Pucci’s Conjecture and the Alexandrov Inequality for Elliptic PDEs in the Plane

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Abstract

The inequality of Alexandrov, Bakel’man and Pucci is a basic tool in the theory of linear elliptic partial differential equations (PDEs) which are not in divergence form as well as in the more general theory of nonlinear elliptic PDEs. Here, in two dimensions, we prove the sharp form of the maximum principle as conjectured by Pucci in 1966, give sharp forms of removable singularity results and prove a number of results for the degenerate elliptic setting. These results make use of the substantial recent advances in the planar theory of quasiconformal mappings.

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1 Introduction

The inequality of Alexandrov, Bakel’man and Pucci is a basic tool in the theory of linear elliptic partial differential equations (PDEs) which are not in divergence form, as well as in the more general theory of nonlinear elliptic PDEs, see [8, 11]. In its usual formulation the inequality is concerned with uniformly elliptic second order differential operators of the form

\[(\mathcal{L}_A u)(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) \]  

(1)

defined for functions of the Sobolev class \(W^{2,p}_{\text{loc}}(\Omega)\) in a bounded subdomain \(\Omega \subset \mathbb{R}^n\). Here the coefficient matrix

\[A = A(x) = [a_{ij}(x)]\]

is a symmetric positive definite matrix function with real valued measurable entries \(a_{ij}(x)\). Thus

\[\langle A(x)\zeta, \zeta \rangle > 0, \quad \zeta \in \mathbb{R}^n \setminus \{0\}\]  

(2)

We abbreviate \(\frac{\partial^2}{\partial x_i \partial x_j} u(x)\) to \(u_{ij}\) and write

\[\mathcal{L}_A u = \text{Tr} \left[ A(x) D^2 u \right]\]

where \(D^2 u\) is the Hessian matrix

\[D^2 u = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{bmatrix}\]  

(3)

For rather obvious reasons the natural domain of definition of the operator \(\mathcal{L}_A\) is the Sobolev class \(W^{2,n}_{\text{loc}}(\Omega)\) for in this class the determinant of the Hessian matrix is locally integrable,

\[\mathcal{H}u \overset{\text{def}}{=} \det D^2 u \in L^1_{\text{loc}}(\Omega)\]  

(4)

As a matter of fact the condition (4) is less restrictive than the assumption that the Sobolev class is \(W^{2,n}_{\text{loc}}(\Omega)\), when uniform ellipticity bounds are lost. As far as degenerate equations are concerned, we shall see that (4) is a natural restriction. A fact worth noting at this point is that

\[W^{2,s}_{\text{loc}}(\Omega) \subset \mathcal{H}(\Omega), \quad \text{whenever } s > n/2\]

We point this out as we will be interested in studying the operator \(\mathcal{L}_A\) in weaker domains of definition than \(W^{2,n}_{\text{loc}}\), mainly the Sobolev classes \(W^{2,s}_{\text{loc}}(\Omega)\) with \(s < n\).

The classical maximum principle of Alexandrov [11, Theorem 9.1] reads as:
Theorem 1.1. There is a constant $C_n$ such that for every $u \in H^{2,n}(\Omega)$ and every relatively compact subdomain $\Omega' \subset \Omega$ we have the estimate

$$\max_{\Omega'} |u| \leq \max_{\partial \Omega'} |u| + C_n \text{diam}(\Omega) \left( \int_{\Omega} |L_A u(x)|^n \det A^{-1}(x) \, dx \right)^{1/n}$$  \tag{5}

Of course as a special case of this result when $A(x) = I$ and $L_A$ is the Laplacian one knows that the estimate remains valid for $u \in H^{2,s}(\Omega)$, for all $s > n/2$, where it reads as

$$\max_{\Omega'} |u| \leq \max_{\partial \Omega'} |u| + C(s,n)(\text{diam} \Omega)^2^{-\frac{n}{s}} \left( \int_{\Omega} |\Delta u(x)|^s \, dx \right)^{1/s}$$  \tag{6}

Such an estimate cannot be achieved at the critical exponent $s = n/2$, see e.g. Section 7 below.

This paper is concerned, in two dimensions, with interpolation between the extremes of the estimate (5) for general elliptic operators and the estimate (6) for the Laplacian. Other recent extensions of this maximum principle, in a different direction and for all dimensions can be found in [23].

It is only natural from the point of view of applications and the theory of weak solutions to PDE’s, to try and extend the range of exponents for which the estimate (5) holds when one has control of the ellipticity constant. In particular one would seek to replace the $L^n(\Omega)$-norm of $L_A u$ by some $L^p(\Omega)$-norm, $p < n$. But as shown by Alexandrov [2] and Pucci [20] without any restriction on the ellipticity constant the $L^n(\Omega)$ estimate in (5) is best possible.

However if one fixes the ellipticity constant, say $K$, for the equation, then it is known (at least in the plane) that there is $\epsilon = \epsilon(K) > 0$ such that one can obtain estimates in $L^p(\Omega)$ for all $p > 2 - \epsilon$. In fact in 1966 Pucci conjectured [21] Theorem 3.5 below, which gives the sharp range of values for $p$ for which an estimate of the form (5) holds.

It should come as no surprise that the recent developments in the theory of planar quasiconformal mappings, such as optimal regularity (the area distortion theorem [3]) and invertibility of Beltrami operators [5] will play a significant role in the proof of this result. In fact the well–known, but not often appreciated, connections between quasiconformal mappings and elliptic PDEs are the focus of our recent monograph [4] where much of the background behind the material used here can be found. Further, recent developments concerning the Beltrami equation in the degenerate elliptic setting [15] allow an extension of the Alexandrov inequality to certain degenerate equations, with estimates in Zygmund spaces around $L^2(\Omega)$.

2 Preliminaries

In this section we shall introduce some basic notation and set up some necessary preliminaries.
2.1 Distortion Functions

We shall see that general improvements of the Alexandrov estimate (5) to lower exponents such as (6) will depend on the distortion function

$$K_A(x) = \|A(x)\| \|A^{-1}(x)\|$$

(7)

where, as usual, $\|A\|$ denotes the operator norm of the matrix $A$, $\|A\| = \sup_{\|h\|=1} |Ah|$. Thus $K_A(x) \geq 1$ is the ratio of the largest and smallest eigenvalues of $A(x)$. In many situations there is no loss of generality in assuming that

$$\det A(x) = 1, \quad \text{almost every } x \in \Omega$$

(8)

This can be achieved by multiplying $A$ by a suitable factor and will not affect the distortion function $K_A(x)$. Such a normalisation has the added benefit of simplifying the estimate (5). Under such a normalisation the usual ellipticity bounds for $A(x)$ take the form

$$\frac{|\zeta|^2}{\sqrt{K_A(x)}} \leq \langle A(x)\zeta, \zeta \rangle \leq \sqrt{K_A(x)} |\zeta|^2$$

(9)

In what follows we shall, however, not make the normalisation (8), and we emphasize explicitly that will instead define the ellipticity bounds through the condition at (9).

2.2 Uniform Ellipticity

To be able to make $L^s$ estimates for the operator $L_A$, below the natural exponent $s = n$, certain degree of regularity of the distortion function is necessary. As a first step we assume that $K_A(x) \in L^\infty(\Omega)$, say

$$K_A(x) \leq K, \quad \text{for almost every } x \in \Omega$$

(10)

where $1 \leq K < \infty$. This is usually referred to as the uniformly elliptic setting and $K$ is called the ellipticity constant.

2.3 Critical Exponents, $n = 2$

It appears that the $L^p$ theory of the operator $L_A$ can be developed in a meaningful way only in a specific range of exponents $p$ which depend on the ellipticity constant $K$. Precisely we shall assume henceforth that $p$ lies in the interval

$$\frac{2K}{K+1} < p < \frac{2K}{K-1}$$

(11)

We call $\frac{2K}{K+1}$ the upper, and $\frac{2K}{K-1}$ the lower, critical exponent respectively. Note that these critical exponents are Hölder conjugate.
2.4 Very Weak Solutions, $n = 2$

The term “very weak solution” pertains to a solution of the (possibly) inhomogeneous equation $\mathcal{L}_A u = h$, where

$$u \in W^{1,s}_{\text{loc}}(\Omega), \quad \frac{1}{2} < s < 2$$

3 Results

Our first result concerns the interior regularity of solutions to $\mathcal{L}_A u = h$ in the complex plane.

