Schottky’s Theorem on Conformal Mappings Between Annuli

A Play of Derivatives and Integrals

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1 Introduction

In this note we present a direct and fairly elementary proof of a classical result in complex analysis and conformal geometry. It asserts that,

**Theorem 1.1 (F.H. Schottky, 1877).** An annulus $A = \{ z; r < |z| < R \}$ can be mapped conformally onto the annulus $A' = \{ z'; r' < |z'| < R' \}$ if and only if $R/r = R'/r'$. Moreover, every conformal mapping $f : A \to A'$ takes the form $f(z) = \lambda z^{\pm 1}$, where $|\lambda| = r'/r$ or $|\lambda| = r'R$ as the case may be.

The study of conformal deformations of annuli and more general multiply connected domains goes back to the doctoral dissertation of F.H. Schottky, a student of Weierstrass in Berlin [7], see also the work of H. Grötzsch in 1935 [4], a student of P. Koebe. By the Riemann mapping theorem, annuli are the first place one meets nontrivial conformal invariants such as moduli - obstructions to the existence of conformal mappings.

Our new arguments for this elegant theorem came from [2] and [5] where we initiated the study of extremal mappings with integrable distortion, whereas classical Teichmüller theory concerns itself with extremal mappings of bounded distortion. Specifically, here the integration of certain nonlinear differential expressions we call *invariable forms* together with sharp estimates yields the result.

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The underlying ideas differ very much from those used in the extremal length method [1] and [6]. Other approaches can be based around the reflection principle given knowledge of the conformal automorphisms of the disk, something one meets in any first course in complex analysis, usually as an application of the Schwarz lemma.

We prove the theorem without any reference to analytic functions and we do not appeal to any significant part of the theory partial differential equations (PDEs). In fact the Cauchy-Riemann equations are used only as to define conformality of a map $f : \mathbb{A} \to \mathbb{A}'$. In a similar fashion we could consider more general elliptic systems of first order PDEs or differential inequalities. In particular, our method applies to quasiconformal deformations of annuli. However, in the interest of clarity, and without loss of the general insight, we present the arguments only for the Cauchy-Riemann equations. Yet we carry out these arguments under minimal regularity hypotheses, that is in the Sobolev class $W^{1,1}_{\text{loc}}(\mathbb{A}, \mathbb{A}')$. Such relaxation of regularity hypotheses, although insignificant in case of Cauchy-Riemann equations (where one has the Weyl Lemma or the Looman-Menchoff Theorem), is fundamentally important when studying deformations with unbounded distortion and where the governing equations have only measurable coefficients [3], [2] and [5]. The reader satisfied with $C^1$-deformations of annuli may skip subtle adjustments necessary to fit the computation to Sobolev mappings.

2 Some topological observations

We consider the family $\mathcal{F}(\mathbb{A}, \mathbb{A}')$ of all orientation preserving homeomorphisms $f : \mathbb{A} \to \mathbb{A}'$ in the Sobolev class $W^{1,1}_{\text{loc}}(\mathbb{A}, \mathbb{A}')$ whose distributional first order derivatives are locally integrable. Since $f : \mathbb{A} \to \mathbb{A}'$ is a homeomorphism, we find that the function $z \to |f(z)|$ extends continuously to the closure of $\mathbb{A}$ with two possibilities:

$$
\begin{align*}
|f(z)| &= r' \quad \text{for } |z| = r, \\
|f(z)| &= R' \quad \text{for } |z| = R,
\end{align*}
$$

or

$$
\begin{align*}
|f(z)| &= R' \quad \text{for } |z| = r, \\
|f(z)| &= r' \quad \text{for } |z| = R.
\end{align*}
$$

