- **Example 3.6: For kappa defined by 3x3 matrices**

  A = -diag (1,2,3), B = Id, C =D = 0

  we compute the dimension of Vxi (see beginnin of Section 3) at the singular point and on the coordinate planes.

In[1]:=
```
SetDirectory["~/writing/WIP/KappaLib/"];
<< KappaLib.m
```

```
KappaLib v1.1
```

- **Fresnel surface**

In[3]:=
```
Ax = - DiagonalMatrix[{1, 2, 3}];
Bx = DiagonalMatrix[{1, 1, 1}];
Cx = 0 IdentityMatrix[3];
Dx = 0 IdentityMatrix[3];

kappa = emABCDToKappa[Ax, Bx, Cx, Dx];
```

In[8]:=
```
xi = {xi0, xi1, xi2, xi3};
fresnel = Simplify[emKappaToFresnel[kappa, xi]];
```

- **Compute matrix representation of ast o L_xi where ast is Hodge operator for Euclidean metric.**

In[10]:=
```
Mat[xx_] := Table[

   Sum[
    1 / 2 xx[[r]] xx[[l]] emReadNormal[kappa, l, m, c, d] Signature[{r, c, d, p}],
    {r, 1, 4}, {l, 1, 4}, {c, 1, 4}, {d, 1, 4}, {f, 1, 4}
   ]

   , {m, 1, 4}, {p, 1, 4}
  ];
```

- **Define singular point**

In[11]:= $xi = \left\{1, \sqrt{\frac{3}{2}}, 0, \frac{1}{\sqrt{2}}\right\};$

In[12]:=
```
QQ = Mat[xi];
QQ // MatrixForm
```

Out[13]//MatrixForm=

$$\begin{pmatrix} -12 & 2\sqrt{6} & 0 & 6\sqrt{2} \\ 2\sqrt{6} & -2 & 0 & -2\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 6\sqrt{2} & -2\sqrt{3} & 0 & -6 \end{pmatrix}$$

- **Case 1: Check that Dim V_xi = 1 when xi1= 0**

In[14]:=
```
Case1 = Mat[{1, 0, xi2, xi3}];
Case1 // MatrixForm
Union[Flatten[Case1 - Transpose[Case1]]]
```

Out[15]//MatrixForm=

$$\begin{pmatrix} -8\,xi2^2 - 12\,xi3^2 & 0 & 8\,xi2 & 12\,xi3 \\ 0 & -4 + 4\,xi2^2 + 4\,xi3^2 & 0 & 0 \\ 8\,xi2 & 0 & -8 + 4\,xi3^2 & -4\,xi2\,xi3 \\ 12\,xi3 & 0 & -4\,xi2\,xi3 & -12 + 4\,xi2^2 \end{pmatrix}$$

Out[16]= {0}

In[17]:=
```
Eigen1 = Simplify[Eigenvalues[Case1] /. {xi0 → 1, xi1 → 0}]
```

Out[17]=
$$\left\{0,\ 4\left(-1+\text{xi2}^2+\text{xi3}^2\right),\ -2\left(5+\text{xi2}^2+2\,\text{xi3}^2+\sqrt{9\,\text{xi2}^4+6\,\text{xi2}^2\left(-1+4\,\text{xi3}^2\right)+\left(1+4\,\text{xi3}^2\right)^2}\right),\ 2\left(-5-\text{xi2}^2-2\,\text{xi3}^2+\sqrt{9\,\text{xi2}^4+6\,\text{xi2}^2\left(-1+4\,\text{xi3}^2\right)+\left(1+4\,\text{xi3}^2\right)^2}\right)\right\}$$

In[18]:=
```
FullSimplify[fresnel /. {xi0 → 1, xi1 → 0}]
```

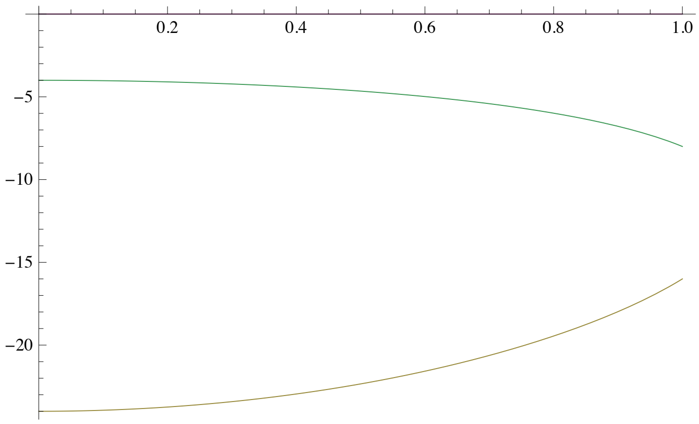
Out[18]=
$$-\left(-1+\text{xi2}^2+\text{xi3}^2\right)\left(-6+2\,\text{xi2}^2+3\,\text{xi3}^2\right)$$