3.1 Interior Regularity

**Theorem 3.1.** Suppose that $u \in W^{2,q}(\Omega)$ whereas $\mathcal{L}_A u \in L^p(\Omega)$ for some pair of exponents

$$\frac{2K}{K+1} < q \leq \frac{2K}{K-1}$$

Then $u \in W^{2,p}_{\text{loc}}(\Omega)$. Moreover, for every relatively compact subdomain $\Omega' \subset \Omega$ we have the uniform estimate

$$\|D^2 u\|_{L^p(\Omega')} \leq C\|u\|_{W^{1,2}(\Omega)} + C\|\mathcal{L}_A u\|_{L^p(\Omega)}$$

where $C = C(p, q, \Omega', \Omega)$.

The bounds on the exponents are sharp. In fact the example of section 7 demonstrates that the estimate (13) fails in the borderline case $q = \frac{2K}{K+1}$. Similarly using the recent work of Faraco [10], it is shown in [6] that the $L^p$ regularity for the operator $2\frac{\partial}{\partial z} + \mu \frac{\partial}{\partial z} + \bar{\mu} \frac{\partial}{\partial z}$ no longer applies at the other borderline case $p = \frac{2K}{K-1}$. Our arguments at Section 5 show that hence also (13) fails at $p = \frac{2K}{K-1}$.

We remark at this point that the Sobolev embedding theorem implies that solutions $u \in W^{2,2}_{\text{loc}}(\Omega)$ of the homogeneous equation $\mathcal{L}_A u = 0$ have Hölder continuous first derivatives, $u \in C^{1,\alpha}(\Omega)$ for every $0 < \alpha < \frac{1}{K}$. As a matter of fact the borderline exponent $\alpha = \frac{1}{K}$ is also right and we have sharp energy estimates over measurable sets.

**Theorem 3.2.** Let $u \in W^{2,2}_{\text{loc}}(\Omega)$ solve $\mathcal{L}_A u = 0$. Then $u \in C^{1,\alpha}(\Omega)$ with $\alpha = \frac{1}{K}$. Moreover to every compact $\Omega' \subset \Omega$ there corresponds a constant $C = C(K, \Omega', \Omega)$ such that

$$\iint_E |D^2 u|^2 \leq C|E|^\frac{1}{K}$$

for every measurable subset $E \subset \Omega'$. 

3.2 Removable Singularities

We now turn to discuss the removability of singularities of solutions to the equation \( L_A u = h \).

**Theorem 3.3.** Let \( E \subset \mathbb{C} \) be a closed set of \( \alpha \)-Hausdorff measure zero,

\[
H^\alpha(E) = 0, \quad \alpha = \frac{2}{K+1}
\]  

and consider a function of Sobolev class

\[
u \in \mathcal{W}^{1,\infty}_{\text{loc}}(\Omega) \cap \mathcal{W}^{2,p}_{\text{loc}}(\Omega \setminus E), \quad \frac{2K}{K+1} < p < \frac{2K}{K-1}
\]

If \( L_A u \in \mathcal{L}^p_{\text{loc}}(\Omega) \), then \( u \in \mathcal{W}^{2,p}_{\text{loc}}(\Omega) \).

Examples given in [3] suggest that sets \( E \subset \mathbb{C} \) of Hausdorff dimension greater than \( 2/(K+1) \) need not be removable. Further, this result can be generalised by weakening the requirement that the gradient of \( u \) is bounded. We need only assume a certain degree of integrability of the gradient. However, under these circumstances we must assume that the dimension of \( E \) is smaller,

**Theorem 3.4.** Let \( E \subset \mathbb{C} \) be a closed set of Hausdorff dimension \( \alpha \), precisely

\[
dim_H(E) = \alpha < 2 \left( 1 + \frac{sK}{s-2K} \right)^{-1}, \quad s > 2K
\]  

and consider a function of Sobolev class

\[
u \in \mathcal{W}^{1,s}_{\text{loc}}(\Omega) \cap \mathcal{W}^{2,p}_{\text{loc}}(\Omega \setminus E), \quad \frac{2K}{K+1} < p < \frac{2K}{K-1}
\]

If \( L_A u \in \mathcal{L}^p_{\text{loc}}(\Omega) \), then \( u \in \mathcal{W}^{2,p}_{\text{loc}}(\Omega) \).

3.3 Maximum Principle: Pucci’s Conjecture

The main result of this paper is the following sharp generalisation of Alexandrov’s estimate conjectured by Pucci [21] in 1966.

**Theorem 3.5.** Let \( u \in \mathcal{W}^{2,p}(\Omega), \ p > \frac{2K}{K+1} \). Then for every relative compact subdomain \( \Omega' \subset \Omega \) we have

\[
\max_{\Omega'} |u| \leq \max_{\partial\Omega'} |u| + C_p(\Omega) \| L_A u \|_{\mathcal{L}^p(\Omega)}
\]  

where

\[
C_p(\Omega) \leq C(p)|\Omega|^{1-1/p} \log \left( \frac{\text{diam} \Omega}{|\Omega|} \right) \leq C(p) (\text{diam} \Omega)^{2-2/p}
\]  

Pucci’s conjecture for equations with first order terms will be discussed later in Theorem 6.1. We are unsure as to whether the logarithmic factor in (18) is in fact necessary. It is shown in [7] that for \( p = 2 \) one can do without this term. Again, the example of section 7 shows that the estimate (17) fails for the lower critical exponent \( p = \frac{2K}{K+1} \).
3.4 The Dirichlet Problem

We illustrate for the unit disk $D = \{ z : |z| < 1 \}$ how our estimates yield the solution of the Dirichlet problem in $\mathcal{W}^{2,p}(D)$.

**Theorem 3.6.** Let $b \in \mathcal{L}^s(D, \mathbb{R}^2)$ and $h \in \mathcal{L}^p(D)$ for some $s > 2$ and $\frac{2K}{K-1} < p < \frac{2K}{K+1}$. Then the equation

$$\text{Tr}[A D^2 u] + \langle b, \nabla u \rangle = h$$

has a unique solution $u \in \mathcal{W}^{2,p}(D)$ vanishing on $\partial D$. Moreover, we have the uniform estimate

$$\| D^2 u \|_{\mathcal{W}^p(D)} \leq C \| h \|_{\mathcal{L}^p(D)}$$

where $C$ does not depend on $h$.

3.5 The Degenerate Elliptic Case

There are interesting new developments in Geometric Function Theory concerning mappings with unbounded distortion, see [15] for an overview. Some of the results obtained so far can be applied in the setting here to solutions of the homogeneous equation. As above we shall assume the ellipticity condition

$$|\zeta|^2 / \sqrt{K(z)} \leq \langle A(z), \zeta \rangle \leq |\zeta|^2 \sqrt{K(z)}, \quad \text{almost every } z \in \Omega$$

and $K(z)$ finite almost everywhere, but we do not assume $K \in \mathcal{L}^{\infty}(\Omega)$ so there is no uniform ellipticity. In this setting the equation $\mathcal{L}_A u = h$ is referred to as a general degenerate elliptic equation. It is important to realise that we are dealing with the genuine anisotropic situation in which the ratio of the largest to the smallest eigenvalue is allowed to be unbounded.

The first result we mention here is an immediate consequence of a well known theorem of Goldstein and Vodopyanov, [12]. Precise estimates for the modulus of continuity of the gradient are to be found in [15].

**Theorem 3.7.** Let $K(z)$ be finite almost everywhere in $\Omega$. Then every solution to the homogeneous equation $\mathcal{L}_A u = 0$ for $u \in \mathcal{W}^{2,2}(\Omega)$ belongs to $C^1(\Omega)$. Furthermore, if $z_1, z_2$ lie in a disk $B$, with $2B \subset \Omega$, then we have the following uniform estimate

$$|\nabla u(z_1) - \nabla u(z_2)|^2 \leq \frac{C \int_{2B} |D^2 u|^2}{\log \left( e + \frac{\text{diam } B}{|z_1 - z_2|} \right)}$$

for the modulus of continuity of the gradient.

3.6 Exponentially Integrable Distortion

We say that the distortion function $K(z)$ is exponentially integrable and write $K \in \text{Exp}(\Omega)$ if there is some positive $\lambda$ such that

$$\int_\Omega e^{\lambda K(z)} dz < \infty$$
In the case that the operator $L_{\lambda}$ has exponentially integrable distortion, the a priori regularity assumption $u \in W^{2,2}_{\text{loc}}(\Omega)$ can be weakened. We require that $u \in W^{2,1}_{\text{loc}}(\Omega)$ and that

$$\mathcal{H}u = \det D^2u = u_{xx}u_{yy} - u_{xy}^2 \in L^1_{\text{loc}}(\Omega),$$

instead of $|D^2u|^2 = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \in L^1_{\text{loc}}(\Omega)$.

