t)=\tfrac{\beta A}{2-\beta}\frac{t_0^2}{1+\frac{2(\beta-1)}{2-\beta}t_0}=\tfrac{\beta B}{2-\beta}t_0^2(1-t_0)^{\frac{2}{\beta-2}}=M(t_0),$$

where t_0 is the unique solution of

$$(1-t_0)^{\frac{2}{2-\beta}}=\tfrac{B}{A}\big(1+\tfrac{2(\beta-1)}{2-\beta}t_0\big).\tag{3.8}$$

From (3.8) we obtain

$$\tfrac{B}{A}\big(1+\tfrac{2(\beta-1)}{2-\beta}t_0\big)=(1-t_0)^{\frac{2}{2-\beta}}\ge1-\tfrac{2t_0}{2-\beta},$$

from which it follows that

$$t_0\ge\tfrac{2-\beta}{2}\tfrac{A-B}{(\beta-1)B+A}\ge\tfrac{2-\beta}{2\beta}\tfrac{A-B}{A}$$

and consequently

$$M(t_0)=\big(\tfrac{\beta A}{2-\beta}\big)\frac{t_0^2}{1+\frac{2(\beta-1)}{2-\beta}t_0}\ge\big(\tfrac{\beta A}{2-\beta}\big)\tfrac{t_0^2}{1+\frac{2(\beta-1)}{2-\beta}}\ge At_0^2\ge\big(\tfrac{2-\beta}{2\beta}\big)^2\tfrac{(A-B)^2}{A}.$$

This leads to

$$\left(\int_\Omega \Big(c - \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}}\big(\tfrac{2}{\beta}-1\big)b^{\frac{2}{2-\beta}}\Big)\,\phi^2\,dz\right)^2 \le \big(\tfrac{2\beta}{2-\beta}\big)^2\left(\int_\Omega c\phi^2\,dz\right)\left(\int_\Omega |\nabla_\gamma\phi|^2\,dz\right).$$

For the case $0<\beta\le 1$, from (3.8) we conclude that $(1-t_0)^{\frac{2}{2-\beta}} \le \frac{B}{A}$, or equivalently,

$$t_0 \ge 1 - \big(\tfrac{B}{A}\big)^{\frac{2-\beta}{2}}.$$

Since $M(t_0) > \big(\frac{\beta A}{2-\beta}\big)t_0^2$, we obtain

$$\tfrac{\beta A}{2-\beta}\Big(1-\big(\tfrac{B}{A}\big)^{\frac{2-\beta}{2}}\Big)^2 \le \int_\Omega |\nabla_\gamma\phi|^2 dz,$$

from which it follows that

$$\tfrac{\beta}{2-\beta}A^{\beta-1}\big(A^{\frac{2-\beta}{2}} - B^{\frac{2-\beta}{2}}\big)^2 \le \int_\Omega |\nabla_\gamma\phi|^2\,dz,$$

We obtain (3.6) by inserting A and B.

To prove the last part, testing (3.5) with ϕ_R as in the proof of Proposition 2.3 gives

$$\frac{\inf_{R<\rho<dR}\frac{\rho^{2+2\gamma}}{|x|^{2\gamma}}b}{\sup_{\frac{R}{2}<\rho<2dR}\frac{\rho^{1+\gamma}}{|x|^{\gamma}}\sqrt{c}} \le \tfrac{2\beta}{2-\beta}\big(\tfrac{2(\delta+4)^2\ln 2}{\ln d}+\delta^2\big).$$

This implies that (1.1) does not have positive supersolutions, if (3.7) holds. □

4. Liouville-type results

Based on the estimates above, we present several nonexistence results on both bounded domains and unbounded exterior domains. We begin by considering the case where the coefficients $c(x)\equiv c\in\mathbb{R}$ and $b(x)\equiv b\in\mathbb{R}$ are constants, that is,

$$-\Delta_\gamma u + \frac{b}{u^{\beta-1}}|\nabla_\gamma u|^\beta = cu \quad \text{in } \Omega, \tag{4.1}$$

where $\Omega\subset\mathbb{R}^N$ is a domain and $0<\beta\le 2$.

Consider the eigenvalue problem

$$\begin{cases} -\Delta_\gamma u = \lambda u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set such that $0 \in \Omega$ and denote by $H^\gamma(\Omega)$ the completion of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_\gamma = \left(\int_\Omega |\nabla_\gamma u|^2 \, dz \right)^{\frac{1}{2}}.$$

It turns out that $H^\gamma(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v \rangle_\gamma = \int_\Omega \nabla_\gamma u \cdot \nabla_\gamma v \, dz.$$

Let $2^*_\gamma = \frac{2Q}{Q-2}$ be the critical Sobolev exponent. It is known that the embedding $H^\gamma(\Omega) \hookrightarrow L^{2^*_\gamma}(\Omega)$ is continuous, see [44, 15]. Moreover, if $\Omega \subset \mathbb{R}^N$ is bounded, then the embedding $H^\gamma(\Omega) \hookrightarrow L^p(\Omega)$ is compact for every $p \in [1, 2^*_\gamma)$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. The first eigenvalue $\lambda_1^\gamma(\Omega)$, and the related eigenfunctions of $-\Delta_\gamma$, are characterized by the variational formula

$$\lambda_1^\gamma(\Omega) = \min_{u \in H^\gamma(\Omega)} \frac{\int_\Omega |\nabla_\gamma u|^2 \, dz}{\int_\Omega |u|^2 \, dz},$$

and there exists a positive function $e_1 \in H^\gamma(\Omega)$, which is an eigenfunction corresponding to λ_1^γ, normalized such that $\|e_1\|_{L^2(\Omega)} = 1$. Moreover, the first eigenvalue $\lambda_1^\gamma(\Omega)$ is simple.