We say that $f$ preserves or reverses the order of the boundary components respectively. In each case there is a positive or negative increment of the argument of $f$ along Jordan curves $\gamma \subset \mathbb{A}$ separating the boundary components. That is,

$$
\Delta_{z \in \gamma} \text{Arg } f(z) = \pm 2\pi,
$$

respectively.
3 Invariable integrals

The term invariable integral refers to differential expressions (linear or nonlinear) defined on mappings \( f \in \mathcal{F}(A, A') \) whose integral mean over \( A \) does not depend on \( f \). Naturally, polar coordinates \( z = \rho e^{i\theta}, \ r < \rho < R, \ 0 \leq \theta < 2\pi \), are best suited for dealing with mappings of annuli. Through every point of \( A \) there passes exactly one circle centered at the origin and so we can speak of the normal and tangential derivatives of \( f \) at this point. Precisely,

\[
\begin{align*}
    f_n(z) & \overset{\text{def}}{=} \frac{\partial f}{\partial \rho}, \quad \text{and} \quad f_t(z) \overset{\text{def}}{=} \frac{1}{\rho} \frac{\partial f}{\partial \theta}, \quad \text{at} \quad z = \rho e^{i\theta}
\end{align*}
\]

We can now make the concept of invariable integrals clear with two examples. According to whether \( f \) preserves or reverses the order of the boundary components, we have:

\[
\Re \iint_A \frac{f_n}{\rho f} = \int_0^{2\pi} \left[ \int_r^R \frac{R' \log |f|}{\rho} \, d\rho \right] \, d\theta = \pm \int_0^{2\pi} \log \frac{R'}{r'} \, d\theta = \pm 2\pi \log \frac{R'}{r'} \overset{\text{def}}{=} \pm \text{Mod } A'
\]

A dual computation results in the modulus of \( A \),

\[
\Im \iint_A \frac{f_t}{\rho f} = \int_r^R \left[ \int_0^{2\pi} \frac{\partial \text{Arg } f}{\partial \theta} \, d\theta \right] \, d\rho = \pm \int_r^R \frac{2\pi \, d\rho}{\rho} = \pm 2\pi \log \frac{R}{r} \overset{\text{def}}{=} \pm \text{Mod } A
\]

The third example involves the Jacobian determinant of \( f \), usually denoted by \( J(z, f) \). Let \( dz \) and \( dz' \) denote the Lebesgue area elements on \( A \) and \( A' \), respectively. If \( f \) is sufficiently smooth, we may apply change of variables to obtain

\[
\iint_A \frac{J(z, f)}{|f(z)|^2} \, dz = \iint_{A'} \frac{dz'}{|z'|^2} = \text{Mod } A'
\]

As a matter of fact this invariable integral identity holds for orientation preserving homeomorphisms \( f : A \to A' \) of Sobolev class \( W^{1,2}_{\text{loc}}(A, A') \), but it fails for \( f \in \mathcal{F}(A, A') \) as Lusin’s property fails to hold; sets of zero measure can be mapped onto sets of positive measure. Fortunately, this still holds as a sharp inequality in the direction needed for our proof.

\[
\iint_A \frac{J(z, f)}{|f(z)|^2} \, dz \leq \iint_{A'} \frac{dz'}{|z'|^2} = \text{Mod } A', \quad \text{for all } f \in \mathcal{F}(A, A')
\]
see [3], [2] and [5] for references and applications to extremal problems.

4 Conformality versus the distortion inequality

The analytic description of conformality goes via the Cauchy-Riemann equations which we may state in polar coordinates

$$\frac{1}{\rho} \frac{\partial f}{\partial \theta} = i \frac{\partial f}{\partial \rho}, \text{ equivalently } f_r(z) = i f_n(z) \text{ for almost every } z = \rho e^{i\theta}$$

The tangential and normal derivatives of $f$ represent orthogonal vectors of the same length, equal to the square root of the Jacobian determinant:

$$J(z, f) = \Im (f_r f_n) = |f_n|^2 = |f_r|^2$$

We now take advantage of these formulas by distorting with a factor $1 \leq K < \infty$.