Our next result asserts that the Hessian determinant of a solution to the homogeneous equation satisfies a Zygmund class estimate, slightly better than $L^1$. Actually, in the case of exponentially integrable distortion we can make estimates in the inhomogeneous setting as well. We make a few remarks along these lines in the proofs of the stated results. The basic requirement is the existence theory for solutions of the Beltrami equation in the degenerate elliptic setting as discussed in [9], [15].

**Theorem 3.8.** Under the assumptions at (23) and (24) the solutions to the equation $L_{\lambda}u = 0$ of Sobolev class $W^{2,1}_{\text{loc}}(\Omega)$ satisfy the following uniform estimates

$$\int_{\Omega'} \frac{|D^2u|^2}{\log(e + |D^2u|)} < \infty \text{ and } \int_{\Omega'} (\mathcal{H}u) \log \log(e + \mathcal{H}u) < \infty$$

for every relatively compact domain $\Omega' \subset \Omega$. Furthermore, if $z_1, z_2$ lie in a disk $B$ with $2B \subset \Omega$, we have the estimate

$$|\nabla u(z_1) - \nabla u(z_2)|^2 \leq \frac{C \int_{2B} |D^2u|^2}{\log \log \left( e + \frac{\text{diam } B}{|z_1 - z_2|} \right)}$$

for the modulus of continuity of the gradient. Here the constant depends on the value of the integral at (23).

One can expect an improvement of the regularity if the degree of integrability of $K$ is better. If the parameter $\lambda$ in (23) is sufficiently large, then in fact we have the following $W^{2,2}_{\text{loc}}(\Omega)$ regularity result.

**Theorem 3.9.** There is a constant $\lambda_0 \geq 1$ with the following property. Suppose $u \in W^{2,1}_{\text{loc}}(\Omega)$ solves the homogeneous equation $L_{\lambda}u = 0$, suppose $\mathcal{H}u = \det D^2u$ is locally integrable and suppose the distortion function of $A$ satisfies

$$\int_{\Omega} e^{\lambda K(z)} dz < \infty$$

for some $\lambda > \lambda_0$. Then

$$u \in W^{2,2}_{\text{loc}}(\Omega)$$

Moreover the Hessian determinant locally belongs to the Zygmund class $L^{\log} \mathcal{L}$, precisely

$$\|\mathcal{H}u\|_{L^{\log} \mathcal{L}(B)} \overset{\text{def}}{=} \int_B |\mathcal{H}u| \log \left( e + \frac{\mathcal{H}u}{|B|} \int_B \mathcal{H}u \right) \leq C \int_{2B} |D^2u|^2$$

for every disk $B$ with $2B \subset \Omega$. 9
The underlying philosophy here is that the integrability of the second derivatives improve as $\lambda \to \infty$ in (23). In general, when the Hessian determinant $\mathcal{H}u \in \mathcal{L}^1_{\text{loc}}(\Omega)$ then for each $\alpha \geq -1$ we have

$$\int_{\Omega'} |D^2u|^2 \log^\alpha (e + |D^2u|) < \infty$$

as soon as $\lambda > \lambda(\alpha)$, sufficiently large. Conjecturally $\lambda(\alpha) = 1 + \alpha$. More details along these lines for mappings of finite distortion can be found in [15] and [13]. It is worthwhile noticing here that the inequality (29) for $\alpha > 1$ implies the continuity of the gradient of $u$.

### 3.7 Locally Integrable Distortion

We have already seen in Theorem 3.7 that $\mathcal{H}^{2,2}_{\text{loc}}(\Omega)$-solutions to the homogeneous equation $\mathcal{L}_\alpha u = 0$ belong to $\mathcal{C}^1(\Omega)$ if $K(z)$ is only assumed finite almost everywhere. Some additional properties of the gradient can be obtained if we assume that

$$K \in \mathcal{L}^1_{\text{loc}}(\Omega)$$

The following is a simple consequence of the factorisation theorem developed in [17].

**Theorem 3.10.** Suppose $u \in \mathcal{H}^{2,2}_{\text{loc}}(\Omega)$ solves the equation $\mathcal{L}_\alpha u = 0$ and that the distortion function $K \in \mathcal{L}^1_{\text{loc}}(\Omega)$. Then the complex gradient of $u$ can be factored as

$$\frac{\partial u}{\partial z} = \frac{1}{2} \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) = F \circ \chi$$

where $\chi : \Omega \to \mathbb{C}$ is a homeomorphism and $F : \chi(\Omega) \to \mathbb{C}$ is holomorphic. In particular if $u$ is nonconstant, then the critical points of $u$ (points where $\nabla u = 0$) are isolated.

### 4 Proofs: Unbounded Distortion

We begin with a few elementary remarks and algebraic computations which aim to reduce the equations $\mathcal{L}_\alpha u = h$ to a system of first order PDEs where we can use the modern theory of quasiregular mappings.

#### 4.1 Complex Gradients and Quasiregular Mappings

Let $u \in \mathcal{H}^{2,1}_{\text{loc}}(\Omega)$. The complex gradient of $u$ is defined as

$$\frac{\partial u}{\partial z} = \frac{1}{2} \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right)$$
The second order complex derivatives are
\[
\begin{align*}
    u_{zz} &= \frac{1}{4}(u_{xx} - u_{yy} - 2iu_{xy}) \\
    u_{z\bar{z}} &= \frac{1}{4}(u_{xx} + u_{yy}) = \frac{1}{4}\Delta u
\end{align*}
\]  (33)
We rearrange these to obtain the equivalent
\[
\begin{align*}
    u_{xx} &= 2u_{z\bar{z}} + u_{zz} + \overline{u_{zz}} \\
    u_{yy} &= 2u_{z\bar{z}} - u_{zz} - \overline{u_{zz}} \\
    u_{xy} &= i(u_{zz} - \overline{u_{zz}})
\end{align*}
\]  (34)
We now wish to express the operator \( L_A \) in complex notation. Suppose that
\[
A = \begin{bmatrix}
    a_{11}(z) & a_{12}(z) \\
    a_{21}(z) & a_{22}(z)
\end{bmatrix}, \quad a_{12} = a_{21}
\]
Then in terms of the complex derivatives,
\[
L_A u = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy}
\]
\[
= 2(a_{11} + a_{22})u_{z\bar{z}} + (a_{11} - a_{22} + 2ia_{12})u_{zz} + (a_{11} - a_{22} - 2ia_{12})\overline{u_{zz}}
\]
\[
= \text{Tr} A(z) [2u_{z\bar{z}} + \mu u_{zz} + \overline{\mu \overline{u_{zz}}}]
\]
where
\[
\mu(z) = \frac{a_{11} - a_{22} + 2ia_{12}}{a_{11} + a_{22}}
\]  (35)
Hence,
\[
|\mu(z)|^2 = 1 - \frac{4\det A}{(\text{Tr} A)^2} < 1
\]
In terms of the eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \) of the matrix \( A \) we see that
\[
|\mu(z)|^2 = \left( \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} \right)^2
\]
and hence
\[
|\mu(z)| = \frac{K_A(z) - 1}{K_A(z) + 1} \leq \frac{K - 1}{K + 1}
\]  (36)
It is common to denote this last term by \( k \),
\[
k = \frac{K - 1}{K + 1}
\]
We summarize the above in the next lemma
**Lemma 4.1.** For every \( u \in \mathcal{W}^{2,1}_{\text{loc}}(\Omega) \) we have
\[
\frac{1}{\text{Tr} A} L_A u = 2u_{z\bar{z}} + \mu(z)u_{zz} + \overline{\mu(z)\overline{u_{zz}}}
\]  (37)
where the complex coefficient \( \mu = \mu(z) \) satisfies
\[
|\mu(z)| = \frac{K_A(z) - 1}{K_A(z) + 1}
\]  (38)
At this point we wish to recall the following definition.
4.2 Mappings of Finite Distortion

A function \( F \in W^{1,1}_{\text{loc}}(\Omega) \) is said to have finite distortion \( K(z) \), where \( 1 \leq K(z) < \infty \) for almost every \( z \in \Omega \), if

1. \[ |F_z(z)| \leq \frac{K(z)-1}{K(z)+1} |F_z(z)| \]

2. The Jacobian determinant \( J(z, F) = |F_z|^2 - |F_{\bar{z}}|^2 \) is locally integrable.

Further, such a function \( F \) is said to be \( K \)-quasiregular if \( K(z) \leq K \) for almost every \( z \in \Omega \). Then of course

\[ |DF(z)|^2 = (|F_z| + |F_{\bar{z}}|)^2 \leq KJ(z, F) \]  

(39)

and so a priori for quasiregular mappings we have \( F \in W^{1,2}_{\text{loc}}(\Omega) \).