Remark 4.1. We note that the simplicity of $\lambda_1^\gamma(\Omega)$ can also be directly deduced from our Hardy-type inequalities. Assume that u and v are two solutions of (4.2). Then from (2.3) we obtain

$$\lambda_1 \int_\Omega \phi^2 \, dz + \alpha(1-\alpha) \int_\Omega \left| \nabla_\gamma \log \tfrac{u}{v} \right|^2 \phi^2 \, dz \le \int_\Omega |\nabla_\gamma \phi|^2 \, dz.$$

By choosing $\phi = e_1$, the first eigenfunction, we have

$$\int_\Omega \left| \nabla_\gamma \log \tfrac{u}{v} \right|^2 e_1^2 \, dz \le 0,$$

which implies that $\frac{u}{v} \equiv c$.

Proposition 4.2. *Consider* (4.1) *with* $0 < \beta \le 2$ *in a domain* $\Omega \subset \mathbb{R}^N$*. If* $0 < \beta < 2$*, we assume that*

$$c - \left(\tfrac{\beta}{2}\right)^{\frac{2}{2-\beta}} \left(\tfrac{2}{\beta} - 1\right) b^{\frac{2}{2-\beta}} > 0 \tag{4.3}$$

and if $\beta = 2$*, we assume that* $0 \le b < 1$*.*

(i) *If* $\Omega \subset \mathbb{R}^N$ *is a bounded domain, then* (4.1) *does not have positive supersolutions provided that*

$$\sup_{0<t<1} t\Big(c - (1-t)^{\frac{\beta}{\beta-2}}\big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta b}{2}\big)^{\frac{2}{2-\beta}}\Big) > \lambda_1^\gamma(\Omega), \tag{4.4}$$

if $0<\beta<2$, *and*

$$(1-b)c > \lambda_1(\Omega), \tag{4.5}$$

if $\beta = 2$. *In particular, if* $\beta = 1$, *the condition reduces to* $\sqrt{c} - \frac{b}{2} > \sqrt{\lambda_1^\gamma(\Omega)}$.

(ii) *If* Ω *is an exterior domain in* $\mathbb{R}^N$, *then* (4.1) *does not have positive supersolutions.*

Proof of Proposition 4.2. (i) Let $u > 0$ be a supersolution of (4.1). If $0 < \beta < 2$, from (3.1) we have

$$\sup_{0<t<1} t\Big(c - (1-t)^{\frac{\beta}{\beta-2}}\big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta b}{2}\big)^{\frac{2}{2-\beta}}\Big) \int_\Omega \phi^2\,dz \le \int_\Omega |\nabla_\gamma \phi|^2 dz, \tag{4.6}$$

for all $\phi \in C_c^\infty(\Omega)$. Dividing both sides by $\int_\Omega \phi^2 dz$ and taking infimum over all $0 \ne \phi \in C_c^\infty(\Omega)$ we obtain

$$\sup_{0<t<1} t\Big(c - (1-t)^{\frac{\beta}{\beta-2}}\big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta b}{2}\big)^{\frac{2}{2-\beta}}\Big) \int_\Omega \phi^2\,dz \le \lambda_1^\gamma(\Omega).$$

Hence, positive supersolutions do not exist, if (4.4) holds. If $\beta = 2$, by (3.2) we obtain

$$(1-b)c\int_\Omega \phi^2\,dz \le \int_\Omega |\nabla_\gamma\phi|^2\,dz,$$

and similarly to the previous case, the equation does not have positive supersolutions, if (4.5) holds. Note also that, for the special case $\beta = 1$, we have

$$\sup_{0<t<1} t\big(c - \tfrac{b^2}{4(1-t)}\big) = \big(\sqrt{c} - \tfrac{b}{2}\big)^2.$$

(ii) Let $\Omega \subset \mathbb{R}^N$ be an exterior domain. By (4.3) there exists t_0 such that

$$D_0 = c - (1-t_0)^{\frac{\beta}{\beta-2}}\big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta b}{2}\big)^{\frac{2}{2-\beta}} > 0.$$

Then, by (4.6) and Proposition 2.3, we obtain

$$t_0 D_0 \liminf_{\rho\to\infty} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} \le \big(\tfrac{Q-2}{2}\big)^2,$$

which leads to a contradiction since

$$\liminf_{\rho\to\infty} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} = \infty.$$

Similarly, in the case $\beta = 2$ by (4.6) and Proposition 2.3 we have

$$(1-b)c \liminf_{\rho\to\infty} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} \leq \big(\tfrac{Q-2}{2}\big)^2,$$

which leads to a contradiction, if $(1-b)c > 0$. $\square$

Remark 4.3. Using Corollary 3.2, it is possible to derive an explicit (though generally weaker) criterion, in place of inequality (4.4), for the nonexistence of positive supersolutions to (4.1). For instance, if $0 < \beta \leq 1$, applying (3.6) and following an argument similar to the one above, we find that there does not exist positive supersolutions, if the constants c and b satisfy

$$c^{\frac{2-\beta}{2}} - \tfrac{\beta}{2}\big(\tfrac{2}{\beta}-1\big)^{\frac{2-\beta}{2}} b > \sqrt{\tfrac{2-\beta}{\beta}}\, c^{\frac{1-\beta}{2}} \sqrt{\lambda_1^\gamma(\Omega)}.$$