- **Subcase A: $\left(-1+\text{xi2}^2+\text{xi3}^2\right)=0$**

In[19]:=
```
tmp = Simplify[Eigen1 /. {xi3^2 → 1 - xi2^2, xi3^4 → (1 - xi2^2)^2}]
```

Out[19]=
$$\left\{0,\ 0,\ -2\left(7-\text{xi2}^2+\sqrt{25-22\,\text{xi2}^2+\text{xi2}^4}\right),\ 2\left(-7+\text{xi2}^2+\sqrt{25-22\,\text{xi2}^2+\text{xi2}^4}\right)\right\}$$

In[20]:=
```
Plot[%, {xi2, 0, 1}]
```

Out[20]=


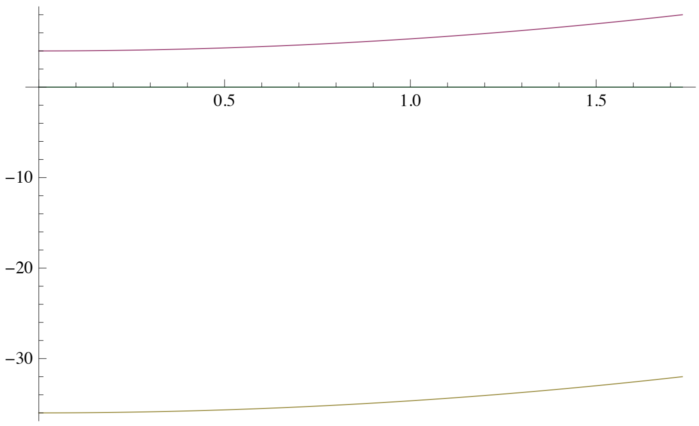
- **Subcase B: $\left(-6+2\,\text{xi2}^2+3\,\text{xi3}^2\right)==0$**

xi2 = 0..sqrt(3)
xi3 = 0..sqrt(2)

In[21]:=
```
tmp = Simplify[Eigen1 /. {xi3^2 → (6 - 2 xi2^2) / 3, xi3^4 → ((6 - 2 xi2^2) / 3)^2}]
```

Out[21]=
$$\left\{0,\ \frac{4}{3}\left(3+\text{xi2}^2\right),\ -\frac{2}{3}\left(27-\text{xi2}^2+\sqrt{\left(-27+\text{xi2}^2\right)^2}\right),\ \frac{2}{3}\left(-27+\text{xi2}^2+\sqrt{\left(-27+\text{xi2}^2\right)^2}\right)\right\}$$

In[22]:=
```
Plot[tmp, {xi2, 0, Sqrt[3]}]
```

Out[22]=


■ Case 2: Check that Dim V_xi = 1 when xi2 = 0 except at the singular point where dim V_xi = 2

In[23]:=
```
Case2 = Mat[{1, xi1, 0, xi3}];
Case2 // MatrixForm
Union[Flatten[Case2 - Transpose[Case2]]]
```

Out[24]//MatrixForm=

$$\begin{pmatrix} -4\,xi1^2 - 12\,xi3^2 & 4\,xi1 & 0 & 12\,xi3 \\ 4\,xi1 & -4 + 4\,xi3^2 & 0 & -4\,xi1\,xi3 \\ 0 & 0 & -8 + 4\,xi1^2 + 4\,xi3^2 & 0 \\ 12\,xi3 & -4\,xi1\,xi3 & 0 & -12 + 4\,xi1^2 \end{pmatrix}$$