As a particular case let \( F = u_z \) be the complex gradient of the homogeneous equation \( L_A u = 0 \). From (37) and (38) we obtain

**Corollary 4.1.** The complex gradient \( F = \frac{\partial u}{\partial z} \) of a solution \( u \in W^{2,1}_{\text{loc}}(\Omega) \) of the homogeneous equation \( L_A u = 0 \) whose Hessian determinant \( \det D^2 u \) is locally integrable is a mapping of finite distortion. The distortion function of \( F \) is equal to \( K_A(z) \) pointwise.

4.3 Proof for Theorems 3.7, 3.8, 3.9 and 3.10

We begin with the results concerning the unbounded distortion since after applying Corollary 4.1, in most cases this requires only pointing out the appropriate results for mappings of finite distortion when applied to the complex gradient \( f = u_z \). Most of these results can be found in [15].

First, Theorem 3.7 is concerned with the continuity of \( f \). As we mentioned at the time this fact was first established in [12]. Further refinements can be found in [14] (monotonicity and continuity of \( f \)) as well as [15, Corollary 7.5.1] where the modulus of continuity estimate is given. The point basically is that under the assumptions, \( u_x \) and \( u_y \) are weakly monotone functions.

Next, for Theorem 3.8 we need only apply the results of [15, Theorem 7.6.1, (7.78)] for (26) above. That the determinant of the Hessian enjoys a higher degree of integrability, \( \det D^2 u \in \mathcal{L} \log \log \mathcal{L} \) is a direct consequence of a result of Moscariello, [18].

Next, the \( W^{2,2}(\Omega) \) regularity of Theorem 3.9 is proven in [15, Theorem 18.5.1]. Then the \( \mathcal{L} \log \mathcal{L} \) integrability of the Hessian determinant is immediate from a result of Müller, [19]. The precise estimate of (28) can be found in [13] where in fact these sorts of results can be found in all dimensions.

Finally, Theorem 3.10 is a direct consequence of [17].
5 Proofs: Bounded Distortion

We now shall consider the case that the distortion function is uniformly bounded. Thus we assume
\[ K_A(z) \leq K, \quad \text{almost every } z \in \Omega \quad (40) \]
The heart of the matter in establishing the sharp estimates goes back to the “Area Distortion Theorem” [3] and its consequences concerning the invertibility of the Beltrami operator [5].

5.1 The Beltrami Operator

To some considerable extent the regularity theory of the operator \( L_A : W^{2,p}_{\text{loc}}(\Omega) \rightarrow L^p_{\text{loc}}(\Omega) \) is controlled by the invertibility properties of the singular integral operator
\[ B = 2I + \mu S + \overline{\mu S} : L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C}) \quad (41) \]
known as a Beltrami operator. Here \( S \) is the classical Beurling transform, as introduced by Beurling and Ahlfors [1], defined as a singular integral of Calderón–Zygmund type,
\[ S\omega(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\omega(\zeta)d\zeta}{(\zeta - z)^2} \quad (42) \]
for all functions \( \omega \in L^p(\mathbb{C}) \), \( 1 < p < \infty \). More precisely, the integral is understood by means of the Cauchy principal value. Note that \( S \) is bounded for all \( 1 < p < \infty \) and is an isometry in \( L^2(\mathbb{C}) \). We denote the \( p \)-norms of the operator \( S \) by \( \|S\|_p \), so \( \|S\|_2 = 1 \). The characteristic property of this operator, and the property which makes it very important to complex analysis, is that it relates the \( \partial \) and \( \bar{\partial} \) derivatives,
\[ S \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} : W^{1,p}(\mathbb{C}) \rightarrow W^{1,p}(\mathbb{C}) \quad (43) \]
The operator \( B \) defined at (41) originates from Lemma 4.1 and our desire to study the equation
\[ 2U_{\bar{z}z} + \mu(z)U_{zz} + \overline{\mu(z)U_{zz}} = H(z), \quad z \in \mathbb{C} \quad (44) \]
which will be auxiliary to the equation \( \mathcal{L}_A u = h \) in a bounded domain. We are going to assume that \( H \in L^p(\mathbb{C}) \) with \( p \) in the critical interval, \( 2K/(K+1) < p < 2K/(K-1) \) and we will be looking for solutions \( U \in W^{2,p}_{\text{loc}}(\mathbb{C}) \) for which \( \partial^2 U \in L^p_{\text{loc}}(\mathbb{C}) \). To simplify matters we shall assume that both \( \mu \) and \( H \) are compactly supported in a bounded domain \( \Omega \subset \mathbb{C} \).

A solution \( U \) should be of the form
\[ U(z) = \frac{2}{\pi} \int_{\Omega} \omega(\tau) \log \frac{|z - \tau|}{(\text{diam } \Omega)} \, d\tau \quad (45) \]
for some function \( \omega \in \mathcal{L}^p(\mathbb{C}) \) supported in \( \Omega \). The complex gradient of such a solution will be given by the Cauchy formula

\[
\Phi(z) = \frac{\partial U}{\partial z} = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\omega(\tau) d\tau}{z - \tau} = C\omega
\]

which upon further differentiation yields

\[
\begin{align*}
U_{z\bar{z}} &= \frac{\partial \Phi}{\partial \bar{z}} = \omega \in \mathcal{L}^p(\mathbb{C}) \\
U_{zz} &= \frac{\partial \Phi}{\partial z} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\omega(\tau) d\tau}{(z - \tau)^2} \\
&= S\omega \in \mathcal{L}^p(\mathbb{C})
\end{align*}
\]

Now of course, equation (44) takes the form

\[
\mathcal{B}\omega = H
\]

The key ingredient is then the invertibility theorem [5] which we recall.

**Theorem 5.1.** The Beltrami operator defined at (41) has a continuous inverse for all \( p \) in the critical interval,

\[
\mathcal{B}^{-1} : \mathcal{L}^p(\mathbb{C}) \to \mathcal{L}^p(\mathbb{C}), \quad \frac{2K}{K + 1} < p < \frac{2K}{K - 1}
\]

We mention here that the operator \( \mathcal{B} \) is in fact injective (but not surjective) at the critical endpoints. Injectivity was established in [5] for \( p = \frac{2K}{K - 1} \) and by Petermichl–Volberg [22] for \( p = \frac{2K}{K + 1} \).

We can rephrase Theorem 5.1 and (50) as

\[
\|U_{z\bar{z}}\|_{\mathcal{L}^p(\mathbb{C})} + \|U_{zz}\|_{\mathcal{L}^p(\mathbb{C})} \leq C_p \|H\|_{\mathcal{L}^p(\mathbb{C})}
\]

In particular, using (45)

\[
\|U\|_{\mathcal{L}^\infty(\Omega)} \leq C_{14} \|\omega\|_{\mathcal{L}^p(\Omega)} \leq C_p \|H\|_{\mathcal{L}^p(\Omega)}
\]

### 5.2 Caccioppoli estimates of second order

As in Corollary 4.1 we see that if \( u \in \mathcal{W}^{2,2}_\text{loc}(\Omega) \) is a solution to the equation \( \mathcal{L}_u u = 0 \), then in the uniformly elliptic case the complex gradient \( f = u_x = \frac{1}{2}(u_x - iu_y) \) is a quasiregular mapping. This fact makes necessary a brief discussion of quasiregular gradient mappings. It makes no difference to consider here a general \( n \)-dimensional situation.