Proposition 4.4. *Let $b, c \in C(\mathbb{R}^N \setminus B^\rho_{R_0})$, $0 < \beta \leq 2$,*

$$\alpha_+ = \limsup_{|z|\to\infty} \tfrac{\rho(z)^{2+2\gamma}}{|x|^{2\gamma}} c(z), \quad \alpha_- = \liminf_{|z|\to\infty} \tfrac{\rho(z)^{2+2\gamma}}{|x|^{2\gamma}} c(z),$$

and

$$\tau_+ = \limsup_{|z|\to\infty} \big(\tfrac{\rho(z)^{1+\gamma}}{|x|^{\gamma}}\big)^{2-\beta} b(z), \quad \tau_- = \liminf_{|z|\to\infty} \big(\tfrac{\rho(z)^{1+\gamma}}{|x|^{\gamma}}\big)^{2-\beta} b(z).$$

(i) *If $\tau_+ < \infty$ (when $\beta = 2$ we assume $\tau_+ < 1$), then (1.1) does not have positive supersolutions provided*

$$\alpha_- > \big(\tfrac{Q-2}{2}\big)^2 \Big(\big(\tfrac{2}{\beta}-1\big)T_0^2 + \tfrac{2}{\beta}T_0 + 1\Big), \tag{4.7}$$

where T_0 is the unique solution of

$$\frac{2}{\beta}\big(\tfrac{Q-2}{2}\big)^{2-\beta} \tfrac{T_0}{(T_0+1)^{\beta-1}} = \tau_+. \tag{4.8}$$

(ii) *The above results is sharp in the sense that, if $\tau_- < \infty$ (when $\beta = 2$ we assume $\tau_- < 1$) and*

$$\alpha_+ < \big(\tfrac{Q-2}{2}\big)^2 \Big(\big(\tfrac{2}{\beta}-1\big)T_0^2 + \tfrac{2}{\beta}T_0 + 1\Big), \tag{4.9}$$

where T_0 is the unique solution of

$$\tfrac{2}{\beta}\big(\tfrac{Q-2}{2}\big)^{2-\beta} \tfrac{T_0}{(T_0+1)^{\beta-1}} = \tau_-, \tag{4.10}$$

then (1.1) has positive supersolutions in $\mathbb{R}^N \setminus B^\rho_R$ for sufficently large R.

(iii) *If* $0 < \beta < 2$ *and* $\tau_+ = \infty$*, then* (1.1) *does not have positive supersolutions, if*

$$\limsup_{R\to\infty}\left(\frac{\inf_{R<\rho(z)<2R}\frac{\rho(z)^{2+2\gamma}}{|x|^{2\gamma}}c(z)}{\sup_{\frac{R}{2}<\rho(z)<4R}\frac{\rho(z)^{2+2\gamma}}{|x|^{2\gamma}}b(z)^{\frac{2}{2-\beta}}}\right) = \infty. \tag{4.11}$$

Similarly, for $\beta = 2$ *the same conclusion holds, if*

$$\limsup_{R\to\infty}\left(\left(1 - \sup_{\frac{R}{2}<\rho(z)<4R} b(z)\right)\inf_{R<\rho(z)<2R}\frac{\rho(z)^{2+2\gamma}}{|x|^{2\gamma}}c(z)\right) = \infty.$$

Remark 4.5. If $\beta = 1$, the value of T_0 in the proposition above can be computed explicitly. From (4.8) we obtain $T_0 = \frac{\tau_+}{Q-2}$ and substituting it into (4.7), the equation does not have positive supersolutions, if

$$\alpha_- > \left(\tfrac{Q-2+\tau_+}{2}\right)^2.$$

Similarly, from (4.9) and (4.10) a positive supersolution exists in $\mathbb{R}^N \setminus B_R$ for R sufficently large, if

$$\alpha_+ < \left(\tfrac{Q-2+\tau_-}{2}\right)^2.$$

Moreover, if $\beta = 2$, then $T_0 = \frac{\tau_+}{1-\tau_+}$, and the equation does not have supersolutions, if

$$(1-\tau_+)\alpha_- > \left(\tfrac{Q-2}{2}\right)^2$$

while a positive supersolution exists in $\mathbb{R}^N \setminus B_R$ for R sufficently large, if

$$(1-\tau_-)\alpha_+ < \left(\tfrac{Q-2}{2}\right)^2.$$

Proof of Proposition 4.4. (i) Consider first the case $\beta \neq 2$, and suppose that (1.1) has a supersolution $u > 0$. From Proposition 3.1, the coefficient functions $b = b(z)$ and $c = c(z)$ must satisfy (3.1). Testing this inequality with $\phi = \phi_R$ and using the gradient bound in (2.8), we obtain

$$\int_\Omega |\nabla_\gamma \phi|^2\,dz \le 2C(\delta+4)^2\ln 2 + C\delta^2\ln d.$$