Out[25]= {0}

In[26]:=
```
Eigen2 = Simplify[Eigenvalues[Case2] /. {xi0 → 1, xi2 → 0}]
```

Out[26]=

$$\left\{0,\ 4\left(-2 + xi1^2 + xi3^2\right),\ -4\left(2 + xi3^2 + \sqrt{xi1^4 + \left(1 + 2\,xi3^2\right)^2 + xi1^2\left(-2 + 4\,xi3^2\right)}\right),\ 4\left(-2 - xi3^2 + \sqrt{xi1^4 + \left(1 + 2\,xi3^2\right)^2 + xi1^2\left(-2 + 4\,xi3^2\right)}\right)\right\}$$

In[27]:=
```
FullSimplify[fresnel /. {xi0 → 1, xi2 → 0}]
```

Out[27]= $-\left(-2 + xi1^2 + xi3^2\right)\left(-3 + xi1^2 + 3\,xi3^2\right)$

■ Subcase A: $\left(-2 + xi1^2 + xi3^2\right)=0$

xi1=0..Sqrt(2)
xi3=0..Sqrt(2)

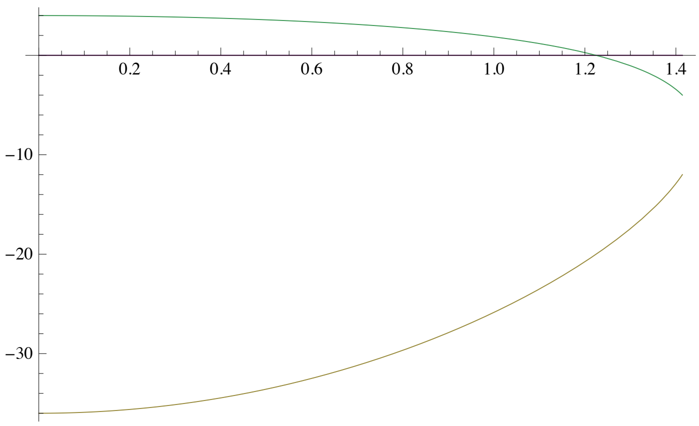
In[28]:=
```
tmp = Simplify[Eigen2 /. {xi3^2 → 2 - xi1^2, xi3^4 → (2 - xi1^2)^2}]
```

Out[28]=

$$\left\{0,\ 0,\ -4\left(4 - xi1^2 + \sqrt{25 - 14\,xi1^2 + xi1^4}\right),\ 4\left(-4 + xi1^2 + \sqrt{25 - 14\,xi1^2 + xi1^4}\right)\right\}$$

In[29]:=
```
Plot[tmp, {xi1, 0, Sqrt[2]}]
```

Out[29]=



In[30]:=
```
Simplify[Eigen2 /. {xi1 → Sqrt[3 / 2], xi3 → Sqrt[1 / 2]}]
```

Out[30]= {0, 0, -20, 0}

In[31]:=
```
Sqrt[3 / 2] // N
```

Out[31]= 1.22474

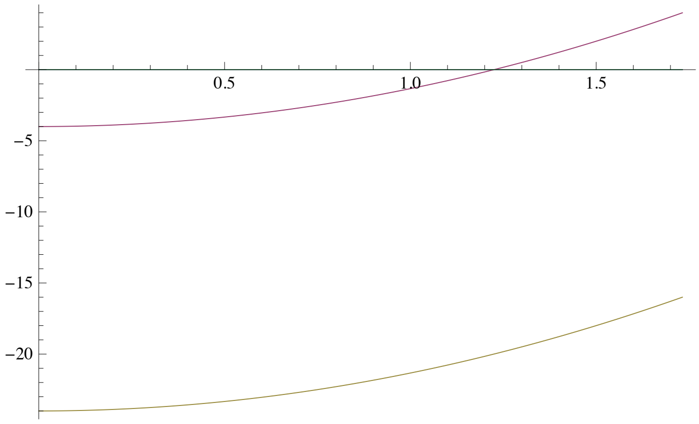
▪ Subcase A: $(-3 + xi1^2 + 3\ xi3^2)=0$

xi1 = 0..sqrt(3)
xi3 = 0..1

In[32]:=
```
tmp = Simplify[Eigen2 /. {xi3^2 → (3 - xi1^2) / 3, xi3^4 → ((3 - xi1^2) / 3)^2}]
```

Out[32]= $\left\{0,\ -4 + \frac{8\ xi1^2}{3},\ -\frac{4}{3}\left(9 - xi1^2 + \sqrt{\left(-9 + xi1^2\right)^2}\right),\ \frac{4}{3}\left(-9 + xi1^2 + \sqrt{\left(-9 + xi1^2\right)^2}\right)\right\}$

