MATH 3305 General Relativity Problem sheet 7

Please hand in your solutions to exercises 1-4 by Friday, 4th December 2009. The bonus problem can be returned along with any of the remaining problem sheets.

Problem 1 (15 points) Suppose g_{ij} is a metric tensor.

(a) Write out
$$\nabla_a A^{ij}, \quad \nabla_a A^i{}_j$$
using Christoffel symbols of g_{ij}.

(b) Show that
$$\nabla I^i{}_j = 0$$
when $I^i{}_j$ is the $\binom{1}{1}$-tensor $I^i{}_j = \delta^i_j$.

(c) For a $\binom{0}{1}$-tensor A_i and a $\binom{1}{0}$-tensor B^i show that
$$\nabla_c(A^i B_i) = (\nabla_c A^i)B_i + A^i \nabla_c B_i = W^i{}_{ci}$$
when $W^i{}_{cd}$ is the $\binom{1}{2}$-tensor $W^i{}_{cd} = \nabla_c(A^i B_d)$.

Problem 2 (15 points) Suppose $X^i(\lambda)$ is a geodesic for a metric tensor g_{ij}, and $V^i(\lambda)$ is a $\binom{1}{0}$-tensor along X^i. Write out
$$\frac{D}{d\lambda}\frac{D}{d\lambda}V^i + R_{cbd}{}^i \frac{\partial X^c}{\partial \lambda}\frac{\partial X^d}{\partial \lambda}V^b = 0$$
in terms of Christoffel symbols of g_{ij}.

Problem 3 (35 points) Consider the line-element
$$ds^2 = -A(r)dt^2 + B(r)dr^2.$$
The non-vanishing components of the Christoffel symbol are
$$\Gamma^r_{tt} = \frac{A'}{2B} \qquad \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{A'}{2A} \qquad \Gamma^r_{rr} = \frac{B'}{2B}.$$
In two dimensions the Riemann curvature tensor has only one independent component. Show that
$$R_{trtr} = \frac{1}{4}\left[2A'' - A'\left(\frac{A'}{A} + \frac{B'}{B}\right)\right].$$

Problem 4 (35 points) Show that
$$\nabla_a \nabla_b W_c - \nabla_b \nabla_a W_c = R_{abc}{}^d W_d,$$
when W_i is a $\binom{0}{1}$-tensor W_i, and $R_{abc}{}^d$ is the Riemann curvature tensor of a metric tensor g_{ij}.

Bonus Problem (40 points) Give a detailed argument for the identity

$$R_{abcd} + R_{bcad} + R_{cabd} = 0,$$

when $R_{abcd} = R_{abc}{}^{s} g_{sd}$ and $R_{abc}{}^{s}$ is the Riemann curvature tensor.