On the other hand, we have

$$\begin{aligned}\int_\Omega c\phi^2\,dz &\ge \int_{R<\rho<dR} c\rho^{-2\delta}\,dz\\ &= \int_{R<\rho<dR}\left(\tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}}c\right)\left(|x|^{2\gamma}\rho^{-2\delta-2-2\gamma}\right)dz\end{aligned}$$

$$
\begin{aligned}
&\geq \inf_{R<\rho<dR} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} c \int_{R<\rho<dR} |x|^{2\gamma} \rho^{-Q-2\gamma}\, dz \\
&= C \ln d \inf_{R<\rho<dR} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} c.
\end{aligned}
$$

For the remaining term in (3.1) we have

$$
\begin{aligned}
\int_\Omega \big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta b}{2}\big)^{\frac{2}{2-\beta}} \phi^2\, dz &\leq \big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \int_{\frac{R}{2}<\rho<2dR} b^{\frac{2}{2-\beta}}\, dz \\
&\leq \big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \sup_{\frac{R}{2}<\rho<2dR} \frac{\rho^{2+2\gamma}}{|x|^{2\gamma}} b^{\frac{2}{2-\beta}} \int_{\frac{R}{2}<\rho<2dR} \tfrac{|x|^{2\gamma}}{\rho^{2+2\gamma}} \phi^2\, dz.
\end{aligned}
$$

By applying the Hardy-type inequality in (2.2), we obtain

$$
\begin{aligned}
\int_\Omega \big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta b}{2}\big)^{\frac{2}{2-\beta}} \phi^2\, dz &\leq \big(\tfrac{2}{\beta}-1\big)\left(\tfrac{\beta}{2}\right)^{\frac{2}{2-\beta}} \\
&\cdot \big(\tfrac{2}{Q-2}\big)^2 \sup_{\frac{R}{2}<\rho<2dR} \tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}} b^{\frac{2}{2-\beta}} \int_{\frac{R}{2}<\rho<2dR} |\nabla_\gamma \phi|^2\, dz.
\end{aligned}
$$

Combining the estimates in (3.1), we obtain

$$
\begin{aligned}
&\frac{t \inf_{R<\rho<dR} \frac{\rho^{2+2\gamma}}{|x|^{2\gamma}} c}{t(1-t)^{\frac{\beta}{\beta-2}} \Big(\big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \big(\tfrac{2}{Q-2}\big)^2 \sup_{\frac{R}{2}<\rho<2dR} \frac{\rho^{2+2\gamma}}{|x|^{2\gamma}} b^{\frac{2}{2-\beta}}\Big) + 1} \qquad (4.12)\\
&\quad \leq \tfrac{2(\delta+4)^2 \ln 2}{\ln d} + \delta^2,
\end{aligned}
$$

for every $t \in (0,1)$. Taking the limits $d \to \infty$ and then $R \to \infty$, we arrive at

$$
\frac{t\alpha_-}{t(1-t)^{\frac{\beta}{\beta-2}} \big(\frac{2}{\beta}-1\big)\big(\frac{\beta}{2}\big)^{\frac{2}{2-\beta}} \big(\frac{2}{Q-2}\big)^2 \tau_+^{\frac{2}{2-\beta}} + 1} \leq \big(\tfrac{Q-2}{2}\big)^2,
$$

or equivalently,

$$
\alpha_- \leq (1-t)^{\frac{\beta}{\beta-2}} \big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \tau_+^{\frac{2}{2-\beta}} + \tfrac{1}{t}\big(\tfrac{Q-2}{2}\big)^2.
$$

Hence there is no solution, if

$$
\alpha_- > \inf_{0<t<1} \Big((1-t)^{\frac{\beta}{\beta-2}} \big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \tau_+^{\frac{2}{2-\beta}} + \tfrac{1}{t}\big(\tfrac{Q-2}{2}\big)^2\Big).
$$

Letting $t = \frac{1}{1+T}$, $T > 0$, we rewrite the display above as

$$
\alpha_- > \inf_{T>0} \Big(\big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \tau_+^{\frac{2}{2-\beta}} \big(\tfrac{T+1}{T}\big)^{\frac{\beta}{2-\beta}} + (T+1)\big(\tfrac{Q-2}{2}\big)^2\Big).
$$

Let

$$f(T)=\big(\tfrac{2}{\beta}-1\big)\big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}}\tau_+^{\frac{2}{2-\beta}}\big(\tfrac{T+1}{T}\big)^{\frac{\beta}{2-\beta}}+(T+1)\big(\tfrac{Q-2}{2}\big)^2,$$

where $0<T<\infty$. Since $f(T)\to\infty$ as $T\to 0$ or $T\to\infty$, f attains a minimum at a unique point T_0 such that $f'(T_0)=0$, that is,

$$\tfrac{\beta}{2}\tau_+(T_0+1)^{\beta-1}=T_0\big(\tfrac{Q-2}{2}\big)^{2-\beta}.$$

Then

$$f(T_0)=\big(\tfrac{Q-2}{2}\big)^2\Big(\big(\tfrac{2}{\beta}-1\big)T_0^2+\tfrac{2}{\beta}T_0+1\Big),$$

which proves (i) for $\beta\neq 2$.

