Geometry and Electromagnetic Gaussian beams

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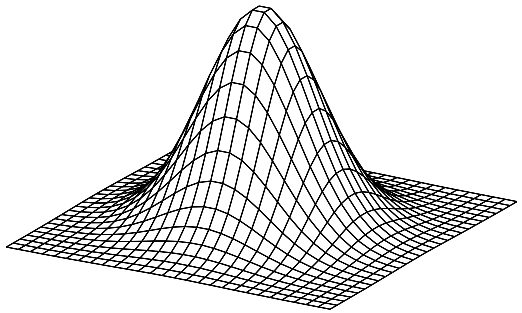
Introduction

- A **Gaussian beam** is an asymptotic solution to Maxwell's equations that is always concentrated in space.



- A Gaussian beam propagates along a curve
- **Motivation:** Fast calculation of qualitative features of solutions to Maxwell's equations in time domain

Initial assumptions

- Everything is smooth
- M open set in $\mathbb{R}^3$ (or a 3-manifold)
- Media is anisotropic, non-homogeneous, no time or frequency dependence
- ε, μ real, symmetric, positive definite 3×3-matrices.
- ε, μ **simultaneously diagonalizable:** For some orthogonal matrix $R = R(x)$,

$$\varepsilon = R^{-1} \cdot \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix} \cdot R, \quad \mu = R^{-1} \cdot \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} \cdot R$$

Asymptotic solutions

$$\textbf{Trial:} \qquad E(x,t) \;=\; \mathrm{Re}\{e^{iP\theta(x,t)}E_0(x,t)\}$$

$P \gg 0$ large constant, θ complex phase function, E_0 complex vector field.

- Interpretation: $(e^{iP\theta} = e^{iP\,\mathrm{Re}\,\theta} \cdot e^{-P\,\mathrm{Im}\,\theta})$
  - $\mathrm{Re}\,\theta$ describes how E propagates
  - $\mathrm{Im}\,\theta \geq 0$ describes shape of E
  - E_0 describes polarization of E
- Plugging E into Maxwell's equations gives:
  - **Hamilton-Jacobi equation** for θ: $\frac{\partial\theta}{\partial t} = h(x, \nabla\theta)$.
    Here $h = h_\pm$ is a **Hamiltonian function** $h\colon T^*M \to \mathbb{R}$, which depend only on media.
  - **Transport equation** for E_0.

Definition of Gaussian beam

Definition: An asymptotic solution $E = \mathrm{Re}\{e^{iP\theta} E_0\}$ is a **Gaussian beam** if there is a curve $c\colon I \to M$ such that

$$\begin{aligned} \theta(x,t) &= \phi(t) + p(t)\cdot z + \frac{1}{2} z^T \cdot S(t) \cdot z + o(|z|^2), \\ z &= z(x,t) = x - c(t) \end{aligned}$$

ϕ, p real, S symmetric, $\mathrm{Im}\, S$ positive definite

Motivation:

$$\begin{aligned} |\exp(iP\theta(x,t))| &\approx \exp\left(-\frac{P}{2}\, z^T \cdot \mathrm{Im}\, S \cdot z\right) \\ &= \text{Gaussian bell curve with centre } c(t) \end{aligned}$$

Phase function for a Gaussian beam:

$$\theta(x,t) = \phi(t) \quad + \quad p(t)\cdot z \quad + \quad \frac{1}{2} z^T \cdot S(t) \cdot z \quad + \quad o(|z|^2),$$

- ϕ is constant
- (c,p) is a solution to **Hamilton's equations**

$$\begin{aligned} \frac{dc^i}{dt} &= \frac{\partial h}{\partial \xi_i} \circ (c,p), \\ \frac{dp_i}{dt} &= -\frac{\partial h}{\partial x^i} \circ (c,p). \end{aligned}$$

- S is a solution to a **complex matrix Riccati equation**

$$\frac{dS}{dt} + BS + SB^T + SCS + D = 0.$$

Geometrization of Gaussian beams

Suppose ε, μ are simultaneously diagonalizable. Then:

Gaussian beams propagate using Riemannian geometry

$\Leftrightarrow$ For some $i \neq j$, $\varepsilon_i \mu_j = \varepsilon_j \mu_i$

Examples of media where Gaussian beams propagate using Riemannian geometry:

- **isotropic media:** $\varepsilon = \varepsilon(x)I$, $\mu = \mu(x)I$,
- $\varepsilon = \rho\mu$ for a positive function $\rho > 0$
- $\mu = \mu_0(x)I, \quad \varepsilon = R^{-1} \cdot \mathrm{diag}(\varepsilon_\perp, \varepsilon_\perp, \varepsilon_\parallel) \cdot R.$

Example

If $\varepsilon_2\mu_3 = \varepsilon_3\mu_2$, then Gaussian beams propagate along geodesics of Riemannian metrics

$$\begin{aligned} g_{+,ij}(x) &= (R^{-1} \cdot \mathrm{diag}\,(\varepsilon_2\mu_3, \varepsilon_1\mu_3, \varepsilon_1\mu_2) \cdot R)_{ij}, \\ g_{-,ij}(x) &= (R^{-1} \cdot \mathrm{diag}\,(\varepsilon_3\mu_2, \varepsilon_3\mu_1, \varepsilon_2\mu_1) \cdot R)_{ij}. \end{aligned}$$