In[1]:=
```
SetDirectory["~/KappaLib/"];
<< kappaLib-1.2.m
<< helper.m
```

```
Loading KappaLib v1.2

Loading helper.m..
```

Notebook to find kappa such that kappa is algebraically decomposable and the equation for D has no solution.

In[4]:=
```
coords = {xi0, xi1, xi2, xi3};
```

In[5]:=
```
(* contract kappa with a bivector from the left *)
contract[biv_, kappa_] := Table[
  1 / 2 Sum[
    biv[[i]][[j]] emReadNormal[kappa, a, b, i, j]
    ,
    {i, 1, 4}, {j, 1, 4}
   ]
  ,
  {a, 1, 4}, {b, 1, 4}
 ]
```

- **Define general kappa and put X variables to zero to simplify the computations.**

  Change X for speed/generality

In[6]:=
```
X = 10;
kappa = emGeneralKappa["k"];
For[ii = 1, ii ≤ X, ii++,
  vKappa = Variables[kappa];
  v = Random[Integer, {1, Length[vKappa]}];
  kappa = kappa /. vKappa[[v]] → 0;
 ];
emKappaToMatrix[kappa] // MatrixForm
```

Out[9]//MatrixForm=

$$\begin{pmatrix} k11 & k12 & k13 & k14 & k15 & k16 \\ k21 & 0 & k23 & k24 & k25 & k26 \\ 0 & 0 & 0 & k34 & 0 & 0 \\ 0 & k42 & k43 & k44 & k45 & k46 \\ k51 & k52 & k53 & k54 & k55 & 0 \\ 0 & k62 & k63 & k64 & 0 & k66 \end{pmatrix}$$

- **Formulate constraints:**

  *** kappa is algebraically decomposable**
  *** beta^2 -alpha = -1**
  *** det(kappa)=1**
  *** trace(kappa)=0**

In[10]:=
```
Abivector = emMatrix["a", 4, Structure → "AntiSymmetric"];
Bbivector = emMatrix["b", 4, Structure → "AntiSymmetric"];

LHS = alpha emIdentityKappa[] +
   beta (kappa + emPoincare[kappa]) + emCompose[emPoincare[kappa], kappa];
RHS = emBiProduct[rho, Abivector, Bbivector] + emBiProduct[rho, Bbivector, Abivector];
eqs = Union[Simplify[Flatten[(LHS - RHS)]]];

eqs = Append[eqs, (beta^2 - alpha) + 1];
eqs = Append[eqs, emDet[kappa] - 1];
eqs = Append[eqs, emTrace[kappa]];
```

▪ Randomly assign some variables and solve the rest.

Note: The below can take a long time.

In[498]:=
```
For[done = 0, done == 0, (* nop *),
 subs = {rho → 1};

 (* pick values for Y variables randomly  *)
 (* Change Y for speed/generality *)
 For[ii = 1, ii ≤ 20, ii++,
  vars = Variables[eqs //. subs];
  v = Random[Integer, {1, Length[vars]}];
  v = vars[[v]];
  subs = Append[subs, v -> Random[Integer, {0, 1}]];
 ];
 vars = Variables[eqs //. subs];

 eqs0 = eqs //. subs;

 sol = TimeConstrained[Solve[toEqs[eqs0], Variables[eqs0]], 2];
 If[Length[sol] == 0,
  ,
  Print["Found " <> ToString[Length[sol]] <> " solutions."];
  done = 1;
 ];
]
subs
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

```
Found 4 solutions.
```

Out[499]=
```
{rho → 1, k14 → 0, a23 → 0, b24 → 1, a13 → 0, a34 → 1,
 k52 → 1, a12 → 0, k55 → 1, k63 → 0, k25 → 1, k12 → 1, b14 → 0,
 k53 → 0, k46 → 0, a24 → 0, k21 → 0, k23 → 1, k13 → 1, k26 → 0, k43 → 0}
```

- **Manually inspect solutions to see if the equation for D has a solution.**

In[500]:=
```
(* select solution *)
index = 1;

(* Compute alpha,beta,A,B,kappa for solution *)
kSubs = Join[subs, sol[[index]]];
kSubs = Append[kSubs, a34 → 1];
kSubs = Append[kSubs, kx64 → 1 / 2];

a0 = alpha //. kSubs;
b0 = beta //. kSubs;
A0 = Abivector //. kSubs;
B0 = Bbivector //. kSubs;
k0 = Simplify[kappa //. kSubs];

(* output *)
Variables[k0]
Print["beta^2-alpha gamma = " <> ToString[b0 ^ 2 - a0 * 1 ]];
{a0, b0}
A0 // MatrixForm
B0 // MatrixForm
emKappaToMatrix[k0] // MatrixForm


Dbivector = emMatrix["dv", 4, Structure → "AntiSymmetric"];
dLHS  = contract[Dbivector, k0 + b0 emIdentityKappa[]];
dRHS  = 1 / 2 emTrace[emBiProduct[rho, Dbivector, Dbivector]] A0 + B0;

deqs = Union[Flatten[Simplify[dLHS - dRHS]]];
deqs = deqs /. kSubs;
dsol = Solve[toEqs[deqs], Variables[deqs]];
Print["Solutions to D:"];
GroebnerBasis[deqs, Variables[deqs]]
Solve[toEqs[deqs], Variables[deqs]]
```

Out[509]= {k24}

```
beta^2-alpha gamma = -1
```

Out[511]= {2, -1}

Out[512]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Out[513]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{8+k24}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} & 1 \\ \frac{1}{2}(-8-k24) & -\frac{1}{4} & 0 & \frac{1}{4}(-17-24\,k24) \\ 0 & -1 & \frac{1}{4}(17+24\,k24) & 0 \end{pmatrix}$$

Out[514]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & k24 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -12 & 1 & 0 \\ 0 & 2 & 0 & -17 & 0 & 2 \end{pmatrix}$$

```
Solutions to D:
```

Out[522]= {1}

Out[523]= {}

- **Note: Finding solutions as above, where the equation for D is not solvable requires patience.**
- **Does the Fresnel polynomial factorise?**

In[524]:=
```
fr = FullSimplify[emKappaToFresnel[k0, coords]]
```

Out[524]=

$$\frac{1}{2}\left(-xi2\,(xi1+xi2)+2\,(6\,xi1+(12+k24)\,xi2)\,xi3+34\,xi3^2+2\,xi0\,(xi1+8\,xi2+2\,xi3)\right)\,(xi0\,xi2+xi3\,(xi1+xi2+3\,xi3))$$

- **Yes. In this case, the Fresnel surface factorises into two quadratic forms.**