If $\beta=2$ we apply (3.2) and obtain

$$\big(1-\sup_{\frac{R}{2}<\rho<2dR} b\big)\int_{\frac{R}{2}<\rho<2dR} c\phi_R^2\,dz\le\int_{\frac{R}{2}<\rho<2dR}|\nabla_\gamma\phi_R|^2\,dz.$$

Using the estimates as before we find that

$$\big(1-\sup_{\frac{R}{2}<\rho<2dR} b\big)\inf_{R<\rho<dR}\tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}}c\le\tfrac{2(\delta+4)^2\ln 2}{\ln d}+\delta^2 \tag{4.13}$$

and letting $d\to\infty$ and $R\to\infty$, we arrive at

$$(1-\tau_+)\alpha_-\le\big(\tfrac{Q-2}{2}\big)^2.$$

Hence there is no solution, if

$$(1-\tau_+)\alpha_->\big(\tfrac{Q-2}{2}\big)^2,$$

which matches with (4.7) and (4.8) as $\beta=2$, completing the proof of (i).

(ii) To prove the existence part, we look for a constant $a>0$ such that the function $u=\rho^{-a}$ is a supersolution of (1.1). We observe that

$$-\Delta_\gamma u=-a(a-Q+2)|x|^{2\gamma}\rho^{-a-2-2\gamma},$$

$$\tfrac{b|\nabla_\gamma u|^\beta}{u^{\beta-1}}=a^\beta|x|^{\gamma\beta}\rho^{-a-\beta(1+\gamma)}b$$

and $cu=c\rho^{-a}$. Thus, we have

$$-\Delta_\gamma u+\tfrac{b|\nabla_\gamma u|^\beta}{u^{\beta-1}}-cu=\rho^{-a-2-2\gamma}|x|^{2\gamma}\Big(a(Q-2-a)+a^\beta\big(\tfrac{\rho^{\gamma+1}}{|x|^\gamma}\big)^{2-\beta}b-\tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}}c\Big).$$

Let $\alpha_1>\alpha_+$ and $\tau_1<\tau_-$. Then for $|z|$ sufficiently large, we have

$$a(Q-2-a)+a^\beta\big(\tfrac{\rho^{\gamma+1}}{|x|^\gamma}\big)^{2-\beta}b-\tfrac{\rho^{2+2\gamma}}{|x|^{2\gamma}}c\ge a(Q-2-a)+a^\beta\tau_1-\alpha_1.$$

Hence, $u=\rho^{-a}$ is a supersolution in $\mathbb{R}^N \setminus B_R$ for large enough R provided

$$\alpha_1 \le (Q-2-a)a + a^\beta \tau_1.$$

Such parameters α_1 and τ_1 exist, if

$$\alpha_+ < \sup_{a>0}((Q-2-a)a + a^\beta \tau_-).$$

Let

$$g(a) = (Q-2-a)a + a^\beta \tau_-, \quad 0<a<\infty.$$

We have $g(0)=0$, $g(\infty)=-\infty$ and $g(a)>0$ for small enough $a>0$, so that g attains supremum at a unique point a_0, where $g'(a_0)=0$, that is,

$$\beta a_0^{\beta-1}\tau_- = 2a_0 - Q + 2.$$

Then

$$g(a_0) = \big(\tfrac{2}{\beta}-1\big)a_0^2 + \big(1-\tfrac{1}{\beta}\big)(Q-2)a_0.$$

Set $a_0 = \big(\frac{Q-2}{2}\big)(T_0+1)$ to match the earlier parametrization. Then

$$g(a_0) = \big(\tfrac{Q-2}{2}\big)^2\big(\big(\tfrac{2}{\beta}-1\big)T_0^2 + \tfrac{2}{\beta}T_0 + 1\big),$$

where

$$\tfrac{\beta}{2}(T_0+1)^{\beta-1}\tau_- = \big(\tfrac{Q-2}{2}\big)^{2-\beta}T_0.$$

This completes the proof of (ii).

(iii) If $0<\beta<2$, it suffices to choose $d=2$ and $t=\frac{1}{2}$ in (4.12) and to apply $\tau_+=\infty$. If $\beta=2$, we may choose $d=2$ in (4.13). □

Corollary 4.6. *Let $b>0$, $c,\mu\in\mathbb{R}$, $0<\beta<2$ and $\lambda > -(2-\beta)(1+\gamma)$. The equation*

$$-\Delta_\gamma u + \frac{b|x|^{(2-\beta)\gamma}\rho^\lambda}{u^{\beta-1}}|\nabla_\gamma u|^\beta = c|x|^{2\gamma}\rho^\mu u \quad \text{in } \mathbb{R}^N \setminus B^\rho_{R_0} \tag{4.14}$$

does not have positive supersolutions in each of the following cases:

(i) $\mu > \frac{2\lambda}{2-\beta}$, $c>0$,

(ii) $\mu = \frac{2\lambda}{2-\beta}$, $c > (\frac{\beta}{2})^{\frac{2}{2-\beta}}(\frac{2}{\beta}-1)b^{\frac{2}{2-\beta}}$.

In all other cases, positive supersolutions exist for sufficiently large R_0.