Let

\[
f = \nabla V = (V_x, V_y, \ldots, V_{xn}) \in \mathcal{W}^{1,n}_\text{loc}(\Omega, \mathbb{R}^n)
\]

where \( V \), called the potential function for \( f \), lies in the Sobolev space \( \mathcal{W}^{2,n}_\text{loc}(\Omega) \). If \( f \) is \( K \)-quasiregular we show how one can control \( |f| \) and \( |Df| \) by integral
averages of the potential $V$ alone. We begin with the classical Caccioppoli inequality

$$\int_{\Omega} |\phi Df|^n \leq C_K(n) \int_{\Omega} |\nabla \phi|^n |f|^n, \quad \phi \in C_0^\infty(\Omega)$$  \hspace{1cm} (54)

see e.g. [15]. Next, given any Lipschitz function $\eta$ with compact support in $\Omega$, we use integration by parts to obtain

$$\int |\eta f|^n = \int |\eta|^n (\nabla V, |f|^{n-2} f)$$

$$\leq n \int |\eta|^{n-1} |f|^{n-1} |\nabla \eta| |V| + (n-1) \int |\eta|^{n-2} |f|^{n-2} |\eta|^2 |Df| |V|$$

With the aid of Young's inequality we obtain after arranging the terms

$$\int |\eta f|^n \leq C_n \int |\nabla \eta|^n |V|^n + C_n \int |\eta|^n |Df|^{\frac{n}{2}} |V|^{\frac{n}{2}}$$  \hspace{1cm} (55)

We choose $\eta = |\phi \nabla \phi|$ and combine this estimate with the Caccioppoli inequality (54), where $\phi$ is replaced by $\phi^2$

$$\int |\phi^{2n} |Df|^n \leq C_K(n) \int |\phi \nabla \phi|^n |f|^n$$

$$\leq C \int \left( |\nabla \phi|^2 + |\phi| |\nabla^2 \phi| \right)^n |V|^n + C \int |\phi|^n |Df|^{\frac{n}{2}} |\nabla \phi|^n |V|^{\frac{n}{2}}$$

By again using Young's inequality we reduce the estimate to

$$\int |\phi^{2n} |Df|^n \leq C_K(n) \int \left( |\nabla \phi|^2 + |\phi \nabla^2 \phi| \right)^n |V|^n$$  \hspace{1cm} (56)

or equivalently

$$|\phi^2 Df|_n \leq C_K(n) \left( |\nabla \phi|^2 + |\phi \nabla^2 \phi| \right) V |_n$$  \hspace{1cm} (57)

This is what we call the **Second Caccioppoli Inequality**.

We now combine the second Caccioppoli inequality with the estimate at (55) and use Hölder's inequality to obtain

$$\int |\eta f|^n \leq C_n \int |\nabla \eta|^n |V|^n$$

$$+ C_K(n) \left[ \int |V|^n \right]^{\frac{1}{2}} \left[ \int \left( |\nabla \eta|^2 + |\eta \nabla^2 \eta| \right)^n |V|^n \right]^{\frac{1}{2}}$$  \hspace{1cm} (58)

Next, we restrict these estimates to concentric balls $B = B(a, R) \subset B(a, \lambda R) = \lambda B$, where $\lambda > 1$. With appropriate choice of a test function $\eta \in C_0^\infty(\lambda B)$ we infer from (58) that

$$\left( \int_{B} |f|^n \right)^{\frac{1}{2}} \leq \frac{C(n, K)\lambda}{(\lambda - 1) R} \left( \int_{\lambda B} |V|^n \right)^{\frac{1}{2}}$$  \hspace{1cm} (59)
We also appeal to the Sobolev-Poincare inequality for $V$ to write (59) as
\[
\left( \int_B |V - V_B|^{2n} \right)^{\frac{1}{2n}} \leq C_n R \left( \int_B |\nabla V|^n \right)^{\frac{1}{n}} \leq \frac{C(n, K) \lambda}{(\lambda - 1)} \left( \int_{\lambda B} |V|^n \right)^{\frac{1}{n}} \tag{60}
\]
where the integral mean can be estimated as
\[
|V_B| \leq \left( \int_B |V|^n \right)^{\frac{1}{n}} \leq \lambda \left( \int_{\lambda B} |V|^n \right)^{\frac{1}{n}} \tag{61}
\]
Hence, we arrive at the so called weak reverse Hölder inequality for the potential function,
\[
\left( \int_B |V|^{2n} \right)^{\frac{1}{2n}} \leq \frac{C(n, K) \lambda}{(\lambda - 1)} \left( \int_{\lambda B} |V|^n \right)^{\frac{1}{n}} \tag{62}
\]
It is well known that the reverse Hölder inequalities yield improved integrability properties; here we use the fact that by an iteration process the exponent $n$ in the right hand side of (62) can be arbitrarily decreased, see [16], Theorem 2. In particular, we obtain
\[
\left( \int_B |V|^n \right)^{\frac{1}{n}} \leq C(n, K) \int_{\lambda B} |V| \tag{63}
\]
As a final conclusion, with this information the inequality (59) and the second Caccioppoli estimate (57) take the elegant form
\[
\left( \int_B |f|^n \right)^{\frac{1}{n}} \leq \frac{C(n, K)}{R} \int_{2B} |V| \tag{64}
\]
and
\[
\left( \int_B |Df|^n \right)^{\frac{1}{n}} \leq \frac{C(n, K)}{R^2} \int_{2B} |V| \tag{65}
\]
Actually, these estimates in turn yield better inequalities for the potential function, e.g.
\[
\max_B |V| \leq C(n, K) \int_{2B} |V|
\]
for every disk $B$ with $2B \subset \Omega$.

5.3 Proof of Theorem 3.1

For proving the optimal interior regularity we need in addition the sharp regularity results from the so-called weakly quasiregular mappings. This term refers to mappings in Sobolev space $W^{1,1}_{\text{loc}}(\Omega)$ for which we have the distortion uniformly bounded as in the definition of quasiregularity, but we do not require the Jacobian determinant $J(z, f)$ to be locally integrable.
Theorem 5.2. Every weakly $K$–quasiregular mapping $F \in W^{1,q}_{\text{loc}}(\Omega)$ belongs to the class $W^{1,p}_{\text{loc}}(\Omega)$ whenever

$$\frac{2K}{K+1} < q \leq p < \frac{2K}{K-1} \tag{66}$$

In particular, under this assumption on $q$, the Jacobian determinant $J(z,F)$ will be locally integrable and the mapping will be $K$–quasiregular. Moreover we have the following Caccioppoli type estimate,

$$\|\phi DF\|_{L^p(\Omega)} \leq C_p(K) \|F\| \|\nabla \phi\|_{L^p(\Omega)} \tag{67}$$

for every test function $\phi \in C_0^\infty(\Omega)$.

The proof of this result can be obtained by using [5] and the nonhomogeneous Beltrami equation

$$(\phi F)_z = \mu (\phi F)_z + (\phi_z - \mu \phi_z) F.$$  

Note that (66) combined with the Poincare-Sobolev inequality gives the reverse Hölder inequalities

$$\left( \int_B |Df|^p \right)^{\frac{1}{p}} \leq C_p(K) \left( \int_{2B} |Df|^2 \right)^{\frac{1}{2}} \tag{68}$$

for all concentric disks $B \subset 2B \subset \Omega$ and all exponents $p < 2K/(K-1)$.

We have an immediate consequence for quasiregular complex gradients.

Corollary 5.1. Assume that the complex gradient $F = v_z$ of the function $v \in W^{2,\bar{q}}(\Omega)$ is weakly $K$–quasiregular. Then for every relatively compact subdomain $\Omega' \subset \Omega$ we have

$$\|DF\|_{L^p(\Omega')} \leq C\|v\|_{L^1(\Omega)} \tag{69}$$

whenever $p$, $q$ and $K$ satisfy (66). Here $C = C(q, p, \Omega, \Omega')$.

For the proof of Theorem 3.1, let us now denote by $h \in L^p(\Omega)$ the function

$$h = \frac{1}{Tr A} \mathcal{L}_A u = 2u_z + \mu(z) u_{zz} + \overline{\mu(z)} u_{\overline{z}z} \tag{70}$$

We extend $\mu$ and $h$ to the entire complex plane $\mathbb{C}$ by defining them to be 0 outside of $\Omega$. We solve the auxiliary equation at (44) with $H = h \chi_\Omega$. From (51) we see that $U \in W^{2,p}_{\text{loc}}(\mathbb{C})$. The function $U - u \in W^{2,q}_{\text{loc}}(\Omega)$ solves the homogeneous equation

$$\mathcal{L}_A(U - u) = 0, \quad \text{in } \Omega \tag{71}$$

Therefore, by Lemma 4.1 its complex gradient $F = U_z - u_z$ satisfies $F_\overline{z} = -\frac{1}{2}(\mu F_z + \overline{\mu} F_{\overline{z}})$ and so is a weakly $K$–quasiregular mapping defined in $\Omega$. Corollary 5.1 tells us that in fact $F \in W^{1,2}_{\text{loc}}(\Omega)$, together with the estimate

$$\|U_z - u_z\|_{L^p(\Omega')} + \|U_{\overline{z}z} - u_{\overline{z}z}\|_{L^p(\Omega')} \leq C\|U - u\|_{L^1(\Omega)}$$
Hence the triangle inequality will imply
\[ \|D^2 u\|_{L^p(\Omega')} \leq C(\|u\|_{L^1(\Omega)} + \|U\|_{L^1(\Omega)} + \|D^2 U\|_{L^p(\Omega)}) \]
The two last terms here are controlled by
\[ \|H\|_{L^p(\mathbb{C})} = \|h\|_{L^p(\Omega)} \leq 2K\|\mathcal{L}_A u\|_{L^p(\Omega)} \]
because of (51) and (52). This completes the proof of Theorem 3.1.

\[ \square \]

5.4 Proof of Theorems 3.2, 3.3 and 3.4

Lemma 4.1 reduces the proof of Theorem 3.2 to the area distortion bounds [3].

