Descending Maps Between Slashed Tangent Bundles

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joint work with

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Mathematical setting

- M is a manifold
- TM is the tangent bundle
- $TM \setminus \{0\}$ is the **slashed tangent bundle**

The Problem

- Suppose F is a diffeomorphism

$$F \colon TM \setminus \{0\} \to TM \setminus \{0\}.$$

Characterize those F that can be written as

$$F = D\phi|_{TM\setminus\{0\}}.$$

for a diffeomorphism $\phi \colon M \to M$.

When ϕ exists, one say that F **descends.**

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- **Note:** Uniqueness result for inverse problems in anisotropic media = existence result of isometry

Canonical involution in the second tangent bundle

Definition: Let M be a manifold. Then the **canonical involution** is the diffeomorphism

$$\kappa\colon\ TTM \ \to\ TTM$$

that is locally given by

$$\kappa(x, y, X, Y) \ = \ (x, X, y, Y).$$

Note:

- $\kappa^2 =$ identity.

First main theorem:

If M is a manifold and F is a diffeomorphism

$$F\colon TM \setminus \{0\} \;\to\; TM \setminus \{0\},$$

then the following are equivalent:

(i) There exists a diffeomorphism $\phi\colon M \to M$ such that

$$F \;=\; D\phi|_{TM\setminus\{0\}}.$$

(ii) $DF \;=\; \kappa \circ DF \circ \kappa$

A related result

Theorem [Robbin-Weinstein-Lie]:

Let F be a diffeomorphism

$$F\colon T^*M \;\to\; T^*M.$$

Then the following are equivalent:

(i) $F = \phi^*$ for a diffeomorphism $\phi\colon M \to M$.

(ii) $F^*\theta = \theta$.

Here:

- ϕ^* = **pullback of** ϕ, $\phi(x,\xi) = \left((\phi^{-1})^i(x), \frac{\partial(\phi^{-1})^i}{\partial x^a}\xi_i\right)$
- θ = **canonical 1-form** $\theta \in \Omega^1(T^*M)$, $\theta = \xi_i dx^i$

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Proof: Let locally $F(x,y) = (F_1(x,y), F_2(x,y))$. Then

$$\begin{aligned} DF(x,y,X,Y) &= \Big(F_1(x,y), F_2(x,y), \frac{\partial F_1}{\partial x^a}(x,y)X^a + \frac{\partial F_1}{\partial y^a}(x,y)Y^a, \\ &\qquad \frac{\partial F_2}{\partial x^a}(x,y)X^a + \frac{\partial F_2}{\partial y^a}(x,y)Y^a\Big), \end{aligned}$$

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$$\begin{aligned} \kappa \circ DF \circ \kappa(x, y, X, Y) &\\ = \Big(F_1(x, X), &\frac{\partial F_1}{\partial x^a}(x, X)y^a + \frac{\partial F_1}{\partial y^a}(x, X)Y^a, F_2(x, X), \\ &\frac{\partial F_2}{\partial x^a}(x, X)y^a + \frac{\partial F_2}{\partial y^a}(x, X)Y^a \Big). \end{aligned}$$

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- First components: $F_1(x,y) = F_1(x,X)$.

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- First components: $F_1(x,y) = F_1(x,X)$. Thus $F_1(x,y)$ does not depend on y. Set $\phi(x) = F_1(x,y)$.
- Second components: $F_2(x,y) = \frac{\partial \phi}{\partial x^a}(x)y^a$. Thus $F = D\phi|_{TM\setminus\{0\}}$.

Second main theorem:

Suppose M is a connected manifold with $\dim M \geq 2$ and F is a diffeomorphism

$$F\colon TM \setminus \{0\} \;\to\; TM \setminus \{0\}.$$

If M has two complete Riemann metrics g and $\widetilde{g}$ such that

(i) g has a **trapping hypersurface** $\Sigma \subset M \Leftrightarrow$ Every geodesic of g intersects Σ

(ii) for all $p \in \Sigma$,

$$\begin{aligned} g_{ij}(p) &= \widetilde{g}_{ij}(p), \quad \Gamma^i_{jk}(p) = \widetilde{\Gamma}^i_{jk}(p), \\ DF(\xi) &= \xi, \qquad\qquad\qquad\qquad \xi \in T(T_pM \setminus \{0\}) \end{aligned}$$

(iii) If $J\colon I \to TM \setminus \{0\}$ is a Jacobi field for g then $F \circ J\colon I \to TM \setminus \{0\}$ is a Jacobi field for $\widetilde{g}$.

Then there exists a diffeomorphism $\phi\colon M \to M$ such that $F = D\phi|_{TM\setminus\{0\}}$ and ϕ is an isometry.

Addendum: Outline of proof:

1. F preserves integral curves since:
   - every integral curve is a Jacobi field
   - F preserves Jacobi fields
   - $\Gamma^i_{jk} = \widetilde{\Gamma}^i_{jk}$ and $DF = \mathrm{Id}$ on Σ

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4. ϕ is totally gedesic since:
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5. **Proposition:** Let M be a connected manifold with two Riemann metrics. If $\phi\colon M \to M$ is totally geodesic and ϕ is an isometry at one point, then ϕ is an isometry.