If, instead of being conformal, a mapping $f \in F(A, A')$ is $K$-quasiconformal then

$$|f_n|^2 \leq K \cdot J(z, f) \text{ and } |f_r|^2 \leq K \cdot J(z, f) \quad (4)$$

5 Proof of the theorem

Given the above preliminaries our proof takes a few lines.

$$(\text{Mod } A) \cdot (\text{Mod } A') \geq \iiint_A \frac{dz}{|z|^2} \cdot \iiint_A \frac{J(z, f) \, dz}{|f(z)|^2} \geq \left( \iiint_A \sqrt{J(z, f)} \, dz \right)^2$$

$$= \left\{ \left( \iint_A \frac{|f_n|}{|f|^2} \right)^2 \right\} \geq \left\{ \left( \Re \iint_A \frac{f_n}{|f|^2} \right)^2 \right\} = \left\{ \left( \Im \iint_A \frac{f_r}{|f|^2} \right)^2 \right\} = \left( \text{Mod } A' \right)^2 = \left( \text{Mod } A \right)^2$$

Hence a necessary condition for the existence of a conformal map $f : \mathbb{A} \to \mathbb{A}'$ is that $\text{Mod } A = \text{Mod } A'$. Once this condition is satisfied every conformal map $f : \mathbb{A} \to \mathbb{A}'$ must give equality in every step of the above computation. A close
inspection of these inequalities reveals that
\begin{align*}
\frac{\mathcal{J}(z, f)}{|f(z)|^2} &= \frac{m^2}{|z|^2}, \quad \text{and} \quad \begin{cases} 
|\frac{\partial f}{\partial \rho}| = \pm \frac{m}{\rho} \\
|\frac{\partial f}{\partial \theta}| = \pm i \frac{m}{\rho}
\end{cases}
\end{align*}
where \( m \) is a real number. The sign in each equation remains at our choice but must be the same for all points in \( A \). This can easily be summarized in two differential equations
\begin{align*}
\begin{cases}
\frac{\partial f}{\partial \rho} &= m f \\
\frac{\partial f}{\partial \theta} &= i m f
\end{cases}
\end{align*}
for some constant \( m \in \mathbb{R} \) \text{ (5)}
Solving these equations poses no difficulty. First the real constant \( m \) can be identified from the second equation via the argument principle as follows
\begin{align*}
m &\equiv 3m \left( \frac{1}{f} \frac{\partial f}{\partial \theta} \right) = \frac{\partial \text{Arg} f}{\partial \theta} = \frac{1}{2\pi} \frac{\Delta}{|z| = \rho} \text{Arg} f(z) = \pm 1
\end{align*}
The plus sign applies when \( f \) preserves the order of the boundary components and the minus sign otherwise. Now the general solution takes the form \( f(z) = \lambda z^{\pm 1} \), where \( \lambda \) is a complex number whose modulus is uniquely determined by requiring that \( |\lambda| r = r' \) or \( R' \), respectively.

6 \text{ } K\text{-quasiconformal deformations}

The above arguments work just as well for the differential inequalities at (4), in particular, for \( K \)-quasiconformal mappings. Actually, our computation gives an even more precise conclusion. Namely, one obtains results under two independent distortion conditions:
\begin{itemize}
  \item \textit{Case 1:} the distortion inequality \( |f_N|^2 \leq K \cdot \mathcal{J}(z, f) \) yields
    \[ \text{Mod } A' \leq K \cdot \text{Mod } A \]
  \item \textit{Case 2:} the distortion inequality \( |f_T|^2 \leq K \cdot \mathcal{J}(z, f) \) yields
    \[ \text{Mod } A \leq K \cdot \text{Mod } A' \]
\end{itemize}
It is therefore of interest to look at two extremal situations. In the first case we assume that \( \text{Mod } A' = K \cdot \text{Mod } A \) and conclude, just as above, that the only solutions are power functions of the form \( f(\rho e^{i\theta}) = \lambda \rho K e^{i\theta} \). In the second case
we assume that $\text{Mod}\, \mathcal{A} = K \cdot \text{Mod}\, \mathcal{A}'$ and identify the only solutions $f(\rho e^{i\theta}) = \lambda \rho^{1/K} e^{i\theta}$. Here in both cases we have assumed that $f$ preserves the order of the boundary components. If it reverses the order then the reciprocals of these functions are the only solutions.

There are many more $K$-quasiconformal deformations $f : \mathcal{A} \rightarrow \mathcal{A}'$ if $K^{-1} \cdot \text{Mod}\, \mathcal{A} < \text{Mod}\, \mathcal{A}' < K \cdot \text{Mod}\, \mathcal{A}$.

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