For the proof of Theorem 3.3 we continue with the notation of the previous subsection. What we need to do here is to show that
\[ U - u \in W^{2,p}_{\text{loc}}(\Omega) \]
We are able to assume the complex gradient \( F = U_z - u_z \) is a locally bounded mapping of class \( W^{1,p}_{\text{loc}}(\Omega \setminus E) \), for \( 2K/(K+1) \leq p < 2K/(K-1) \). Moreover the complex gradient \( F \) is a weakly quasiregular mapping in \( \Omega \setminus E \), and so in fact \( K \)-quasiregular in \( \Omega \setminus E \). It was proven in [3] that \( E \) is removable if its Hausdorff dimension is less than \( (1/2)(K+1) \). The assumption that the Hausdorff measure \( H^{2/(K+1)}(E) = 0 \) is of course weaker, but still enough to conclude that \( E \) is removable, [4]. Thus \( F \) is \( K \)-quasiregular in \( \Omega \), and in particular \( F \in W^{1,p}_{\text{loc}}(\Omega) \) which in turn implies \( U - u \in W^{2,p}_{\text{loc}}(\Omega) \) as desired.

\[ \square \]

Concerning Theorem 3.4 we use the same lines of reasoning as above and need only establish that the weaker assumption \( F \in \mathcal{L}^s_{\text{loc}}(\Omega) \), \( s > 2K \), and the stronger condition at (15) are enough to force the removability of \( E \). This is shown in [15, Corollary 17.3.1].

\[ \square \]

5.5 Proof of the Pucci Conjecture, Theorem 3.5

Finally we come to the proof of our main result. There is no loss of generality in assuming that \( p \) lies in the critical interval. As before we solve the auxiliary equation at (44) with \( H = (\mathcal{L}_A u)\chi_U \) for a function \( U \in W^{2,p}_{\text{loc}}(\mathbb{C}) \). The difference \( U - u \) solves in \( \Omega \) the homogeneous equation
\[ \mathcal{L}_A(U - u) = 0 \]
and therefore is a \( W^{2,2}_{\text{loc}}(\Omega) \)-function by Theorem 3.1. We are now in the situation where we may apply Alexandrov’s maximum principle, Theorem 1.1. Accordingly,
\[ \max_{\Omega'} |U - u| \leq \max_{\partial \Omega'} |U - u| \]
from which the triangle inequality yields
\[ \max_{\Omega'} |u| \leq \max_{\partial \Omega'} |u| + \max_{\partial \Omega'} |U| + \max_{\Omega'} |U| \leq \max_{\partial \Omega'} |u| + 2\max_{\Omega'} |U| \]
Of course what remains is to estimate the last term here by quantities involving $L_A u$. Recall that $U$ is defined by (45) and (49). From Hölder’s inequality, with $q = p/(p - 1)$ we have

$$
\max_{\Omega} |U| \leq \frac{2}{\pi} \|\omega\|_{s^p(\Omega)} \max_{z \in \Omega} \left\| \log \frac{|z - \tau|}{\text{diam}(\Omega)} \right\|_{s^q(\Omega, d\tau)}
$$

where from (52),

$$
\|\omega\|_{s^p(\Omega)} \leq C_p \|H\|_{s^p(\Omega)} = C_p \|L_A u\|_{s^p(\Omega)}
$$

We therefore arrive at the inequality at (17) with the constant

$$
C_p(\Omega) = \max_{z \in \Omega} \left( \int \int_{\Omega} \log \frac{|z - \tau|}{\text{diam}(\Omega)} d\tau \right)^{1/q}
$$

Analysis of this constant is of independent interest. Because of the homogeneity we can certainly assume that $|\Omega| = 1$ by rescaling $z$ and $\tau$ above. The isodiametric inequality shows

$$(\text{diam}(\Omega))^2 \geq \frac{4}{\pi} > 1$$

and thus $\log \frac{\text{diam}(\Omega)}{|z - \tau|}$ increases when $|z - \tau|$ decreases. We can apply the method of symmetrization to identify that the integral at (73) assumes its largest value when $\Omega$ is a disk of area 1. Then

$$
C_p(\Omega) \leq \left( \int \int_{\pi|\zeta| < 1} \left( \log \frac{\text{diam}(\Omega)}{|\zeta|} \right)^q d\zeta \right)^{1/q}
$$

$$
\leq \left( \int \int_{\pi|\zeta| < 1} \log(\text{diam}(\Omega))^q d\zeta \right)^{1/q} + \left( \int \int_{\pi|\zeta| < 1} \log^q \frac{1}{|\zeta|} d\zeta \right)^{1/q}
$$

$$
\leq \log(\text{diam}(\Omega)) + C_q \leq C \log(\text{diam}(\Omega)) \leq C \log \frac{(\text{diam}(\Omega))^2}{|\Omega|}
$$

which, when the normalisation of area is removed, gives us the constant in the statement of the theorem. □

### 6 Equations with Lower Order Terms

Alexandrov’s estimate at (5) actually holds when lower order terms are added to the operator $L_A$. In particular let us consider the operator

$$
Q_A u = L_A u + \langle b, \nabla u \rangle
$$

defined for $u \in \mathcal{H}^{2,p}(\Omega)$, where $p > 2K/(K+1)$. Here the vector valued function $b(z) = (b_1(z), b_2(z))$ is assumed to lie in the space $\mathcal{L}^2(\Omega)$. For the classical case $p \geq 2$ and $s = 2$ the generalisation of (5) reads as

$$
\max_{\Omega} |u| \leq \max_{\partial \Omega} |u| + C (\text{diam}(\Omega)) \|Q_A u\|_{s^2(\Omega)}
$$
see [11]. We shall then consider the case $2K/(K + 1) < p < 2$. Note that by the embedding theorem $\nabla u \in L^{2p/(2-p)}_{loc}(\Omega)$ so that in fact

$$\langle b, \nabla u \rangle \in L^{r}_{loc}(\Omega), \quad r = \frac{2sp}{2p + 2s - sp}$$

as $\frac{1}{s} + \frac{2-p}{2p} = \frac{1}{r}$. As we are going to consider the function $L_{\lambda}u$ in the $L^p(C)$ space it is natural to assume that $s \geq 2$. We shall see in a moment that the case $s = 2$ is not sufficient for our arguments below. Thus we shall assume henceforth

$$s > 2, \quad \text{which implies } r > p \quad (76)$$

We now have the following more general answer to the Pucci conjecture.

**Theorem 6.1.** Let $u \in \mathcal{H}^{2,p}(\Omega)$, $p > \frac{2K}{K+1}$. Then for every relative compact subdomain $\Omega' \subset \Omega$ we have

$$\max_{\Omega'} |u| \leq \max_{\partial\Omega'} |u| + C_p(\Omega)\|Q_{\lambda}u\|_{L^p(\Omega)} \quad (77)$$

where

$$C_p(\Omega) \leq C(p)|\Omega|^{1-1/p} \log \left(\frac{(\text{diam } \Omega)^2}{|\Omega|}\right) \leq C(p)(\text{diam } \Omega)^{2-2/p} \quad (78)$$

**Proof.** There is no loss of generality in assuming that $Tr A = 1$, see Lemma 4.1. Then we are led to consider the equation

$$h = 2u_{zz} + \mu u_{z\bar{z}} + \bar{\mu} u_{\bar{z}z} + b u_z + \bar{b} \bar{u}_z$$

where we have chosen to write $b = b_1 + ib_2$. This equation defines the function $h \in L^p(C)$, once $u$ is given.