In[33]:=
```
Plot[tmp, {xi1, 0, Sqrt[3]}]
```

Out[33]=


▪ Case 3: Check that Dim V_xi = 1 when xi3 = 0

In[34]:=
```
Case = Mat[{1, xi1, xi2, 0}];
Case // MatrixForm
Union[Flatten[Case - Transpose[Case]]]
```

Out[35]//MatrixForm=
$$\begin{pmatrix} -4\ xi1^2 - 8\ xi2^2 & 4\ xi1 & 8\ xi2 & 0 \\ 4\ xi1 & -4 + 4\ xi2^2 & -4\ xi1\ xi2 & 0 \\ 8\ xi2 & -4\ xi1\ xi2 & -8 + 4\ xi1^2 & 0 \\ 0 & 0 & 0 & -12 + 4\ xi1^2 + 4\ xi2^2 \end{pmatrix}$$

Out[36]= {0}

In[37]:=
```
Eigen = Simplify[Eigenvalues[Case]]
```

Out[37]= $\left\{0,\ 4\left(-3 + xi1^2 + xi2^2\right),\ -2\left(3 + xi2^2 + \sqrt{4\ xi1^4 + 4\ xi1^2\left(-1 + 3\ xi2^2\right) + \left(1 + 3\ xi2^2\right)^2}\right),\ 2\left(-3 - xi2^2 + \sqrt{4\ xi1^4 + 4\ xi1^2\left(-1 + 3\ xi2^2\right) + \left(1 + 3\ xi2^2\right)^2}\right)\right\}$

In[38]:=
```
FullSimplify[fresnel /. {xi0 → 1, xi3 → 0}]
```

Out[38]= $-\left(-3 + xi1^2 + xi2^2\right)\left(-2 + xi1^2 + 2\ xi2^2\right)$

▪ Subcase A: $(-3 + xi1^2 + xi2^2)=0$

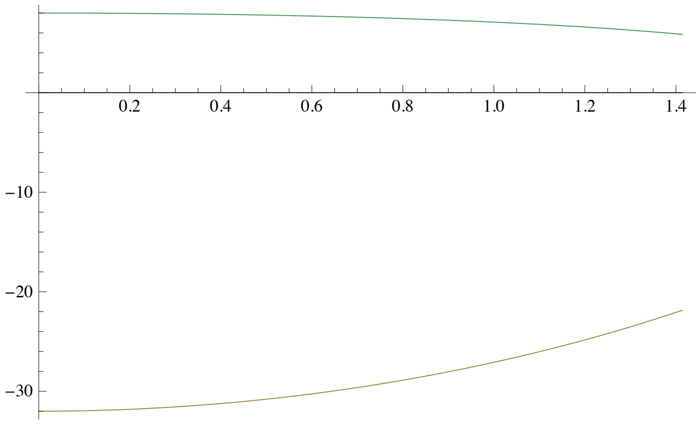
xi1=0..Sqrt(3)
xi2=0..Sqrt(3)

In[39]:=
```
tmp = Simplify[Eigen /. {xi2^2 → 3 - xi1^2, xi2^4 → (3 - xi1^2)^2}]
```

Out[39]= $\left\{0,\ 0,\ -2\left(6 - xi1^2 + \sqrt{100 - 28\ xi1^2 + xi1^4}\right),\ 2\left(-6 + xi1^2 + \sqrt{100 - 28\ xi1^2 + xi1^4}\right)\right\}$

In[40]:= `Plot[tmp, {xi1, 0, Sqrt[2]}]`

Out[40]=



- **Subcase A: $(-2 + xi1^2 + 2\,xi2^2) == 0$**

xi1 = 0..sqrt(2)
xi2 = 0..1

In[41]:= `tmp = Simplify[Eigen /. {xi2^2 → (2 - xi1^2) / 2, xi2^4 → ((2 - xi1^2) / 2)^2}]`

Out[41]= $\left\{0,\ 2\left(-4 + xi1^2\right),\ -8 + xi1^2 - \sqrt{\left(-8 + xi1^2\right)^2},\ -8 + xi1^2 + \sqrt{\left(-8 + xi1^2\right)^2}\right\}$

In[42]:= `Plot[tmp, {xi1, 0, Sqrt[2]}]`

Out[42]=