Remark 4.7. Consider the equation (4.14) with $\gamma=0$ and $\beta=1$, that is,

$$-\Delta u + b|x|^\lambda|\nabla u| = c|x|^\mu u \quad \text{in } \mathbb{R}^N \setminus B^\rho_{R_0}.$$

In [19], nonexistence results were established for $\lambda>0$, assuming that $c - 3eb^2 > 0$ and $\mu = 2\lambda$, though this condition is not optimal. These findings were refined in [3], aligning with our results for $\lambda > 0$. Our work extends these results to include $\lambda > -1$. Furthermore, when $\mu = \frac{2\lambda}{2-\beta}$, we obtain

$$\begin{aligned} D(z) &= c(z) - \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \big(\tfrac{2}{\beta} - 1\big) b(z)^{\frac{2}{2-\beta}} \\ &= \Big(c(z) - \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \big(\tfrac{2}{\beta} - 1\big) b(z)^{\frac{2}{2-\beta}}\Big) |x|^{2\gamma} \rho(z)^{\mu}. \end{aligned}$$

This leads to the degenerate case where $\lim_{|z|\to\infty} D(z) = 0$, provided $\mu + 2\gamma < 0$. By (ii), there does not exist positive supersolutions to (4.14), if $-2\gamma - 2 < \mu < -2\gamma$.

Proof of Corollary 4.6. By the assumption we have

$$\tau_+ = \limsup_{|z|\to\infty} \big(\tfrac{\rho^{1+\gamma}}{|x|^\gamma}\big)^{2-\beta} b = \limsup_{|z|\to\infty} b\rho^{\lambda+(2-\beta)(1+\gamma)} = \infty.$$

Assume that $c > 0$. We note that

$$\frac{\inf\limits_{R<\rho<2R} \frac{\rho^{2+2\gamma}}{|x|^{2\gamma}} c}{\sup\limits_{\frac{R}{2}<\rho<4R} \frac{\rho^{2+2\gamma}}{|x|^{2\gamma}} b^{\frac{2}{2-\beta}}} = \frac{\inf\limits_{R<\rho<2R} c\rho^{2+2\gamma+\mu}}{\sup\limits_{\frac{R}{2}<\rho<4R} b\rho^{2+2\gamma+\frac{2\lambda}{2-\beta}}} \approx C_0 R^{\mu - \frac{2\lambda}{2-\beta}}.$$

Hence, condition (4.11) is satisfied, if $\mu > \frac{2\lambda}{2-\beta}$, and thus the equation does not have positive supersolutions in this case, which proves (i).

If $\mu = \frac{2\lambda}{2-\beta}$, we verify condition (3.7). We note that

$$\begin{aligned} D(z) &= c(z) - \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \big(\tfrac{2}{\beta} - 1\big) b(z)^{\frac{2}{2-\beta}} \\ &= \Big(c - \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \big(\tfrac{2}{\beta} - 1\big) b^{\frac{2}{2-\beta}}\Big) |x|^{2\gamma} \rho(z)^{\mu}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{\inf\limits_{R<\rho<dR} \frac{\rho^{2+2\gamma}}{|x|^{2\gamma}} D(z)}{\sup\limits_{\frac{R}{2}<\rho<2dR} \frac{\rho^{1+\gamma}}{|x|^{\gamma}} \sqrt{c(z)}} &= \frac{\Big(c - \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \big(\tfrac{2}{\beta} - 1\big) b^{\frac{2}{2-\beta}}\Big)}{\sqrt{c}} \frac{\inf\limits_{R<\rho<dR} \rho^{2+2\gamma+\mu}}{\sup\limits_{\frac{R}{2}<\rho<2dR} \rho^{1+\gamma+\frac{\mu}{2}}} \\ &\geq \frac{\Big(c - \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}} \big(\tfrac{2}{\beta} - 1\big) b^{\frac{2}{2-\beta}}\Big)}{\sqrt{c}} C_\gamma R^{1+\gamma+\frac{\mu}{2}}. \end{aligned}$$

Note that the exponent satisfies

$$1 + \gamma + \tfrac{\mu}{2} = 1 + \gamma + \tfrac{\lambda}{2-\beta} > 0.$$

Therefore, condition (3.7) holds, and by Corollary 3.2, the equation in (4.14) does not have positive supersolutions, which proves (ii).

To complete the proof, we construct positive supersolutions in the remaining cases. First, observe that, if $c \leq 0$, then any positive constant function is a supersolution. When $c > 0$, we look for constants $a, t > 0$, to be chosen later, such that the function

$$u = u(z) = e^{-a\rho(z)^t}$$

is a supersolution of (4.14). A direct computation gives

$$\begin{aligned} -\Delta_\gamma u + \tfrac{b|x|^{(2-\beta)\gamma}\rho^\lambda}{u^{\beta-1}}|\nabla_\gamma u|^\beta - c|x|^{2\gamma}\rho^\mu u &= \tfrac{|x|^{2\gamma}e^{-a\rho^t}}{\rho^{2\gamma}}\big(-t^2a^2\rho^{2t-2} \\ &\quad + ta(Q-2+t)\rho^{t-2} + t^\beta a^\beta b\rho^{\lambda+\beta(t-1-\gamma)+2\gamma} - c\rho^\mu\big) \\ &= \tfrac{|x|^{2\gamma}e^{-a\rho^t}}{\rho^{2\gamma+2-2t}}\big(-t^2a^2 + \tfrac{ta(Q-2+t)}{\rho^t} + bt^\beta a^\beta\rho^{\lambda+\beta(t-1-\gamma)+2-2t+2\gamma} \\ &\quad - c\rho^{\mu+2\gamma+2-2t}\big). \end{aligned}$$