As before we consider $h$, $\mu$ and $b$ to be defined in $C$ by setting them to be zero outside $\Omega$. We now solve the auxiliary equation

$$h = 2U_{zz} + \mu U_{z\bar{z}} + \bar{\mu} \bar{U}_{\bar{z}z} + b U_z + \bar{b} \bar{U}_z \quad (79)$$

in the plane $C$ for a function $U \in \mathcal{H}^{2,p}_{loc}(C)$ of the form

$$U(z) = \frac{1}{2\pi} \int \int_{\Omega} \omega(\tau) \log \frac{|z - \tau|}{(\text{diam } \Omega)} \, d\tau \quad (80)$$

where the density $\omega \in L^p(\Omega)$ is to be found. That this is indeed possible we shall see in a moment. Meanwhile, consider the solution $U - u \in \mathcal{H}^{2,p}(\Omega)$ of the homogeneous equation $Q_{\lambda}(U - u) = 0$. Its complex gradient $F = (U - u)_z$ satisfies the equation

$$2F_{\bar{z}} + \mu F_z + \bar{\mu} \bar{F}_z + b F + \bar{b} \bar{F} = 0$$

which we claim implies $F \in \mathcal{H}^{1,2}_{loc}(\Omega)$. To see this we may assume that $2 < s < 2K/(K-1)$. We proceed by induction to show that

$$F \in \mathcal{H}^{1,p}_{loc}(\Omega) \quad (81)$$
with \( p_0 = p \) and
\[
p_n = \frac{2sp_{n-1}}{2s + 2p_{n-1} - sp_{n-1}}
\]
for \( n = 1, 2, \ldots, N + 1 \), where \( N \) is chosen so that
\[
\frac{(2 - p)s}{(s - 2)p} - 1 \leq N < \frac{(2 - p)s}{(s - 2)p}
\]
Note that \( p_0 < p_1 < \cdots < p_N < 2 \leq p_{N+1} < 2K/(K - 1) \). We need to explain the induction step. Suppose \( F \in W^{1,p_n}_\text{loc}(\Omega) \subset L^{2^{p_n}/(2 - p_n)}_\text{loc}(\Omega) \). We may use Hölder’s inequality to infer that \( bF \in L^{p_{n+1}}_\text{loc}(\Omega) \) while by Theorem 5.1 we have \( F \in W^{1,p_{n+1}}_\text{loc}(\Omega) \) as desired.

For later use let us remark that if we begin the induction with \( p_0 \) slightly smaller than \( p \) then we achieve \( p_{N+1} \neq 2 \), when necessary.

In conclusion, we have shown that \( U - u \in W^{2,2}_\text{loc}(\Omega) \). Therefore we may use Alexandrov’s theorem in this setting to get
\[
\max_{\Omega'}|U - u| \leq \max_{\partial\Omega'}|U - u|
\]
As before the triangle inequality and (80) give
\[
\max_{\Omega'}|u| \leq \max_{\partial\Omega'}|u| + 2\max_{\Omega'}|U| \\
\leq \max_{\partial\Omega'}|u| + C_p(\Omega)\|\omega\|_{L^p(\Omega)}
\]
as desired.

We now return to discuss the issue of solving equation (79). The question reduces to the invertibility for the operator
\[
\mathcal{F} = 2I + \mu \mathcal{S} + \bar{\mu} \bar{\mathcal{S}} + b \mathcal{C} + \bar{b} \bar{\mathcal{C}}
\]
where \( \mathcal{C} \) denotes the usual Cauchy transform. It is important to note that the operator
\[
b \mathcal{C} : \mathcal{L}^p(\mathbb{C}) \to \mathcal{L}^p(\mathbb{C})
\]
is compact because \( b \) vanishes outside the bounded set \( \Omega \). As \( \mathcal{B} \) defined at (41) is invertible we see that \( \mathcal{F} \) is Fredholm of index zero. What we need to ensure the invertibility of \( \mathcal{F} : \mathcal{L}^p(\mathbb{C}) \to \mathcal{L}^p(\mathbb{C}) \) is to show that the kernel of \( \mathcal{F} \) is trivial, \( \text{Ker}(\mathcal{F}) = 0 \). Equivalently, we must show that the equation
\[
2g_z + \mu g_z + \bar{\mu} \bar{g}_z + bg + \bar{b} \bar{g} = 0
\]
where \( g = \mathcal{C} \omega \) and \( \omega \in \mathcal{L}^p(\Omega) \), admits only the trivial solution \( g \equiv 0 \).

Assuming that we have a solution to (82), the derivative \( g_z \) has compact support and just as we established (81) one can see that
\[
g \in W^{1,p_{N+1}}_\text{loc}(\mathbb{C}) \cap \mathcal{C}_0(\mathbb{C}), \quad \text{with } p_{N+1} > 2.
\]
Moreover, we have the differential inequality
\[ |g| \leq \frac{K - 1}{K + 1} |g| + |g(z)| \]
where \( \sigma(z) \leq |b(z)| \in \mathcal{L}^{2+\varepsilon}(\mathbb{C}) \cap \mathcal{L}^{2-\varepsilon}(\mathbb{C}) \), for some \( 0 < \varepsilon < 1 \), since \( b \) has compact support. Finally using the version of the Liouville Theorem given in [4, Theorem 10.3.1] we conclude that \( g = 0 \) as desired. \( \square \)

The assumption \( s > 2 \) was needed at several stages of the previous proof. As an example, the reader may wish to verify that the \( \mathcal{W}^{1,2}_{loc} \)-regularity for the equation \( F_{z} + \sigma F = 0 \) with only \( \sigma \in \mathcal{L}^2_{loc}(\Omega) \) fails even if \( F \in \mathcal{W}^{1,p}_{loc}(\Omega) \) for all \( 1 < p < 2 \). Indeed, consider
\[
F(z) = (1 - 2 \log |z|)^{\frac{1}{2}} \quad \text{and} \quad \sigma(z) = \frac{1}{2\pi(1 - 2 \log |z|)}
\]
in the unit disk \( \mathbb{D} \). We find that both \( F_{z} = -1/(2zF) \) and \( F_{z} = -1/(2\pi F) \) lie in every \( \mathcal{L}^p(\mathbb{D}) \), \( 1 < p < 2 \), but not in \( \mathcal{L}^2(\mathbb{D}) \).

### 6.1 Solution to the Dirichlet Problem

We shall now give the proof of Theorem 3.6. Suppose we are given a function \( u \in \mathcal{W}^{2,p}(\mathbb{D}) \subset \mathcal{C}(\mathbb{D}) \) that vanishes on the boundary \( \partial \mathbb{D} \). We consider the differential operator
\[
\mathcal{L}_u u = 2u_{zz} + \mu_{zz} u_{zz} + \mu u_{zz} + bu_z + \overline{b}u_z \overset{\text{def}}{=} h \quad \text{(83)}
\]
defining the function \( h \). We look for estimates for \( D^2u \) in terms of \( h \).

We make the following assumptions on the coefficients,
\[
|\mu(z)| \leq k < 1 \quad \text{almost everywhere on } \mathbb{D} \quad \text{and} \quad b \in \mathcal{L}^s(\mathbb{D}) \quad \text{for some } s > 2. \quad \text{(84, 85)}
\]

To make use of the global estimates developed in the previous sections we extend \( u \) outside \( \mathbb{D} \) by reflection, and set
\[
U(z) = u(z) \text{ for } |z| \leq 1 \quad \text{and} \quad U(z) = -u(z^{-1}) \text{ for } |z| \geq 1.
\]

Clearly \( U \in \mathcal{W}^{2,p}_{loc}(\mathbb{C}) \). A calculation shows that the extended function \( U \) solves the following equation in the entire complex plane \( \mathbb{C} \),
\[
2U_{zz} + \mu_{zz} U_{zz} + \mu U_{zz} + b^* U_z + \overline{b}U_z = h^*,
\]
where \( \mu^*(z) = \mu(z) \) for \( |z| \leq 1 \) and \( \mu^*(z) = z^2 \mathcal{P}^{-2} \mathcal{P}(z^{-1}) \) for \( |z| \geq 1 \). Thus in particular,
\[
|\mu^*(z)| \leq k < 1 \quad \text{almost everywhere on } \mathbb{C}.
\]
The function $b^*$ is defined by

$$b^*(z) = b(z) \text{ for } |z| \leq 1 \quad \text{and} \quad b^*(z) = (2z/\bar{z}^2)\overline{\mathfrak{p}}(1/\bar{z}) - \bar{z}^{-2} \text{ for } |z| \geq 1.$$ 

while the nonhomogeneous part attains the form

$$h^*(z) = h(z) \text{ for } |z| \leq 1 \quad \text{and} \quad h^*(z) = -|z|^{-4}h(\bar{z}^{-1}) \text{ for } |z| \geq 1.$$ 

The point to make here is that we have a uniform control of the norms of $U$ over the double disk $2\mathbb{D}$ in terms of $u(z), z \in \mathbb{D}$.

The first order equation for the complex gradient $F(z) = U_z(z)$ takes now the form

$$2F_z + \mu^* F_z + \overline{\mu}^* F_z + b^* F + \overline{b}^* \overline{F} = h^*.$$ 

(86)

To use the global estimates here fix a nonnegative bump function $\phi \in \mathcal{C}_0^\infty(2\mathbb{D})$ such that $\phi \equiv 1$ on $\mathbb{D}$. Multiply then (86) by $\phi$ to obtain

$$2(\phi F)_z + \mu^* (\phi F)_z + \overline{\mu}^* (\overline{\phi} F) + b^* \phi F + \overline{b}^* \overline{\phi} F$$

$$= \phi h^* + 2\phi_2 F + \mu^* \phi_2 F + \overline{\mu}^* \overline{\phi}_2 F$$

(87)

(88)

It makes no difference if we redefine $b^*$ as equal to zero outside $2\mathbb{D}$. As shown in the proof of Theorem 6.1 the operator

$$\mathcal{F} = 2I + \mu^* S + \overline{\mu}^* \overline{S} + b^* C + \overline{b}^* \overline{C} : \mathcal{L}^p(\mathbb{C}) \rightarrow \mathcal{L}^p(\mathbb{C})$$

(89)

is then invertible and we have the $\mathcal{L}^p$-estimate

$$\|D(\phi F)\|_{\mathcal{L}^p(\mathbb{C})} \leq C\|\phi h^*\|_{\mathcal{L}^p(\mathbb{C})} + C\|\nabla \phi\| \|F\|_{\mathcal{L}^p(\mathbb{C})}$$

for all $p$ in the critical interval. Hence $\|DF\|_{\mathcal{L}^p(\mathbb{D})} \leq C\|h^*\|_{\mathcal{L}^p(2\mathbb{D})} + C\|F\|_{\mathcal{L}^p(2\mathbb{D})}$, which in turn reads in terms of $u$ as

$$\|D^2 u\|_{\mathcal{L}^p(\mathbb{D})} \leq C\|h\|_{\mathcal{L}^p(\mathbb{D})} + C\|\nabla u\|_{\mathcal{L}^p(\mathbb{D})}.$$ 

(90)

We now invoke the well-known interpolation inequality

$$\|\nabla u\|_{\mathcal{L}^p(\mathbb{D})} \leq \varepsilon\|D^2 u\|_{\mathcal{L}^p(\mathbb{D})} + C_\varepsilon\|u\|_{\mathcal{L}^p(\mathbb{D})}$$

Choosing $\varepsilon$ sufficiently small the term with the second derivatives is absorbed by the left hand side of (90) and we arrive at the estimate

$$\|D^2 u\|_{\mathcal{L}^p(\mathbb{D})} \leq C\|h\|_{\mathcal{L}^p(\mathbb{D})} + C\|u\|_{\mathcal{L}^p(\mathbb{D})}.$$ 

Finally by using Theorem 6.1 we see that $\|u\|_{\mathcal{L}^p(\mathbb{D})}$ is dominated by $\|h\|_{\mathcal{L}^p(\mathbb{D})}$. We conclude with the a priori estimate

$$\|D^2 u\|_{\mathcal{L}^p(\mathbb{D})} \leq C\|h\|_{\mathcal{L}^p(\mathbb{D})}.$$ 

(91)

This result has an interesting interpretation. To this effect, given $w \in \mathcal{L}^p(\mathbb{D})$ consider the logarithmic potential

$$u(z) = \frac{2}{\pi} \int_{\mathbb{D}} \int_{\mathbb{D}} \log \left( \frac{|z - \tau|}{|1 - z\bar{\tau}|} \right) w(\tau) d\tau$$

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Then \( u \in \mathcal{C}(\mathbb{D}) \) and \( u \) vanishes on the boundary \( \partial \mathbb{D} \). The complex gradient is given by

\[
u_z(z) = (C_D w)(z) \overset{\text{def}}{=} \frac{1}{\pi} \iint_D \left[ \frac{1}{z - \tau} + \frac{\tau}{1 - z\bar{\tau}} \right] w(\tau) d\tau
\]

while the second derivatives are

\[
u_{zz} = (S_D w)(z) \overset{\text{def}}{=} -\frac{1}{\pi} \iint_D \left[ \frac{1}{(z - \tau)^2} - \frac{\tau^2}{(1 - z\bar{\tau})^2} \right] w(\tau) d\tau, \]

Estimate (91) tells us that \( \|w\|_{L^p} \leq C \|h\|_{L^p} \), when \( h \) is defined through (83). In other words, the operator

\[
F_D = 2I + \mu S_D + \bar{\mu} S_D + bC_D + \bar{b}C_D
\]

is invertible in \( L^p(\mathbb{D}) \), for all \( p \) in the critical interval \( \frac{2K}{K+1} < p < \frac{2K}{K-1} \). Of course, Theorem 3.6 is an immediate consequence of this fact. □

At this point the reader may care to go on and look at questions of removable singularities and so forth in this more general setting using the methods and ideas outlined above. We are content to finish by presenting Pucci’s example next.

### 7 Pucci’s Example

In this section we give the example of Pucci [20] which exhibits sharpness in the theorems we have stated, with regard to the critical exponent. Our domain will be the unit disk \( \mathbb{D} \). To begin with we put for \( z = x + iy \)

\[
A(z) = \frac{z \otimes z}{|z|^2} \left( \sqrt{K} - \frac{1}{\sqrt{K}} \right) + \frac{1}{\sqrt{K}} I, \quad z \otimes z = \begin{bmatrix} x^2 & xy & y^2 \\ xy & y^2 & x^2 \end{bmatrix}
\]

An elementary calculation reveals the ellipticity bounds

\[
\frac{1}{\sqrt{K}} \leq \langle A(z)\zeta, \zeta \rangle \leq \sqrt{K}, \quad |\zeta| = 1
\]

Let \( u(z) = \varphi(|z|) \) be any radial function of Sobolev class \( W^{2,2}_{\text{loc}}(\Omega) \). Then

\[
\mathcal{L}_A u = \text{Tr}[A(z)D^2 u] = \sqrt{K} \varphi''(|z|) + \frac{1}{\sqrt{K}} \frac{\varphi'(|z|)}{|z|}
\]

One then makes a careful choice of the function \( \varphi \). Let first \( K > 1 \) and set

\[
\varphi(r) = \begin{cases} (\log^r)r^{1-1/K} + \left( \log N - \frac{K}{K-1} \right) (r^{1-1/K} - 1), & \text{if } \frac{1}{N} \leq r < 1 \\ -\log N + \frac{K}{K-1}(1 - N^{-1+1/K}), & \text{if } 0 \leq r < \frac{1}{N} \end{cases} \quad (94)
\]

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We compute that
\[
\varphi'(r) = \begin{cases} 
(1 - 1/K) \log(Nr)r^{-1/K}, & \text{if } \frac{1}{N} \leq r < 1 \\
0, & \text{if } 0 \leq r < \frac{1}{N}
\end{cases}
\]
Thus our function \( u(z) = \varphi(|z|) \) has

- \( u \in W^{2,\infty}(D) \)
- \( u|_{\partial D} = 0 \)
- \( \mathcal{L}_u h = \left( \sqrt{K} - \frac{1}{\sqrt{K}} \right)|z|^{-1-1/K} \chi_{1/N < |z| < 1}(x) \)
- \( \|h\|_{L^p(D)} = \left( \sqrt{K} - \frac{1}{\sqrt{K}} \right) (2\pi \log N)^{\frac{K-1}{2N}} \), when \( p = 2K/(K + 1) \)
- \( \|u\|_{L^\infty(D)} \geq \frac{1}{2} \log N \) for all sufficiently large \( N \).
- \( \|D^2u\|_{L^p(D)} \geq C(K)(\log N)^{1 + \frac{K+1}{2K}} \)

It follows that
\[
\frac{\|u\|_{L^\infty}}{\|h\|_{L^p}} \geq c(K)(\log N)^{1 - \frac{K+1}{2N}} = C(\log N)^{\frac{K-1}{2N}} \to \infty \tag{95}
\]
as \( N \to \infty \). Hence (17) fails at \( p = 2K/(K + 1) \). Similarly,
\[
\frac{\|D^2u\|_{L^p(D)}}{\|u\|_{L^\infty}} \geq c(K)(\log N)^{\frac{K+1}{2N}} \to \infty \tag{96}
\]
as \( N \to \infty \) so that (13) fails at \( q = p = 2K/(K + 1) \).

When \( K = 1 \) choose e.g. \( u(z) = \phi(|z|) \), where
\[
\varphi(r) = \begin{cases} 
\log^2(Nr) - \log^2 N & \text{if } \frac{1}{N} \leq r < 1 \\
- \log^2 N & \text{if } 0 \leq r \leq \frac{1}{N}
\end{cases}
\]
Then \( \mathcal{L}_u h = 2|z|^{-2} \chi_{1/N < |z| < 1}(z) \) and \( \|h\|_{L^1(D)} = 4\pi \log N \) while \( \|u\|_{L^\infty(D)} = \log^2 N \). Again \( \|u\|_{L^\infty}/\|h\|_{L^1} = (4\pi)^{-1} \log N \to \infty \) as \( N \to \infty \).

References


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