Let t be such that

$$\lambda + \beta(t-1-\gamma) + 2 - 2t + 2\gamma = 0,$$

from which it follows that

$$t = \tfrac{\lambda+(2-\beta)(1+\gamma)}{2-\beta} > 0.$$

Then u is a supersolution provided

$$-t^2a^2 + \tfrac{ta(Q-2+t)}{\rho^t} + bt^\beta a^\beta - c\rho^{\mu-\frac{2\lambda}{2-\beta}} > 0. \tag{4.15}$$

If $\mu < \frac{2\lambda}{2-\beta}$, the last term vanishes at infinity, and the inequality holds for small $a > 0$ and sufficiently large $|z|$, by taking into account the facts that $t > 0$ and $0 < \beta < 2$. If $\mu = \frac{2\lambda}{2-\beta}$, then (4.15) becomes

$$-t^2a^2 + \tfrac{ta(Q-2+t)}{\rho^t} + bt^\beta a^\beta - c \geq 0,$$

which is the case, if $-t^2a^2 + bt^\beta a^\beta - c \geq 0$.

Letting $T = at$, this becomes

$$f(T) = -T^2 + bT^\beta - c > 0$$

for some $T > 0$. This holds provided that

$$\min_{T>0} f(T) = \big(\tfrac{\beta}{2}\big)^{\frac{2}{2-\beta}}\big(\tfrac{2}{\beta} - 1\big)b^{\frac{2}{2-\beta}} - c > 0.$$

This completes the proof. □

Acknowledgements. This research was carried out during the first author's visit at the Department of Mathematics at Aalto University. He would like to thank the institution and the Nonlinear Partial Differential Equations group for the kind and warm hospitality.

References

[1] L. Aermark and A. Laptev, Hardy's inequality for the Grushin operator with a magnetic field of Aharanov-Bohm type, *St. Petersburg Math. J.*, **23** (2012), no. 2, 203–208.

[2] A. Aghajani and C. Cowan, Explicit estimates on positive supersolutions of nonlinear elliptic equations and applications, *Discrete Contin. Dyn. Syst.* **39** (2019), no. 5, 2731–2742.

[3] S. Alarcón, J. García-Melián and A. Quaas, Liouville type theorems for elliptic equations with gradient terms, *Milan J. Math.* **81** (2013), no. 1, 171–185.

[4] S. Alarcón, J. García-Melián and A. Quaas, Nonexistence of positive supersolutions to some nonlinear elliptic problems, *J. Math. Pures Appl. (9)* **99** (2013), no. 6, 618–634.

[5] D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka, Continuum of solutions for an elliptic problem with critical growth in the gradient, *J. Funct. Anal.* **268** (2015), no. 8, 2298–2335.

[6] A. Bahrouni, V.D. Rădulescu and D. Repovš, Nonvariational and singular double phase problems for the Baouendi–Grushin operator, *J. Differential Equations* **303** (2021), 645–666.

[7] M.S. Baouendi, Sur une classe d'opérateurs elliptiques dégénérés, *Bull. Soc. Math. France* **95** (1967), 45–87.

[8] M. Bardi and A. Goffi, Liouville results for fully nonlinear equations modeled on Hörmander vector fields II: Carnot groups and Grushin geometries, *Adv. Differential Equations* **28** (2023), no. 7-8, 637–684.

[9] H. Berestycki, F. Hamel and N. Nadirashvili, The speed of propagation for KPP type problems I: Periodic framework, *J. Eur. Math. Soc.* **7** (2005), no. 2, 173–213.

[10] H. Berestycki, F. Hamel and L. Rossi, Liouville type results for semilinear elliptic equations in unbounded domains, *Ann. Mat. Pura Appl. (4)* **186** (2007), no. 3, 469–507.

[11] M.F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, A priori estimates for elliptic equations with reaction terms involving the function and its gradient, *Math. Ann.* **378** (2020), no. 1-2, 13–56.

[12] M.F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, Estimates of solutions of elliptic equations with a source reaction term involving the product of the function and its gradient, *Duke Math. J.* **168** (2019), no. 8, 1487–1537.

[13] I. Birindelli, I. Capuzzo Dolcetta and A. Cutrì, Liouville theorems for semilinear equations on the Heisenberg group, *Annales de l'Institut Henri Poincaré C. Analyse Non Linéaire* **14** (1997), no. 3, 295–308.

[14] I. Birindelli and J. Prajapat, Nonlinear Liouville theorems in the Heisenberg group via the moving plane method, *Comm. Partial Differential Equations* **24** (1999), no. 9-10, 1875–1890.

[15] G.M. Bisci, P. Malanchini and S. Secchi, A note on nonlinear critical problems involving the Grushin subelliptic operator: Bifurcation and multiplicity results, *Potential Anal.* (2025). https://doi.org/10.1007/s11118-025-10199-z

[16] I. Capuzzo Dolcetta and A Cutrì, Hadamard and Liouville type results for fully nonlinear partial differential inequalities, *Commun. Contemp. Math.* **5** (2003), no. 3, 435–448.

[17] I. Capuzzo Dolcetta and A. Cutrì, On the Liouville property for the sub-Laplacians, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **25** (1997), no. 1-2. 239–256.

[18] B. Cassano, V. Franceschi, D. Krejčiřík and D. Prandi, Horizontal magnetic fields and improved Hardy inequalities in the Heisenberg group. *Comm. Partial Differential Equations* (2023), no. 5, 711–752.

[19] H. Chen and P. Felmer, On Liouville type theorems for fully nonlinear elliptic equations with gradient term, *J. Differential Equations* **255** (2013), no. 8, 2167–2195.

[20] H. Chen and X. Liao, Liouville theorem for Lane–Emden equation of Baouendi–Grushin operators, *J. Differential Equations* **430** (2025), Paper No. 113201.

[21] C. Cowan, Optimal Hardy inequalities for general elliptic operators with improvements, *Commun. Pure Appl. Anal.* **9** (2010), no. 1, 100–140.

[22] G. M. DallA'ra and A. Martini, A robust approach to sharp multiplier theorems for Grushin operators, *Trans. Amer. Math. Soc.* **373** (2020), no. 11, 7533–7574.

[23] L. D'Ambrosio, Hardy inequalities related to Grushin type operators, *Proc. Amer. Math. Soc.* **132** (2004), no. 3, 725–734.

[24] L. D'Ambrosio, Liouville theorems for anisotropic quasilinear inequalities, *Nonlinear Anal.* **70** (2009), no. 8, 2855–2869.

[25] L. D'Ambrosio, Some Hardy inequalities on the Heisenberg group, *Differ. Equ.* **40** (2004), no. 4, 552–564.

[26] L. D'Ambrosio and S. Lucente, Nonlinear Liouville theorems for Grushin and Tricomi operators, *J. Differential Equations* **193** (2003), no. 2, 511–541.

[27] L. D'Ambrosio and E. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, *Adv. Math.* **224** (2010), no. 3, 967–1020.

[28] B. Devyver, M. Fraas and Y. Pinchover, Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon, *J. Funct. Anal.* **266** (2014), no. 7, 4422–4489.

[29] A.T. Duong and N.T. Nguyen, Liouville type theorems for elliptic equations involving Grushin operator and advection, *Electron. J. Differential Equations* **2017**, Paper No. 108.

[30] A.T. Duong and Q.H. Phan, Liouville type theorem for nonlinear elliptic system involving Grushin operator, *J. Math. Anal. Appl.* **454** (2017), no. 2, 785–801.

[31] P. Felmer, A. Quaas and B. Sirakov, Solvability of nonlinear elliptic equations with gradient terms, *J. Differential Equations* **254** (2013), no. 11, 4327–4346.

[32] N. Garofalo, Unique continuation for a class of elliptic operators which degenerate on a manifold of arbitrary codimension, *J. Differential Equations* **104** (1993), no. 1, 117–146.

[33] N. Garofalo and E. Lanconelli, Existence and nonexistence results for semilinear equations on the Heisenberg group, *Indiana Univ. Math. J.* **41**, (1992), no.1, 71–98.

[34] V.V. Grushin, On a class of elliptic pseudo-differential operators degenerate on a submanifold, *Math. USSR Sb.* **13**(2) (1971), 155–185.

[35] I. Kombe, Hardy and Rellich type inequalities with remainders for Baouendi–Grushin vector fields, *Houston J. Math.* **41** (2015), no. 3, 849–874.

[36] A. Laptev, M. Ruzhansky and N. Yessirkegenov, Hardy inequalities for Landau Hamiltonian and for Baouendi–Grushin operator with Aharonov–Bohm type magnetic field. Part I, *Math. Scand.* **125** (2019), no. 2, 239–269.

[37] A. Loiudice, Asymptotic estimates and nonexistence results for critical problems with Hardy term involving Grushin-type operators, *Ann. Mat. Pura Appl. (4)* **198** (2019), no. 6, 1909–1930.

[38] A. Martini and D. Müller, A sharp multiplier theorem for Grushin operators in arbitrary dimensions, *Rev. Mat. Iberoam.* **30** (2014), no. 4, 1265–1280.

[39] E. Mitidieri, A simple approach to Hardy inequalities, *Math. Notes* **67** (2000), no. 3-4, 479–486.

[40] R. Monti and D. Morbidelli, Kelvin transform for Grushin operators and critical semilinear equations, *Duke Math. J.* **131** (2006), no.1, 167–202.

[41] D.D. Monticelli, Maximum principles and the method of moving planes for a class of degenerate elliptic linear operators, *J. Eur. Math. Soc.* **12** (2010), 611–654.

[42] L. Rossi, Non-existence of positive solutions of fully nonlinear elliptic equations in bounded domains, *Commun. Pure Appl. Anal.* **7** (2008), no. 1, 125–141.

[43] M. Ruzhansky, D. Suragan and N. Yessirkegenov, Sobolev type inequalities, Euler-Hilbert-Sobolev and Sobolev-Lorentz-Zygmund spaces on homogeneous groups *Integral Equations Operator Theory* **90** (2018), no. 1, Paper No. 10.

[44] J. Xu, J. Chen and D. O'Regan, Nontrivial solutions for an asymptotically linear Δ_α-Laplace equation, *Nonlinear Anal. Model. Control* **28** (2023), no. 5. 841–858.

[45] X. Yu, Liouville type theorem for nonlinear elliptic equation involving Grushin operators, *Commun. Contemp. Math.* **17** (2015), no. 5, 1250050.